PCOAST: A Pauli-based Quantum Circuit Optimization Framework

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The PCOAST framework *- in a nutshell*

- **Toolchain** for Pauli-based Cirucit Optimization, Analysis and Synthesis.

- Based on **commutative properties of Pauli strings.**
  (Recall: unitary circuits can be decomposed into Clifford and non-Clifford gates represented by Pauli rotations)

- Technique is adapted to **mixed unitary and non-unitary circuits.**

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Distinguishing optimization procedures

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Local, peephole-style optimization ⟷ **Global optimizations**

Local, peephole-style optimization:

- Replace **local** patterns
- General circuit structure is **preserved**

Global optimizations:

- **Convert** circuit to intermediate mathematical representation (IR)
- **Simplify** IR according to structural rules
- **Synthesize** circuit back

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PCOAST

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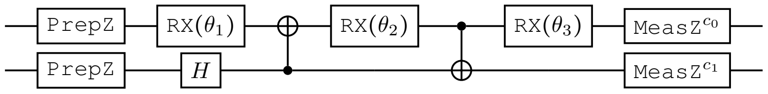
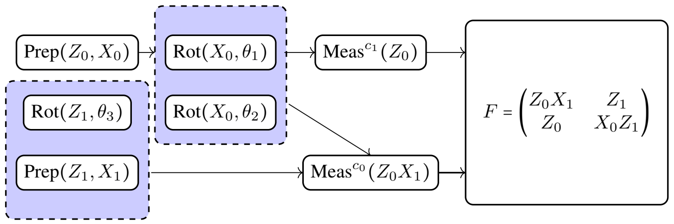
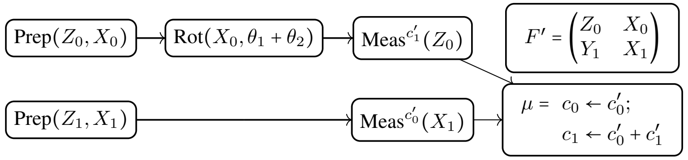
Full PCOAST optimization pass



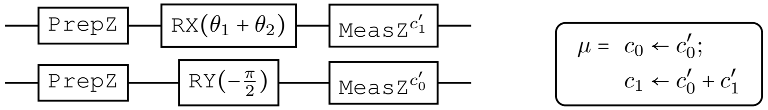
Exemplary initial circuit



Generated (compiled) PCOAST graph



Optimized PCOAST graph



Optimized released circuit which is synthesized back out

(customizable greedy search algorithm finds an efficient gate implementation for the optimized PCOAST graph)

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Classical-quantum states

→ *Operate over mixed classical-quantum states (cq-states)*

$$\gamma \in \mathbf{CQ}^{\mathcal{M}} = \mathcal{M} \to \mathcal{C}^n \times \mathcal{C}^n$$

$$m \mapsto \gamma_m$$

$m \in \mathcal{M}$

classical state

quantum state
partial density matrix with

$0 \leq tr(\gamma_m) \leq 1$

~ prob. of observing m

Semantics of a quantum circuit C can be described by a classical-quantum channel

$$[\![C]\!] : \mathbf{CQ}^{\mathcal{M}_1} \to \mathbf{CQ}^{\mathcal{M}_2}$$

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Optimization variants

Hold outcome:

- optimization **preserves the semantics** of the original circuit precisely

$$\gamma^1 =^{hold} \gamma^2 \iff \forall m \in \mathcal{M} \quad \gamma^1_m = \gamma^2_m$$

Release outcome:

- More **aggressive** optimization can be applied
- **Preserve** same **measurement result** during the optimization strategy
- Drop unitary gates that can be delayed after the measurement
- Produce measurement results that are **statistically equivalent**

$$\gamma^1 =^{release} \gamma^2 \iff \forall m \in \mathcal{M} \quad tr(\gamma^1_m) = tr(\gamma^2_m)$$

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Recap on density matrices

$\rho \longrightarrow U\rho U^\dagger$

Quantum state
(positive semi-definite, hermitian, complex matrix with trace 1)

Unitary transformation
(conjugation by unitary)

$\longrightarrow$ Can formulate the behaviour of a quantum circuit over density matrices as a **quantum channel**

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Pauli group

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Commutativity of a Pauli p given by:

$$\lambda(p_1, p_2) = \begin{cases} 0 & p_1 \perp\!\!\!\perp p_2 \\ 1 & p_1 \not\perp\!\!\!\perp p_2 \end{cases}$$

Generalize to n-qubit Paulis

$$P = \alpha(p_0, \ldots, p_{n-1}) \in \mathcal{P}_n \qquad \alpha \in \{1, -1, i, -i\}$$

Commutativity $\quad P \cdot P' = (-1)^{\lambda(P,P')} P' \cdot P \quad$ with $\quad \lambda(P, P') \in \{0, 1\}$

$$\lambda(\alpha(p_0, \ldots, p_{n-1}), \alpha'(p'_0, \ldots, p'_{n-1})) = \sum_{i=0}^{n-1} \lambda(p_i, p'_i) \mod 2$$

Binary function

→ $P \perp\!\!\!\perp P'$ if $\lambda(P, P') = 0$ and $P \not\perp\!\!\!\perp P'$ if $\lambda(P, P') = 1$

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Pauli tableau / Pauli frame

→ Use this compact representation to represent Cliffords

Clifford gate set = {CNOT, H, S}

$$U \in Cliffords, \qquad P \in Paulis \implies UPU^{\dagger} \in Paulis$$

j	$U^{\dagger} Z_j U$	$U^{\dagger} X_j U$	$U^{\dagger} Y_j U$
0	Z_0	$X_0 X_1$	$Y_0 X_1$
1	$Z_0 Z_1$	X_1	$Z_0 Y_1$

Compact representation of a Clifford unitary $U = CNOT_{0,1}$

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$Y = -i\, Z \cdot X$

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1	$Z_0 Z_1$	X_1	~~$Z_0 Y_1$~~

$Y = -i\, Z \cdot X$

Compact representation of a Clifford unitary $U = CNOT_{0,1}$

$$\begin{pmatrix} Z_0 & X_0 X_1 \\ Z_0 Z_1 & X_1 \end{pmatrix}$$

n - qubit Clifford is represented by a n x 2 Pauli frame

$$F = \begin{pmatrix} \mathrm{eff} Z_0 & \mathrm{eff} X_0 \\ \vdots & \vdots \\ \mathrm{eff} Z_{n-1} & \mathrm{eff} X_{n-1} \end{pmatrix}$$

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→ n - qubit Clifford is represented by a $(n\ x\ 2)$ Pauli frame

$$F = \begin{pmatrix} \mathrm{eff}Z_0 & \mathrm{eff}X_0 \\ \vdots & \vdots \\ \mathrm{eff}Z_{n-1} & \mathrm{eff}X_{n-1} \end{pmatrix}$$

↘ Respect the follwing commutativity relations:

$$\lambda(\mathrm{eff}Z_i, \mathrm{eff}Z_j) = \lambda(\mathrm{eff}X_i, \mathrm{eff}X_j) = 0 \quad \textit{commute}$$

$$\lambda(\mathrm{eff}Z_i, \mathrm{eff}X_j) = \lambda(\mathrm{eff}X_j, \mathrm{eff}Z_i) = \delta_{i,j} \quad \textit{anti-commute}$$

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→ n - qubit Clifford is represented by a $(n\ x\ 2)$ Pauli frame

A *lookup action* on F on $P = \alpha(p_0, \dots, p_{k-1})$ written $\vec{F}(P)$, is given by:

$$\overrightarrow{F}(\alpha(p_0, \dots, p_{n-1})) = \alpha \prod_j \mathrm{eff} p_j$$

with $\mathrm{eff} I_j = I$ and $\mathrm{eff} Y_j = \mathrm{eff} Z_j \odot \mathrm{eff} X_j$

For every Pauli frame F there is a Clifford unitary U^F unique up to an overall phase satisfying $\vec{F}(P) = (U^F)^\dagger\, P\, U^F$ for any Pauli P.

→ The semantics of a Pauli frame is given by the Pauli map: $[\![F]\!](Q) = U^F Q (U^F)^\dagger$

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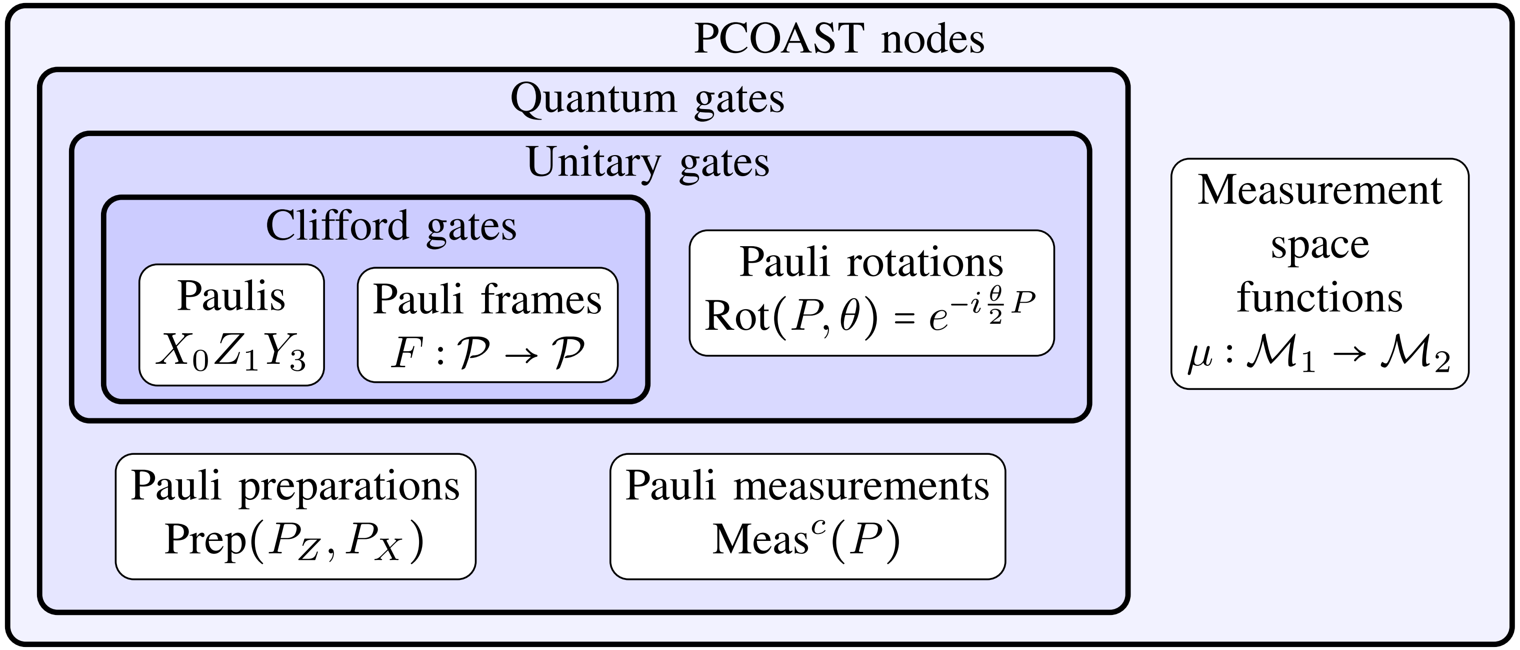
Types of PCOAST nodes and the PCOAST data structure



$$[\![n]\!] : \mathcal{P}_k \to \mathrm{CQ}$$

PCOAST nodes as Pauli maps
(can be lifted to cq-channels)

$$\mu : \mathcal{M}_1 \to \mathcal{M}_2$$

Measurement space functions are directly defined as cq-channels

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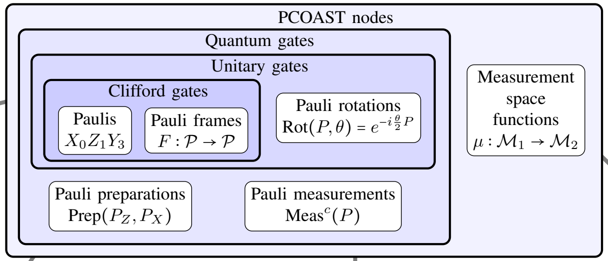
PCOAST data structures



Pauli rotation

$$[\![\mathrm{Rot}(P,\theta)]\!](Q)$$

$$= [\mathrm{Rot}(P,\theta)]Q[\mathrm{Rot}(P,-\theta)]$$

Measurement space functions

$$\mu : \mathcal{M}_1 \to \mathcal{M}_2$$

$$\longrightarrow \quad [\![\mu]\!] : \mathrm{CQ}^{\mathcal{M}_1} \to \mathrm{CQ}^{\mathcal{M}_2}$$

Pauli measurement

$$[\![\mathrm{Meas}^c(P)]\!](Q)$$

$$= m \mapsto \begin{cases} \frac{1}{4}(I+P)Q(I+P) & m[c] = 0 \\ \frac{1}{4}(I-P)Q(I-P) & m[c] = 1 \end{cases}$$

Pauli preparation

$$[\![\mathrm{Prep}(P_Z, P_X)]\!](Q)$$

$$= \frac{1}{4}\left((I+P_Z)Q(I+P_Z) + P_X(I-P_Z)Q(I-P_Z)P_X\right)$$

Pauli frames

$$F = \begin{pmatrix} \mathrm{eff}Z_0 & \mathrm{eff}X_0 \\ \vdots & \vdots \\ \mathrm{eff}Z_{n-1} & \mathrm{eff}X_{n-1} \end{pmatrix}$$

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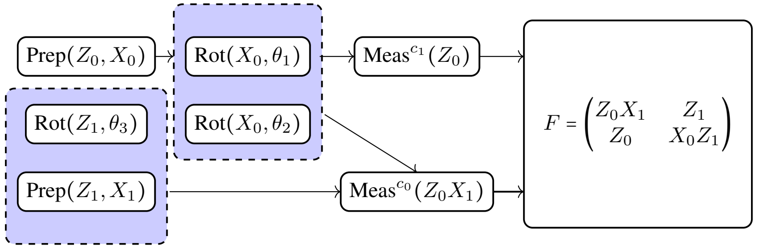
PCOAST graph

PCOAST graph $G = (V, E)$



Exemplary PCOAST graph

- Directed acyclic graph with **vertices V that are PCOAST nodes**
- For any nodes n_1 and n_2 that **do not commute**, there is a **edge** from n_1 to n_2 or vice versa
- There are **no edges between commuting vertices**

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PCOAST optimization techniques

Compiling circuits to PCOAST graphs

↓

Internal optimization on PCOAST graphs

↓

Synthesizing circuits from PCOAST graphs
(depending on whether *hold* or *release* is chosen)

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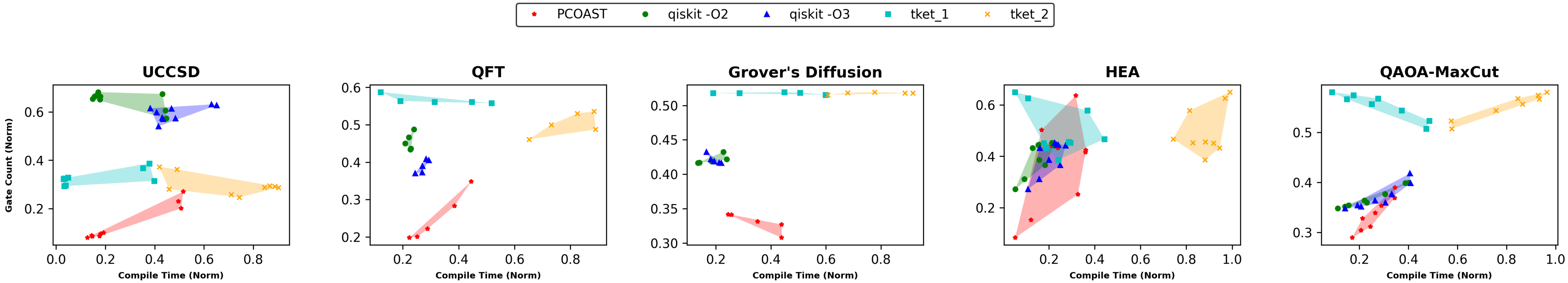
Performance Evaluation: Gate count vs. Compile time



Evaluated against Qiskit and t|ket⟩: reduces **total gate count** by 32.53% and 33.33% on average, **two-qubit gates** by 29.22% and 20.58%, and **circuit depth** by 42.02% and 51.27%.

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Summary

- **A novel optimization framework** for mixed unitary and non-unitary quantum circuits is presented.

- Adapts the **commutativity properties** of Cliffords and Pauli strings to preparation and measurement gates in the PCOAST graph.

- **Good performance** is shown by evaluation against Qiskit and t|ket⟩.

- Implemented as core optimization of the **Intel SDK**, it can be enabled via the **(-O1) flag**.

Thank you!

Are there any questions?

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