Gravitational-Wave Observables:
PM expansion vs soft theorems

Meeting on *Infrared Surprises of Scattering Amplitudes*

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Based on:

- 2306.16488: Report on the gravitational eikonal
  Paolo Di Vecchia, CH, Rodolfo Russo, Gabriele Veneziano
- 2312.07452, 2402.06361: Analysis of the NLO waveform
  Alessandro Georgoudis, CH, Rodolfo Russo
- 2406.03937: Angular momentum losses from the NLO waveform
  CH, Rodolfo Russo
- 2407.04128: Logarithmic soft theorems and soft spectra
  Francesco Alessio, Paolo Di Vecchia, CH
- 2501.02904: Radiation-Reaction and Angular Momentum Loss at $\mathcal{O}(G^4)$
  CH

Outline

Introduction

Warm-Up: Elastic Eikonal and Deflection Angle

Eikonal Operator and Gravitational Waveform

Soft Theorems, Soft Energy Spectrum, Static Angular Momentum Loss

Integrated Energy and Angular Momentum Losses

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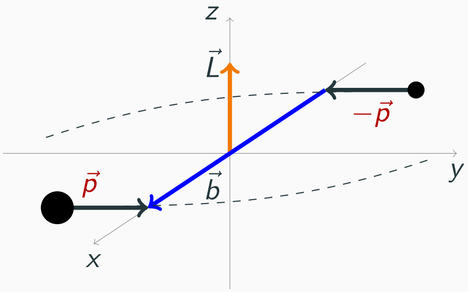
Two-Body Problem: Analytical Approximation Methods

- Post-Newtonian (PN): expansion
  "for small G and small v"

$$\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \ll 1\,.$$

- Post-Minkowskian (PM): expansion
  "for small G"

$$\frac{Gm}{rc^2} \ll 1\,, \qquad \text{generic } \frac{v^2}{c^2}\,.$$



- Self-Force: expansion
  in the near-probe limit $m_2 \ll m_1$ or

$$m = m_1 + m_2\,, \qquad \nu = \frac{m_1 m_2}{m^2} \ll 1\,.$$

- Soft limit: expansion
  in the limit of small frequencies

$$\omega \ll \frac{v}{r}\,.$$

General Relativity from Scattering Amplitudes

Key Idea: Extract the PM gravitational dynamics from scattering amplitudes.

- Weak-coupling expansion ↔ PM expansion

Weak-coupling:	$\mathcal{A}_0 = \mathcal{O}(G)$	$\mathcal{A}_1 = \mathcal{O}(G^2)$	$\mathcal{A}_2 = \mathcal{O}(G^3)$	$\mathcal{A}_3 = \mathcal{O}(G^4)$
PM:	1PM	2PM	3PM	4PM State of the art: [[Driesse et al. '24; Bern et al. '24] 5PM, 1SF from WQFT]

- Lorentz invariance ↔ generic velocities
- Study scattering events, then export to bound trajectories (V_{eff}, analytic continuation...) [Kälin, Porto '19; Saketh, Steinhoff, Vines, Buonanno '21; Cho, Kälin, Porto '21]

Classical Soft Theorems

[Sahoo, Sen '18; Saha, Sahoo, Sen '19; Sahoo '20; Sahoo, Sen '21; Sahoo, Krishna '23]

- Universal constraints on the soft expansion $\omega \to 0$ of the gravitational waveform:

$$\tilde{w}^{\mu\nu} = -\frac{i}{\omega}\, \omega^{2iGE\omega} \sum_{n=0}^{\infty} \frac{1}{n!} (-i\omega \log \omega)^n \, a_n^{\mu\nu} + \cdots$$

where $\cdots \sim \omega^{n-1}(\log \omega)^m$ and $0 \leq m \leq n-1$, e.g. ω^0 or $\omega \log \omega$.

- Explicitly, for $\frac{1}{\omega}$, $\log \omega$, $\omega(\log \omega)^2$,

$$a_0^{\mu\nu} = \sum_a \frac{p_a^\mu p_a^\nu}{p_a \cdot n}\,, \quad a_1^{\mu\nu} = G \sum_{a,b} \frac{\tau_{ab}^{(\eta)} p_a^\mu}{p_a \cdot n} n_\rho p_{[b}^\rho p_{a]}^\nu\,, \quad a_2^{\mu\nu} = G^2 \sum_{a,b,c} \frac{\tau_{ab}^{(\eta)} \tau_{ac}^{(\eta)}}{p_a \cdot n} n_\rho p_{[b}^\rho p_{a]}^\mu n_\sigma p_{[c}^\sigma p_{a]}^\nu$$

and $\tau_{ab}^{(\eta)}$ is a function of the **invariants** $\sigma_{ab} = -\eta_a \eta_b p_a \cdot p_b/(m_a m_b)$ (with $\eta_a = +$ if the hard state is outoing, -1 if it is incoming)

$$\tau_{ab}^{(\eta)} = |\eta_a + \eta_b|\, \tau_{ab}\,, \qquad \tau_{ab} = -\frac{\sigma_{ab}(\sigma_{ab}^2 - \frac{3}{2})}{(\sigma_{ab}^2 - 1)^{3/2}} \quad \text{for GR}\,.$$

In general, it is fixed by the IR divergences of the theory.

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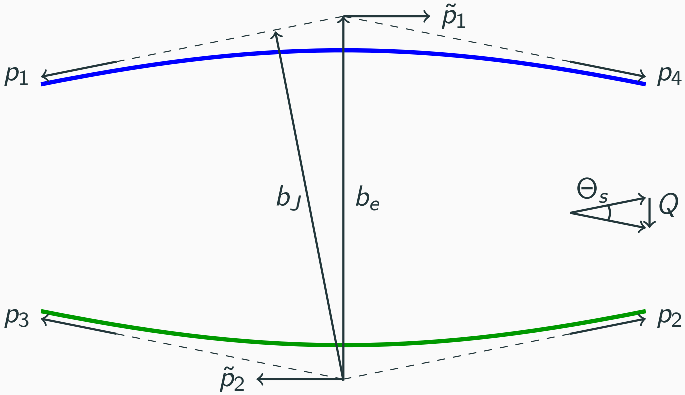
Kinematics of Classical Post-Minkowskian (PM) Scattering

$$\tilde{p}_1^\mu = m_1 \tilde{u}_1^\mu = \frac{1}{2}(p_4^\mu - p_1^\mu)$$

$$\tilde{p}_2^\mu = m_2 \tilde{u}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$

$$Q^\mu = p_1^\mu + p_4^\mu = -p_2^\mu - p_3^\mu$$

$$b_e^\mu = b_J^\mu - \left(\frac{\check{v}_1^\mu}{2m_1} - \frac{\check{v}_2^\mu}{2m_2}\right) Qb$$

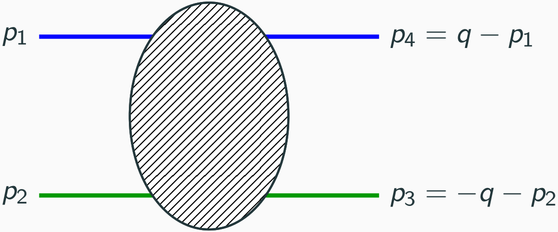


In this way, $v_1 \cdot b_J = v_2 \cdot b_J = 0$ and $\tilde{u}_1 \cdot b_e = \tilde{u}_2 \cdot b_e = 0$. Classical PM regime:

$$\frac{Gm^2}{\hbar} \underset{CL}{\gg} 1\,, \qquad \frac{Gm}{b} \underset{\text{PM}}{\ll} 1\,, \qquad \boxed{\frac{\hbar}{m} \ll Gm \ll b} \qquad \sigma = \frac{1}{\sqrt{1-v^2}} \geq 1 \text{ (generic)}.$$

Kinematics of the Elastic $2 \to 2$ Amplitude

$$\bar{p}_1^\mu = \frac{1}{2}(p_4^\mu - p_1^\mu)$$
$$\bar{p}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$
$$\boxed{q^\mu} = p_1^\mu + p_4^\mu = -p_2^\mu - p_3^\mu$$



Defining velocities by $p_1^\mu = -m_1 v_1^\mu$, $p_2^\mu = -m_2 v_2^\mu$

$$\boxed{\sigma} = -v_1 \cdot v_2 = \frac{1}{\sqrt{1 - v^2}}$$

with v the speed of either object as measured by the other one.

Dual velocities: $v_1^\mu = \sigma \check{v}_2^\mu + \check{v}_1^\mu$, $v_2^\mu = \sigma \check{v}_1^\mu + \check{v}_2^\mu$ obey $\check{v}_i \cdot v_j = -\delta_{ij}$.

The Elastic Eikonal

- From q to b: Fourier transform $[q \sim \mathcal{O}(\frac{\hbar}{b})]$

$$\tilde{\mathcal{A}}^{(4)}(b) = \frac{1}{4Ep} \int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{ib\cdot q} \mathcal{A}^{(4)}(q)\,, \qquad \boxed{1 + i\tilde{\mathcal{A}}^{(4)}(b) = e^{2i\delta(b)}}$$

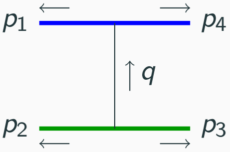
with $2\delta = 2\delta_0 + 2\delta_1 + 2\delta_2 + \cdots \sim \frac{Gm^2}{\hbar}\left(\log b + \frac{Gm}{b} + \left(\frac{Gm}{b}\right)^2 + \cdots\right)$

- From b to Q: stationary-phase approximation $[Q \sim \mathcal{O}(p \cdot \frac{Gm}{b})]$

$$\int d^{D-2}b\, e^{-ib\cdot Q} e^{i2\delta(b)} \implies Q_\mu = \frac{\partial \,\mathrm{Re}\, 2\delta}{\partial b_e^\mu}$$

Tree-Level Amplitude and 1PM Impulse

- Tree-level amplitude in $D = 4 - 2\epsilon$ dimensions



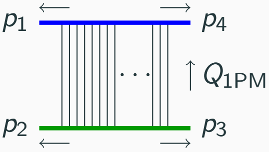
$$\mathcal{A}_0^{(4)}(q) = \frac{32\pi G m_1^2 m_2^2 (\sigma^2 - \frac{1}{2-2\epsilon})}{q^2} + \cdots$$

$$\tilde{\mathcal{A}}_0^{(4)}(b) = \frac{4 G m_1 m_2 (\sigma^2 - \frac{1}{2-2\epsilon})}{2\sqrt{\sigma^2 - 1}} \frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}} .$$

- Matching to the eikonal exponentiation [Kabat, Ortiz '92; Bjerrum-Bohr et al. '18]

$$e^{2i\delta_0} \xrightarrow[\text{“small } G\text{”}]{} 1 + i\tilde{\mathcal{A}}_0^{(4)} \implies 2\delta_0 = \tilde{\mathcal{A}}_0^{(4)} .$$

- From $2\delta_0$, we obtain the leading-order deflection

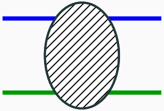


$$Q_{1\text{PM}} = -\frac{\partial 2\delta_0}{\partial b} = \frac{4 G m_1 m_2 \left(\sigma^2 - \frac{1}{2}\right)}{b\sqrt{\sigma^2 - 1}}$$

$$\Theta_{1\text{PM}} = \frac{4GE\left(\sigma^2 - \frac{1}{2}\right)}{b(\sigma^2 - 1)} .$$

Impulse from the Eikonal Phase up to One Loop

[Collado, Di Vecchia, Russo '19]

$1 + i\,\mathrm{FT}$  $\sim e^{2i\delta}\,, \qquad 2\delta = 2\delta_0 + 2\delta_1 + \cdots \qquad Q_\mu = \dfrac{\partial 2\delta}{\partial b_e^\mu}$

- Tree level: $i\tilde{\mathcal{A}}_0 = 2i\delta_0$, so

$$2\delta_0 = \tilde{\mathcal{A}}_0^{(4)} = \frac{2Gm^2\nu(\sigma^2 - \frac{1}{2-2\epsilon})}{\sqrt{\sigma^2-1}}\frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}}\,, \qquad Q^\mu_{\mathrm{1PM}} = -\frac{4Gm^2\nu(\sigma^2-\frac{1}{2})}{b\sqrt{\sigma^2-1}}\frac{b_e^\mu}{b}\,.$$

- One loop: By the unitarity, $i\tilde{\mathcal{A}}_1 - \frac{1}{2!}(2i\delta_0)^2 = i\,\mathrm{Re}\,\tilde{\mathcal{A}}_1 = 2i\delta_1$, so

$$2\delta_1 = \mathrm{Re}\,\tilde{\mathcal{A}}_1^{(4)} = \frac{3\pi G^2m^3\nu\left(5\sigma^2-1\right)}{4b\sqrt{\sigma^2-1}}\,, \qquad Q^\mu_{\mathrm{2PM}} = -\frac{3\pi G^2m^3\nu\left(5\sigma^2-1\right)}{4b^2\sqrt{\sigma^2-1}}\frac{b_e^\mu}{b}\,.$$

The 3PM Eikonal in General Relativity [Di Vecchia, CH, Russo, Veneziano '20, '21]

[Related work at 3PM: Bern, Cheung, Roiban, Shen, Solon, Zeng '19; Damour '20; Herrmann, Parra-Martinez, Ruf, Zeng '21, Bjerrum-Bohr, Damgaard, Planté, Vanhove '21; Brandhuber, Chen, Travaglini, Wen '21]

- Eikonal phase:

$$\mathrm{Re}\,2\delta_2 = \frac{4G^3 m_1^2 m_2^2}{b^2}\left[\frac{s\left(12\sigma^4-10\sigma^2+1\right)}{2m_1 m_2\left(\sigma^2-1\right)^{\frac{3}{2}}} - \frac{\sigma\left(14\sigma^2+25\right)}{3\sqrt{\sigma^2-1}} - \frac{4\sigma^4-12\sigma^2-3}{\sigma^2-1}\,\mathrm{arccosh}\sigma\right] + \mathrm{Re}\,2\delta_2^{\mathrm{RR}}\,,$$

$$\mathrm{Re}\,2\delta_2^{\mathrm{RR}} = \frac{G}{4}Q_{1\mathrm{PM}}^2\mathcal{I}(\sigma)\,,\quad \mathcal{I}(\sigma)\equiv\frac{2(8-5\sigma^2)}{3(\sigma^2-1)}+\frac{2\sigma\left(2\sigma^2-3\right)}{(\sigma^2-1)^{3/2}}\,\mathrm{arccosh}\,\sigma\,.$$

- Infrared divergent exponential suppression:

$$\mathrm{Im}\,2\delta_2 = \frac{1}{\pi}\left[-\frac{1}{\epsilon}+\log(\sigma^2-1)\right]\mathrm{Re}\,2\delta_2^{\mathrm{RR}}+\cdots$$

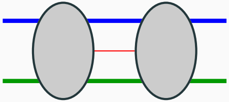
- $\mathrm{Re}\,2\delta_2^{\mathrm{RR}}$ contributes half-odd-PN corrections (odd in velocity) to $\Theta_{3\mathrm{PM}}$

Unitarity and Analyticity Fix the Radiation-Reaction Contribution

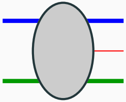
[Di Vecchia, CH, Russo, Veneziano '21]

- Unitarity determines the imaginary part of the two-loop eikonal,

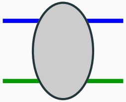
$$2\,\mathrm{Im}\,2\delta_2 = \mathrm{FT}$$



- IR divergence comes from low frequencies, use the soft graviton theorem:



$$\sim \sqrt{8\pi G}\sum_a \frac{p_a^\mu p_a^\nu}{p_a \cdot k}$$



as $k^\alpha \to 0$

- Then, using the natural upper cutoff $\omega^* \simeq \frac{v}{b}$, we find

$$\mathrm{Im}\,2\delta_2 = \frac{G}{2\pi}\left[-\frac{1}{2\epsilon} + \log\sqrt{\sigma^2 - 1}\right] Q_{1\mathrm{PM}}^2\,\mathcal{I}(\sigma) + \cdots$$

- By analyticity, $i\log(1-\sigma^2 - i0) = i\log(\sigma^2 - 1) + \pi$, hence

$$\mathrm{Re}\,2\delta_2^{\mathrm{RR}} = \lim_{\epsilon\to 0}\left[-\pi\epsilon\,\mathrm{Im}\,2\delta_2\right] = \frac{G}{4}\,Q_{1\mathrm{PM}}^2\,\mathcal{I}(\sigma)\,.$$

Smoothness and Universality of $\mathrm{Re}\, 2\delta_2$ at High Energy [Di Vecchia, CH, Russo, Veneziano '20, '21]

At high energy, as $\sigma \to \infty$ and $s \sim 2 m_1 m_2 \sigma$, i.e. in the massless limit:

- The *complete* eikonal phase is smooth, although the conservative and radiation-reaction parts separately diverge like $\log \sigma$
- Its expression is the same in $\mathcal{N} = 8$ supergravity and in GR,

$$\mathrm{Re}\, 2\delta_2 \sim Gs \frac{\Theta_s^2}{4} , \qquad \Theta_s \sim \frac{4G\sqrt{s}}{b}$$

in agreement with [Amati, Ciafaloni, Veneziano '90].

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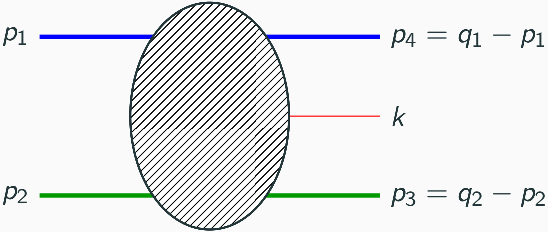
$$\bar{p}_1^\mu = \frac{1}{2}(p_4^\mu - p_1^\mu)$$

$$\bar{p}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$

$$\boxed{q_1^\mu} = p_1^\mu + p_4^\mu$$

$$\boxed{q_2^\mu} = p_2^\mu + p_3^\mu$$

$$0 = q_1^\mu + q_2^\mu + k^\mu$$



More invariants, besides q_1^2, q_2^2, also

$$\boxed{\sigma} = -v_1 \cdot v_2\,, \qquad \boxed{\omega_1} = -v_1 \cdot k\,, \qquad \boxed{\omega_2} = -v_2 \cdot k\,.$$

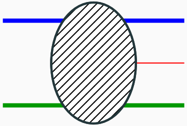
We denote by E, ω the total energy and the graviton frequency in the CoM frame,

$$E = \sqrt{-(p_1+p_2)^2}\,, \qquad \omega = \frac{1}{E}(p_1+p_2)\cdot k = \frac{1}{E}(m_1\omega_1 + m_2\omega_2)\,, \qquad \alpha_{1,2} = \frac{\omega_{1,2}}{\omega}\,.$$

$2 \to 3$ Amplitude up to One Loop

[Brandhuber et al. '23; Herderschee, Roiban, Teng 23; Elkhidir, O'Connell, Sergola, Vazquez-Holm '23] [Georgoudis, CH, Vazquez-Holm '23]

$$\mathcal{A} =$$



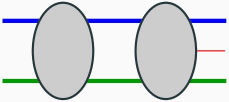
$$= \mathcal{A}_0 + \mathcal{A}_1 + \cdots$$

with $\mathcal{A}_0$ the tree-level amplitude, and

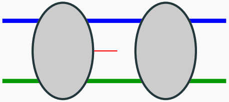
$$\mathcal{A}_1 = \mathcal{B}_1 + \frac{i}{2}(s + s') + \frac{i}{2}(c_1 + c_2)\,.$$

where $\mathcal{B}_1 = \operatorname{Re}\mathcal{A}_1$ and the unitarity cuts can be depicted as follows,

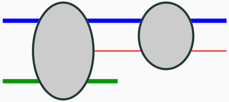
$s =$



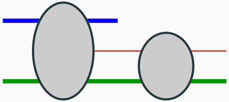
$s' =$



$c_1 =$



$c_2 =$



Inelastic Final State [Di Vecchia, CH, Russo, Veneziano '22]

[cf. Kosower, Maybee, O'Connell '18; Damgaard, Planté, Vanhove '21; Cristofoli et al. '21]

Eikonal Exponentiation of Graviton Exchanges + Coherent Radiation:

$$e^{2i\hat{\delta}(b_1,b_2)} = e^{i\,\mathrm{Re}\,2\delta(b)}e^{i\int_k\left[\tilde{W}(k)a^\dagger(k)+\tilde{W}^*(k)a(k)\right]}\,.$$

- Final state, schematically:

$$|\text{out}\rangle = e^{2i\hat{\delta}(b_1,b_2)}|\text{in}\rangle$$

- Unitarity:

$$\langle\text{out}|\text{out}\rangle = \langle\text{in}|\text{in}\rangle = 1$$

- The asymptotic metric fluctuation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ sourced by the scattering (the waveform) is expressed formally as

$$h_{\mu\nu}(x) = \sqrt{32\pi G}\,\langle\text{out}|\hat{H}_{\mu\nu}(x)|\text{out}\rangle \sim \frac{4G}{\kappa r}\int_0^\infty e^{-i\omega U}\,\tilde{W}_{\mu\nu}(\omega n)\,\frac{d\omega}{2\pi} + (\text{c.c.})$$

where $\kappa = \sqrt{8\pi G}$, r is the distance from the observer and U the retarded time. Normalization $\tilde{W}^{\mu\nu} = \kappa\,\tilde{w}^{\mu\nu}$.

The Waveform Kernel

[Caron-Huot, Giroux, Hannesdottir, Mizera '23] [Georgoudis, CH, Russo, '23, '24]

- Working with "eikonal" variables, we can use the following radiation kernel,

$$W = \mathcal{A}_0 + \left[\mathcal{B}_1 + \tfrac{i}{2}\left(c_1 + c_2\right)\right].$$

- Tree level: $\mathcal{A}_0$ is a relatively simple rational function [Luna, Nicholson, O'Connell, White '17]
- One loop: We isolate the even and odd parts of $\mathcal{B}_1$ under $\omega_{1,2} \mapsto -\omega_{1,2}$,

$$\mathcal{B}_1 = \mathcal{B}_{1O} + \mathcal{B}_{1E}\,,$$

and $\mathcal{B}_{1O} = \mathcal{B}_{1O}^{(h)} + \mathcal{B}_{1O}^{(i)}$ is fixed by unitarity and analyticity in terms of the tree-level amplitude,

$$\mathcal{B}_{1O}^{(h)} = \pi G E \omega\, \mathcal{A}_0\,, \qquad \mathcal{B}_{1O}^{(i)} = -\frac{\sigma\left(\sigma^2 - \frac{3}{2}\right)}{\left(\sigma^2 - 1\right)^{3/2}}\, \pi G E \omega\, \mathcal{A}_0$$

while $\mathcal{B}_{1E}$ and c_1, c_2 represent new one-loop data.

Infrared Divergences Revisited

[Georgoudis, CH, Russo '24] [Bini et al. '24]

- IR divergences due to c_1, c_2,

$$\frac{i}{2}\, c_1 = 2iGm_1\omega_1 \left(-\frac{1}{2\epsilon} + \log\frac{\omega_1}{\mu} \right) \mathcal{A}_0 + \frac{i}{2}\, c_1^{(\text{reg})}$$

exponentiate in momentum space,

$$W = e^{-\frac{i}{\epsilon} GE\omega} \left[\mathcal{A}_0 + \mathcal{B}_1 + \tfrac{i}{2}\, \mathcal{C} \right] = e^{-\frac{i}{\epsilon} GE\omega}\, W^{\text{reg}} \,,$$

where $\frac{i}{2}\,\mathcal{C} = \sum_{a=1,2} \left(2iGm_a\omega_a \log\frac{\omega_a}{\mu} + \frac{i}{2} c_a^{(\text{reg})} \right)$

- Cancel the divergence by redefining the origin of retarded time [Goldberger, Ross '10]

$$h_{\mu\nu}(x) \sim \frac{4G}{\kappa r} \int_0^\infty e^{-i\omega U}\, \tilde{W}^{\text{reg}}_{\mu\nu}(\omega n)\, \frac{d\omega}{2\pi} + (\text{c.c.})$$

- It **resums velocity corrections** to the Einstein quadrupole formula up to $\mathcal{O}(G^3)$:
  - $\mathcal{A}_0$, $\mathcal{B}_{1O}^{(h)}$ and $\mathcal{B}_E$ give integer PN corrections (even powers of v)
  - $\mathcal{B}_{1O}^{(i)}$ and $c_1^{(\text{reg})}$, $c_2^{(\text{reg})}$ give half-odd PN corrections (odd powers of v)

Universal Terms ω^{-1}, $\log\omega$, $\omega(\log\omega)^2$

- Leading $1/\omega$ soft term (memory effect in time domain) [matches Weinberg '64; Sahoo, Sen '18; '21]

$$\tilde{W}^{[\omega^{-1}]} = \frac{i\kappa Q}{b\omega\tilde{\alpha}_1^2\tilde{\alpha}_2^2}(\tilde{\alpha}_1\tilde{u}_2\cdot\varepsilon - \tilde{\alpha}_2\tilde{u}_1\cdot\varepsilon)(2\tilde{\alpha}_1\tilde{\alpha}_2 b_e\cdot\varepsilon + b_e\cdot n(\tilde{\alpha}_1\tilde{u}_2\cdot\varepsilon + \tilde{\alpha}_2\tilde{u}_1\cdot\varepsilon))$$

- Subleading $\log\omega$ soft term [matches Sahoo, Sen '18; '21]

$$\begin{aligned}\tilde{W}^{[\log\omega]} &= \kappa\frac{2Gm_1m_2\sigma(2\sigma^2-3)}{\tilde{\alpha}_1\tilde{\alpha}_2(\sigma^2-1)^{3/2}}(\tilde{\alpha}_1\tilde{u}_2\cdot\varepsilon - \tilde{\alpha}_2\tilde{u}_1\cdot\varepsilon)^2\log\left(\frac{\omega b\, e^{\gamma}}{2\sqrt{\sigma^2-1}}\right)\\ &\quad + 2iGE\omega\,\tilde{W}_0^{[\omega^{-1}]}\log\omega\end{aligned}$$

- Sub-subleading $\omega(\log\omega)^2$ soft term [matches Sahoo, Sen '18; '21]

$$\tilde{W}^{[\omega(\log\omega)^2]} = 2iGE\omega\tilde{W}_0^{[\log\omega]}\log\omega$$

ω^0

[Georgoudis, CH, Russo '23]

- Non-universal ω^0 piece of the tree-level result,

$$\tilde{W}_0^{[\omega^0]} = \kappa(\tilde{\alpha}_1 \tilde{u}_2 \cdot \varepsilon - \tilde{\alpha}_2 \tilde{u}_1 \cdot \varepsilon)^2 \left[\frac{G m_1 m_2 \sigma (2\sigma^2 - 3)}{\tilde{\alpha}_1 \tilde{\alpha}_2 (\sigma^2 - 1)^{3/2}} \log(\tilde{\alpha}_1 \tilde{\alpha}_2) - \frac{2 G m_1 m_2 (2\sigma^2 - 1)}{\tilde{\mathcal{P}} \sqrt{\sigma^2 - 1}} \right]$$
$$+ \frac{4 G m_1 m_2}{\tilde{\mathcal{P}}} \left[\frac{(\tilde{\alpha}_1 \tilde{u}_2 \cdot \varepsilon - \tilde{\alpha}_2 \tilde{u}_1 \cdot \varepsilon)^2}{\tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\mathcal{P}}} \left(g_3 \operatorname{arccosh} \sigma + g_2 \log \frac{\tilde{\alpha}_1}{\tilde{\alpha}_2} \right) \right.$$
$$\left. + \frac{2\sigma^2 - 1}{2 b^2 \tilde{\alpha}_1^2 \sqrt{\sigma^2 - 1}} g_1 \right] + i b_2 \cdot n \, \omega \tilde{W}_0^{[\omega^{-1}]} .$$

- For this one, when expanding for small frequencies, two regions in the Fourier integral are needed.

$\omega \log \omega$

[Georgoudis, CH, Russo '23]

- Tree-level $\omega \log \omega$ piece [matches Ghosh, Sahoo '21]

$$\tilde{W}_0^{[\omega \log \omega]} = \kappa \frac{2iGm_1 m_2 \sigma(2\sigma^2 - 3)}{\tilde{\alpha}_1 \tilde{\alpha}_2 (\sigma^2 - 1)^{3/2}} (\tilde{\alpha}_1\, \tilde{u}_2 \cdot \varepsilon - \tilde{\alpha}_2\, \tilde{u}_1 \cdot \varepsilon)$$
$$\times \left[\tilde{\alpha}_1 \tilde{\alpha}_2\, b_e \cdot \varepsilon + \tilde{\alpha}_2 (b_1 \cdot n)(\tilde{u}_1 \cdot \varepsilon) - \tilde{\alpha}_1 (b_2 \cdot n)(\tilde{u}_2 \cdot \varepsilon)\right] \omega \log \omega$$

- Non-universal one-loop $\omega \log \omega$ piece. $\mathcal{B}_{1E}$ does not contribute.

$$\frac{i}{2}(\tilde{c}_1 + \tilde{c}_2)^{[\omega \log \omega]} = iGE \left[-\frac{1}{\epsilon} + \log \frac{\alpha_1 \alpha_2}{\mu_{\text{IR}}^2}\right] \omega \tilde{W}_0^{[\log \omega]} + 2iGE\omega \log \omega\, \tilde{W}_0^{[\omega^0]} + i\tilde{\mathcal{M}}_1^{[\omega \log \omega]}$$

with

$$i\tilde{\mathcal{M}}_1^{[\omega \log \omega]} = i\kappa\omega \log \omega\, G^2\, m_1^2 m_2 \frac{2\sigma(\alpha_1\, u_2 \cdot \varepsilon - \alpha_2\, u_1 \cdot \varepsilon)^2}{(\sigma^2 - 1)^{3/2} \tilde{\mathcal{P}}}$$
$$\times \left[\frac{2\sigma^2 - 3}{\tilde{\mathcal{P}}} \left(f_3 \frac{\operatorname{arccosh} \sigma}{(\sigma^2 - 1)^{3/2}} + f_2 \frac{1}{\alpha_2} \log \frac{\alpha_1}{\alpha_2}\right) - \frac{f_1}{\alpha_2(\sigma^2 - 1)}\right] + (1 \leftrightarrow 2)\,.$$

Comparison with Predictions from MPM Formalism

[Georgoudis, CH, Russo '23]

- The result for the $\omega \log \omega$ term was given explicitly in the PN expansion using the Multipolar post-Minkowskian (MPM) formalism in [Bini, Damour, Geralico '23], where a mismatch was found when comparing with the amplitude-based result starting at 2.5PN ($\sim 1/c^5$)
- We find that agreement is restored after performing the following supertranslation [Veneziano, Vilkovisky '22]

$$U \mapsto U - T(n)\,, \qquad T(n) = 2G(m_1\alpha_1 \log \alpha_1 + m_2\alpha_2 \log \alpha_2)$$

or more precisely

$$\delta_T h_{AB} = -T(n)\,\partial_u h_{AB} + r\,[2D_A D_B - \gamma_{AB}\Delta]\,T(n)$$

where only the first term on the RHS (the non-static one) matters.
Here, $n^\mu = (1, \hat{n})$, $e_A^\mu = \partial_A n^\mu$, $h_{AB} = r^2 e_A^\mu e_B^\nu h_{\mu\nu}$, $\gamma_{AB} = e_A \cdot e_B$, D_A is the associated covariant derivative, $\Delta = D_A D^A$.

The PN Limit

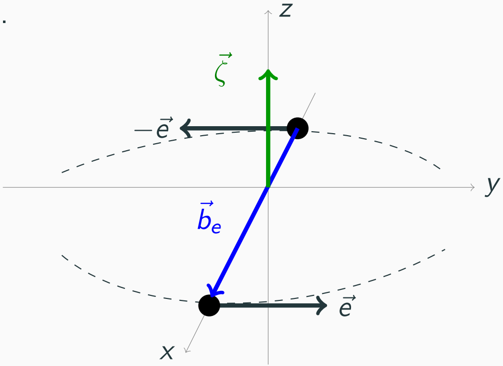
- The PN expansion is defined by the limit

$$p_\infty = \sqrt{\sigma^2 - 1} = \mathcal{O}(\lambda)\,, \qquad \omega = \mathcal{O}(\lambda) \qquad \text{as } \lambda \to 0$$

- Each instance of the Newton constant G increases the PN order by one unit.
- Each power of λ increases it by half a unit.

Reference vectors in the CoM frame:

$$\begin{aligned} t^\alpha &= (1,0,0,0) \\ b_e^\alpha &= (0,b,0,0) \\ e^\alpha &= (0,0,1,0) = \tilde{p}^\alpha/p \\ \zeta^\mu &= (0,0,0,1) \end{aligned}$$



Multipolar Decomposition

[Blanchet, Damour '85; Blanchet '13]

- We define the dimensionless frequency

$$u = \frac{\omega b}{p_\infty}\,,$$

which does not scale in the PN limit.

- It is convenient to express the waveform in terms of "multipoles", i.e. symmetric trace-free (STF) tensors $\mathrm{U}_L(u)$, $\mathrm{V}_L(u)$,

$$h_{ij}^{\mathsf{TT}} = \frac{4G}{r}\sum_{\ell=2}^{\infty}\frac{1}{\ell!}\left[n_{L-2}\,\mathrm{U}_{ijL-2}(u) - \frac{2\ell}{\ell+1}\,n_{cL-2}\epsilon_{cd(i}\,\mathrm{V}_{j)dL-2}(u)\right]^{\mathsf{TT}}$$

(decomposition into symmetric, traceless tensors with definite Δ-eigenvalue)

- Order by order in the PN expansion, only the first few U_L, V_L show up.

Examples

[Georgoudis, CH, Russo '24]

Newtonian quadrupole at tree level,

$$\mathrm{U}_{11}^{\mathrm{LO}} = -\frac{4Gm^2\nu}{3p_\infty}(K_0(u) + 3uK_1(u))\,,$$

$$\mathrm{U}_{12}^{\mathrm{LO}} = -\frac{4iGm^2\nu}{p_\infty}(uK_0(u) + K_1(u))\,,$$

$$\mathrm{U}_{22}^{\mathrm{LO}} = \frac{4Gm^2\nu}{3p_\infty}(2K_0(u) + 3uK_1(u))\,,$$

$$\mathrm{U}_{33}^{\mathrm{LO}} = -\frac{4Gm^2\nu K_0(u)}{3p_\infty}$$

1PN correction to the quadrupole due to $\mathcal{B}_{1E}$,

$$\mathrm{U}_{E11} = -\mathrm{U}_{E22} = -\frac{6\pi G^2 m^3\nu}{bp_\infty}(1+u)\,e^{-u}\,,$$

$$\mathrm{U}_{E12} = -\frac{6i\pi G^2 m^3\nu}{bp_\infty}\left(\frac{1}{u} + 1 + u\right)e^{-u}\,,$$

while e.g. one component at 2PN is

$$\mathrm{U}_{E33}^{\mathrm{NLO}} = -\frac{\pi G^2 m^3\nu p_\infty}{b}(2\nu-5)(u+1)\,e^{-u}\,.$$

Comparison with MPM Formalism

[Georgoudis, CH, Russo '24]

- Integer PN terms arise from various corrections to the trajectories.
- Half-odd PN: **Tail formula**

$$\mathrm{U}_L^{\text{tail}} = \frac{2GE}{c^3} i\omega \mathrm{U}_L^{\text{tree}} \left(\log \frac{\omega}{\mu_{\text{IR}}} - \kappa_\ell - \frac{i\pi}{2} \right)$$

(similarly for $\mathrm{V}_L(u)$ with π_ℓ)

- Half-odd PN: **Nonlinear** effects, e.g.

$$U_{ij}^{QQ} = \frac{G}{c^5} \left[\frac{1}{7} I_{a\langle i}^{(5)} I_{j\rangle a} - \frac{5}{7} I_{a\langle i}^{(4)} I_{j\rangle a}^{(1)} - \frac{2}{7} I_{a\langle i}^{(3)} I_{j\rangle a}^{(2)} \right]$$

- Half-odd PN: **Radiation-reaction**

$$x_{RR}^{\mu} = \frac{8G^2 m^2 p_\infty \nu}{5b^2 r} \left(b^2 e^{\mu} - (r + p_\infty t) b_e^{\mu} \right) \qquad U_{ij}^{RR} = 2m\nu \frac{d^2}{dt^2} \left(x_{\langle i} x_{j\rangle}^{RR} \right)$$

We checked that C^{reg} completely agrees with the MPM prediction given by tail+nonlinear+radiation-reaction up to and including 2.5PN.

See also [Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, Teng '24]

Outline

Introduction

Warm-Up: Elastic Eikonal and Deflection Angle

Eikonal Operator and Gravitational Waveform

Soft Theorems, Soft Energy Spectrum, Static Angular Momentum Loss

Integrated Energy and Angular Momentum Losses

Emitted Energy-Momentum

[Georgoudis, CH, Russo ’24]

- The operator insertion $\langle \text{out}|\hat{P}^\alpha|\text{out}\rangle = P^\alpha$ leads to leads to

$$P^\alpha = \int_k k^\alpha \rho(k)\,, \qquad \int_k = \int 2\pi\theta(k^0)\,\delta(k^2)\,\frac{d^D k}{(2\pi)^D}$$

where the spectral emission rate ρ is given by

$$\rho = \tilde{w}^{\text{TT}*}_{\mu\nu}\,\tilde{w}^{\text{TT}\mu\nu} = \tilde{W}^*_{\mu\nu}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{D-2}\,\eta^{\mu\nu}\eta^{\rho\sigma}\right)\tilde{W}_{\rho\sigma}$$

Note the equivalence between the two expressions, with

$$\tilde{w}^{TT}_{\mu\nu} = \Pi^{\text{TT}}_{\mu\nu\rho\sigma}\tilde{W}^{\rho\sigma}\,, \qquad k_\mu \tilde{W}^{\mu\nu}(k) = 0\,.$$

- We can choose the TT projector to be space-like in the CoM frame, so that

$$\kappa^2 P^0 \equiv \kappa^2 E_{\text{rad}} = G\int_0^\infty \frac{d\omega}{\pi}\oint\frac{d\Omega}{2\pi}\,\omega^2\,\tilde{w}^{\text{TT}*}_{ab}\,\tilde{w}^{\text{TT}}_{ab}\,,$$

$$\kappa^2 P^i = G\int_0^\infty \frac{d\omega}{\pi}\oint\frac{d\Omega}{2\pi}\,\omega^2 n^i\,\tilde{w}^{\text{TT}*}_{ab}\,\tilde{w}^{\text{TT}}_{ab}\,,$$

Zero-Frequency Limit (ZFL) of the Energy Spectrum

[Di Vecchia, CH, Russo, Veneziano '22]

- Inserting the leading soft theorem and performing the angular integrals, one finds

$$\left(\frac{dE}{d\omega}\right)_{\text{ZFL}} = \frac{2G}{\pi}\sum_{a,b} m_a m_b \left(\sigma_{ab}^2 - \tfrac{1}{2}\right)\eta_a\eta_b \frac{\operatorname{arccosh}\sigma_{ab}}{\sqrt{\sigma_{ab}^2-1}}$$

- **PM expansion** $Q = 2p\sin\frac{\Theta}{2} \ll p \sim m_{1,2}$

$$\left(\frac{dE}{d\omega}\right)_{\text{ZFL}} = \frac{GQ^2}{\pi}\mathcal{I}(\sigma) - \frac{GQ^4}{\pi m_1 m_2(\sigma^2-1)^2}\left[\frac{3}{2}\frac{\operatorname{arccosh}\sigma}{\sqrt{\sigma^2-1}} + \frac{\sigma}{2}(2\sigma^2-5) + \frac{2}{5}\frac{m_1^2+m_2^2}{m_1 m_2}(\sigma^2-1)^2\right] + \mathcal{O}(G^6)\,.$$

- Ultrarelativistic limit $m_{1,2} \ll Q = 2p\sin\frac{\Theta}{2}$ (so $\sqrt{s} = E \simeq 2p$) [Addazi, Bianchi, Veneziano '19]

$$\left(\frac{dE}{d\omega}\right)_{\text{ZFL}} = \frac{4G}{\pi}\left[Q^2\log\left(\frac{s}{Q^2}-1\right) - s\log\left(1-\frac{Q^2}{s}\right)\right]$$

- The small-$m_{1,2}$ of the PM expansion is singular, while the $\Theta \ll 1$ limit is smooth in the ultrarelativistic regime $\left(\frac{dE}{d\omega}\right)_{\text{ZFL}} \simeq \frac{Gs\Theta^2}{\pi}\log\frac{4e}{\Theta^2}$

Nonlinear memory contribution?

[Alessio, Di Vecchia, CH '24]

- Take outgoing gravitons into account by

$$\sum_a \mapsto \sum_{a_m} + \int_k \rho(k)$$

where a_m runs over massive states, $\rho(k)$ is the distribution of emitted gravitons.

- This is the operation that gives the nonlinear memory effect,

$$a_0^{\mu\nu} \mapsto a_0^{\mu\nu} + \delta a_0^{\mu\nu}\,, \qquad \delta a_0^{\mu\nu} = \int_k \rho(k)\,\frac{k^\mu k^\nu}{k\cdot n}\,.$$

- For the ZFL of the energy spectrum, it gives

$$\delta\left(\frac{dE}{d\omega}\right)_{\text{ZFL}} = -\frac{4G}{\pi}\int_k \rho(k)\sum_a p_a\cdot k\log\left(-\eta_a\,\frac{p_a\cdot k}{m_a\Lambda}\right) \simeq -\frac{4G}{\pi}\int_k \rho(k) Q\cdot k\log\frac{\tilde{u}_1\cdot k}{\tilde{u}_2\cdot k}$$

- The $\mathcal{O}(G^5)$ **vanishes** since $\rho(k)$ is invariant under $b\cdot k \to -b\cdot k$ to leading order in G owing to the reality of the tree-level amplitude.

A Bound on the PM Expansion

[Di Vecchia, CH, Russo, Veneziano '22]

- The explicit nonperturbative expression for the ZFL reads

$$\left(\frac{dE}{d\omega}\right)_{\text{ZFL}} = \frac{4G}{\pi}\Bigg\{2m_1m_2\left(\sigma^2-\tfrac{1}{2}\right)\frac{\operatorname{arccosh}\sigma}{\sqrt{\sigma^2-1}} - 2m_1m_2\left(\sigma_Q^2-\tfrac{1}{2}\right)\frac{\operatorname{arccosh}\sigma_Q}{\sqrt{\sigma_Q^2-1}}$$

$$+\sum_{a=1,2}\left[\frac{m_a^2}{2}-m_a^2\left(\left(1+\tfrac{Q^2}{2m_a^2}\right)^2-\tfrac{1}{2}\right)\frac{\operatorname{arccosh}\left(1+\frac{Q^2}{2m_a^2}\right)}{\sqrt{\left(1+\frac{Q^2}{2m_a^2}\right)^2-1}}\right]\Bigg\}\,.$$

- Note the presence of branch points at (recall also $Q \sim \sqrt{m_1m_2\sigma}\,\Theta$)

$$Q^2 = -4m_a^2$$

  corresponding to the t-channel thresholds (outside the physical region)

- The PM expansion converges for [D'Eath '76; Kovacs, Thorne '77,'78]

$$\sigma \lesssim \frac{1}{\Theta^2}$$

NLO, $\omega^2(\log\omega)^2$, in the Energy Spectrum

[Alessio, Di Vecchia, CH '24]

- We specify a reference frame: center-of-mass
- Inserting the $\log\omega$ and $\omega(\log\omega)^2$ soft theorems in the expression for the spectrum, we obtain a general prediction for the $\omega^2(\log\omega)^2$ contribution
- **PM expansion** $Q = 2p\sin\frac{\Theta}{2} \ll p \sim m_{1,2}$

$$\left(\frac{dE}{d\omega}\right)_{\omega^2(\log\omega)^2} = (GEh(\sigma))^2\,\frac{G}{\pi}\,\mathcal{H}(m_1, m_2, \sigma) + 2\,(GEh(\sigma))^2\,\frac{GQ^2}{\pi}\,\mathcal{I}(\sigma) + \mathcal{O}(G^6)$$

- Ultrarelativistic limit $m_{1,2} \ll Q = 2p\sin\frac{\Theta}{2}$ (so $\sqrt{s} = E \simeq 2p$) [Sahoo, Sen '19]

$$\left(\frac{dE}{d\omega}\right)_{\omega^2(\log\omega)^2} = s\,\frac{16G^3}{\pi}\left[s + 2s\log\left(1 - \frac{Q^2}{s}\right) - 2Q^2\log\left(\frac{s}{Q^2} - 1\right)\right]$$

- For $\Theta \ll 1$, one finds $\left(\frac{dE}{d\omega}\right)_{\omega^2(\log\omega)^2} = \frac{16G^3}{\pi}\,s^2\left[1 - \frac{2Q^2}{s}\log\frac{s}{Q^2} + \cdots\right]$.
- Again, (only) when a log appears, the UR limit of the PM expansion is singular

The functions $\mathcal{H}(m_1, m_2, \sigma)$ and $\mathcal{I}(\sigma)$ appearing in the low-frequency spectrum

$$\mathcal{H}(\sigma, m_1, m_2) = \left[2(s - m_1 m_2 \sigma) + \frac{m_2^2(2m_1\sigma + m_2)}{m_1\sqrt{\sigma^2-1}}\,\ell_1 + \frac{m_1^2(2m_2\sigma + m_1)}{m_2\sqrt{\sigma^2-1}}\,\ell_2\right],$$

with

$$\ell_1 = \log\left(\frac{x\,(m_1 x + m_2)}{m_2 x + m_1}\right), \qquad \ell_2 = \log\left(\frac{x\,(m_2 x + m_1)}{m_1 x + m_2}\right), \qquad x = \sigma - \sqrt{\sigma^2-1}\,,$$

while

$$\mathcal{I}(\sigma) = \frac{2}{\sigma^2-1}\left[\frac{8-5\sigma^2}{3} + \frac{\sigma(2\sigma^2-3)\,\operatorname{arccosh}\sigma}{\sqrt{\sigma^2-1}}\right].$$

Log-Resummed Waveform

[Alessio, CH, Di Vecchia '24]

- Considering an elastic $2 \to 2$ hard process, let us define

$$E = (p_1{+}p_2)\cdot n\,, \quad B^{\mu\nu}(p_1,p_2) = (p_1{+}p_2)\cdot n \left(\frac{p_1^\mu p_1^\nu}{p_1 \cdot n} + \frac{p_2^\mu p_2^\nu}{p_2 \cdot n} \right) - (p_1^\mu{+}p_2^\mu)(p_1^\nu{+}p_2^\nu)\,.$$

- Then, the known soft theorems [Sahoo, Sen '18: '21] for $\ell = 0, 1, 2$ reduce to (define $h(\sigma) = \sigma(2\sigma^2 - 3)/(\sigma^2 - 1)^{3/2}$)

$$a_\ell^{\mu\nu} = \frac{1}{E}(-GEh(\sigma))^\ell \left[B^{\mu\nu}(p_1,p_2) - (-1)^\ell B^{\mu\nu}(p_3,p_4) \right]$$

- We **conjecture** that this expression generalizes to any $\ell \geq 0$.
- Frequency-domain resummation

$$\tilde{w}^{\mu\nu} = -\frac{i}{E\omega}\,\omega^{2iGE\omega} \left[\omega^{iGE\omega h(\sigma)} B^{\mu\nu}(p_1,p_2) - \omega^{-iGE\omega h(\sigma)} B^{\mu\nu}(p_3,p_4) \right] + \cdots$$

- Proofs: (1) at Newtonian level as $p_\infty \to 0$ for generic GM/bp_∞^2 [Alessio, CH, Di Vecchia '24]

  (2) in the near-probe limit $\nu \to 0$ [Fucito, Morales, Russo '24]
- Cross-check: 2PN approximation up to $\mathcal{O}(G^3)$ [Bini, Damour, Geralico '24]

Log-Resummed Soft Spectra

[Alessio, Di Vecchia, CH '24]

- The resummed waveform in the soft limit gives universal results for the "leading logs" (LL) of the type $(\omega \log \omega)^n$ in the energy emission spectrum $dE/d\omega$.
- In the CoM frame we find, expanding for small deflections $Q \to 0$,

$$\begin{aligned}\left(\frac{dE}{d\omega}\right)_{\rm LL} &= \left[1 - \cos\left(2GEh(\sigma)\omega \log \omega\right)\right] \frac{2G}{\pi} \mathcal{H}(m_1, m_2, \sigma) \\ &\quad + \cos\left(2GEh(\sigma)\omega \log \omega\right) \frac{GQ^2}{\pi} \mathcal{I}(\sigma) + \cdots\end{aligned}$$

  fixing $G^{2n+1}(\omega \log \omega)^{2n}$ for $n = 1, 2, \ldots$ and $G^{2n+3}(\omega \log \omega)^{2n}$ for $n = 0, 1, 2, \ldots$ (see the additional material for the functions $\mathcal{H}(m_1, m_2, \sigma)$ and $\mathcal{I}(\sigma)$).
- In the ultrarelativistic limit instead

$$\begin{aligned}\left(\frac{dE}{d\omega}\right)_{\rm LL} &= \frac{4G}{\pi} \left[\sin(2G\sqrt{s}\, \omega \log \omega)\right]^2 s \\ &\quad + \frac{4G}{\pi} \cos(4G\sqrt{s}\, \omega \log \omega) \left[Q^2 \log\left(\frac{s}{Q^2} - 1\right) - s \log\left(1 - \frac{Q^2}{s}\right)\right] + \cdots\end{aligned}$$

Emitted Angular Momentum

- $\langle \text{out}|\hat{J}_{\alpha\beta}|\text{out}\rangle = \boldsymbol{J}_{\alpha\beta} + \mathcal{J}^{\text{tot}}_{\alpha\beta}$
- Radiative contribution

$$i\boldsymbol{J}_{\alpha\beta} = \int_k \left[\tfrac{1}{2}\left(\tilde{w}^{\mathrm{TT}*}_{\mu\nu} k_{[\alpha}\frac{\partial \tilde{w}^{\mathrm{TT}\mu\nu}}{\partial k^{\beta]}} - \tilde{w}^{\mathrm{TT}\mu\nu} k_{[\alpha}\frac{\partial \tilde{w}^{\mathrm{TT}*}_{\mu\nu}}{\partial k^{\beta]}}\right) + 2\tilde{w}^{\mathrm{TT}*}_{\mu[\alpha}\tilde{w}^{\mathrm{TT}\mu}_{\beta]}\right]$$

or equivalently [Manohar, Ridgway, Shen '22] [Di Vecchia, CH, Russo '22]

$$i\boldsymbol{J}_{\alpha\beta} = \int_k \left[\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{D-2}\,\eta^{\mu\nu}\eta^{\rho\sigma}\right)\tilde{W}^{*}_{\mu\nu} k_{[\alpha}\frac{\overset{\leftrightarrow}{\partial}\,\tilde{W}_{\rho\sigma}}{\partial k^{\beta]}} + 2\eta^{\mu\nu}\tilde{W}^{*}_{\mu[\alpha}\tilde{W}_{\beta]\nu}\right].$$

- In particular, for the spatial components in the CoM frame [Compère, Oliveri, Seraj '19],

$$\kappa^2\boldsymbol{J}^{ij} = G\int_0^\infty \frac{d\omega}{i\pi}\oint\frac{d\Omega}{2\pi}\,\tilde{w}^{\mathrm{TT}*}_{ab}\partial_A\tilde{w}^{\mathrm{TT}}_{ab}\,\omega\,\gamma^{AB}n^{[i}\partial_B n^{j]} + 2G\int_0^\infty\frac{d\omega}{i\pi}\oint\frac{d\Omega}{2\pi}\,\tilde{w}^{*a[i}_{\mathrm{TT}}\tilde{w}^{j]a}_{\mathrm{TT}}\,\omega\,.$$

Eikonal Operator in the ZFL

[Di Vecchia, CH, Russo, Veneziano '22]

Exponential dressing of the eikonal operator

We can include static/Coulombic modes by letting $e^{2i\hat{\delta}(b_1,b_2)} \mapsto S_{s.r.} e^{2i\hat{\delta}(b_1,b_2)}$ with

$$S_{s.r.} = e^{\int_k^* \left[F^{\mu\nu}(k) a^{\dagger}_{\mu\nu}(k) - F^{*\mu\nu}(k) a_{\mu\nu}(k)\right]}$$

where [Weinberg '64,'65] [Manohar, Ridgway, Shen '22] [Di Vecchia, CH, Russo, Veneziano '22]

$$F^{\mu\nu}(k) = \sum_a \frac{\sqrt{8\pi G}\, p_a^\mu p_a^\nu}{p_a \cdot k - i0}\,, \qquad n_\mu F^{\mu\nu}(k) = i\pi\sqrt{8\pi G} \sum_{a\in\text{in}} p_a^\mu \delta(\omega) \neq 0$$

and $\int_k^* = \int_k \theta(\omega^* - k^0)$, with ω^* a cutoff (to be sent to zero).

Angular Momentum of the Static Gravitational Field $\mathcal{J}_{\alpha\beta}$

[Di Vecchia, CH, Russo '22] [see also: Veneziano, Vilkovisky '22; Javadinezhad, Porrati '22, '23; Riva, Vernizzi, Wong '23]

This leads to

$$i\mathcal{J}_{\alpha\beta} = \int_k \left[\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{D-2}\,\eta^{\mu\nu}\eta^{\rho\sigma} \right) F^*_{\mu\nu} k_{[\alpha} \frac{\overleftrightarrow{\partial} F_{\rho\sigma}}{\partial k^{\beta]}} + 2\eta^{\mu\nu} F^*_{\mu[\alpha} F_{\beta]\nu} \right] .$$

Angular momentum loss due to static modes

$$\mathcal{J}^{\alpha\beta} = \frac{G}{2} \sum_{a,b} c(\sigma_{ab})\,(\eta_a - \eta_b)\, p_a^{[\alpha} p_b^{\beta]} , \quad c(\sigma_{ab}) = - \left[\left(\frac{\sigma_{ab}^2 - \frac{3}{2}}{\sigma_{ab}^2 - 1} \right) \frac{\sigma_{ab} \operatorname{arccosh} \sigma_{ab}}{\sqrt{\sigma_{ab}^2 - 1}} + \frac{\sigma_{ab}^2 - \frac{1}{2}}{\sigma_{ab}^2 - 1} \right]$$

- Match with [Damour '20; Manohar, Ridgway, Shen '22; Bini, Damour '22] up to $\mathcal{O}(G^3)$ upon expanding

$$\mathcal{J}^{\alpha\beta} = -\frac{G}{2}(p_1 - p_2)^{[\alpha} Q^{\beta]} \mathcal{I}(\sigma) + \mathcal{O}(G^4) , \qquad Q^\mu = Q^\mu_{1\mathrm{PM}} + Q^\mu_{2\mathrm{PM}} + \mathcal{O}(G^3)$$

- **Easy** to include tidal [CH '22] and spin [Alessio, Di Vecchia '22] [CH '23] effects, via Q^α.

$\mathcal{J}_{\alpha\beta}$ VS $\mathcal{J}^{\mathrm{TT}}_{\alpha\beta}$

- Since $n_\mu F^{\mu\nu} \neq 0$, the formula used above is *in general* not equivalent to the one obtained by using the TT-projected static field $f_{\mu\nu} = \Pi_{\mu\nu\alpha\beta} F^{\alpha\beta}$,

$$i\mathcal{J}^{\mathrm{TT}}_{\alpha\beta} = \int_k \left[\tfrac{1}{2} \left(f^*_{\mu\nu} k_{[\alpha} \frac{\partial f^{\mu\nu}}{\partial k^{\beta]}} - f^{\mu\nu} k_{[\alpha} \frac{\partial f^*_{\mu\nu}}{\partial k^{\beta]}} \right) + 2 f^*_{\mu[\alpha} f^{\mu}_{\beta]} \right],$$

- In the CoM frame, letting $\sigma_a = \eta_a p^0_a / m_a$, we find [CH, Russo '24]

$$\mathcal{J}^{\mathrm{TT}\alpha\beta} = \frac{G}{2} \sum_{a,b} c(\sigma_{ab}) \left(\eta_a - \eta_b\right) p_a^{[\alpha} p_b^{\beta]} + 2G \sum_a c(\sigma_a) \sum_{b \in \mathrm{in}} p_b^{[\alpha} p_a^{\beta]} \,.$$

- So, in the CoM frame, $\mathcal{J}^{\mathrm{TT}}_{ij} = \mathcal{J}_{ij}$, but $\mathcal{J}^{\mathrm{TT}}_{0i} \neq \mathcal{J}_{0i}$.

$\mathcal{J}_{\alpha\beta}$ **VS** $\mathcal{J}_{\alpha\beta}^{\mathsf{TT}}$

- We note that $\mathcal{J}_{0i}^{\mathsf{TT}}$ admits a smooth high-energy limit,

$$\mathcal{J}^{\mathsf{TT}\alpha\beta} = -2G \log\left(\frac{s}{Q^2} - 1\right)(p_1 - p_2)^{[\alpha} Q^{\beta]} \,.$$

- The TT contribution due to nonlinear memory is cutoff-independent

$$\delta\mathcal{J}^{\mathsf{TT}\alpha\beta} = 2G \int_k \rho(k) \sum_{a\in\text{in}} p_a^{[\alpha} k^{\beta]} \log \frac{p_a \cdot n}{m_a}$$

with $k^\mu = \omega\, n^\mu$ as defined in the CoM frame.

Nonlinear memory contribution?

[Heissenberg, Russo '24]

- Take outgoing gravitons into account as before by

$$\sum_a \mapsto \sum_{a_m} + \int_k \rho(k)$$

where a_m runs over massive states, $\rho(k)$ is the distribution of emitted gravitons.

- For the static contribution to the angular momentum, it gives

$$\delta \mathcal{J}^{\alpha\beta} = 2G \int_k \rho(k) \sum_{a \in \text{in}} p_a^{[\alpha} k^{\beta]} \log \frac{p_a \cdot k}{m_a \Lambda}$$

and Λ is an energy scale introduced to regulate the collinear divergence. It amounts to a time-translation ambiguity in the mass-dipole components.

- To leading PM order, $\mathcal{O}(G^4)$,

$$\delta \mathcal{J}_{xy} = \frac{b_\alpha}{b} \delta \mathcal{J}^{\alpha\beta} \frac{p_\beta}{p} \simeq \frac{2Gp}{b} \int_k \rho(k)(b \cdot k) \log \frac{p_1 \cdot k}{p_2 \cdot k}$$

which **vanishes** by parity.

Explicit expression of $\mathcal{J} = \mathcal{J}_{xy}$ up to $\mathcal{O}(G^4)$

[CH, Russo '24]

In conclusion the complete static contribution up to and including $\mathcal{O}(G^4)$ reads

$$
\begin{aligned}
\mathcal{J} &= G\,p\,Q(b)\,\mathcal{I}(\sigma) \\
&\quad - \frac{G Q(b)^3}{E\sqrt{\sigma^2-1}}\left[\frac{E^2}{8 m_1 m_2}\left(\mathcal{I}(\sigma) + \frac{32}{5}\left(\sigma^2-1\right)\right) + \frac{1}{2}\left(\mathcal{I}'(\sigma) - \frac{16}{5}\sigma\right)\left(\sigma^2-1\right)\right] \\
&\quad + \mathcal{O}(G^5)\,,
\end{aligned}
$$

with [Bern et al. '19, Damour '20]

$$
\begin{aligned}
Q(b) &= \frac{4 G m_1 m_2 \left(\sigma^2 - \frac{1}{2}\right)}{b\sqrt{\sigma^2-1}} + \frac{3\pi G^2 m_1 m_2 \left(m_1+m_2\right)\left(5\sigma^2-1\right)}{4 b^2\sqrt{\sigma^2-1}} + \frac{8 G^3 m_1^2 m_2^2}{b^3} \\
&\quad \times \left(\frac{\left(-4\sigma^4+12\sigma^2+3\right)\operatorname{arccosh}\sigma}{\sigma^2-1} + \frac{E^2\left(12\sigma^4-10\sigma^2+1\right)}{2 m_1 m_2\left(\sigma^2-1\right)^{3/2}} - \frac{\sigma\left(14\sigma^2+25\right)}{3\sqrt{\sigma^2-1}}\right) \\
&\quad + \frac{G}{2b}\left(\frac{4 G m_1 m_2\left(\sigma^2-\frac{1}{2}\right)}{b\sqrt{\sigma^2-1}}\right)^2 \mathcal{I}(\sigma) - \frac{G^3 m_1 m_2 E^2\left(2\sigma^2-1\right)^3}{b^3\left(\sigma^2-1\right)^{5/2}} + \mathcal{O}(G^4)\,.
\end{aligned}
$$

Angular momentum loss in the PN expansion

[CH, Russo '24]

Combining everything, $J_{xy} = \boldsymbol{J}_{xy} + \mathcal{J}_{xy} + \mathcal{O}(G^5)$ with

$$\begin{aligned}
J_{xy} = {} & \frac{G^2 m^3}{b} p_\infty^2 \nu^2 \left[\frac{16}{5} + \left(\frac{176}{35} - \frac{8}{5}\,\nu\right) p_\infty^2 + \mathcal{O}(p_\infty^4)\right] \\
& + \frac{G^3 \pi m^4}{b^2} \nu^2 \left[\frac{28}{5} + \left(\frac{739}{84} - \frac{79}{15}\,\nu\right) p_\infty^2 + \mathcal{O}(p_\infty^4)\right] \\
& + \frac{G^4 m^5}{b^3 p_\infty^2} \nu^2 \left[\frac{176}{5} + \left(\frac{8144}{105} - \frac{2984}{45}\,\nu\right) p_\infty^2 + \mathcal{O}(p_\infty^4)\right] \\
& + \frac{G^4 m^5}{b^3} p_\infty \nu^2 \left[\frac{448}{5} + \left(\frac{1184}{21} - \frac{220256}{1575}\,\nu\right) p_\infty^2 + \mathcal{O}(p_\infty^4)\right] + \mathcal{O}(G^5).
\end{aligned}$$

- The first two lines reproduce the small-velocity expansion of the $\mathcal{O}(G^2)$ [Damour '20] and $\mathcal{O}(G^3)$ [Manohar, Ridgway, Shen '22] [Di Vecchia, CH, Russo, Veneziano '22] results
- The last two lines are in perfect agreement with the 0PN, 1PN, 1.5PN and 2.5PN contributions at $\mathcal{O}(G^4)$ [Bini, Damour, Geralico '21; '22]
- Later, we will be able to resum the last line!

Outline

Emitted Energy-Momentum and Angular Momentum

[Herrmann, Parra-Martinez, Ruf, Zeng '21; Manohar, Ridgway, Shen '22] [Di Vecchia, CH, Russo, Veneziano '22]

- We define for later convenience the notation

$$\boldsymbol{K}_\alpha[\tilde{X},\tilde{Y}] = D^{\mu\nu,\rho\sigma} k_\alpha \tilde{X}^*_{\mu\nu} \tilde{Y}_{\rho\sigma}\,, \qquad \boldsymbol{O}_{\alpha\beta}[\tilde{X},\tilde{Y}] = D^{\mu\nu,\rho\sigma} \tilde{X}^*_{\mu\nu} k_{[\alpha} \frac{\overset{\leftrightarrow}{\partial}}{\partial k^{\beta]}} \tilde{Y}_{\rho\sigma} + 2\tilde{X}^*_{\mu[\alpha} \tilde{Y}^{\mu}_{\beta]}$$

- The operator insertion for the energy-momentum $\langle \text{out}|\hat{P}^\alpha|\text{out}\rangle = P^\alpha$ leads to leads to

$$\boldsymbol{P}^\alpha = \int_k \boldsymbol{K}^\alpha[\tilde{W},\tilde{W}]\,, \qquad \int_k = \int 2\pi\theta(k^0)\,\delta(k^2)\,\frac{d^Dk}{(2\pi)^D}$$

- For the angular momentum, one has $\langle \text{out}|\hat{J}_{\alpha\beta}|\text{out}\rangle = \boldsymbol{J}_{\alpha\beta}$ with

$$\boldsymbol{J}^{\alpha\beta} = -i\int_k \boldsymbol{O}^{\alpha\beta}[\tilde{W},\tilde{W}]$$

Emitted Energy-Momentum at $\mathcal{O}(G^3)$ (Two Loops)

[Heissenberg '25]

- To leading order

$$\boldsymbol{P}^{\alpha}_{\mathcal{O}(G^3)} = \int_k \boldsymbol{K}^{\alpha}_0\,, \qquad \boldsymbol{K}^{\alpha}_0 = \boldsymbol{K}^{\alpha}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{A}}_0]$$

- Note that $\tilde{\mathcal{A}}_0^* = \tilde{\mathcal{A}}_0\big|_{b\mapsto -b}$ (the tree-level amplitude is real!). So,

$$\boldsymbol{K}^{\alpha}_0 = \boldsymbol{K}^{\alpha}_0\big|_{b\to -b}\,. \tag{1}$$

- Writing $\boldsymbol{K}^{\alpha}_0 = f_{u_1}\,\check{u}_1^{\alpha} + f_{u_2}\,\check{u}_2^{\alpha} + f_b\,b^{\alpha} + f_k\,k^{\alpha}$ (here $\check{u}_i \cdot u_j = -\delta_{ij}$) we deduce

$$f_{u_{1,2}}(-b\cdot k) = +f_{u_{1,2}}(b\cdot k)\,, \quad f_b(-b\cdot k) = -f_b(b\cdot k)\,, \quad f_k(-b\cdot k) = +f_k(b\cdot k)\,.$$

- Therefore, the integrand $b\cdot \boldsymbol{K}_0 = f_b\,b^2 + f_k\,b\cdot k$ is odd under $b\cdot k \mapsto -b\cdot k$,

$$b\cdot \boldsymbol{P}_{\mathcal{O}(G^3)} = 0$$

in agreement with the explicit result [Herrmann, Parra-Martinez, Ruf, Zeng '21]

Take-home message: Some components vanish by analyticity considerations.

Emitted Angular Momentum at $\mathcal{O}(G^3)$ (Two Loops)

[Heissenberg '25]

- To leading order

$$\boldsymbol{J}^{\alpha\beta}_{\mathcal{O}(G^3)} = -i \int_k \boldsymbol{O}^{\alpha\beta}_0 \,, \qquad \boldsymbol{O}^{\alpha\beta}_0 = \boldsymbol{O}^{\alpha\beta}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{A}}_0]$$

- . . . [intermediate steps left to the reader as an exercise!]
- We can show that the integrand $u_1 \cdot \boldsymbol{O}_0 \cdot u_2$ is odd under $b \cdot k \mapsto -b \cdot k$,

$$u_1 \cdot \boldsymbol{J}_{\mathcal{O}(G^3)} \cdot u_2 = 0$$

  in agreement with the explicit result [Manohar, Ridgway, Shen '22]

The nontrivial components: $u_{1,2} \cdot \boldsymbol{P}_{\mathcal{O}(G^3)}$ and $b \cdot \boldsymbol{J}_{\mathcal{O}(G^3)} \cdot u_{1,2}$ can be evaluated by reducing them to (cut) **two-loop** integrals

[Herrmann, Parra-Martinez, Ruf, Zeng '21; Manohar, Ridgway, Shen '22] [Di Vecchia, CH, Russo, Veneziano '22].

Emitted Energy and Angular Momentum at $\mathcal{O}(G^4)$

[Heissenberg '25]

To next-to-leading order, we split $\boldsymbol{P}^{\alpha}_{\mathcal{O}(G^4)} = \boldsymbol{P}^{\alpha}_{\text{1rad}} + \boldsymbol{P}^{\alpha}_{\text{2rad}}$ and $\boldsymbol{J}^{\alpha\beta}_{\mathcal{O}(G^4)} = \boldsymbol{J}^{\alpha\beta}_{\text{1rad}} + \boldsymbol{J}^{\alpha\beta}_{\text{2rad}}$

- Integer-PN contributions (even in velocity):

$$\boldsymbol{P}^{\alpha}_{\text{1rad}} = 2\int_k \operatorname{Re}\boldsymbol{K}^{\alpha}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}^{(i)}_{1O} + \tilde{\mathcal{B}}_{1E}]\,, \qquad \boldsymbol{J}^{\alpha\beta}_{\text{1rad}} = 2\int_k \operatorname{Im}\boldsymbol{O}^{\alpha\beta}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}^{(i)}_{1O} + \tilde{\mathcal{B}}_{1E}]$$

- Half-odd-PN contributions (odd in velocity):

$$\boldsymbol{P}^{\alpha}_{\text{2rad}} = \int_k \left(2\operatorname{Re}\boldsymbol{K}^{\alpha}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}^{(h)}_{1O}] - \operatorname{Im}\boldsymbol{K}^{\alpha}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{C}}]\right),$$

$$\boldsymbol{J}^{\alpha\beta}_{\text{2rad}} = \int_k \left(2\operatorname{Im}\boldsymbol{O}^{\alpha\beta}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}^{(h)}_{1O}] + \operatorname{Re}\boldsymbol{O}^{\alpha\beta}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{C}}]\right)$$

These (naively) involve **three-loop** integrals.

Can their analytic structure dictated by unitarity help us?

Emitted Energy and Angular Momentum at $\mathcal{O}(G^4)$

[Heissenberg '25]

- The integer-PN contributions (even in velocity) behave as the $\mathcal{O}(G^3)$ ones,

$$b \cdot \boldsymbol{P}_{1\text{rad}} = 0\,, \qquad u_1 \cdot \boldsymbol{J}_{1\text{rad}} \cdot u_2 = 0$$

  and $u_{1,2} \cdot \boldsymbol{P}_{1\text{rad}}$ and $b \cdot \boldsymbol{J}_{1\text{rad}} \cdot u_{1,2}$ indeed involve integrals at **tree loops**

- For the half-odd-PN contributions (odd in velocity) we find instead

$$u_{1,2} \cdot \boldsymbol{P}_{2\text{rad}} = 2u^{\alpha}_{1,2} \int_k \text{Re}\, \boldsymbol{K}_{\alpha}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}^{(h)}_{1O}] \qquad \textbf{two loops}$$

$$b \cdot \boldsymbol{P}_{2\text{rad}} = -b^{\alpha} \int_k \text{Im}\, \boldsymbol{K}_{\alpha}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{C}}] \qquad \textbf{three loops}$$

$$u_{1,2} \cdot \boldsymbol{J}_{2\text{rad}} \cdot b = 2u^{\alpha}_{1,2} b^{\beta} \int_k \text{Im}\, \boldsymbol{O}_{\alpha\beta}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}^{(h)}_{1O}] \qquad \textbf{two loops}$$

$$u_1 \cdot \boldsymbol{J}_{2\text{rad}} \cdot u_2 = u^{\alpha}_1 u^{\beta}_2 \int_k \text{Re}\, \boldsymbol{O}_{\alpha\beta}[\tilde{\mathcal{A}}_0, \tilde{\mathcal{C}}] \qquad \textbf{three loops}$$

The 2rad energy and angular momentum (in the CM) only involve two-loop integrals!

Results: Half-Odd-PN Emitted Energy

[Heissenberg '25]

- **Warm-up**: 2rad emitted energy, $\boldsymbol{P}^{\alpha}_{\text{2rad}} = \boldsymbol{P}^{\alpha}_{\parallel} + (\cdots)b^{\alpha}$

$$\boldsymbol{P}^{\alpha}_{\parallel} = \frac{G^4 m_1^2 m_2^2}{b^4}\left[m_1(\mathcal{E}^{(1)}\check{u}_1^{\alpha} + \mathcal{E}^{(2)}\check{u}_2^{\alpha}) + (1 \leftrightarrow 2)\right]$$

with

$$\mathcal{E}^{(i)} = \frac{f_1^{(i)}}{\sigma^2 - 1} + f_2^{(i)}\,\frac{\operatorname{arccosh}\sigma}{(\sigma^2-1)^{3/2}} + f_3^{(i)}\,\frac{(\operatorname{arccosh}\sigma)^2}{(\sigma^2-1)^2}$$

for $i = 1, 2$ and polynomials in σ denoted by $f^{(i)}_{1,2,3}$ (here omitted for brevity)

- Perfectly matches [Dlapa, Kälin, Liu, Porto '22], where Q_1^{α} and Q_2^{α} where calculated up to $\mathcal{O}(G^4)$, using

$$\boldsymbol{P}^{\alpha} = -Q_1^{\alpha} - Q_2^{\alpha}\,.$$

Results: Half-Odd-PN Angular Momentum Loss

[Heissenberg '25]

- **New result**: 2rad emitted angular momentum, $\boldsymbol{J}^{\alpha\beta}_{\text{2rad}} = \boldsymbol{J}^{\alpha\beta}_{\perp} + (\cdots) u_1^{[\alpha} u_2^{\beta]}$

$$\boldsymbol{J}^{\alpha\beta}_{\perp} = \frac{G^4 m_1^2 m_2^2}{b^3}\left[m_1\left(\mathcal{F}^{(1)} b^{[\alpha} u_1^{\beta]} + \mathcal{F}^{(2)} b^{[\alpha} u_2^{\beta]}\right) + (1 \leftrightarrow 2)\right]$$

with

$$\mathcal{F}^{(i)} = \frac{g_1^{(i)}}{(\sigma^2-1)^2} + g_2^{(i)} \frac{\operatorname{arccosh}\sigma}{(\sigma^2-1)^{5/2}} + g_3^{(i)} \frac{(\operatorname{arccosh}\sigma)^2}{(\sigma^2-1)^3}$$

for $i = 1, 2$ and polynomials in σ denoted by $g^{(i)}_{1,2,3}$ (here omitted for brevity)

- Adding the static contribution [CH, Russo '24]

$$J_{\text{2rad}} = \boldsymbol{J}_{\text{2rad}} + \mathcal{J}_{\text{2rad}}\,, \qquad \mathcal{J}_{\text{2rad}} = \frac{G^2 p}{2b}\, Q^2_{\text{1PM}}\, \mathcal{I}(\sigma)^2$$

we obtain the 2rad angular momentum loss in the CM frame J_{2rad}

- The first few terms in its PN expansion agree with [Bini, Damour, Geralico '21, '22]

$$J_{\text{2rad}} = \frac{G^4 M^5}{b^3} \nu^2 p_\infty \left[\frac{448}{5} + \left(\frac{1184}{21} - \frac{220256}{1575}\nu\right) p_\infty^2 + \cdots\right]$$

Summary and Outlook

- The **eikonal approach** provides a framework to calculate scattering observables, including the **impulse**, the **waveform** and the emitted **energy and angular momentum**.
- PM expansion and soft theorems provide nicely **complementary** approaches
- The unitarity and analyticity properties of the amplitude/waveform can **simplify** the calculation of half-odd-PN (2rad) contributions to via **radiation-reaction** (**higher loops** → **lower loops**)

For the future:

- Amplitude derivation of the log-resummed waveform?
- Calculate the integer-PN (1rad) contribution to $J_{\alpha\beta}$
- NNLO waveform? Nonlinear memory effect