Analytic properties of infrared-finite amplitudes in theories with long-range forces

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Motivation

Infrared divergences occur in four-dimensional theories with massless particles making them relevant to a wide range of research programs: collider physics, gravitational-waves, S-matrix bootstrap....

Despite their prevalence and long-history these divergences are not fully under control:

1. There are very few **explicitly calculated** IR-finite amplitudes [Hannesdottir, Schwartz '19], [Forde, Signer '04].
2. Fundamental challenges remain in proving theorems on how **causality**, **unitarity**, and **crossing symmetry** constrain scattering amplitudes.

e.g. causality (Steinmann-relations), unitarity (Froissart-Martin bound), crossing symmetry ($2 \to 2$ proof requires mass-gap).

Motivation

Consequently S-matrix bootstrap encounters difficulties in $d = 4$ [Guerrieri,Murali,Penedones, Vieira,'22], [Caron-Huot, Li, Parra-Martinez, Simmons-Duffin '22]

"No long-range interactions"

Overview

IR-divergences occur because standard perturbation theory **incorrectly** assumes that the **in and out states** behave as **free states** at asymptotic times.
This is clearly wrong as can even been seen classically
$\ddot{\vec{x}} \sim \frac{1}{|\vec{x}|^2} \sim \frac{1}{|\vec{p}|^2 t^2} \implies$

$$x^\mu(\tau) = b^\mu + \frac{p^\mu}{m}\tau + c^\mu \log(\tau). \tag{1}$$

[Sahoo, Sen '19].This is reflected in the Feynman rules where we use **free-plane waves** $e^{ip\cdot x}$ meeting at interaction vertices. Instead it is more natural to use

$$e^{-iEt+i\vec{p}\cdot\vec{x}+i\gamma\log(|\vec{p}||\vec{x}|-\vec{p}\cdot\vec{x})}, \tag{2}$$

distorted plane waves which encode the asymptotic trajectories [Rowe '85]. Successfully applied in **non-relativistic** quantum mechanics for atomic collisions.

Overview

However, it is challenging to generalize this to **relativistic** QFT because there are **no completeness relations** known for the distorted wavefunctions. However there are completeness relations for the **relativistic Coulomb wavefunctions**.

$$f_{\text{in}}(x,p) = \Gamma(1+i\gamma)e^{-\frac{\pi\gamma}{2}}e^{-iEt+i\vec{p}\cdot\vec{x}}{}_1F_1\Big(-i\gamma, 1, i(|\vec{p}||\vec{x}| - \vec{p}\cdot\vec{x})\Big), \tag{3}$$

$$\gamma(p) = \frac{e_1 e_2 E}{4\pi|\vec{p}|}, \qquad E = \sqrt{|\vec{p}|^2 + m^2} \tag{4}$$

The large $|\vec{x}|$ expansion of this coincides with the distorted plane waves and hence encode the asymptotic trajectories. Completely **eliminates Coulomb phase divergence in NRQM** (will review), and will demonstrate how it eliminates it in a simple relativistic setting of $2 \to 2$ charged particle scattering with one of the particles **infinitely massive**. Integrals are **simpler** than distorted plane waves, and I would argue that they are **much simpler than the infinite sum of Feynman diagrams they encode**.

Overview

But what about the **real radiative divergence**?I will review that the real radiative divergence and Coulomb phase divergence are **related by analytic properties** such as **crossing symmetry**, and the absence of the **pseudothreshold** singularity on the physical Riemann sheet.
Consequently, *if* either of these properties hold for the IR-finite S-matrix, then an unambiguous treatment of the Coulomb phase divergence could be leveraged to gain insight into taming the real radiative.

Outline

- A very brief history of IR-divergences.
- Review of non-perturbative solution to Coulomb scattering in NRQM.
- Modified connectedness structure in theories with long range forces. Consequently a modified general optical theorem.
- Factorization on the bound state poles.
- Diagonalizing the asymptotic Hamiltonian.
- Modified LSZ reduction.
- Analytic relation between Coulomb phase and real radiative divergence.
- Future directions.

Brief history of IR divergences

- 1937 Bloch and Nordsieck (BN) taught us that **inclusive cross sections** are IR-finite.
- I will **not** discuss massless charged particles (gives rise to **collinear divergences**), and I will only discuss theories with genuinely long-range forces, so **not QCD**.For massless charged particles BN cancellation fails, and one resorts to the **KLN** theorem. In QCD one focuses on **infrared safe** observables which factorize into a perturbatively calculable part and a non-perturbative part extracted from experiment.
- To study unitarity, causality, and crossing symmetry, it is preferable to alleviate divergences on the level of **amplitudes**.During the 60's the efforts of many culminated in the **Faddeev-Kulish** approach.

Brief history of IR-divergences

The FK approach factors out the asymptotic evolution

$$S_{\mathsf{FK}} = \lim_{t_\pm \to \pm\infty} e^{-R(t_+)} e^{-i\Phi(t_+)} \mathcal{S} e^{-i\Phi(t_-)} e^{R(t_-)}, \tag{5}$$

where

$$\Phi(t) = \frac{\alpha}{2} \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} : \rho(p)\rho(q) : \frac{p \cdot q}{\sqrt{(p \cdot q)^2 - m^4}} \ln |t/t_0|, \tag{6}$$

The FK approach has provided insight into the infrared structure of gauge theories and gravity [Choi, Akhoury '17],however in the 50 years since the original paper, **very few explicit amplitudes** have been calculated. There is an **inherent ambiguity** in the approach, e.g. choices for t_0 can affect the analytic properties of the resultant amplitudes.Practical insights into **analytic properties** of amplitudes has been limited.

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An important lesson from NRQM

When solving the non-relativistic Coulomb scattering problem **non-perturbatively**, **no ambiguous scales** appear in the scattering amplitudes.

$$H = \frac{p^2}{2m} + \frac{e_1 e_2}{4\pi}\frac{1}{r} \tag{7}$$

The **exact in/out** states with eigenvalue $E = \frac{|\vec{p}|^2}{2m}$ are known

$$\phi_{\text{in}}(\vec{x},\vec{p}) = \Gamma(1+i\gamma)e^{-\frac{\pi\gamma}{2}}e^{-iEt+i\vec{p}\cdot\vec{x}}\,{}_1F_1\Big(-i\gamma, 1, i(|\vec{p}||\vec{x}| - \vec{p}\cdot\vec{x})\Big) \tag{8}$$

where $\gamma = \frac{e_1 e_2 m}{4\pi|\vec{p}|}$. Doing a large $|\vec{x}|$ is insightful

$$\lim_{|\vec{x}|\to\infty}\phi_{\text{in}}(\vec{x},\vec{p}) = e^{-iEt}\Big(e^{i\vec{p}\cdot\vec{x}+i\gamma\log(|\vec{p}||\vec{x}|-\vec{p}\cdot\vec{x})} - \gamma\frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)}\frac{e^{i(|\vec{p}||\vec{x}|-\gamma\log(|\vec{p}||\vec{x}|-\vec{p}\cdot\vec{x}))}}{|\vec{p}||\vec{x}|-\vec{p}\cdot\vec{x}}\Big) \tag{9}$$

An important lesson from NRQM

The **exact** amplitude is given by the inner product of the in state with the out state

$$A^{\mathsf{NRQM}}(\vec{p}_1, \vec{p}_2) = \int \mathrm{d}^3x\, \phi^\star_{\mathsf{out}}(\vec{x}, \vec{p}_2)\phi_{\mathsf{in}}(\vec{x}, \vec{p}_1) \tag{10}$$

$$= -\lim_{\epsilon_+ \to 0} \frac{i\pi}{2} \frac{e_1 e_2}{|\vec{p}_1|^2} \delta(E_1 - E_2) \frac{\Gamma(1 + i\gamma)}{\Gamma(1 - i\gamma)} \left(\frac{4|\vec{p}_1|^2}{|\vec{p}_1 - \vec{p}_2|^2} \right)^{1 + i\gamma_1 - \epsilon_+}, \tag{11}$$

Note that there are **no ambiguous scales**.Will review: IR divergence arise purely because of **incorrect assumptions** of standard perturbation theory. Standard perturbation theory **does not resum** to the exact amplitude. Important: no theorems from scattering theory assumed! Note that there is no $\mathbb{1} + iT$, instead there is a ϵ_+ which represents the unique distributional interpretation of the amplitude.

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Modified connectedness structure

There have been brief mentions in the literature that $S = \mathbb{1} + iT$ decomposition only reflects the connectedness in theories with short range forces [Eden, Landshoff, Olive, Polkinghorne '66].Lets verify that the exact NRQM amplitude satisfies unitarity,

$$\int \mathrm{d}^3 l \; A(p_1 \to l) A^\star(p_2 \to l) = \delta^{(3)}(\vec{p}_1 - \vec{p}_2) \tag{12}$$

The proof will confirm that the inner product of the in-state with the out-state does not contain the usual disconnected component,

$$\int \mathrm{d}^3 l \; A(p_1 \to l) A^\star(p_2 \to l) = \frac{\pi \gamma^2}{|\vec{p}|} \delta(E_1 - E_2) I(\hat{p}_1, \hat{p}_2) \tag{13}$$

$$I(\hat{p}_1, \hat{p}_2) = \lim_{\epsilon_+ \to 0} \int \mathrm{d}^2 \Omega_l \left(\frac{2}{1 - \hat{p}_1 \cdot \hat{p}_l} \right)^{1 + i\gamma - \epsilon_+} \left(\frac{2}{1 - \hat{p}_2 \cdot \hat{p}_l} \right)^{1 - i\gamma - \epsilon_+} \tag{14}$$

Modified connectedness structure

The integral $I(\hat{p}_1, \hat{p}_2)$ actually has a **2D conformal symmetry** (Runge-Lenz) symmetry that makes it trivial to evaluate,

$$\hat{k} = \frac{1}{1+|z|^2}\left(z+\bar{z}, i(\bar{z}-z), 1-|z|^2\right) \tag{15}$$

$$z = e^{i\phi}\tan\frac{\theta}{2}\,. \tag{16}$$

So the integral becomes

$$I(\hat{p}_1, \hat{p}_2) = 4(1+|z_1|^2)^{1+i\gamma}(1+|z_2|^2)^{1-i\gamma}$$

$$\lim_{\epsilon_+\to 0}\int \mathrm{d}^2 z_l \frac{1}{|z_l - z_1|^{2+2i\gamma-\epsilon_+}|z_l - z_2|^{2-2i\gamma-\epsilon_+}}. \tag{17}$$

$$= \frac{4\pi^2}{\gamma^2}(1+|z_1|^2)^2\delta^{(2)}(z_1 - z_2) \tag{18}$$

[Dolan, Osborn '12]. Combining this $\delta^{(2)}(z_1 - z_2)$ with $\delta(E_1 - E_2)$ demonstrates that **unitarity is satisfied**.

Modified connectedness structure

We have verified

$$\int d^3l \; A(p_1 \to l) A^\star(p_2 \to l) = \delta^{(3)}(\vec{p}_1 - \vec{p}_2) \tag{19}$$

for

$$A(\vec{p}_1 \to \vec{p}_2) = \lim_{\epsilon_+ \to 0} \frac{i\pi}{2} \frac{e_1 e_2}{|\vec{p}_1|^2} \delta(E_1 - E_2) \frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)} \left(\frac{4|\vec{p}_1|^2}{|\vec{p}_1 - \vec{p}_2|^2} \right)^{1+i\gamma_1-\epsilon_+} \tag{20}$$

Note that if $\langle \text{out}|\text{in}\rangle = \delta_{\text{in,out}} + iA$ then unitarity would have instead required $A - A^\star = iAA^\star$. [Herbst '73] proven that the ϵ_+ prescription is the **unique** distributional completion in the forward direction consistent with unitarity. Suggests that the connectedness structure may differ in SQED.

Modified optical theorem

The usual $S = \mathbb{1} + iT$ decomposition leads to the general optical theorem $T - T^\dagger = iTT^\dagger$. We can derive an analog of this.Consider NRQM scattering on

$$H = \frac{p^2}{2m} + \frac{\alpha}{r} + \frac{\alpha^2}{2m}\frac{1}{r^2} \tag{21}$$

The scattering amplitude is the inner product of the in-state with the out-state of this Hamiltonian

$$P(p_1 \to p_2) := \langle \psi_{\mathsf{out}}(p_2) | \psi_{\mathsf{in}}(p_1) \rangle \tag{22}$$

We can then decompose $P(p_1 \to p_2)$ into the Coulomb amplitude (no $\frac{\alpha^2}{2m}\frac{1}{r^2}$) plus corrections

$$\underbrace{P(p_1 \to p_2)}_{\text{exact amplitude}} = \underbrace{A(p_1 \to p_2)}_{\text{Coulomb amplitude}} + \underbrace{T(p_1 \to p_2)}_{\frac{1}{r^2}\text{ corrections}} \tag{23}$$

Modified optical theorem

$$\underbrace{P(p_1 \to p_2)}_{\text{exact amplitude}} = \underbrace{A(p_1 \to p_2)}_{\text{Coulomb amplitude}} + \underbrace{T(p_1 \to p_2)}_{\frac{1}{r^2}\text{ corrections}} \tag{24}$$

Unitarity requires

$$\int \mathrm{d}^3 l \; P(p_1 \to l) P^\star(p_2 \to l) = \delta^{(3)}(p_1 - p_2) \tag{25}$$

But $A(p_1 \to p_2)$ **already satisfies the unitarity relation** (much like $\mathbb{1}$ in the short range case). Plugging the decomposition

$$\int \mathrm{d}^3 k \Big(A(p_1 \to k) T^\star(p_2 \to k) + A^\star(p_2 \to k) T(p_1 \to k) + T(p_1 \to k) T^\star(p_2 \to k) \Big) = 0. \tag{26}$$

Relates different numbers of $\frac{\alpha^2}{r^2}$ interactions. Verified explicitly up to order α^4.

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Factorization on the bound state poles

An intriguing feature of the exact NRQM amplitude ($\gamma = \alpha \frac{m}{|\vec{p}|}$)

$$\mathcal{A}^{\text{exact}}(E,\theta) = -i\gamma \frac{4\pi^2}{|\vec{p}_1|}\delta(E_1 - E_2)\frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)}\left(\frac{2|\vec{p}_1|}{|\vec{p}_1 - \vec{p}_2|}\right)^{2+2i\gamma}, \tag{27}$$

is that comes equipped with a specific scale $|\vec{p}|$. Whereas using standard perturbation theory with a regulator such as a photon mass μ instead gives

$$\mathcal{A}^{\text{reg.}}(E,\theta) = -i\gamma \frac{4\pi^2}{|\vec{p}_1|}\delta(E_1 - E_2)\frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)}\left(\frac{2\mu}{|\vec{p}_1 - \vec{p}_2|}\right)^{2+2i\gamma}. \tag{28}$$

Well known that the $\Gamma(1+i\gamma)$ encode the **bound state spectrum of hydrogen** $1 + i\gamma = -n$.Note that $|\vec{p}|$ acquires a specific value at these poles.Thus there is a difference between the two for how the amplitude factorizes on its bound state poles.

Factorization on the bound state poles

Evaluating the residue of the exact amplitude on the bound state poles, one sees that it factorizes into a product of continuum to bound state transitions

$$\underset{E\to E_n}{\text{Res}}\,\mathcal{A}^{\text{exact}}(E,\theta)$$
$$= 2\pi i \sum_{l=0}^{n-1}\sum_{m=-l}^{m=l}\left(\underset{E\to E_n}{\text{Res}}\,\langle p_2^{\text{out}}|B_{nlm}\rangle\right)\left(\underset{E\to E_n}{\text{Res}}\,\langle B_{nlm}|p_1^{\text{in}}\rangle\right) \quad (29)$$

whereas the μ regulated amplitude exhibits a similar relation but with a factor of $\left(n\mu/\alpha E_n\right)^n$, indicating that the exact amplitude has some more natural analytic properties.

More broadly, in any scattering amplitude involving a regulator μ, different choices of μ - as functions of the external momenta - **can affect important analytic properties of the amplitude**. These include whether the **Steinmann** relations hold, the locations of **branch points**, **crossing symmetry**, **factorization** structure at bound-state poles...

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Moving now towards QFT

Review the root cause of IR-divergences.Standard perturbation theory follows from the following distributional equality

$$\lim_{t\to-\infty} e^{-i\hat{H}t}\,|\psi_{\mathsf{in}}(p_H,p_L)\rangle = \lim_{t\to-\infty} e^{-i\hat{H}_0 t}\,|(p_H,p_L)_{\mathsf{free}}\rangle\,. \qquad (30)$$

From this equality follows standard perturbation theory (the Feynman rules)

$$\begin{aligned}&\langle\psi_{\mathsf{out}}(k_H,k_L)|\psi_{\mathsf{in}}(p_H,p_L)\rangle\\ &= \langle(k_H,k_L)_{\mathsf{free}}|\mathsf{T}\exp\left(-i\int_{-\infty}^{\infty}\mathrm{d}t\,V(t)\right)|(p_H,p_L)_{\mathsf{free}}\rangle \qquad (31)\end{aligned}$$

We can see explicitly in the exactly solvable NRQM example that the **distributional equality (30) does not hold**

$$\lim_{|\vec{x}|\to\infty}\phi_{\mathsf{in}}(\vec{x},\vec{p}) = e^{-iEt}\Big(e^{i\vec{p}\cdot\vec{x}+i\gamma\log(|\vec{p}||\vec{x}|-\vec{p}\cdot\vec{x})} + \mathcal{O}(1/r)\Big). \qquad (32)$$

Moving now towards QFT

Culprit is the distributional equality. Consider simplest relativistic QFT setup where the Feynman rules lead to a **Coulomb phase divergence**. Consider $2 \to 2$ scattering in scalar QED, where one of the particles is **infinitely massive**, and we **ignore soft radiation**.Then we expect a correct distributional equality to be

$$\lim_{t\to-\infty} e^{-i\hat{H}t} |\psi_{\mathsf{in}}(p_H, p_L)\rangle = \lim_{t\to-\infty} e^{-i\hat{H}_{\mathsf{as}}t} |\phi_{\mathsf{in}}(p_H, p_L)\rangle \,. \quad (33)$$

where H_{as} is the Hamiltonian describing the light particle moving in the Coulomb background of the heavy particle, and $|\phi_{\mathsf{in}}(p_H, p_L)\rangle$ are eigenstates of the asymptotic Hamiltonian.(33) is the starting point for a perturbation theory for SQED with no Coulomb phase divergence. Prerequisite is to fully solve RHS. Contrast with FK approach.

Diagonalizing the asymptotic Hamiltonian

Asymptotic Hamiltonian density

$$\mathcal{H}_{\mathsf{as}} = \pi^\dagger\pi + ieA^0(\pi^\dagger\phi^\dagger - \phi\pi) + \phi^\dagger(-\vec{D}^2 + m^2)\phi + e^2 A^\mu A_\mu \phi^\dagger\phi,$$

$$A^\mu(x, u_s) = \frac{e_s}{4\pi}\frac{u_H^\mu}{\sqrt{(u_H\cdot x)^2 - x^2}}$$

We expand the quantum field in terms of Coulomb wavefunction modes (using completeness of $f_{\mathsf{in}}(x,p), B_{nlm}$)

$$\hat{\phi}(\vec{x},t) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p}\left(\hat{a}_{\mathsf{in}}(\vec{p})f_{\mathsf{in},+}(x,p) + \hat{b}^\dagger_{\mathsf{in}}(\vec{p})f_{\mathsf{in},-}(x,p)\right) + \sum_{n,l,m}\hat{c}_{nlm}B_{nlm}(x). \tag{34}$$

where $f_{\mathsf{in}} \sim e^{ip\cdot x}{}_1F_1(...)$.

Diagonalizing the asymptotic Hamiltonian

Plugging in this mode expansion, diagonalizes Hamiltonian

$$\hat{H}_{\mathsf{as}}(u_s^r) = \frac{1}{2}\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\Big(\hat{a}_{\mathsf{in}}^\dagger(\vec{p})\hat{a}_{\mathsf{in}}(\vec{p}) + \hat{b}_{\mathsf{in}}^\dagger(\vec{p})\hat{b}_{\mathsf{in}}(\vec{p})\Big) + \sum_{nlm} E_n \hat{c}_{nlm}^\dagger \hat{c}_{nlm},$$

Impose the canonical commutation relations

$$\begin{aligned}
[\hat{a}_{\mathsf{in}}(p_1), \hat{a}_{\mathsf{in}}^\dagger(p_2)] \quad &= [\hat{b}_{\mathsf{in}}(p_1), \hat{b}_{\mathsf{in}}^\dagger(p_2)] \quad = (2\pi)^3 2E_p \delta^3(\vec{p}_1 - \vec{p}_2)\\
[\hat{a}_{\mathsf{out}}(p_1), \hat{a}_{\mathsf{out}}^\dagger(p_2)] &= [\hat{b}_{\mathsf{out}}(p_1), \hat{b}_{\mathsf{out}}^\dagger(p_2)] = (2\pi)^3 2E_p \delta^3(\vec{p}_1 - \vec{p}_2),
\end{aligned}$$

Using which one can verify

$$[\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \tag{35}$$

Computing the Bogoliubov coefficients one derives the in-out commutation relations

$$[\hat{a}_{\mathsf{out}}(p_1), \hat{a}_{\mathsf{in}}^\dagger(p_2)] = \mathcal{A}_0^+(p_2 \to p_1)$$

$$= i\pi\ ee_s \delta\big(u_s \cdot (p_1 - p_2)\big) \frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)} \frac{p_1 \cdot u_s}{m^2 - (p_1 \cdot u_s)^2} \left(\frac{4\Big(m^2 - (p_1 \cdot u_s)^2\Big)}{(p_1 - p_2)^2}\right)^{1+i\gamma}$$

Defining the states

One can define the in/out vacua

$$\hat{a}_{\mathsf{in}}(p)\,|\Omega_{\mathsf{in}}\rangle = \hat{b}_{\mathsf{in}}(p)\,|\Omega_{\mathsf{in}}\rangle = \hat{c}_{nlm}\,|\Omega_{\mathsf{in}}\rangle = 0 \tag{36}$$

$$\hat{a}_{\mathsf{out}}(p)\,|\Omega_{\mathsf{out}}\rangle = \hat{b}_{\mathsf{out}}(p)\,|\Omega_{\mathsf{out}}\rangle = \hat{c}_{nlm}\,|\Omega_{\mathsf{out}}\rangle = 0, \tag{37}$$

From which one can build a Fock space of states

$$|\Phi^{+}_{\mathsf{in}}(p)\rangle = \hat{a}^{\dagger}_{\mathsf{in}}(\vec{p})\,|\Omega_{\mathsf{in}}\rangle \quad , \quad |\Phi^{-}_{\mathsf{in}}(p)\rangle = \hat{b}^{\dagger}_{\mathsf{in}}(\vec{p})\,|\Omega_{\mathsf{in}}\rangle\,, \tag{38}$$

$$|\Phi^{+}_{\mathsf{out}}(p)\rangle = \hat{a}^{\dagger}_{\mathsf{out}}(\vec{p})\,|\Omega_{\mathsf{out}}\rangle \quad , \quad |\Phi^{-}_{\mathsf{out}}(p)\rangle = \hat{b}^{\dagger}_{\mathsf{out}}(\vec{p})\,|\Omega_{\mathsf{out}}\rangle\,. \tag{39}$$

Using the commutation relations one verifies

$$\hat{H}_{\mathsf{as}}\,|\Phi^{+}_{\mathsf{in}}(p)\rangle = E_p\,|\Phi^{+}_{\mathsf{in}}(p)\rangle\,, \tag{40}$$

Scattering amplitudes

Computing the Bogoliubov coefficients relating the in to the out creation/annihilation operators one can show that the in and out **vacua** can be identified up to a phase

$$|\langle \Omega_{\text{out}}|\Omega_{\text{in}}\rangle| = 1, \qquad \langle \Omega_{\text{out}}|\Omega_{\text{in}}\rangle = c_v. \tag{41}$$

One can also calculate the scattering amplitudes

$$\langle \Phi^+_{\text{out}}(q)|\Phi^+_{\text{in}}(p)\rangle\, c_v^{-1} = c_v^{-1}\, \langle \Omega_{\text{out}}|\hat{a}_{\text{out}}(q)\hat{a}^\dagger_{\text{in}}(p)|\Omega_{\text{in}}\rangle \tag{42}$$

$$= i\pi\; ee_s\delta\big(u_s\cdot(p_1-p_2)\big)\frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)}\frac{p_1\cdot u_s}{m^2-(p_1\cdot u_s)^2}\left(\frac{4\big(m^2-(p_1\cdot u_s)^2\big)}{(p_1-p_2)^2}\right)^{1+i\gamma} \tag{43}$$

which are IR-finite.

Correlation functions

Later we will revisit LSZ reduction so we will need the time-ordered correlator

$$G_F(x,y) = \langle \Omega | T(\phi^\dagger(x)\phi(y)) | \Omega \rangle$$
$$= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} \left(\theta(x^0 - y^0) f^\star_{\mathsf{in},-}(x,p) f_{\mathsf{in},-}(y,p) + \theta(y^0 - x^0) f^\star_{\mathsf{in},+}(x,p) f_{\mathsf{in},+}(y,p) \right)$$
$$+\theta(y^0 - x^0) \sum_{nlm} B_{nlm}(y) B^\star_{nlm}(x).$$

which is verified to satisfy Green's function equation in a Coulomb background. I also demonstrate that the field commutators vanish outside the lightcone thus satisfying this **causality** requirement

$$\langle \Omega | [\phi^\dagger(x), \phi(y)] | \Omega \rangle = 0 \quad \text{for} \quad (x-y)^\mu \quad \text{spacelike}$$

Summary of the exactly solvable asymptotic dynamics

One can exactly solve the QFT on the RHS

$$\lim_{t\to-\infty} e^{-i\hat{H}t}\,|\psi_{\mathsf{in}}(p_H, p_L)\rangle = \lim_{t\to-\infty} e^{-i\hat{H}_{\mathsf{as}}t}\,|\phi_{\mathsf{in}}(p_H, p_L)\rangle\,. \quad (44)$$

describing the lighter particle moving in the heavier particle's Coulomb field.This distributional equality can serve as a starting point for a perturbation theory for $2 \to 2$ scattering where one of the particles is infinitely massive, which does not have the Coulomb phase divergence.A nice feature of this approach is that the **tree-level** amplitude resums **infinitely many Feynman diagrams** — though only over a restricted region of loop momentum space.

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LSZ reduction

Many theorems on the analytic properties of scattering amplitudes rely on the LSZ reduction formula, which is known not to apply in theories with long-range forces [Fradkin '67][Kibble '68].Instead of exhibiting a simple $(p^2 - m^2)^{-1}$ pole, the off shell Fourier transform of position space correlation functions instead exhibit a branch point

$$(p^2 - m^2)^{-1+i\gamma+\beta} \tag{45}$$

The starting point of the original LSZ paper is

$$\begin{aligned}&\langle p_1, ..., \mathsf{out}|\hat{\phi}(x_2)...|q_1, q_2, ...\mathsf{in}\rangle \\ &= -i \lim_{x_1^0 \to -\infty} Z^{-\frac{1}{2}} \int_{x_1^0} \mathrm{d}^3\vec{x}_1 \, e^{-iq_1 \cdot x_1} \overleftrightarrow{\partial}_{x_1^0} \langle p_1, ..., \mathsf{out}|\hat{\phi}(x_1)\hat{\phi}(x_2)...|q_2, ...\mathsf{in}\rangle .\end{aligned}$$

The **existence** of this **asymptotic-time limit** precisely captures the LSZ assumption that quantum fields behave as **free fields** at asymptotic times.

LSZ reduction

Lets revisit the intuition for the starting point of the LSZ paper

$$\langle p_1, ..., \mathsf{out}|\hat{\phi}(x_2)...|q_1, q_2, ...\mathsf{in}\rangle \tag{46}$$
$$= -i \lim_{x_1^0 \to -\infty} Z^{-\frac{1}{2}} \int_{x_1^0} \mathrm{d}^3\vec{x}_1 \, e^{-iq_1\cdot x_1} \overleftrightarrow{\partial}_{x_1^0} \langle p_1, ..., \mathsf{out}|\hat{\phi}(x_1)\hat{\phi}(x_2)...|q_2, ...\mathsf{in}\rangle.$$

At any time x_1^0, one can expand the $\vec{x}_1$ dependence of the correlator in a plane-wave basis $e^{ip\cdot x}$.The KG inner product on RHS extracts the co-efficient $\hat{a}^\dagger_{\vec{q}_1}(x_1^0)$.(46) claims that this tends to a constant. Using Schwinger-Dyson in $A^\mu \sim \frac{1}{|\vec{x}|}$ background

$$\lim_{|t|\to\infty} \dot{a}_{\vec{p}} \sim i\frac{\alpha}{|\vec{x}|} a_{\vec{p}} \sim i\alpha\frac{E_p}{|\vec{p}|t} a_{\vec{p}}, \implies \tag{47}$$

$$a_{\vec{p}}(t) = a_{\vec{p}}(t_0)\left(\frac{t}{t_0}\right)^{i\gamma} \tag{48}$$

Which should be compared to the short-ranged case $A^\mu \sim \frac{1}{|\vec{x}|^2}$

$$\lim_{|t|\to\infty} \dot{a}_{\vec{p}}(t) \sim i\frac{\delta}{t^2} a_{\vec{p}}(t), \implies a_{\vec{p}}(t) = C\, e^{-i\frac{\delta}{t}}, \tag{49}$$

LSZ reduction: quick necessary interlude

Just as onshell-planes satisfy an equal time completeness relation

$$\delta(x_0 - y_0)\tfrac{1}{2}\int \tfrac{\mathrm{d}^3 p}{(2\pi)^3}\ e^{ip\cdot(x-y)} + e^{ip\cdot(x-y)}\big|_{p_0=-\sqrt{|\vec{p}|^2+m^2}} = \delta^{(4)}(x-y)$$

and are orthonormal with respect to the KG inner product

$$i\int \mathrm{d}^3\Sigma^\mu\ e^{-ip.x}\overleftrightarrow{\partial}_\mu e^{iq\cdot x} = (2\pi)^3 2E_p \delta^{(3)}(\vec{p}-\vec{q}),$$

the Coulomb wavefunctions satisfy an equal time-completeness relation

$$\delta(x_0 - x_0')\left(\textstyle\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{n=l+1}^{\infty} E_n B_{nlm}(x) B^\star_{nlm}(x') + \tfrac{1}{2}\sum_{\rho=\pm}\int \tfrac{\mathrm{d}^3\vec{p}}{(2\pi)^3} f^\rho_{\mathsf{in}}(x,\vec{p}) f^{\star\rho}_{\mathsf{in}}(x',\vec{p})\right)$$

$$= \delta^4(x-x').$$

and are orthonormal with respect to the SQED inner product

$$i\int \mathrm{d}^3\Sigma^\mu\ f^\star_{\mathsf{in}}(p,x)\overleftrightarrow{D}_\mu f_{\mathsf{in}}(p,x) = (2\pi)^3 2E_p \delta^{(3)}(\vec{p}-\vec{q})$$

where $D_\mu = \partial_\mu - ieA_\mu$.

LSZ reduction

To go from our position space correlation function

$$G(x_1, \ldots, x_n) = \langle \Omega | \mathrm{T} \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) | \Omega \rangle \,, \tag{50}$$

to a scattering amplitude , we should expand the x_i dependence in a complete orthonormal basis, which resemble plane-waves at large distances as closely as possible,but with the property that the **mode coefficients tend to a constant at asymptotic times**.We now do this in an example.

IR-finite LSZ reduction example

For our 2-pt function in a background Coulomb field generated by a heavy source

$$G_F(x,y) = \langle \Omega | T(\phi^\dagger(x)\phi(y)) | \Omega \rangle$$

$$= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} \left(\theta(x^0 - y^0) f^\star_{\mathsf{in},-}(x,p) f_{\mathsf{in},-}(y,p) + \theta(y^0 - x^0) f^\star_{\mathsf{in},+}(x,p) f_{\mathsf{in},+}(y,p) \right)$$

$$+\theta(y^0 - x^0) \sum_{nlm} B_{nlm}(y) B^\star_{nlm}(x).$$

The mode coefficients can be extracted via the SQED inner product, and tend to a constant

$$\int_{x^0=+\infty} \mathrm{d}^3\vec{x} \;\; f^{+,\star}_{\mathsf{out}}(x,p_2) \overleftrightarrow{D}_{x^0} \int_{y^0=-\infty} \mathrm{d}^3\vec{y} \;\; f^{+}_{\mathrm{in}}(p_1,y) \overleftrightarrow{D}_{y^0} G_F(x,y) \quad (51)$$

$$= i\pi\, e_1 e_s \delta\Big(u_s \cdot (p_1 - p_2)\Big) \frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)} \frac{p_1 \cdot u_s}{m^2 - (p_1 \cdot u_s)^2} \left(\frac{4\Big(m^2 - (p_1\cdot u_s)^2\Big)}{(p_1 - p_2)^2} \right)^{1+i\gamma} \quad (52)$$

which coincides with $\langle \Phi^+_{\mathsf{out}}(p_2) | \Phi^+_{\mathsf{in}}(p_1) \rangle$. Direct transition from IR-finite correlator to IR-finite amplitude.

LSZ summary

- In this simple example, we have seen how LSZ reduction can be done with no IR-divergences, or arbitrary scales $(\Lambda_{\text{IR}}, \mu)$ entering, in contrast to the Faddeev-Kulish approach where ambiguous scales always enter.
- In summary in the original LSZ starting formula, we promote the KG inner product to the SQED inner product, and we project onto Coulomb wavefunctions instead of plane waves, because this is a complete orthonormal basis for which the late time limits exist.
- Including the real radiative effects, and moving away from the infinite mass case to the generic $2 \to 2$ case are the next steps.

Outline

- A very brief history of IR-divergences.
- Review of non-perturbative solution to Coulomb scattering in NRQM.
- Modified connectedness structure in theories with long range forces. Consequently a modified general optical theorem.
- Factorization on the bound state poles.
- Diagonalizing the asymptotic Hamiltonian.
- Modified LSZ reduction.
- Analytic relation between Coulomb phase and real radiative divergence.
- Future directions.

Analytic relations between real radiative and Coulomb divergences

- There are 2 IR-divergences in SQED: Coulomb phase divergence (simple) + real radiative divergence (subtle).
- Will now review: these are tightly related via analytic properties like crossing symmetry.
- Consequently, completely eliminating the Coulomb phase divergence may provide insight into how to eliminate the real radiative divergence by exploiting this analytic connection.

Review: crossing symmetry relation between real radiative and Coulomb phase divergence

Abelian exponentiation theorem

$$\mathcal{A} = \left(\frac{\Lambda_{\mathsf{IR}}}{\lambda_{\mathsf{IR}}}\right)^{R+I} \mathcal{A}_0(\Lambda_{\mathsf{IR}}), \tag{53}$$

where the real R and imaginary I exponents are given by

$$R = \underbrace{\frac{1}{16\pi^2}\sum_{i,j}\frac{\eta_i\eta_j e_i e_j}{\beta_{ij}}\log\left(\frac{1+\beta_{ij}}{1-\beta_{ij}}\right)}_{\text{real radiative divergence}} \quad , \quad I = \underbrace{-\frac{i}{8\pi}\sum_{\eta_i\eta_j=+}\frac{e_i e_j}{\beta_{ij}}}_{\text{Coulomb phase}}, \tag{54}$$

$$\beta_{ij} = \sqrt{1 - \frac{m_i^2 m_j^2}{(p_i \cdot p_j)^2}}, \tag{55}$$

where $\eta = +/-$ for incoming/outgoing states.Note that Coulomb phase would manifestly **break crossing symmetry** by itself.

Review: crossing symmetry relation between real radiative and Coulomb phase divergence

In paper, I review the analytic continuation relating the two divergences for massive particles. But its easy to see in the massless case (Lorenzo's talk)

$$\beta_{ij} = \sqrt{1 - \frac{m_i^2 m_j^2}{(p_i \cdot p_j)^2}} \xrightarrow{m_i \to 0} 1 - \frac{1}{2}\frac{m_i^2 m_j^2}{(p_i \cdot p_j)^2} \tag{56}$$

So we have

$$\underbrace{R = \frac{1}{8\pi^2}\sum_{i,j} \eta_i \eta_j e_i e_j \log\left(\frac{2p_i \cdot p_j}{m_i m_j}\right)}_{\text{real radiative divergence}} \quad , \quad \underbrace{I = -\frac{i}{8\pi}\sum_{\eta_i\eta_j=+} e_i e_j}_{\text{Coulomb phase}} \tag{57}$$

Crossing $(p_i \cdot p_j) \to -(p_i \cdot p_j)$ gives $\log(-1) = i\pi$.Only the combination of the two which satisfies crossing.

Analytic relation between real radiative and Coulomb phase

Can rewrite

$$\beta_{ij} = \sqrt{1 - \frac{m_i^2 m_j^2}{(p_i \cdot p_j)^2}}$$
$$= \frac{1}{2(p_i \cdot p_j)} \sqrt{\left[s_{ij} - (m_i - m_j)^2\right]\left[s_{ij} - (m_i + m_j)^2\right]} \quad (58)$$

where $s_{ij} = (p_i + p_j)^2$. Exhibits a branch cut at the **pseudothreshold**,

$$s_{ij} - (m_i - m_j)^2 = 0 \quad (59)$$

However, the **combination** of the real radiative and Coulomb phase divergence conspire so that their combination **does not exhibit this branch point**. Again, *if* the IR-finite amplitude is postulated to have this property, then we could leverage our understanding of the Coulomb phase to gain insight into the real radiative term.

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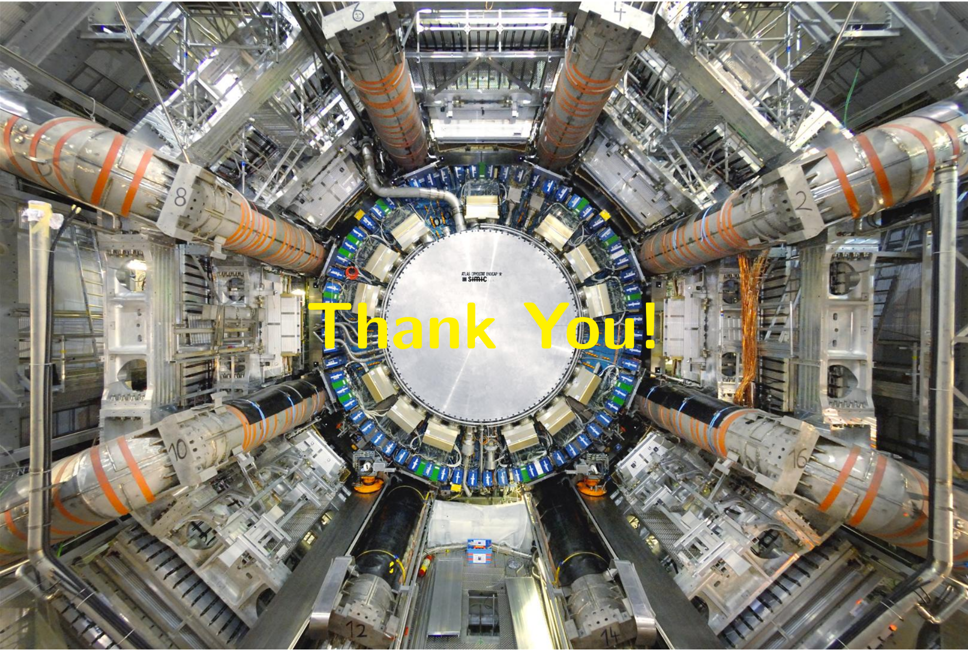
Summary and future directions

Origin of Coulomb phase divergence is simple: incorrect assumption about straight line trajectories. Initial steps to construct a perturbation theory which does not contain the Coulomb phase divergence. Examined modifications to:

- Connectedness structure
- General optical theorem
- LSZ reduction

We have also demonstrated that it is possible to **diagonalize the asymptotic Hamiltonian** and **fully solve** it's associated quantum theory. Future directions

1. Use the distributional equality to compute amplitudes.
2. Relax infinite-mass limit, scattering generic-mass particles using two-particle relativistic quantum mechanics [Sazdjian '86].
3. Soft radiation (bootstrap via crossing symmetry+pseudothreshold? eigenstates of radiation and absorption? $U(1)$ analog of BMS representations?)


Thank You!