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NNNLO zero-jettiness soft function

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1. Introduction and motivation

2. Ingredients of the final result

3. One-loop corrections with two soft emissions

4. Triple real soft emissions

5. Final result and applications

Introduction and motivation

Motivation

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- Differential calculation require a good handle of IR divergences, many schemes exist at NNLO
- Slicing scheme seems to be more feasible at N3LO due to non existence of subtraction schemes

$$\sigma(\mathrm{O}) = \int_0 \mathrm{d}\tau \frac{\mathrm{d}\sigma(\mathrm{O})}{\mathrm{d}\tau} = \int_0^{\tau_0} \mathrm{d}\tau \frac{\mathrm{d}\sigma(\mathrm{O})}{\mathrm{d}\tau} + \int_{\tau_0} \mathrm{d}\tau \frac{\mathrm{d}\sigma(\mathrm{O})}{\mathrm{d}\tau}$$

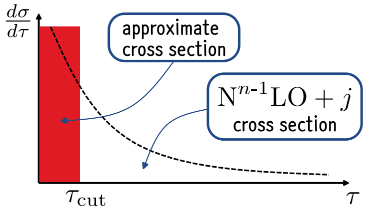
  - q_T slicing scheme [Catani,Grazzini'07]
  - N-jettiness slicing scheme [Boughezal et al.'15][Gaunt et al.'15]

- SCET factorization theorem motivates us to consider jettiness as a convenient slicing variable for processes with jets in the final state

$$\lim_{\tau\to 0} \mathrm{d}\sigma(\mathrm{O}) = \mathrm{B}_\tau \otimes \mathrm{B}_\tau \otimes \mathrm{S}_\tau \otimes \mathrm{H}_\tau \otimes \mathrm{d}\sigma_{\mathrm{LO}}$$

Slicing scheme ingredients

- A phase space is split according to a slicing variable
- Possible to use any lower order calculation with additional jet in the $\tau > \tau_{\rm cut}$ region



To apply at the NNNLO level:

- Existing NNLO+j calculations
- Many efficient NNLO subtraction schemes

- Approximate cross section in the singular region from the factorisation formula

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \mathrm{H}_\tau \otimes \{\mathrm{B}_\tau\} \otimes \{\mathrm{J}_\tau\} \otimes \mathrm{S}_\tau \otimes \frac{\mathrm{d}\sigma_0}{\mathrm{d}\tau} + \mathscr{O}(\tau)$$

- Hard function H_τ
- Beam function B_τ, jet function J_τ
- Soft function S_τ

Zero-jettiness measurement function

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- For two hard partons with momenta p_a and p_b jettiness is defined as follows

$$\mathscr{T}_0 = \sum_{i=1}^{m} \min\left\{ \frac{2p_a \cdot k_i}{Q}, \frac{2p_b \cdot k_i}{Q} \right\}, \quad k_i - \text{are soft partons}$$

- It is possible to rescale $p_a = \frac{\sqrt{s_{ab}}}{2} n, p_b = \frac{\sqrt{s_{ab}}}{2} \bar{n}$ and go to the frame where n and $\bar{n}$ are back-to-back
- Eikonal factors $E(k,l)$ have uniform scaling: rescale integration momenta $q_i = q_i' \frac{Q\tau}{\sqrt{s_{ab}}}, q_i \in \{k,l\}$

$$S(\tau) \sim \int \underbrace{[d^d k]^m}_{\text{ext}} \underbrace{[d^d l]^n}_{\text{loop}} \delta(\tau - \mathscr{T}_0) E(k,l) \to \frac{1}{\tau} \left(\frac{s_{ab}}{Q^2 \tau^2} \right)^{\varepsilon(m+n)} \int [d^d k']^m [d^d l']^n \delta\left(1 - \sum_{i=1}^{m} \min\{\alpha_i, \beta_i\} \right) E(k', l')$$

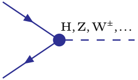
Sudakov decomposition

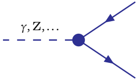
$$k_i = \frac{\alpha_i}{2} n + \frac{\beta_i}{2} \bar{n} + k_{i,\perp}, \quad k_i \cdot n = \beta_i, \quad k_i \cdot \bar{n} = \alpha_i, \quad n \cdot \bar{n} = 2, \quad n^2 = \bar{n}^2 = 0$$

Ingredients of the final result

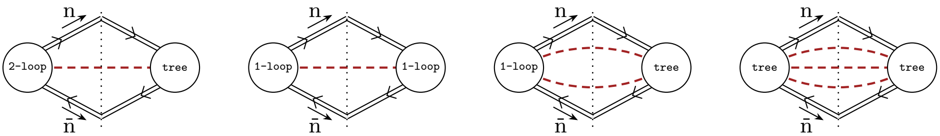
What is actually calculated?

- 0-jettiness in hadronic collisions is equal to Thrust or 2-jettiness in e^+e^- annihilation or Higgs decay





- The limit $\tau \to 0$ corresponds to the soft limit of the squared amplitude - eikonal Feynman rules
- Need to include all possible real and virtual corrections to the amplitude squared



- Possible to combine different measurement function terms into unique configurations
- Perform integration over highly non-trivial region - all kinds of divergencies are possible

From measurement function to configurations

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- Minimum function is a problem for analytic calculation
- Definition which is more friendly for phase-space integration generates many configurations

$$\delta\left(1-\sum_{i=1}^{m}\min\{\alpha_i,\beta_i\}\right)=\delta(1-\beta_1-\beta_2-\ldots)\theta(\alpha_1-\beta_1)\theta(\alpha_2-\beta_2)\cdots+\delta(1-\beta_1-\alpha_2-\ldots)\theta(\alpha_1-\beta_1)\theta(\beta_2-\alpha_2)\ldots$$

- Configurations can be mapped to the minimal set due to symmetries of Eikonal factor and $\delta(1-\{\alpha,\beta\})$
- RVV single configuration with $\delta(1-k\cdot n)$, trivial phase-space integration
  - Two-loop soft current is known [Duhr, Gehrmann'13]
- RRV two configurations nn and $n\bar{n}$
  - Emission of gluons and quark pair [Chen et al.'22] [Baranowski et al.'24]
- RRR two configurations nnn and $nn\bar{n}$
  - Same hemisphere gluon emission [Baranowski et al.'22]
  - Different hemispheres configuration $nn\bar{n}$ and quark pair emission in nnn configuration - this work

Calculation strategy

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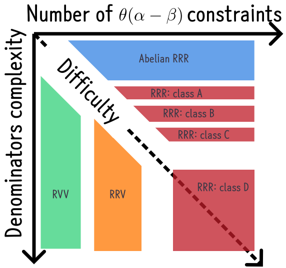
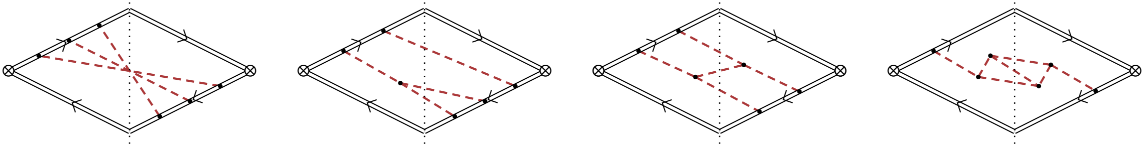
1. There are many highly non-trivial integrals, which we can calculate with direct integration
   - All integrations are divergent at the boundaries only
   - All integrals are linear reducible, GPLs only at all steps
   - Once there is a way to subtract divergencies integrals calculated with `HyperInt` [Panzer'15]

2. Utilization of the modern multi-loop calculation techniques to reduce the problem to (1)
   - Reduction of integrals to the minimal set of master integrals
   - Differential equations for integrals at the expense of introducing new parameters
   - Symmetry relations between integrals
   - Input expression organization in ”diagram”-like structures

Relative complexity of ingredients



- For each soft emission we have one θ-function in the measurement function making integration more complicated
- For complicated denominators in the RRR case make direct integration is impossible
- Complicated one-loop sub-integrals in the RRV make direct integration impossible
- Unregulated divergencies in the RRR case

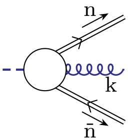


A

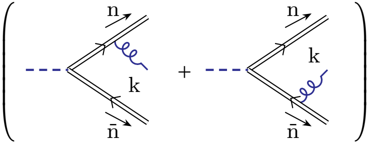
$$B \sim \frac{1}{k_1 \cdot k_2} \qquad C \sim \frac{1}{(k_1 \cdot k_2)(k_1 \cdot k_3)} \qquad D \sim \frac{1}{(k_1 + k_2 + k_3)^2}$$

RVV corrections

Two-loop corrections $r_S^{(2)}$ to single gluon emission soft current are known exactly in ε [Duhr, Gehrmann'13]



$= r_S(k)$

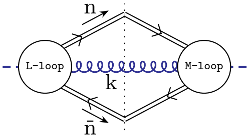


$$r_S(k) = 1 + \sum_{l=1}^{\infty} A_s^l \left[\frac{-(n\cdot\bar{n})}{2(k\cdot n)(k\cdot\bar{n})}\right]^{l\varepsilon} r_S^{(l)}$$

Two contributions from different hemisphere emissions need to be integrated, $S_g^{(3)} = s_{2,0} + s_{1,1} + s_{0,2}$

$$s_{l,m} = \int \frac{d^d k}{(2\pi)^{d-1}} \delta^+(k^2)\left[\delta(1-k\cdot n)\theta(k\cdot\bar{n}-k\cdot n) + \delta(1-k\cdot\bar{n})\theta(k\cdot n-k\cdot\bar{n})\right] w_{L,M}(k)$$

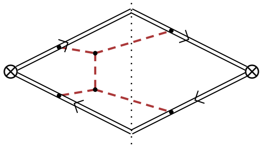
$w_{L,M}(k) = \mathrm{Re}\left[J_L^\dagger(k) J_M(k)\right] =$



- Linear propagators only
- Factorisation of k-dependent part of soft current

One-loop corrections with two soft emissions

One-loop corrections with double emission



- RRV squared amplitudes generated from scratch
- Results for one-loop soft current are known [Zhu'20][Czakon et al.'22]
- RRV result for gg final state were computed earlier [Chen,Feng,Jia,Liue'22]
- Recalculation in the unified way including $q\bar{q}$ final state [Baranowski et al.'24]

Multi-loop calculations inspired approach

- Reduction to the minimal set of master integrals with loop and phase-space integration
- Differential equations from IBP reduction - parameter to differentiate is needed

Modified reverse unitarity to deal with θ-integrals

- In dimensional regularisation system of IBP equation can be constructed by differentiation under integral sign

$$\int d^d l \frac{\partial}{\partial l_\mu}\left[v_\mu \cdot f(\{l\})\right], \qquad \frac{\partial}{\partial k\cdot\bar{n}}\theta(k\cdot\bar{n}-k\cdot n)=\delta(k\cdot\bar{n}-k\cdot n)$$

- IBP for integrals with θ-functions generate new auxiliary topologies, partial fractioning required

$$\frac{\theta(k\cdot\bar{n}-k\cdot n)}{(k\cdot\bar{n})^a(k\cdot n)^b}\to\frac{\delta(k\cdot\bar{n}-k\cdot n)}{(k\cdot\bar{n})^a(k\cdot n)^b}$$

  - RRR $\underbrace{\theta\theta\theta}_{\text{Level 3}}\to\underbrace{\delta\theta\theta+\theta\delta\theta+\theta\theta\delta}_{\text{Level 2}}\to\underbrace{\delta\delta\theta+\delta\theta\delta+\theta\delta\delta}_{\text{Level 1}}\to\underbrace{\delta\delta\delta}_{\text{Level 0}}$
  - RRV $\underbrace{\theta\theta}_{\text{Level 2}}\to\underbrace{\delta\theta+\theta\delta}_{\text{Level 1}}\to\underbrace{\delta\delta}_{\text{Level 0}}$

RRV master integrals calculation

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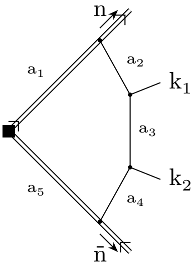


- Number of MIs after IBP reduction of both configurations in RRV case

$\delta\delta$	$\delta\theta + \theta\delta$	$\theta\theta$
8	36	15

- Direct integration possible, except pentagon and box with $a_3 = 0$
- DE in auxiliary parameters for most complicated integrals

Original integrals from DE solution

- Additional parameter z is not needed - utilize variables from integral representation
- To recover integrals of interest I instead of taking limit $I = \lim_{z \to z_0} J(z)$ we integrate $I = \int dz\, J(z)$

RRV master integrals from differential equations

- For $\delta\delta$ integrals we introduce auxiliary parameter x and solve DE system $\partial_x J(x) = M(\varepsilon, x)J(x)$

$$I_{\delta\delta} = \int d(k_1 \cdot k_2) f(k_1 \cdot k_2) = \int_0^1 dx \int d(k_1 \cdot k_2) \delta(k_1 \cdot k_2 - \frac{x}{2}) f(k_1 \cdot k_2) = \int_0^1 J(x) dx$$

- For $\delta\theta$ and $\theta\delta$ we use integral representation for θ-function and solve DE system $\partial_z J(z) = M(\varepsilon, z)J(z)$

$$\theta(b - a) = \int_0^1 b\delta(zb - a)dz, \quad I_{\delta\theta} = \int_0^1 J(z)dz$$

- For $\theta\theta$ integrals PDE system in two variables z_1, z_2, no IBP reduction with θ-functions needed

$$I_{\theta\theta} = \int_0^1 dz_1 \int_0^1 dz_2 \, J(z_1, z_2)$$

Differential equations in canonical form

- For all auxiliary integrals it is possible to find alternative basis of integrals, such ε dependence of the DE system matrix factorizes completely: $\mathrm{M}(\varepsilon) \to \varepsilon \mathrm{A}$ [Henn'13]
- Straightforward solution for integrals in canonical basis in terms of GPLs
- Simpler boundary conditions fixing due to known general form of expansion near singular points

$$g(z) = z^{a_1+b_1\varepsilon}(c_1 + \mathcal{O}(z)) + z^{a_2+b_2\varepsilon}(c_2 + \mathcal{O}(z)) + \ldots$$

- Construction of subtraction terms to remove endpoint singularities in the final integration

$$\int_0^1 J(z)dz = \int_0^1 \underbrace{\left[J(z) - z^{a_i+b_i\varepsilon}j_0(z) - (1-z)^{a_k+b_k\varepsilon}j_1(z)\right]}_{\varepsilon-\text{expanded}} dz + \int_0^1 \underbrace{\left(z^{a_i+b_i\varepsilon}j_0(z) - (1-z)^{a_k+b_k\varepsilon}j_1(z)\right)}_{\varepsilon-\text{exact}} dz$$

Summary: real-real-virtual contributions

- IBP reduction of integrals with θ-functions and loop integration can be efficiently implemented
- Differential equations for auxiliary integrals can be constructed and solved analytically
- Auxiliary integrals are simplified in the limit, and all required boundary constants can be calculated

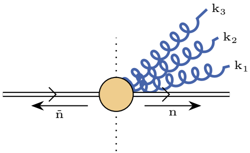
Triple real soft emissions

Triple real emissions

Recalculated input for eikonal factors with partial fractioning and topology mapping

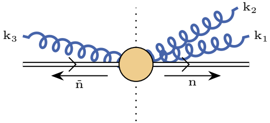
- $\mathrm{ggg} = \mathrm{ggg} + \mathrm{gc\bar{c}}$, coincides with the known expression in physical gauge [Catani,Colferai,Torrini'19]
- $\mathrm{gq\bar{q}}$ in agreement with [Del Duca,Duhr,Haindl,Liu'23]

Same hemisphere



$\delta(\tau - \beta_1 - \beta_2 - \beta_3)$

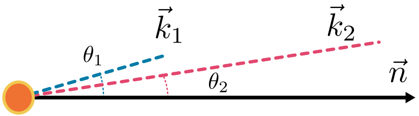
Different hemispheres



$\delta(\tau - \beta_1 - \beta_2 - \alpha_3)$

- Same hemisphere result for ggg final state is known [Baranowski et al.'22]

Divergences unregulated dimensionally



- Same hemisphere emission of $\mathrm{k}_1, \mathrm{k}_2$ partons
- Integration in the region $\underbrace{\beta_1}_{\sim\lambda} << \underbrace{\alpha_1, \beta_2}_{\sim 1} << \underbrace{\alpha_2}_{\sim 1/\lambda}$
- Both are close to the $\vec{\mathrm{n}}$ direction $\cos\theta_1 \sim \cos\theta_2 \sim 1 + \mathcal{O}(\lambda)$
- And large energies difference $\omega_1 \sim 1 << \omega_2 \sim 1/\lambda$

Possible cases for integrals in the potentially unregulated region

- Integrals in the region with scaleless integrations — safe
- Integrals with zero sum of two contributions from $\theta_1 > \theta_2$ and $\theta_1 < \theta_2$ parts — safe
- Rare cases of integrals with non-trivial region contribution — Additional regulator needed

Additional regulator in action

- Example region k_1, k_2: $\beta_1 \sim \lambda$ and $\alpha_2 \sim 1/\lambda$ change of variables $\beta_1 = \xi_1 \alpha_1$ and $\alpha_2 = \beta_2/\xi_2$
- Our choice for regulator to modify integration measure for each $dk_i \theta(a_i - b_i) \to dk_i \theta(a_i - b_i) b_i^\nu$

$$\int \frac{d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 (\beta_1\beta_2)^\nu}{(\alpha_1\beta_1\alpha_2\beta_2)^\varepsilon} \to \begin{cases} \int d\alpha_1 d\beta_2 dx d\xi_2 \frac{(\alpha_1\beta_2)^{1-2\varepsilon+\nu}}{\xi_2^{1-\nu} x^{\varepsilon-\nu}} & , \xi_1 < \xi_2, \xi_1 = x\xi_2 \\ \int d\alpha_1 d\beta_2 dx d\xi_1 \frac{(\alpha_1\beta_2)^{1-2\varepsilon+\nu}}{\xi_1^{1-\nu} x^{2-\varepsilon}} & , \xi_2 < \xi_1, \xi_2 = x\xi_1 \end{cases}$$

- Additional complications due to a new regulator
  - More complicated reduction due to an additional parameter in the problem
  - Master integrals calculation is more difficult due to the need to consider the double limit $\varepsilon, \nu \to 0$

Reduction of ν-regulated integrals

Approaches to ν-dependent IBP reduction problem (IBP with ν is available)

1. Direct ν-dependent reduction with additional variable
   - ✘ Time consuming and not flexible especially if basis change needed
   - ✔ Minimal set of master integrals and full ν-dependent solution
2. Filtering - remove all equations with potentially divergent integrals from the IBP system
   - ✔ Very fast compared to the full ν-dependent reduction
   - ✘ Potentially unreduced integrals, needs divergencies analysis for all integrals in the IBP system
3. Expansion - rewrite IBP system as a new system for $1/\nu$ expansion coefficients of integrals
   - ✔ Fast reduction with control of divergencies
   - ✘ Additional divergent parts of integrals from the intermediate steps of IBP reduction can appear

Importance of a good master integrals basis

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- From the analysis of possible divergencies we consider ansatz $J_a = \sum_{k=k_0}^{\infty} J_a^{(k)} \nu^k$ with $k_0 = -1$
- Solution of the IBP reduction problem for regular-ν integrals I_a has the form

$$I_a^{(0)} = R_{ab}\, J_b^{(0)} + D_{ab}\, \tilde{J}_b^{(-1)}$$

- We require a "good" basis to fulfill the following conditions:
  - Coefficients in front of master integrals do not contain $1/\nu$ poles
  - Each master integral is a member of only one set J_b or $\tilde{J}_b$
  - Candidates for the set J_b can be found from the $\nu = 0$ reduction
- Regular integrals $J_b^{(0)}$ are calculated in a standard way, calculation of needed divergent parts $\tilde{J}_b^{(-1)}$ is simplified, since only specific regions contribute

DE for RRR integrals with auxiliary mass

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- Integrals for both nnn and $\mathrm{nn\bar{n}}$ configurations with denominator $1/\mathrm{k}_{123}^2$ are difficult to calculate
- Since integrals are single scale, auxiliary parameter is needed to construct the system of DE $\mathrm{I} \to \mathrm{J}(\mathrm{m}^2)$
- We modify the most complicated propagator $\frac{1}{(\mathrm{k}_1+\mathrm{k}_2+\mathrm{k}_3)^2} \to \frac{1}{(\mathrm{k}_1+\mathrm{k}_2+\mathrm{k}_3)^2+\mathrm{m}^2}$
- Calculation of boundary conditions is possible in the limit $\mathrm{m}^2 \to \infty$, but still very difficult
- Massless integrals I are obtained from the solution for $\mathrm{J}(\mathrm{m}^2)$ in the limit $\mathrm{m}^2 \to 0$, which is not trivial

Difficulties of the chosen strategy

- Both points $\mathrm{m}^2 \to 0$ and $\mathrm{m}^2 \to \infty$ are singular points of the DE system
- Solution of the DE for integrals with massive denominator is only possible numerically

Details of the DE solution

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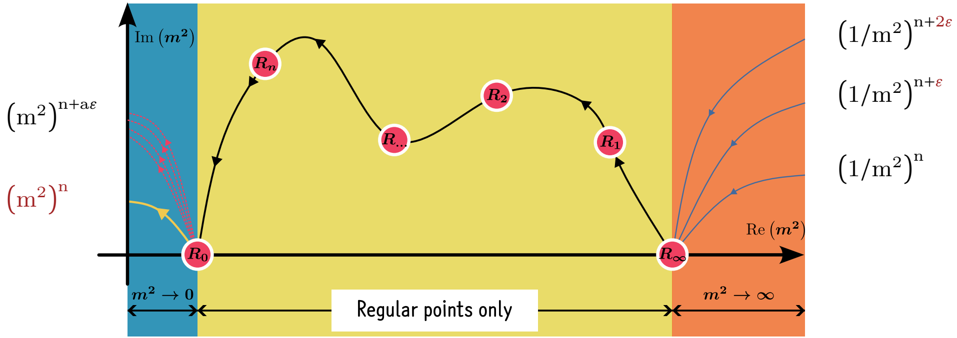
- A much larger DE system, ~ 650 equations are needed for $\mathrm{nn\bar{n}}$ configuration compared to ~ 150 for nnn
- Need to calculate all contributing regions into boundary conditions in the $\mathrm{m}^2 \to \infty$ limit

$\sim (\mathrm{m}^2)^0$	$\sim (\mathrm{m}^2)^{-\varepsilon}$	$\sim (\mathrm{m}^2)^{-2\varepsilon}$
$1/\mathrm{m}^2$	$\alpha_\mathrm{i} \sim \mathrm{m}^2$	$\alpha_\mathrm{i}, \alpha_\mathrm{j} \sim \mathrm{m}^2$

- For each large parameter $\alpha_\mathrm{i} \sim \mathrm{m}^2$ we remove $\theta \implies$ additional IBP reduction of boundary conditions integrals possible
- Numerical solution of the DE system as a sequence of series expansions [Liu et al.'18][Chen et al.'22]

From boundaries at $m^2 \to \infty$ to $m^2 \to 0$ solution



- Sum of all regions at $m^2 \to \infty$ to get high precision numerical solution at the first regular point R_∞
- High precision numerical solution of the DE between seqence of regular point $R_\infty \to R_1 \dots R_n \to R_0$
- Final result - Taylor branch of the generalized $m^2 \to 0$ expansion gives the required result

Nice features of the DE and its solution

- Numerical DE solution at finite m^2
  - Independent numerical checks at finite m^2
- Local Fuchsian form of the DE near singular points $\mathrm{m}^2 \to 0$ and $\mathrm{m}^2 \to \infty$
  - Matrix solution and generalized power series expansions
  - Minimal set of independent boundary constants to calculate
- Self-consistency checks of the DE solution and boundaries
  - Unphysical branches disappear after boundaries substitution
  - On the real axis $\mathrm{m}^2 \in (0, \infty)$ all integrals have zero imaginary parts
- Relations between specific branch expansion coefficients and IBP reduction of boundary constants
- Massless integrals we are interested in are extracted from the specific branch of $\mathrm{m}^2 \to 0$ DE solution

Boundaries at $m^2 \to \infty$ and series expansions

- Local Fuchsian form of the transformed DE with $\vec{f} = T\vec{g}$ and $y = y(m^2)$

$$\frac{\partial \vec{g}}{\partial y} = \left[\frac{A_0}{y} + \sum_i \frac{A_i}{P_i(y)}\right]\vec{g}, \quad P_i(0) \neq 0$$

- Leading order matrix solution $\vec{g}(y) = U(y)\vec{B}$ directly read from the Fuchsian DE: $U(y \to 0) \sim y^{A_0}$

Specific branch y^λ expansions, $\lambda = b\varepsilon$

$$J_1^{(\lambda)} = y^{a_1+\lambda}\left(c^\lambda_{1,0} + c^\lambda_{1,1}y^1 + c^\lambda_{1,2}y^2 + \dots\right)$$

$$\vdots$$

$$J_n^{(\lambda)} = y^{a_n+\lambda}\left(c^\lambda_{n,0} + c^\lambda_{n,1}y^1 + c^\lambda_{n,2}y^2 + \dots\right)$$

- We are interested in $y = m^2$ and $y = 1/m^2$
- Minimal vector $\vec{B}$ is a subset of $\bigcup_\lambda \{c^\lambda_{1,0}, \dots, c^\lambda_{n,0}\}$
- All $c^\lambda_{i,j}$ with $j > 0$ through subset of $c^\lambda_{i,0}$
- Reducible integrals expansion coefficients reduction

IBP reduction of boundary constants at $m^2 \to \infty$

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Local Fuchsian form ⇒ Matrix series solution ⇒ IBP for constants

1. Available IBP reduction tables for massive integrals $X_i = \sum_k R_{i,k}(m^2) J_k$
2. Deep enough $1/m^2$ expansions for master integrals J_k due to possible poles/zeroes in $R_{i,k}(m^2)$
3. Substitution of expanded MIs and unknown integrals $X_i = \sum_\lambda X_i^{(\lambda)}$ to IBP tables provides relations between leading expansion coefficients $x^\lambda_{i,0}$ and $c^\lambda_{j,0}$ valid for each branch $(m^2)^\lambda$ independently

$$X_i^{(\lambda)} = (m^2)^{a_1+\lambda} \left(x^\lambda_{i,0} + \frac{x^\lambda_{i,1}}{m^2} + \frac{x^\lambda_{i,2}}{m^4} + \dots \right)$$

- In each region additional boundary constants calculated and checked against reduction prediction
- Due to huge difference in calculation complexity possible to select simpler/less divergent integrals

Boundary integrals simplification

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- Main difficulty comes from the dependence of $k_{123}^2 + m^2$ on three angles, but in specific regions simplifications occur

$$k_{123}^2 + m^2 = \sum_{i \neq j} \alpha_i \beta_j - \sqrt{\alpha_i \beta_i \alpha_j \beta_j} \cos\left(k_{i,\perp}, k_{j,\perp}\right) + m^2$$

- Region $\left(m^2\right)^{-\varepsilon}$, single large parameter e.g. $\alpha_1 \sim m^2$

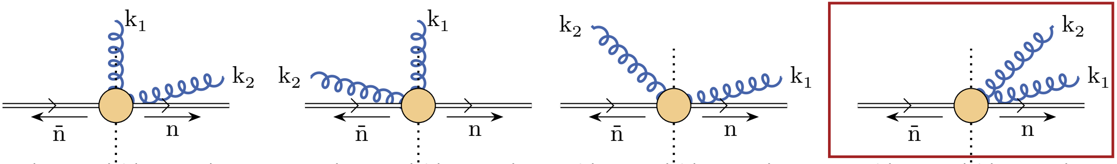
$$k_{123}^2 + m^2 \to \alpha_1 (\beta_2 + \beta_3) + m^2$$

- Region $\left(m^2\right)^{-2\varepsilon}$, pair of large parameters e.g. $\alpha_1 \sim \alpha_2 \sim m^2$, angle dependence remains since $k_1 \cdot k_2 \sim m^2$

$$k_{123}^2 + m^2 \to k_{12} + (\alpha_1 + \alpha_2)\beta_3 + m^2$$

Boundary constants in the region $(m^2)^{-\varepsilon}$

- Dependence on angles disappears in $k_{123}^2 + m^2 \to \alpha_i \left(\beta_j + \beta_k\right) + m^2$ in the $m^2 \to \infty$ limit
- Only non-trivial scalar product for e.g. $\alpha_1 \sim m^2$ is $(k_2 \cdot k_3)$ and $\theta(\alpha_1 - \beta_1) \to 1$
- Integration over the relative angle between soft partons in terms of ${}_2F_1$, function of argument dependent on $r_i = \frac{\beta_i}{\alpha_i}\theta(\alpha_i - \beta_i) + \frac{\alpha_i}{\beta_i}\theta(\beta_i - \alpha_i)$
- For same-hemisphere emissions we split integration region into $r_i > r_j$ and $r_i < r_j$



$\delta(\alpha_1 - \beta_1)\theta(\alpha_2 - \beta_2)$ $\delta(\alpha_1 - \beta_1)\theta(\beta_2 - \alpha_2)$ $\theta(\alpha_1 - \beta_1)\theta(\beta_2 - \alpha_2)$ $\theta(\alpha_1 - \beta_1)\theta(\alpha_2 - \beta_2)$

Boundary constants in the region $(m^2)^{-2\varepsilon}$

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- For two large parameters, say $\alpha_1 \sim \alpha_2 \sim m^2$ integrations become unconstrained $\theta(\alpha_1 - \beta_1)\theta(\alpha_2 - \beta_2) \to 1$
- Turn boundary integrals into ordinary PS integral J using $1 = \int dq \delta(q - k_1 - k_2)$ insertion

$$I_{-2\varepsilon} = \int \frac{dq dk_3 \delta(1 - \beta_q - \beta_3) \mathscr{C}_3}{q^2 + \alpha_q \beta_3 + m^2} \times \frac{1}{\prod_i D_i(\alpha_q, \beta_q, q^2, \alpha_3, \beta_3)} \times J_{a_1 \ldots a_6}(\beta_3, \alpha_q, \beta_q, q^2)$$

$$J_{a_1 \ldots a_6} = \int \frac{[dk_1][dk_2] \delta(k_1^2) \delta(k_2^2) \delta^{(d)}(q - k_1 - k_2)}{(k_1 \cdot n)^{a_1} (k_2 \cdot n)^{a_2} (k_1 \cdot \bar{n})^{a_3} (k_2 \cdot \bar{n})^{a_4} (k_1 \cdot n + \beta_3)^{a_5} (k_2 \cdot n + \beta_3)^{a_6}}$$

- IBP reduction possible, nontrivial part in the angular integral $\Omega_n = \int \frac{d\Omega_k}{(k\cdot v_1)^{a_1}(k\cdot v_2)^{a_2}\ldots(k\cdot v_n)^{a_n}}$
- After partial fractioning only Ω_n with $n = 1, 2$ and maximum single $v_i^2 \neq 0$ and all other $v_j^2 = 0$
- Trivial integration over large parameter $\alpha_q \sim m^2$, linear propagators simplified e.g. $\alpha_1 + \alpha_3 \to \alpha_1$

Direct integration of MIs and boundary constants

- We have calculated ~ 130 integrals without $1/k_{123}^2$ denominator and ~ 100 boundary conditions by direct integration with `HyperInt` [Panzer'15]
- Summary of used techniques

1. Change variables to satisfy all constraints from δ and θ functions
2. Perform as many integrations as possible in terms of ${}_2F_1$ and F_1 functions with known transformation properties
3. Perform remaining integrations in terms of ${}_pF_q$ functions if possible
4. For the final integral representation with minimal number of integrations and minimal set of divergencies - construct subtraction terms
5. Integrand with all divergencies subtracted is expanded in ε and integrated term by term with `HyperInt`
6. Subtraction terms are integrated in the same way

Numerical checks of calculated integrals

- For integrals without $1/k_{123}^2$ denominator use parametrisation similar to one used for analytical calculation
  - Straightforward hyper-cube parametrisation due to simple angle dependence of $1/\left(k_i \cdot k_j\right)$ denominators only
  - Sector decomposition with remapping $x \to 1$ divergencies to $x' \to 0$ with `pySecDec` or `FIESTA`
- For integrals with $1/k_{123}^2$ at $m^2 = 0$ we avoid the need to use angles and construct Mellin-Barnes representation
  - Repeated application of $(A+B)^\lambda \to \int A^{\lambda_1} B^{\lambda_2}$, important to have $A, B > 0$ at each step
  - Angle integration simplified until can be integrated in terms of gamma functions only
  - Analytical continuation with `MBresolve` and numerical integration with `MB`
- Integrals with $1/k_{123}^2$ at finite m^2, which are less divergent due to mass regularization
  - Careful preselection of less divergent integrals using available reduction to prevent SD from complexity explosion
  - For finite integrals or integrals with factorized divergencies direct integration with subtraction
  - Midpoint splitting for $x_i \to 1$ divergencies and sector decomposition for overlapping divergencies using `FIESTA`

Mellin-Barnes representation for angular integral

- First we convert complicated denominator $1/k_{123}^2$ into product of scalar products

$$\frac{1}{(k_1\cdot k_2+k_2\cdot k_3+k_3\cdot k_1)^\lambda}=\frac{1}{\Gamma(\lambda)}\int\limits_{c-i\infty}^{c+i\infty}\frac{dz_1 dz_2}{(2\pi i)^2}\frac{\Gamma(\lambda+z_1+z_2)\Gamma(-z_1)\Gamma(-z_2)}{(k_1\cdot k_2)^{z_1+z_2+\lambda}(k_2\cdot k_3)^{-z_1}(k_3\cdot k_1)^{-z_2}}$$

- Introduce unit length vectors to make standard angular integral structure transparent

$$\frac{1}{(k_i\cdot k_j)^\lambda}=\frac{1}{\Gamma(\lambda)}\int\limits_{c-i\infty}^{c+i\infty}\frac{dz}{2\pi i}\Gamma(-z)\Gamma(\lambda+z)\frac{2^{-z}\left(\sqrt{\alpha_i\beta_j}-\sqrt{\alpha_j\beta_i}\right)^{2z}}{\left(\alpha_i\beta_i\alpha_j\beta_j\right)^{z/2+\lambda/2}}\frac{1}{(\rho_i\cdot\rho_j)^{z+\lambda}},\quad \rho_i=\left(1,\frac{\vec{k}_{i,\perp}}{|\vec{k}_{i,\perp}|}\right)$$

- Final angles integration can be done in closed form well suited for subsequent MB integrations

$$\int\frac{d\Omega_1 d\Omega_2 d\Omega_3}{(\rho_1\cdot\rho_2)^{\lambda_1}(\rho_2\cdot\rho_3)^{\lambda_2}(\rho_3\cdot\rho_1)^{\lambda_3}}=\frac{\Gamma^3(1-\varepsilon)}{\pi^{3/2}2^{6\varepsilon+\lambda}\Gamma(1-2\varepsilon)}\frac{\Gamma(1-2\varepsilon-\lambda)\prod\limits_{i=1}^{3}\Gamma\left(\frac{1}{2}-\varepsilon-\lambda_i\right)}{\prod\limits_{i=1}^{3}\prod\limits_{j=1}^{i-1}\Gamma\left(1-2\varepsilon-\lambda_i-\lambda_j\right)}$$

Finite mass integrals

Use angles between transverse momenta as parameters

$$(k_i \cdot k_j) = 1/2\left(\sqrt{\alpha_i\beta_j} - \sqrt{\alpha_j\beta_i}\right)^2 + \sqrt{\alpha_i\beta_i\alpha_j\beta_j}\,\rho_{ij}$$

$$\rho_{12} = (1-\cos\theta_1) \quad \rho_{13} = (1-\cos\theta_2) \quad \rho_{23} = (1-\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\theta_3)$$

Integral divergences analysis

- $x_i \in [0,1] \to z_i \in [0,\infty)$, div: $\{z\} \to 0$ or $\{z\} \to \infty$
- Possible subsets Z_0 and Z_∞ of $\{z_1,\ldots,z_n\}$
- Do rescalings $z_i \to \lambda z_i, z_i \in Z_0$ and $z_i \to 1/\lambda z_i, z_i \in Z_\infty$
- Divergent if for $\int \frac{dz}{z^a}\prod P(z)^b \to \lambda^w \int \frac{dz}{z^a}\prod P(z)^b$

$$w + \dim(Z_0) - \dim(Z_\infty) \le 0$$

- For all integrals with $Z_\infty \neq \emptyset$ split at point $0 < p < \infty$

$$\int_0^\infty dz f(z) = p\int_0^\infty \frac{dz}{(1+z)^2} f\left(\frac{pz}{1+z}\right) + p\int_0^\infty \frac{dz}{z^2} f\left(\frac{p(1+z)}{z}\right)$$

- Select less divergent integrals determined by all Z_0 sets

Summary: triple-real contributions

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- Additional regulator is required for correct IBP reduction
- Efficient techniques are developed to decrease the complexity of the reduction with additional regulator
- DE for auxiliary m^2 dependent integrals with $1/\mathrm{k}_{123}^2$ propagator makes calculation possible
- DE in addition to numerical solution also provides many important consistency checks and relations
- Integrals are highly non-trivial for numerical checks

Final result and applications

Laplace space and UV renormalization

- Final unrenormalized result for the NNNLO soft function is a sum over configurations C

$$\mathrm{S}_{\tau,\mathrm{B}}^{\mathrm{NNNLO}} = \sum_{\mathrm{C}} \mathrm{S}_{\tau,\mathrm{B}}^{\mathrm{RVV,C}} + \sum_{\mathrm{C}} \mathrm{S}_{\tau,\mathrm{B}}^{\mathrm{RRV,C}} + \sum_{\mathrm{C}} \mathrm{S}_{\tau,\mathrm{B}}^{\mathrm{RRR,C}}$$

- For the renormalization we need the NNLO result expanded to higher orders in ε [Baranowski'20]

$$\mathrm{S}_{\tau,\mathrm{B}} = \delta(\tau) + \frac{\mathrm{a}_{\mathrm{s,B}}}{\tau}\left(\frac{\mathrm{s}_{\mathrm{ab}}}{\mathrm{Q}^2\tau^2}\right)^{\varepsilon}\mathrm{S}_1 + \frac{\mathrm{a}_{\mathrm{s,B}}^2}{\tau}\left(\frac{\mathrm{s}_{\mathrm{ab}}}{\mathrm{Q}^2\tau^2}\right)^{2\varepsilon}\mathrm{S}_2 + \frac{\mathrm{a}_{\mathrm{s,B}}^3}{\tau}\left(\frac{\mathrm{s}_{\mathrm{ab}}}{\mathrm{Q}^2\tau^2}\right)^{3\varepsilon}\mathrm{S}_3 + \mathscr{O}\left(\mathrm{a}_{\mathrm{s,B}}^4\right)$$

- Do strong coupling renormalization $\mathrm{a}_{\mathrm{s,B}} = \mu^{2\varepsilon}\mathrm{Z}_{\mathrm{a_s}}\mathrm{a}_{\mathrm{s}}(\mu)$ and do Laplace transform with parameter $\bar{\mathrm{u}} = \mathrm{u}e^{\gamma_{\mathrm{E}}}$

$$\tilde{\mathrm{S}}_{\mathrm{B}}\left(\mathrm{a}_{\mathrm{s}}(\mu), \mathrm{L}_{\mathrm{S}}\right) = \int_0^\infty \mathrm{d}\tau e^{-\tau \mathrm{u}}\, \mathrm{S}_{\tau,\mathrm{B}}\left(\mathrm{a}_{\mathrm{s,B}} \to \mu^{2\varepsilon}\mathrm{Z}_{\mathrm{a_s}}\mathrm{a}_{\mathrm{s}}(\mu)\right), \quad \mathrm{L}_{\mathrm{S}} = \ln\left(\mu\bar{\mathrm{u}}\frac{\sqrt{\mathrm{s}_{\mathrm{ab}}}}{\mathrm{Q}}\right)$$

- Convenient to consider $\tilde{\mathrm{S}}_{\mathrm{B}}$ because the renormalization in Laplace space is multiplicative

Renormalization and checks from RG equation

- Multiplicative renormalization in the Laplace space with L_S dependent renormalization constant $Z_s(a_s, L_S)$

$$\tilde{S}(a_s, L_S) = Z_s(a_s, L_S)\tilde{S}_B(a_s, L_S) = \mathcal{O}\left(\varepsilon^0\right)$$

  - $L_S = \ln\left(\mu\bar{u}\frac{\sqrt{s_{ab}}}{Q}\right)$
  - μ dependence in $a_s(\mu)$ and L_S

- Z_s determined by the pole part of $\tilde{S}_B$ satisfies RG equation

$$\left(\frac{\partial}{\partial L_S} + \beta(a_s)\frac{\partial}{\partial a_s}\right)\ln Z_s(a_s, L_S) = \Gamma_s(a_s, L_S) = -4\gamma_{cusp}(a_s)L_S - 2\gamma_s(a_s)$$

  - Γ_s is finite
  - Known cusp an.dim γ_{cusp}
  - Known non-cusp an.dim γ_s

- Possible to make prediction for the NNNLO pole part of $\tilde{S}_B$ and therefore for $S_{\tau,B}$ from the NNLO result
- Final form of the renormalized NNNLO soft function can be split into constant and L_S dependent parts

$$\ln\left(\tilde{S}(a_s, L_S)\right) = \sum_{i=1}^{\infty}\sum_{j=0}^{2i} C_{ij} a_s^i L_S^j = \ln\left(\tilde{S}\right) + \sum_{i=1}^{\infty}\sum_{j=1}^{2i} C_{ij} a_s^i L_S^j, \quad \tilde{S} = \tilde{S}(a_s, 0)$$

Result for NNNLO zero-jettiness soft function

- Eikonal line representation dependence completely factorizes at NNNLO due to Casimir scaling

$$\frac{\ln\left(\tilde{S}\right)}{C_R} = -a_s\pi^2 + a_s^2\left[n_f T_F\left(\frac{80}{81}+\frac{154\pi^2}{27}-\frac{104\zeta_3}{9}\right) - C_A\left(\frac{2140}{80}+\frac{871\pi^2}{54}-\frac{286\zeta_3}{9}-\frac{14\pi^4}{15}\right)\right]$$
$$+ a_s^3\left[n_f^2T_F^2\left(\frac{265408}{6561}-\frac{400\pi^2}{243}-\frac{51904\zeta_3}{243}+\frac{328\pi^4}{1215}\right) + n_f T_F\left(C_F X_{FF} + C_A X_{FA}\right) + C_A^2 X_{AA}\right] + \mathcal{O}\left(a_s^4\right)$$

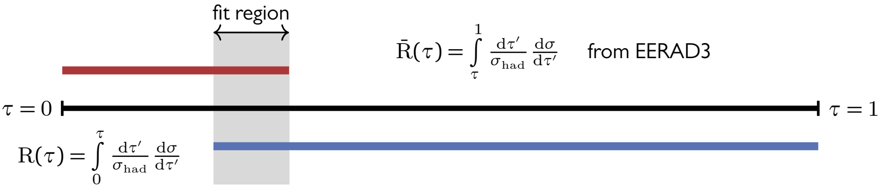
- With $a_s = \frac{\alpha_s}{4\pi}$ and new coefficients calculated numerically with high precision

$$X_{FF} = 68.94258498 \qquad X_{FA} = 839.72385238 \qquad X_{AA} = -753.77578727$$

- Soft function constants in $n_f = 5$ QCD required for resummed predictions $(q : C_R \to C_F)$ and $(g : C_R \to C_A)$

$$c_3^{S,q} = -1369.575849 \qquad c_3^{S,g} = -3541.982541$$

Singular region cross section from MC simulation



- Fit in the region, where NNLO MC predictions and approximate factorization prediction overlap
- From the condition $R(\tau) + \bar{R}(\tau) = 1$ and all C_i, G_{ij} except C_3 known

$$R(\tau) = \left(1 + \sum_{k=1}^{\infty} C_k \left(\frac{\alpha_s}{2\pi}\right)^k\right) \exp\left[\sum_{i=1}^{\infty}\sum_{j=1}^{i+1} G_{ij} \left(\frac{\alpha_s}{2\pi}\right)^i \ln^j \frac{1}{\tau}\right]$$

- Missing C_3 in the parametrisation of dijet region for NNLO Thrust [Monni,Gehrmann,Luisoni'11]

$$C_3 = -1050 \pm 180 \pm 500$$

From soft function to singular cross section

- Coefficient $C3$ is determined by constant parts of Hard(H), Jet(J) and Soft(S) functions
  - N3LO hard function is known $c_3^H = 8998.08$ [Abbate,Fickinger,Hoang et al.'10]
  - N3LO jet function known $c_3^J = -128.651$ [Brüser,Liu,Stahlhofen'18]
- From C_3 value can determine c_3^S, since all other ingredients are known [Brüser,Liu,Stahlhofen'18]

$$c_3^S = \begin{cases} -19988 \pm 1440 \pm 4000 & \text{fit result} \\ -1369.57 & \text{this work, exact} \end{cases}$$

- Inverse of the relation with known c_3^S allows C_3 color structures prediction [Monni,Gehrmann,Luisoni'11]

	$n_f^0 N^2$	$n_f^0 N^0$	$n_f^0 N^{-2}$	$n_f^1 N^1$	$n_f^1 N^{-1}$	$n_f^2 N^0$	sum
From c_3^S	2766.05	-60.1237	0.37891	-1581.01	18.4901	133.47	1277.25
Fit	3541 ± 51	-265 ± 8	-71 ± 3	-5078 ± 145	236 ± 7	95 ± 120	-1543 ± 195

Applications

- Thrust resummation for α_s determination, missing ingredient c_3^S is now available
  - c_2^S numerical fit [Becher,Schwartz'08]
  - c_3^H known, fitted c_3^J, c_3^S [Abbate,Fickinger,Hoang et al.'10]
  - c_3^H, c_3^J known, attempt to extract c_3^S [Bell,Lee,Makris et al.'23]
- Higgs decay to quarks/gluons α_s series convergence restored [Ju,Xu,Yang,Zhou'23]

$$\tilde{s}_g = 1 - 2.36\alpha_s + 1.617\alpha_s^2 - \underbrace{(22.89 \pm 5.67)}_{\text{fit}}\alpha_s^3$$

- Differential N3LO jet production in DIS and VBF

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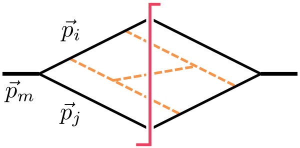
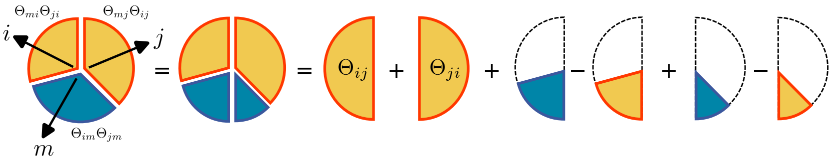
Generalization to N3LO 1-jettiness





- Most complicated real contribution from dipole terms with emissions between i, j lines only
- For each soft momenta k and dipole eikonal factor $\mathrm{S_{ij}}$ dependent on $\mathrm{p_i}, \mathrm{p_j}$ only with $\Theta_{\mathrm{ij}} = \theta\left(\mathrm{k}\cdot\mathrm{p_i} - \mathrm{k}\cdot\mathrm{p_j}\right)$

$$\left[\delta\left(\tau - \mathrm{k}\cdot\mathrm{p_i} - \ldots\right)\Theta_{\mathrm{mi}}\Theta_{\mathrm{ji}} + \delta\left(\tau - \mathrm{k}\cdot\mathrm{p_j} - \ldots\right)\Theta_{\mathrm{mj}}\Theta_{\mathrm{ij}} + \delta\left(\tau - \mathrm{k}\cdot\mathrm{p_m} - \ldots\right)\Theta_{\mathrm{im}}\Theta_{\mathrm{jm}}\right]\mathrm{S_{ij}}$$

- With $\Theta_{\mathrm{mx}} = 1 - \Theta_{\mathrm{xm}}$ most singular contributions coincide with zero-jettiness contributions

$$\left[\delta\left(\tau - \mathrm{k}\cdot\mathrm{p_i} - \ldots\right)\Theta_{\mathrm{ji}} + \delta\left(\tau - \mathrm{k}\cdot\mathrm{p_j} - \ldots\right)\Theta_{\mathrm{ij}}\right]\mathrm{S_{ij}} + \text{less singular}$$

Conclusion

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- Zero-jettiness slicing scheme is pushed from N2LO to N3LO level with the last missing ingredient calculated
  - Thrust resummation in $\mathrm{e}^+\mathrm{e}^-$ annihilation and Higgs decay
  - Differential cross section predictions for DIS and VBF
- Developed techniques
  - For efficient reduction of phase-space integrals with Heaviside θ-functions constraints in the presence of loop corrections and additional regulators
  - For the high precision numerical solution of differential equations for auxiliary integrals, making possible most complicated master integrals computation
  - For calculation of the large number of highly divergent integrals required for boundary conditions and master integrals without complicated dependence on soft partons momenta

Thank you for your attention!