Rigorous approaches to non-perturbative observables in quantum field theories

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Axioms

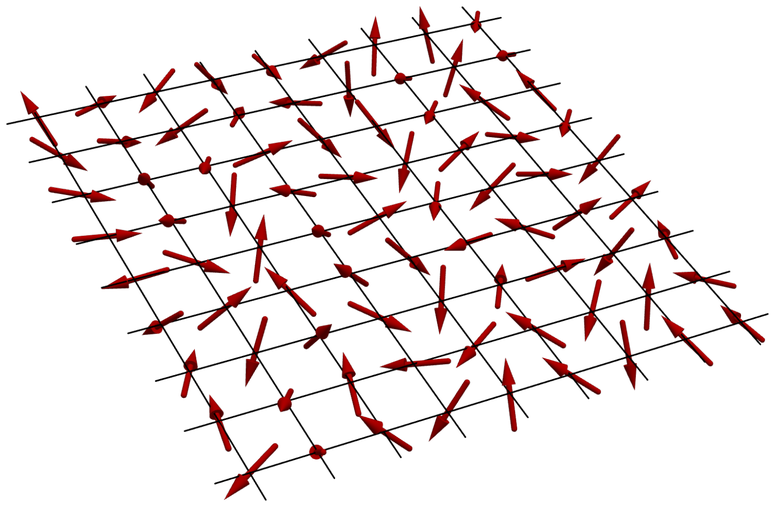
To prove anything rigorously and non-perturbatively,
one first has to choose a starting point:

which axiomatic system do we wish to adopt?

Today: either Wightman axioms or CFT axioms,
but first some context.

First-principles approaches

- Cutoff Λ so the path integral becomes controlled (often finite-dimensional), then investigate $\Lambda \to \infty$ limit.
- Constructive: $\lambda\phi^4$ exists in $d = 3$ [Glimm, Jaffe (1973); Feldman, Osterwalder (1976)] and is trivial in $d = 4$ [Aizenman, Duminil-Copin (2019)]
- Difficult but rewarding, many open questions:
  - Strong coupling? Critical behavior? Gauge theories?
  - Gauge theories in $d = 4$? [Jaffe, Witten (2000)]
  - Existence of $T_{\mu\nu}$? [Cardy 1996]
  - Non-Lagrangian theories?



Algebraic QFT

In the Haag-Kastler (1964) axioms, one postulates a Hilbert space $\mathscr{H}$, and for every open and bounded set $\mathscr{O} \subset \mathbb{M}$, a Von Neumann algebra $\mathscr{A}_{\mathscr{O}}$ respecting

- Isotony: $\mathscr{A}_{\mathscr{O}} \subset \mathscr{A}_{\mathscr{P}}$ if $\mathscr{O} \subset \mathscr{P}$.
- Causality: $[\mathscr{A}_{\mathscr{O}}, \mathscr{A}_{\mathscr{P}}] = 0$ if $\mathscr{O}$ and $\mathscr{P}$ are causally disjoint
- Poincaré covariance: $\alpha(g) : \mathscr{A}[\mathscr{O}] \to \mathscr{A}[g\mathscr{O}]$ for each g in Poincaré group

One also postulates a cyclic and separating *vacuum* $\Omega \in \mathscr{H}$, and a *spectrum condition* on the translation operator P^{μ}.

(See e.g. [Fewster, Rejzner (2019)])

Algebraic QFT

The axioms of algebraic QFT are very general but also quite abstract.

A classical result in AQFT is the Reeh-Schlieder theorem (1961), which states that the vacuum is cyclic for $\mathscr{A}[\mathscr{O}]$ for any open set $\mathscr{O}$, no matter how small.

More recently AQFT has seen a revival, in particular regarding possible applications to holography and the information loss paradox.

(see e.g. [Sorce (2024)])

The results of this talk will however require a little more structure.

Outline

1. Introduction
2. Wightman axioms
   - S-matrix theory, analyticity
   - S-matrix bootstrap
3. CFT axioms
   - Conformal bootstrap
   - Lightcone bootstrap
4. QFT in AdS
   - Connecting bootstraps
5. Outlook

Wightman axioms

The Wightman axioms date back to the 1950s. We postulate a Hilbert space $\mathscr{H}$, on whose states we can act with *operator-valued distributions* $\phi(x)$:

$$\int d^d x\, f(x)\phi(x) \text{ is an operator for any test function } f(x) \in \mathscr{S}(x)$$

Here $\mathscr{S}(x)$ are infinitely smooth test functions: $\forall a, b \in \mathbb{N} : \sup_{x \in \mathbb{M}} |x|^a D^b f(x) < \infty$.

We also postulate causality, Poincaré covariance, a cyclic and separating vacuum $\Omega \in \mathscr{H}$, and a spectrum condition on the translation operator P^μ.

Wightman axioms

- The Wightman axioms are not just obeyed by free field theories…
- Famous consequences of the Wightman axioms are, for example, the reconstruction theorem, cluster decomposition and the spin-statistics theorem. (see e.g. [Streater, Wightman (1964)])
- One also knows that the Wightman distributions are boundary values of the Schwinger functions, which are analytic in a large domain. This leads to a connection with the Osterwalder-Schrader axioms (1973) for Euclidean field theory (although the details are subtle - see *e.g.* [Kravchuk, Qiao, Rychkov (2021)]).
- The Wightman axioms are also the right starting point for S-matrix theory, to which we turn next.

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Haag-Ruelle theory

Haag (1958) - Ruelle (1962) theory explains how to construct S-matrices in the Wightman framework. We review it in [Hebbar, Pochart, BvR (to appear)].

- One postulates an *isolated* single-particle representation, spanned by "vectors" $|\underline{p}\rangle$ with $E_{\underline{p}} = (\underline{p}^2 + m^2)^{1/2}$ and $m > 0$.
- Then one *derives* the existence of multi-particle "in" and "out" states:
$$\int d^d\underline{p}_1 \ldots d^d\underline{p}_n f(\underline{p}_1)\ldots f(\underline{p}_n)|\underline{p}_1, \ldots \underline{p}_n\rangle_{\text{in/out}} = \lim_{t\to\mp\infty} \phi(f_1^{(t)})\ldots\phi(f_1^{(t)})|\Omega\rangle.$$
with free-field inner products. These are well-defined, so $S_{\alpha\beta} = {}_{\text{out}}\langle\alpha|\beta\rangle_{\text{in}}$ also exists.
- The proof is interesting, it relies on a careful choice of wave packets and cluster decomposition.
- Unitarity of the S-matrix is an axiom, known as *asymptotic completeness*.
- Extensions to the massless case exist [Buchholz (1975, 1977)] and deserve more attention.

LSZ formalism

Haag-Ruelle theory also provides the rigorous justification for the LSZ reduction formula (1955):

$$\text{out}\langle \underline{p}_1 \ldots | \ldots \underline{p}_n \rangle_{\text{in}} \simeq (p_1^2 - m^2) \ldots (p_n^2 - m^2) \, \langle T\phi(p_1) \ldots \phi(p_n) \rangle$$

provided the time-ordered products actually exist (and, I think, only for $m > 0$).

Now, as LSZ realized (1957), it is also possible to use other orderings.

We can *e.g.* use a retarded product, $R[\phi(x)\phi(y)] = [\phi(x), \phi(y)]\theta(x^0 - y^0)$, and get:

$$\text{out}\langle \underline{p}_1 \ldots | \ldots \underline{p}_4 \rangle_{\text{in}} \simeq (p_1^2 - m^2) \ldots (p_4^2 - m^2) \, \langle \phi(p_1) R[\phi(p_2)\phi(p_3)] \phi(p_4) \rangle$$

There are in fact *many* prescriptions using variously ordered correlation functions.

Primitive analyticity

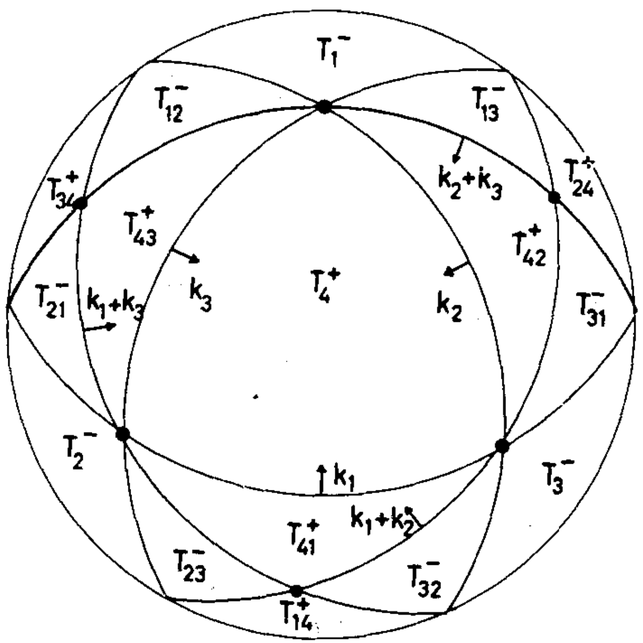
A retarded product like $[\phi(x), \phi(0)]\,\theta(x^0)$ *vanishes* unless x lies in the forward light cone V^+. Correspondingly,

$$\int d^d x\, e^{ip\cdot x}\,[\phi(x), \phi(0)]\,\theta(x^0)$$

is *analytic* for $\mathrm{Im}(p)$ such that $\mathrm{Im}(p)\cdot x > 0$ for all $x \in V^+$. Within an LSZ formula for an n point amplitude, this gives a priori a "primitive" analyticity region, say

$$D_1 \subset \mathbb{C}^{nd} \ni \{p_1, \ldots p_n\}.$$

The many different LSZ prescriptions give many different such primitive domains, say $D_k \subset \mathbb{C}^{nd}$ for $k \in 1,2,\ldots K$.



[Steinmann (1960), …]

Analyticity of amplitudes

Suppose $t(p_1, \ldots p_n) : \mathbb{C}^{nd} \to \mathbb{C}$ is analytic on the primitive domain $D := \bigcup_{k=1}^{K} D_k$.

Mathematical question: where is this function *actually* holomorphic?

This is not obvious. For example, by Hartog's extension theorem every function holomorphic on a thickened $S^3 \subset \mathbb{C}^2$ is also holomorphic in its interior! In other words, not every domain $D \subset \mathbb{C}^{nd}$ is a *domain of holomorphy*.

Even for $2 \to 2$ scattering we <u>do not know</u> the full domain of holomorphy associated to D.

Finding holomorphic extensions is done piecemeal, using *e.g.* the edge-of-the-wedge theorem, or the Jost-Lehmann (1957) - Dyson (1958) representation.

(Another complication: we only care about on-shell momenta $p^2 = m^2$.)

Analyticity

I do not have time to review the known analyticity results (even just for $2 \to 2$).

But consider the standard picture for $A(s, t)$: at a fixed t, a complex s plane with its s-and u-channel cuts. Even this analyticity has been proven only in a few cases [Bogoliubov et al (1958); Bros, Epstein, Glaser (1964); Martin (1966)].

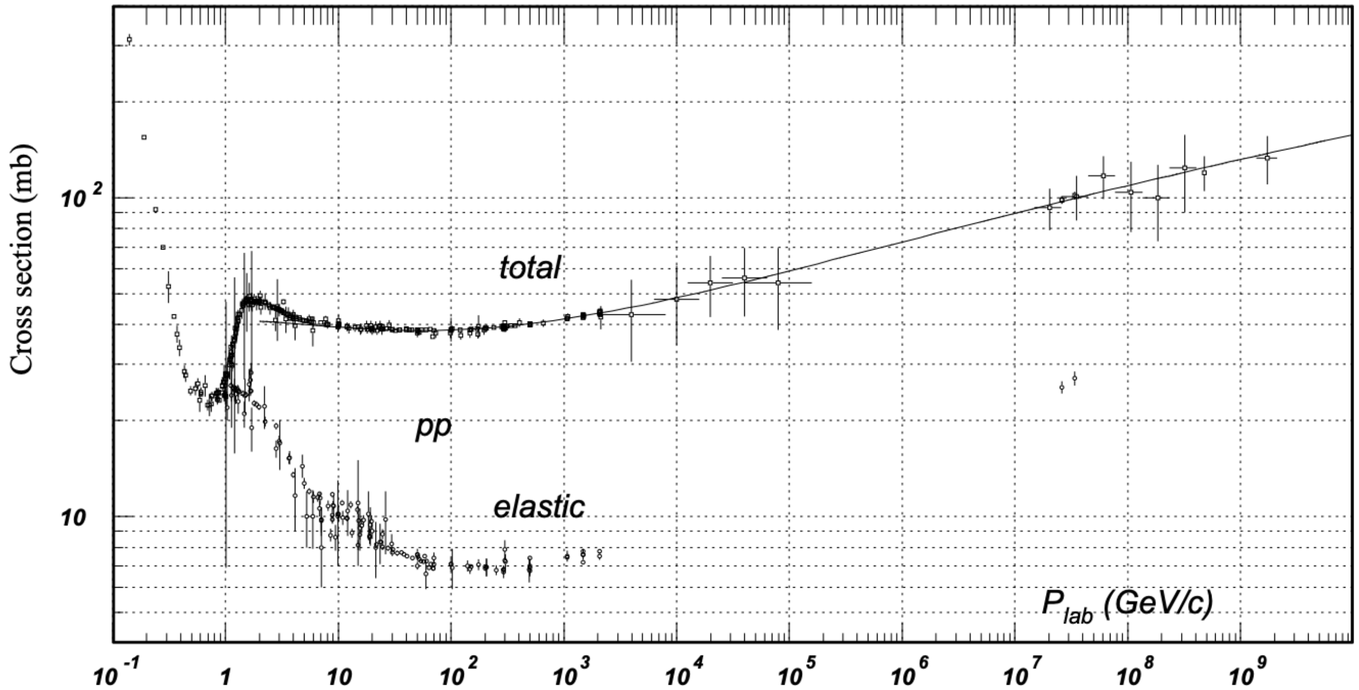
Other results include the Lehmann (1958, 1959) ellipses and some analyticity near every physical point [Bros, Epstein, Glaser (1965)].

Consequences of analyticity?

The most famous consequence is the Froissart (1961) -Martin (1963) bound:

$$A(s,0) \lesssim c\, s \log^2(s)$$

$$\sigma(s) \lesssim c\, \log^2(s)$$



[PDG]

Here $c \leq 16\pi/R$, with R is the largest real value of t for which the amplitude is t-analytic for all physical s. If $R = 4m_\pi^2$ then c is about 10 times too large…

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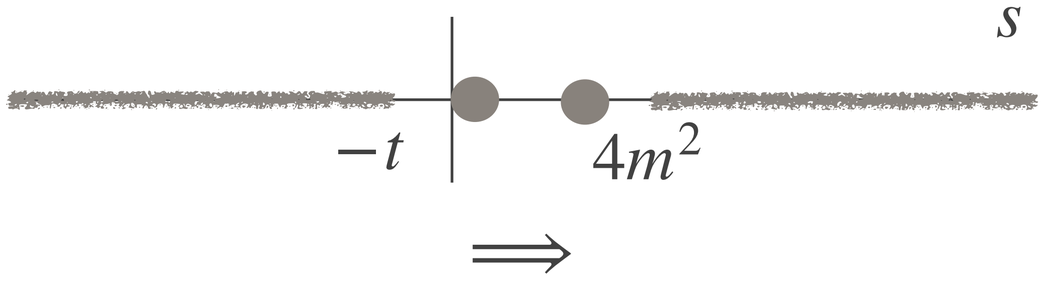
The S-matrix bootstrap

Bootstrap: universal results, from analyticity (crossing symmetry) and unitarity.

First *numerical* approach dates back to Lopez and Mennesier (1975).

More recently a revival, with two methods: either *assuming* a lot of analyticity [PPTvRV (2016-2017); many others] or relying only on rigorously proven analyticity [Guerrieri, Sever (2021); some others].

In some cases the difference is not very big.



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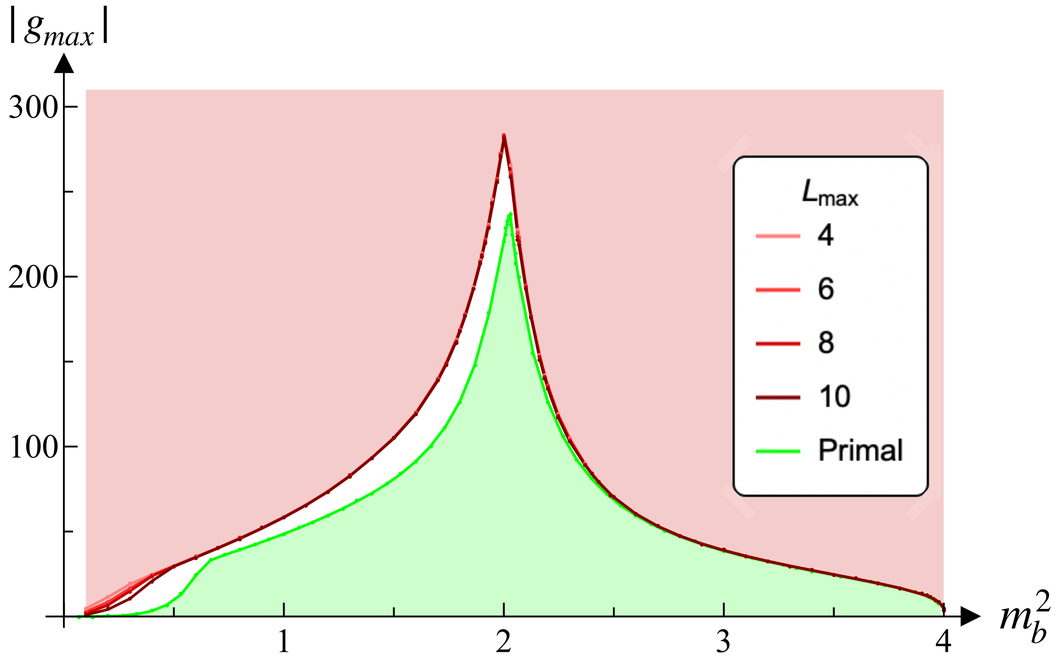

[Guerrier, Hebbar, BvR (2023)]

The S-matrix bootstrap: outlook

- The non-perturbative S-matrix bootstrap continues to be developed:
  - Relations to string- and M-theory [Guerrieri, Penedones, Vieira (2021)]
  - Analysis of pion scattering data [Guerrieri, Häring, Su (2024)]
  - New numerical algorithms [Gumus, Leflot, Tourkine, Zhiboedov (2024)]
  - … (see [Kruczenski, Penedones, BvR (2022)] for a review)
- Better data, experimental or lattice, would help!
- It would be great to understand *analyticity* better…
- … or to get strong *analytic results*. (Can the Froissart bound be improved?)
- For *massless* particles we (or I) know extremely little about analyticity! Nevertheless many promising EFT bounds [too many] so we should do better.
- For *IR divergent* amplitudes we (or I) know even less…

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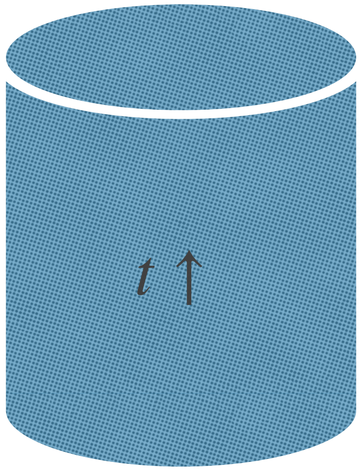
CFT axioms

- The "obvious" axiomatic framework for (higher-dimensional) Euclidean CFTs
- Correlation functions are Poincaré and Weyl covariant
- The natural Hilbert space $\mathscr{H}$ is that of the cylinder $\mathbb{R} \times S^{d-1}$
- There is a *state-operator correspondence, i.e.* completeness of the $\mathcal{O}(0)|0\rangle \in \mathscr{H}$
- There is an algebra of local operators, the operator product expansion:

$$\mathcal{O}_1(x)\mathcal{O}_2(y) = \sum_k \frac{c_{12}{}^k \mathcal{O}_k(y)}{|x-y|^{\Delta_1+\Delta_2-\Delta_k}}$$

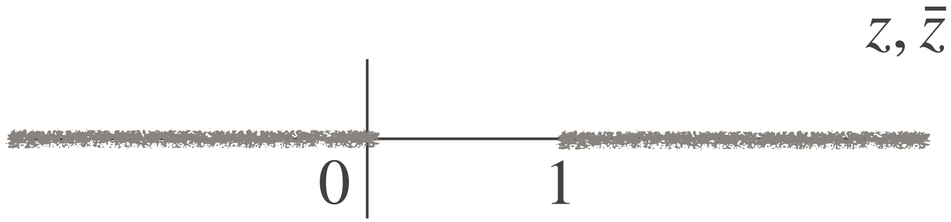
  with finite radius of convergence inside correlation functions.
- A recent write-up can be found in [Kravchuk, Qiao, Rychkov (2021)]



CFT properties

- CFTs correlation functions *immediately* have a big domain of analyticity:

$$\langle \phi(0)\phi(z,\bar{z})\phi(1)\phi(\infty)\rangle = (z\bar{z})^{-\Delta}\left[1 + g(z,\bar{z})\right],$$
$$\text{with } g(z,\bar{z}) = \sum_k \lambda_k^2 G_{\Delta_k}^{(\ell_k)}(z,\bar{z}).$$



- *Crossing symmetry* is also automatic:

$$(z\bar{z})^{-\Delta}\left[1 + g(z,\bar{z})\right] = ((1-z)(1-\bar{z}))^{-\Delta}\left[1 + g(1-z,1-\bar{z})\right]$$

- *Unitarity* now says that Δ_k are bounded from below, and $\lambda_k^2 > 0$.

- Question: what does this imply?

Conformal bootstrap

- Crossing symmetry, unitarity, linear programming $\longrightarrow$ bounds
  [Rattazzi, Rychkov, Tonni, Vichi (2008), many many others]
- Landscape of $d = 3$ theories slowly explored: $O(N)$, $O(N) \times O(M)$, GNY, …
- State of the art: $d = 3$ Ising stress tensor [Chang, Dommes,Erramilli, Homrich, Kravchuk, Liu, Mitchell, Poland, Simmons-Duffin (2024)]. Analysis of all four-point functions involving $T_{\mu\nu}, \sigma, \epsilon$ produces:

$$(\Delta_\sigma, \Delta_\epsilon) = (0.518148806(24), 1.41262528(29)),$$

  with *rigorous* (for physicists) error bars.
- Many open questions remain, most importantly *conformal windows* in $d = 3$ and $d = 4$.
- Finding "disappearing islands" [RRSvR (2024)], possibly with new techniques [RRSSSvR (2021].
- Stay tuned!

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The lightcone bootstrap

- Simplify crossing equation by replacing blocks $G_\Delta^{(\ell)}(z,\bar{z})$ with monomials $z^h\bar{z}^{\bar{h}}$ with $\bar{h}-h=\ell\in\mathbb{N}$. Then take $\bar{z}\to 1$, to get:

$$z^{-\Delta}\bar{z}^{-\Delta}+\sum_{h,\bar{h}}c_{h,\bar{h}}z^{h-\Delta}\bar{z}^{\bar{h}-\Delta}=(1-\bar{z})^{-\Delta}\Big((1-z)^{-\Delta}+O((1-\bar{z})^{\#})\Big)$$

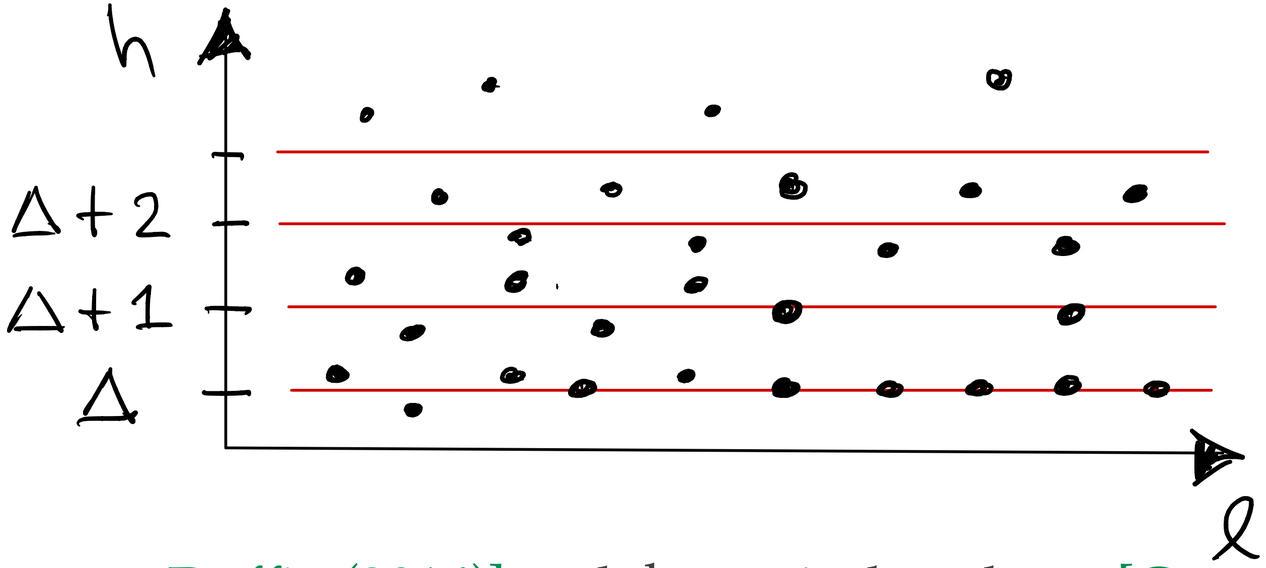
- Conjecture: there exist infinitely many terms $\bar{h}\to\infty$. Also, $z\to 0$ limit indicates that $h\to\Delta+n$ for these terms, with $n=0,1,2,\ldots$.

- Promotion to conformal blocks was done in [Fitzpatrick, Kaplan, Poland, Simmons-Duffin (2012); Komargodski, Zhiboedov (2012)].

- Mathematical question: how true is this?

The lightcone bootstrap

- Some scenarios:



- "Experimental" [Simmons-Duffin (2016)] and theoretical analyses [Caron-Huot (2017)] favor the simplest scenario.
- Important to be more rigorous! Implications for the flat-space limit of QFT in AdS (next), Mellin space [Mack (2009); Penedones, Silva, Zhiboedov (2019)], LLSS, …
- Also, quantitative predictions (!) and applications to numerical bootstrap [Su (2022)].

Rigorous lightcone bootstrap

<u>Theorem</u> [BvR (2024)] : $\forall \epsilon > 0$, there *are* infinitely many operators with $h \in (\Delta + n - \epsilon, \Delta + n + \epsilon)$ and $\bar{h} \sim \ell \to \infty$.

- Extends [Pal, Qiao, Rychkov (2022)] which proved the result for $n = 0$.
- Uses *Tauberian theorems* [many since 2012] but also some complex analysis.
- Direct corollary: *slightly* better control of LLSS.
- Can we improve? Subleading terms? Make ϵ shrink with ℓ, for example?

In the *modular bootstrap* we know [Pal, Qiao, BvR (two weeks ago)] that no spin can misbehave and that the window can shrink like $\ell^{-\eta}$ with $\eta < 1/4$.

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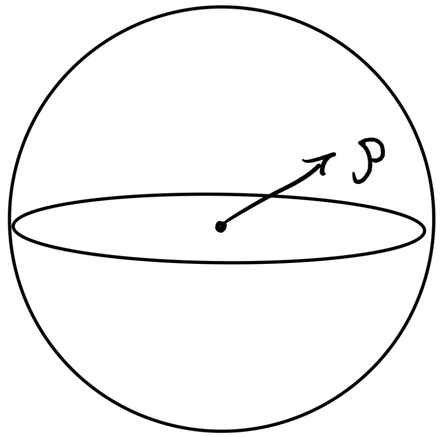
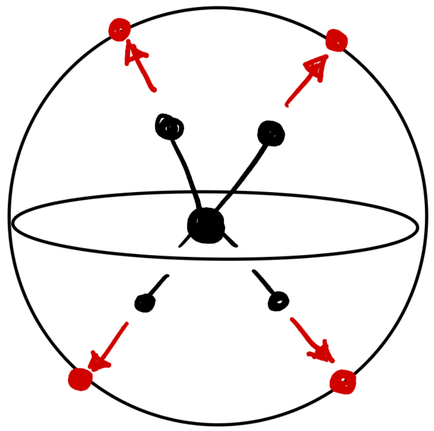
QFT in AdS

- Background: $ds^2 = d\rho^2 + R^2 \sinh^2(\rho/R) d\Omega_d^2$
- Observables:

$$\lim_{\rho\to\infty} \left(\prod_{i=1}^{k} e^{\Delta\rho_i/R} \right) \langle \phi(\rho_1, n_k) \ldots \phi(\rho_k, n_k) \rangle$$

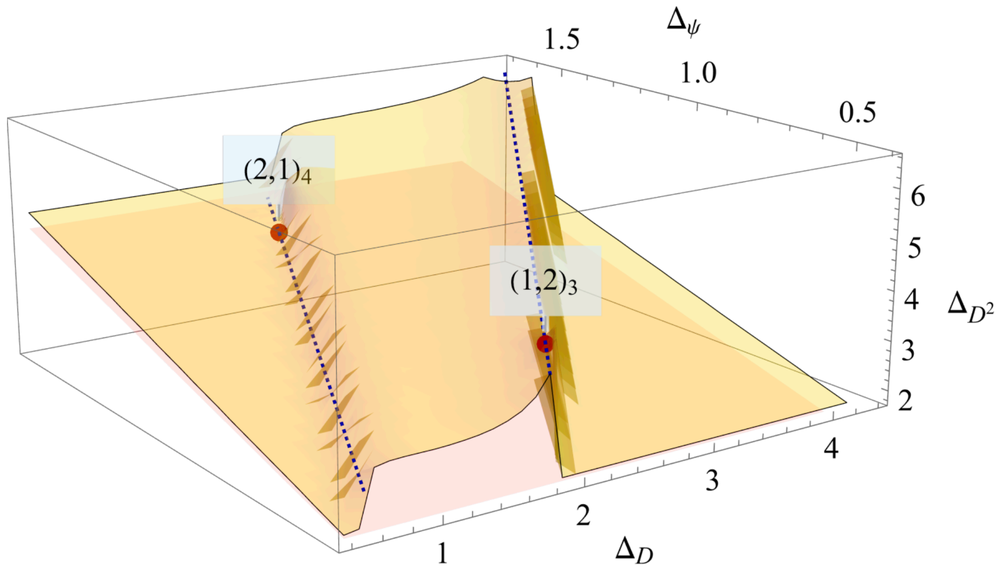
- Claim: these obey CFT axioms!
  [Callan-Wilczek (1990), PPTvRV (2016), …]
- Can we bootstrap QFTs in AdS?





QFT in AdS

- An RG flow in AdS gives rise to a *family* of solutions to crossing symmetry, labeled by μR with μ the RG scale. Can we bootstrap such families?



- Some attempts [Antunes, Costa, Penedones, Salgarkar, BvR (2021), …].

- No "islands" yet, so more control of bulk seems important [Meineri, Penedones, Spirig (2023)]

- Analytic bounds exist, mostly around free solutions [Mazac, Paulos, …]

- (Also: many interesting new physics questions!)

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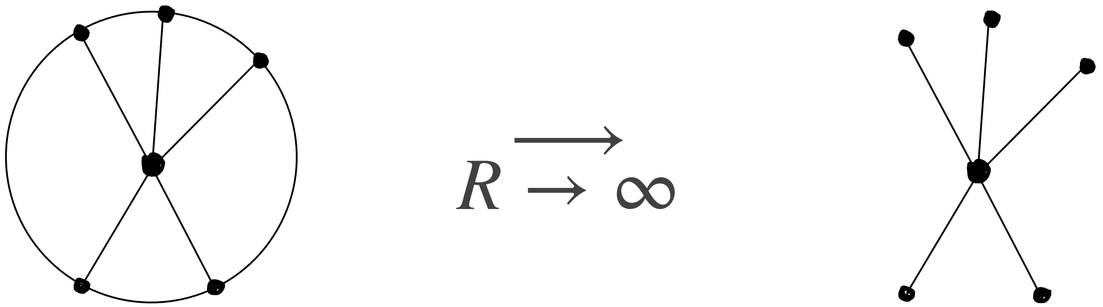
QFT in AdS and the flat-space limit

- Question: what happens to the boundary correlators as $R \to \infty$?
- Physical intuition: they should become *scattering amplitudes*.



- Since $\Delta(\Delta - d) = m^2 R^2$, difference between massless and massive particles.

QFT in AdS and the flat-space limit

- For *massless* particles we have correlation functions with $\Delta = d$.
- The flat-space limit was also studied for AdS/CFT [Polchinski (1999), Giddings, Penedones, …]
- More assumptions are needed to ensure that we are in the flat-space limit.
- Typically "EFT type" assumptions, where the theory is free plus small corrections, possibly with graviton.
- Analytic conformal bootstrap bounds become "rigorous" EFT S-matrix bounds! [Caron-Huot, Mazac, Rastelli, Simmons-Duffin (2021), …]

QFT in AdS and the flat-space limit

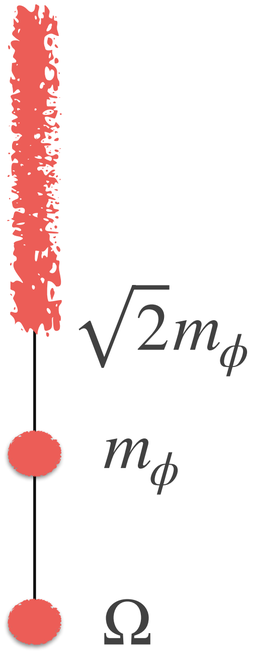
- For *massive* particles we have correlation functions with $\Delta \sim mR \to \infty$.
- We may require *only* assumptions à la Haag-Ruelle!
- Various prescriptions for extracting amplitude: Mellin space [PPTvRV (2017)] or simply as a direct limit [Dubovsky, Gorbenko, Mirbabayi (2017)].
- Question: when does the flat-space limit yield a finite amplitude? If so, does the analyticity of the correlation survive? How does unitarity arise?
- No theorems yet, but I think we are close. [Cordova, He, Paulos (2022)] got far for two-dimensional theories.
- Promising way to get *analyticity of scattering amplitudes* from the CFT axioms "QFT in AdS instead of LSZ" [BvR, Zhao (2022)]

QFT in AdS and the flat-space limit

Almost <u>theorem</u> [BvR, Zhao (2022)]: suppose the spectrum assumptions shown and that:

$$\lim_{R\to\infty} \frac{\langle \phi(x_1)\ldots\phi(x_4)\rangle_c}{(\text{AdS contact diagram})}$$

exists pointwise for some Euclidean kinematics. Then the limit exists in a much bigger domain, and the resulting amplitude has a big domain of analyticity (and obeys unitarity inequality).



- Can we make this a real theorem? Can we bound the correlator?
- Stay tuned!

Conclusions

- The “bootstrap” idea, in its modern incarnation, is going far beyond just numerical analyses of correlation functions and amplitudes.
- It is worth revisiting old ideas. We have learned many things about non-perturbative quantum field theories since the 1960s.
- There is *a lot* to do.
- In my view much of this is worthwhile: it helps us understand what quantum field theory really is, but also because it can lead to quantitative predictions — from a careful axiomatic approach.