Integrable Equation of State and Finite Size Effect

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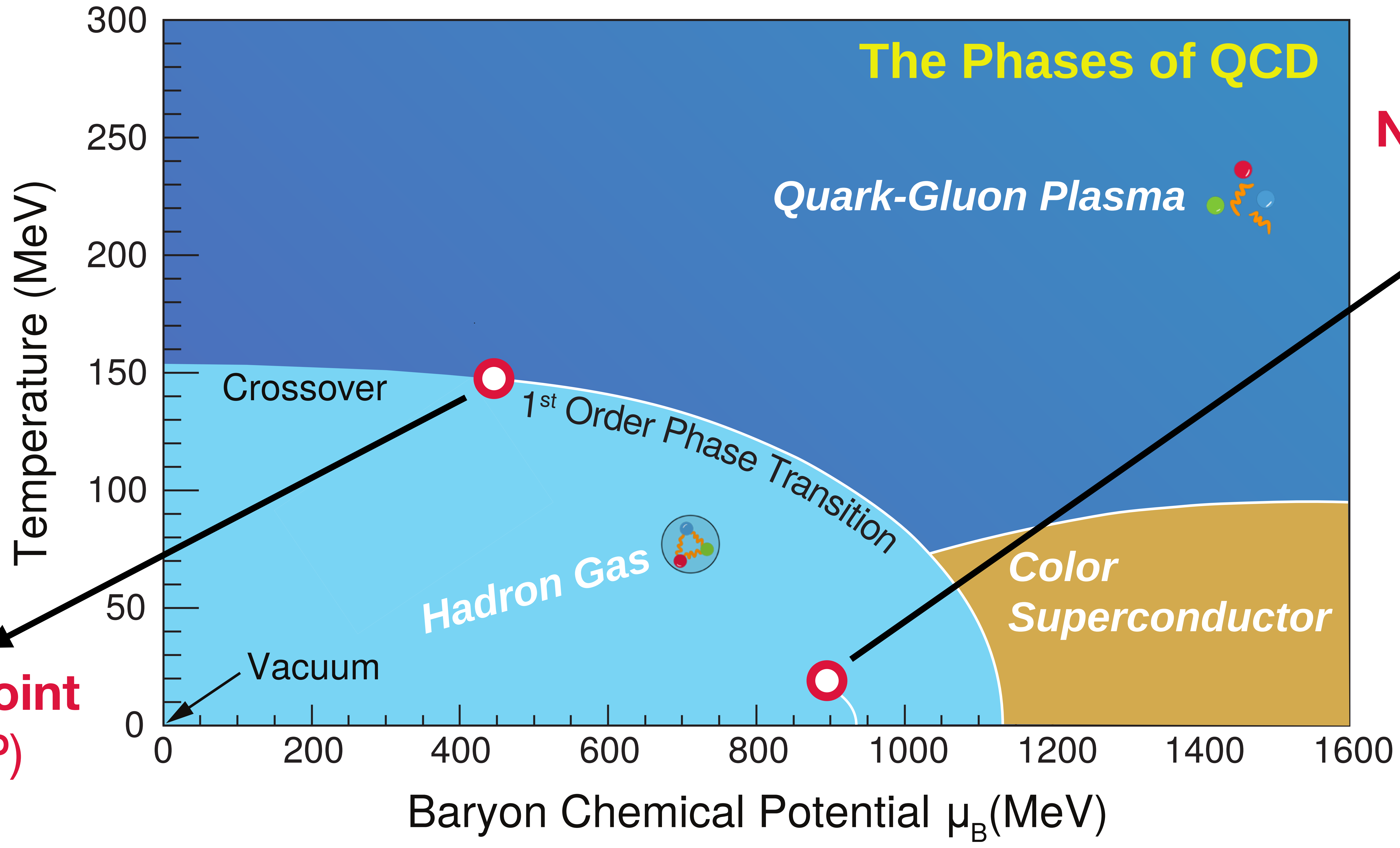
NCBJ

QCD phase diagram

- Heavy-ion collisions → QCD phase diagram — very little is known yet.
  Stephanov, 0402115; Fukushima et al, 1005.4814
- Search for landmark: the critical point where singularity occurs.



Nuclear critical point
(nuclear liquid-gas)
Pochodzalla et al, 1995

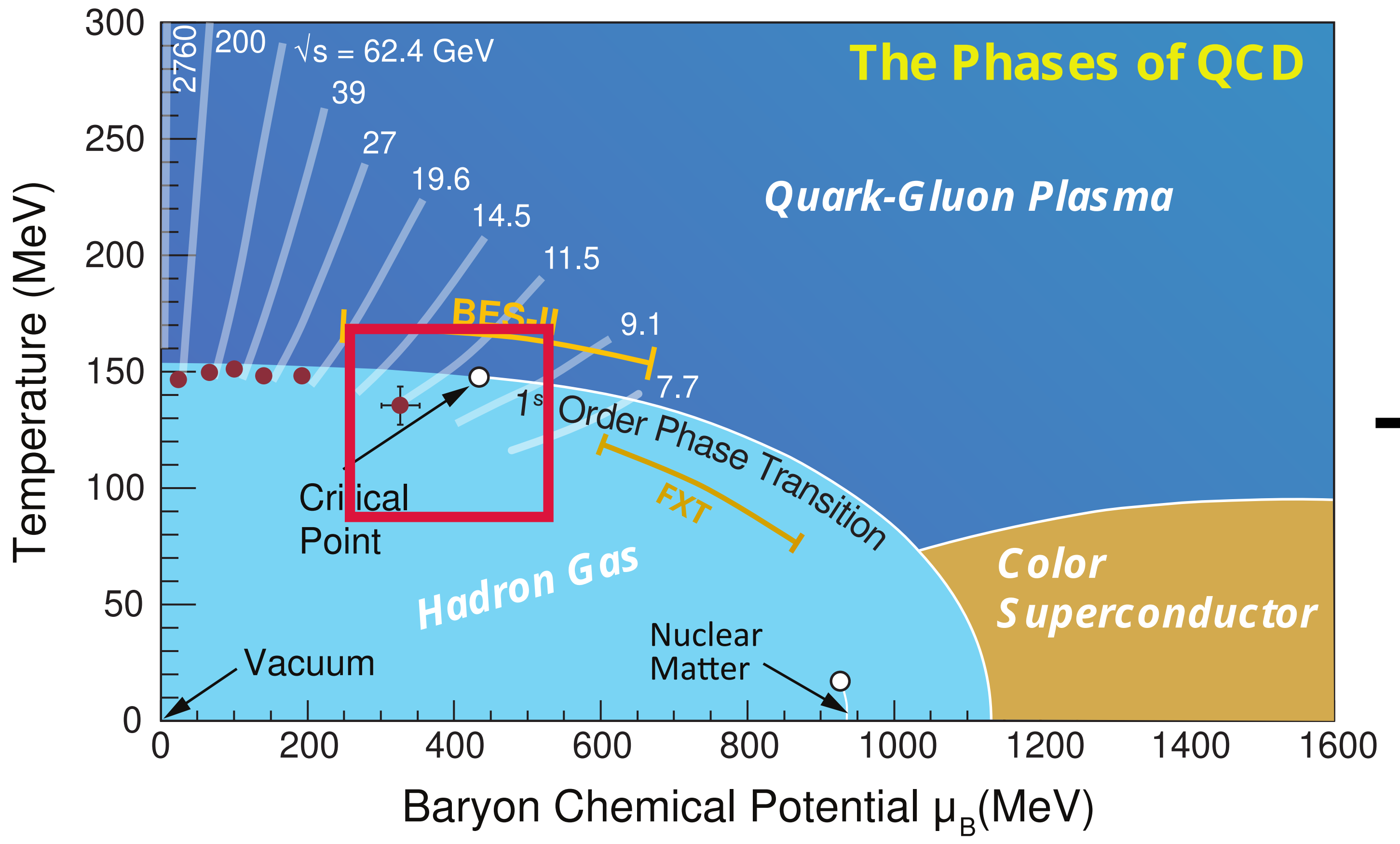
QCD critical point
(hadron-QGP)
TBA

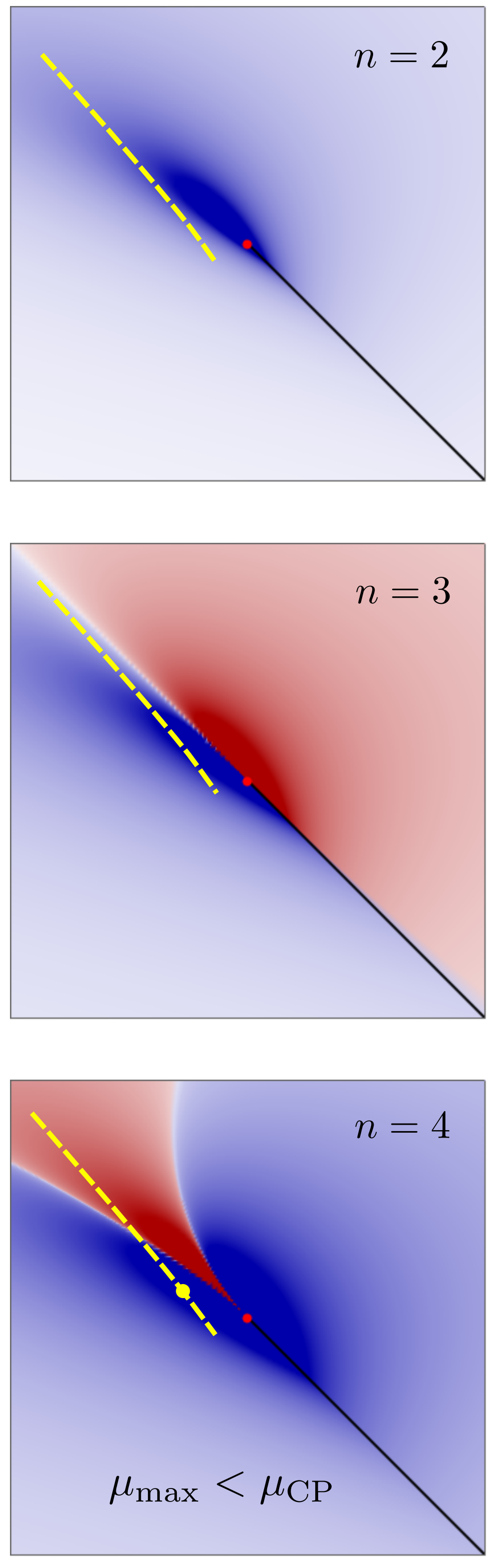
RHIC BES-II data

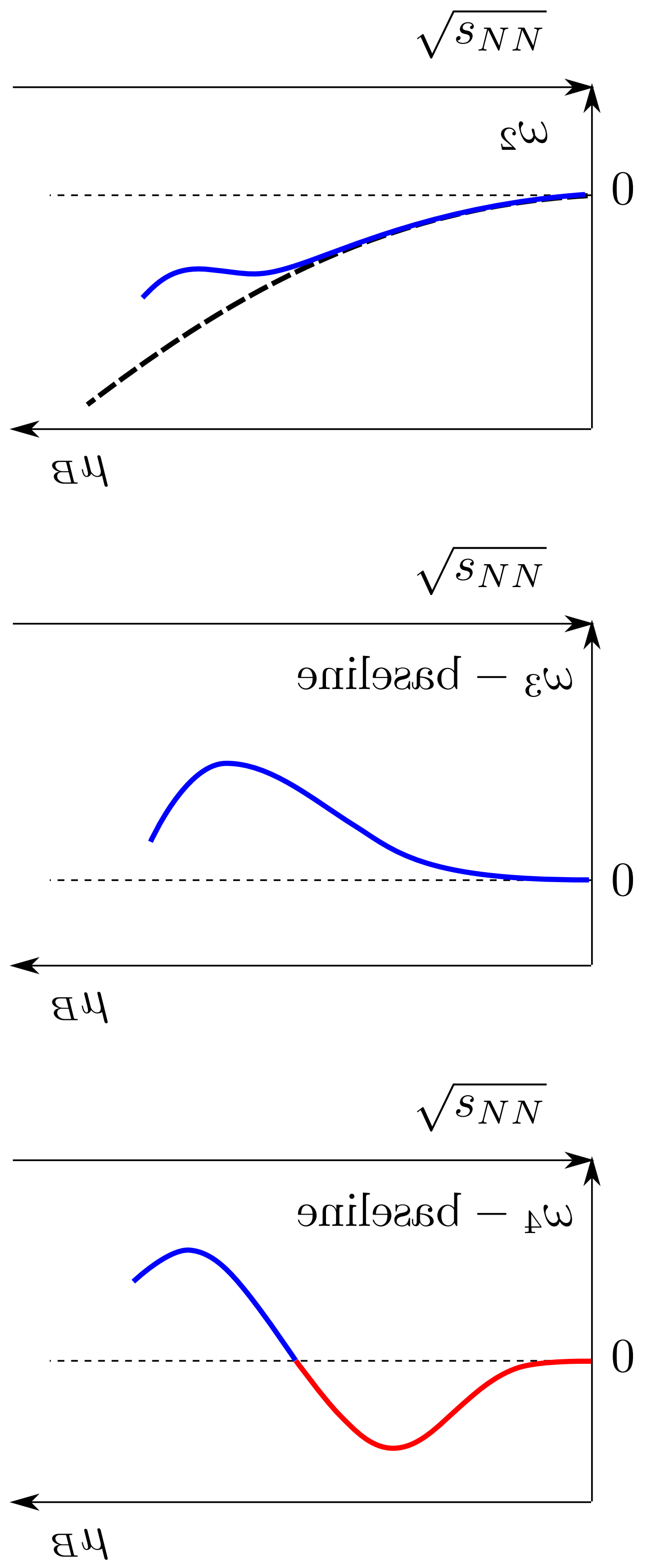
- RHIC BES-II data seem to advocate the intriguing hint of the QCD critical point from BES-I analysis, in a *qualitative* level based on thermodynamics.

STAR, 2112.00240; Stephanov, 1104.1627, SQM24

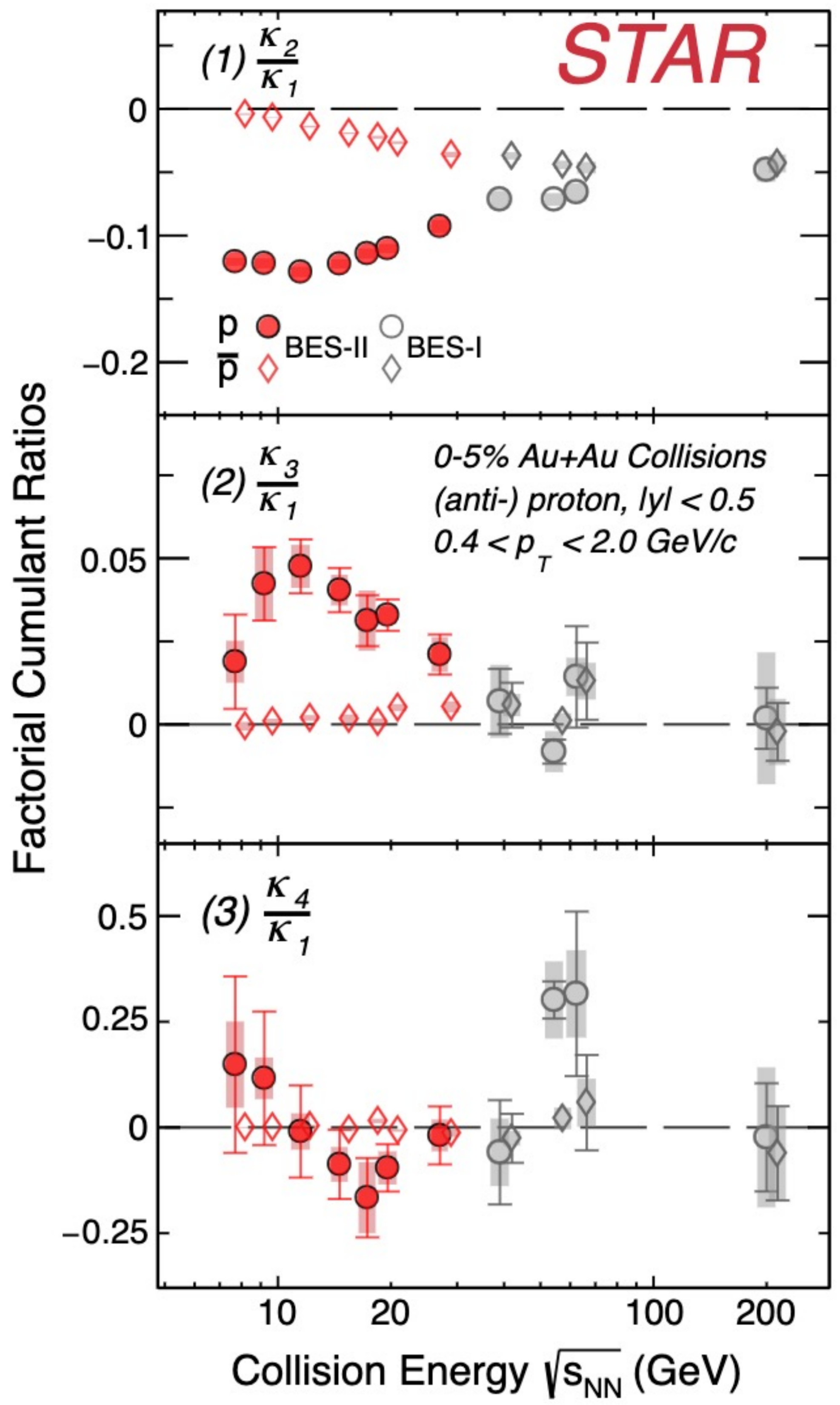
thermodynamics







experiment



However, no significant indication from NA61/SHINE preliminary results. Talk by V. Ortiz

Theory vs experiment

Theoretical idealization

- Infinite system
- Global equilibrium
- Static & homogeneous

Experimental complication

- Finite system
- Local equilibrium
- Dynamic & inhomogeneous

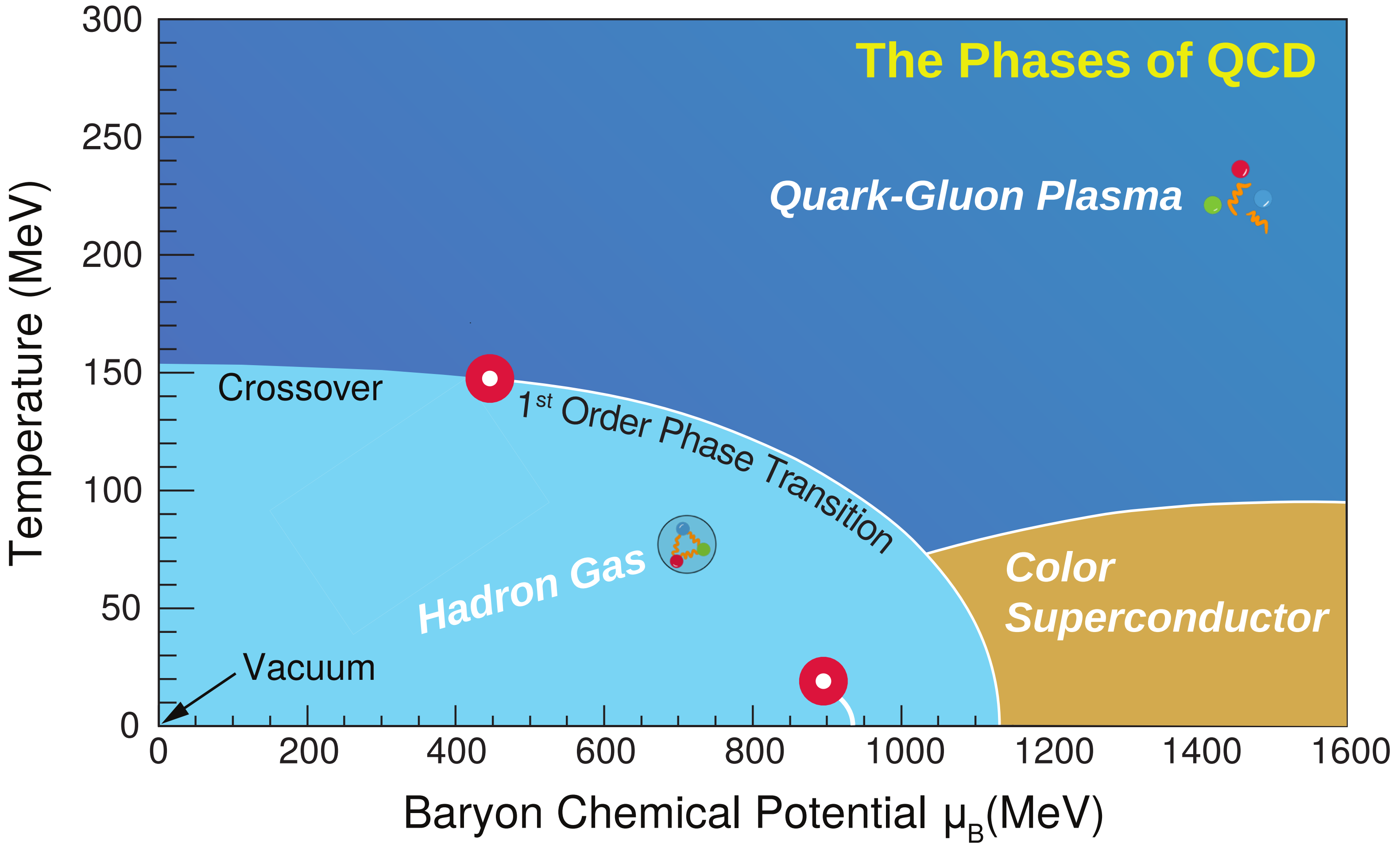
That said, search for QCD critical point is challenging!

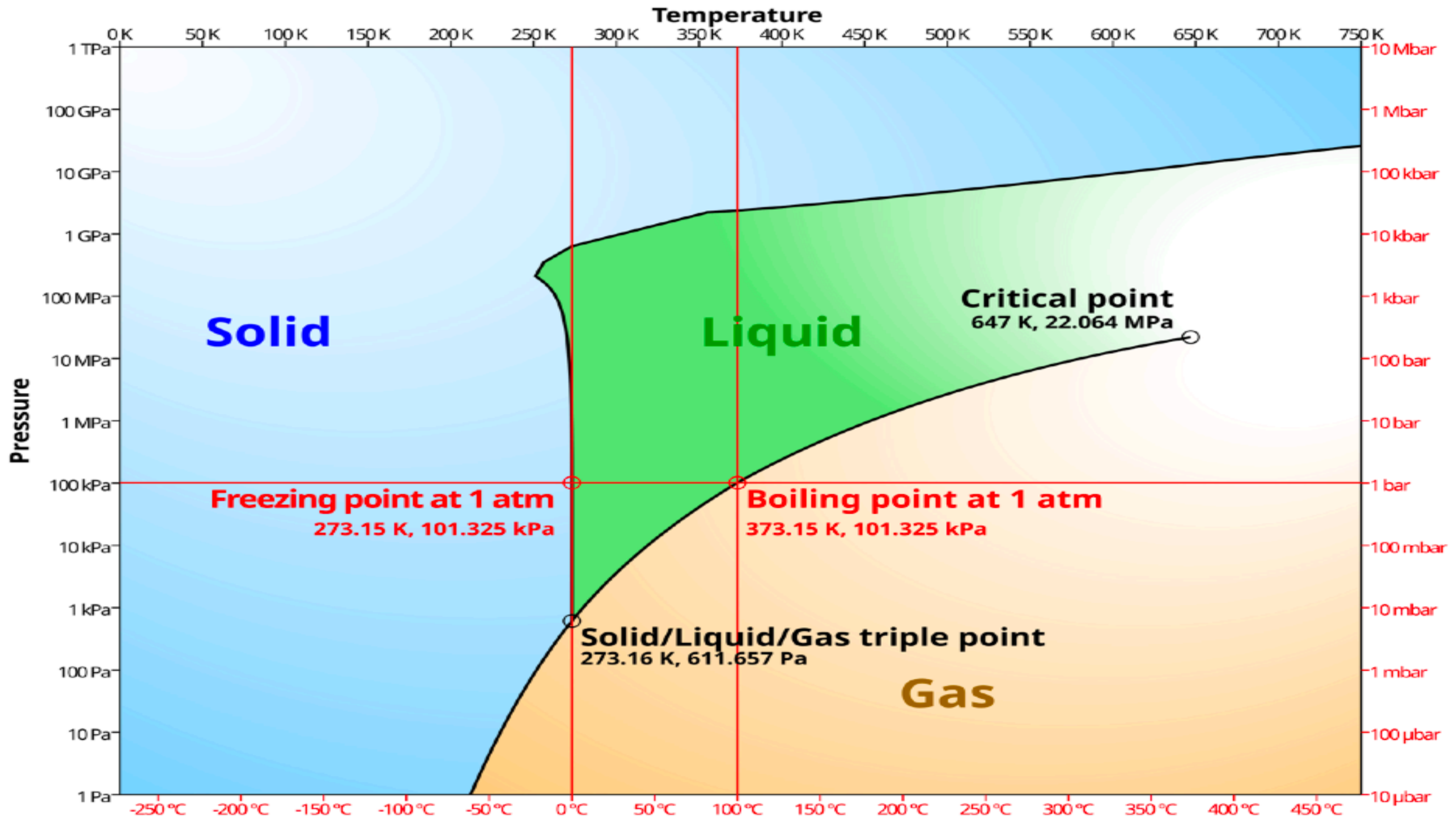
Thermodynamic potentials

- Thermodynamic potential is associated with entropy of the system via

$$\Psi = S - \sum_i J_i \psi_i$$

Landau potential	Planck potential
$\Psi(t, V, y) = S(E, V, N) - tE + yN$	$\Psi(t, x, N) = S(E, V, N) - tE - xV$





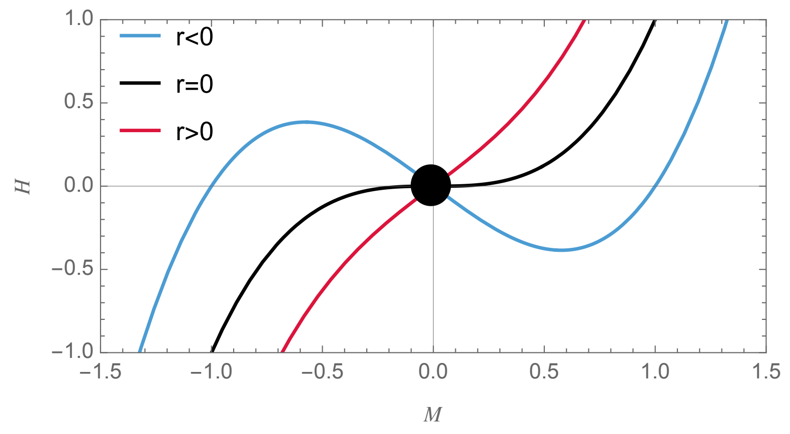
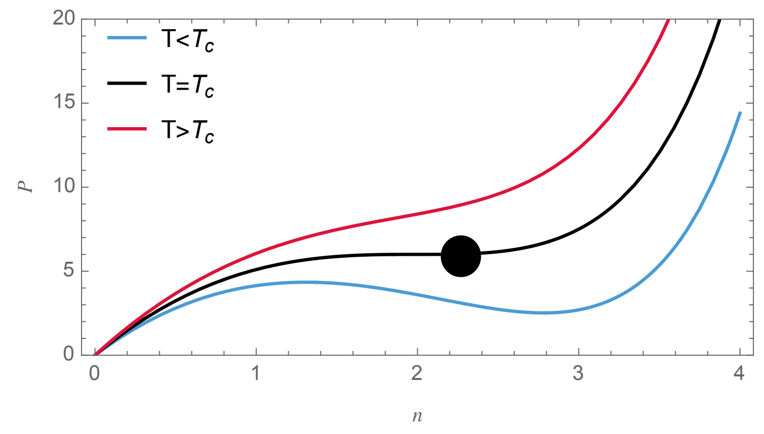
$$t \equiv 1/T, \quad x \equiv P/T, \quad y \equiv \mu/T$$

Mean-field EOS

- Partition function from classical fields to thermodynamic variables:

$$Z[J] = \int \mathscr{D}\psi \, e^{-\int_x S_{\text{eff}}(\psi) - J_i \psi_i} \xrightarrow{\psi(x) = \psi} Z(J) = \int d\psi \, e^{V(S(\psi) + J_i \psi_i)}$$

- Mean-field EOS determined by saddle point: $\left.\frac{\delta S_{\text{eff}}(\psi)}{\delta \psi}\right|_{\psi = \langle\psi\rangle} + J = 0, \quad V \to \infty$

Model	Ising	vdW
$\langle\psi\rangle;\ J$	$M;\ H$	$v;\ 1/T, P/T$
$S_{\text{eff}}(\psi)$	$\frac{1}{2} r\psi^2 + \frac{1}{4} u\psi^4$	$-\log(\psi - b) - \frac{a}{T\psi}$
EOS	$rM + uM^3 + H = 0$ 	$\left(P + av^{-2}\right)(v - b) = T$ 

Extended vdW with multiple critical points

- The partition function for extended vdW:

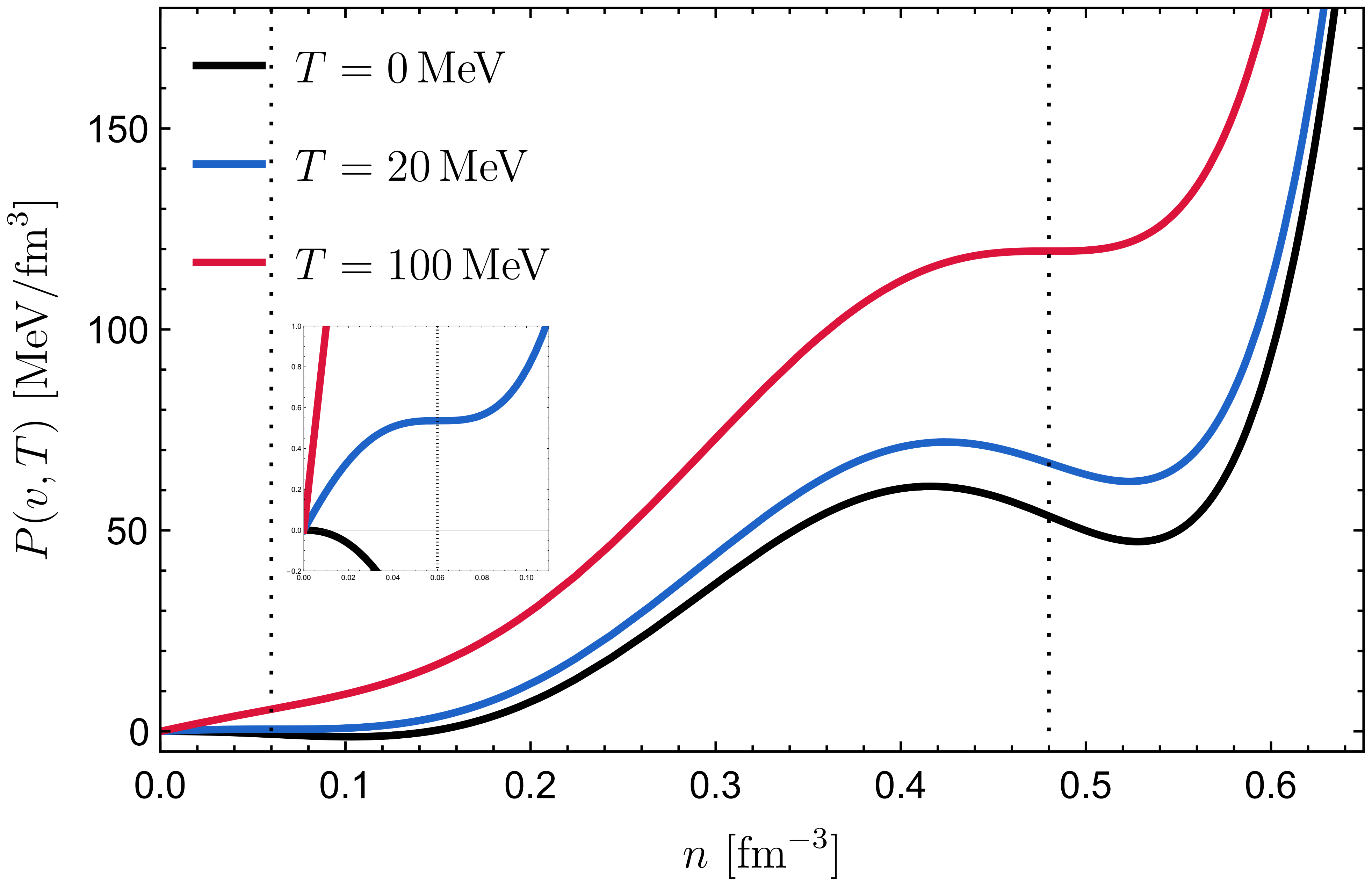
$$Z_N(t,x) \sim \int_b^\infty dv\, e^{N\psi(v)} = \int_b^\infty dv\, e^{N(s(v)-t\epsilon(v)-xv)} \xrightarrow{\text{EOS}} P(v,T) = \frac{T}{v-b} - \sum_{k=2}^{6} \frac{a_k}{v^k}$$

where $s(v) = \log(v-b)$ config. entropy, kinetic part irrelevant

$\epsilon(v) = -\sum_{k=1}^{5} \frac{a_{k+1}}{kv^k}$ virial expansion, multi-particle interactions

We choose

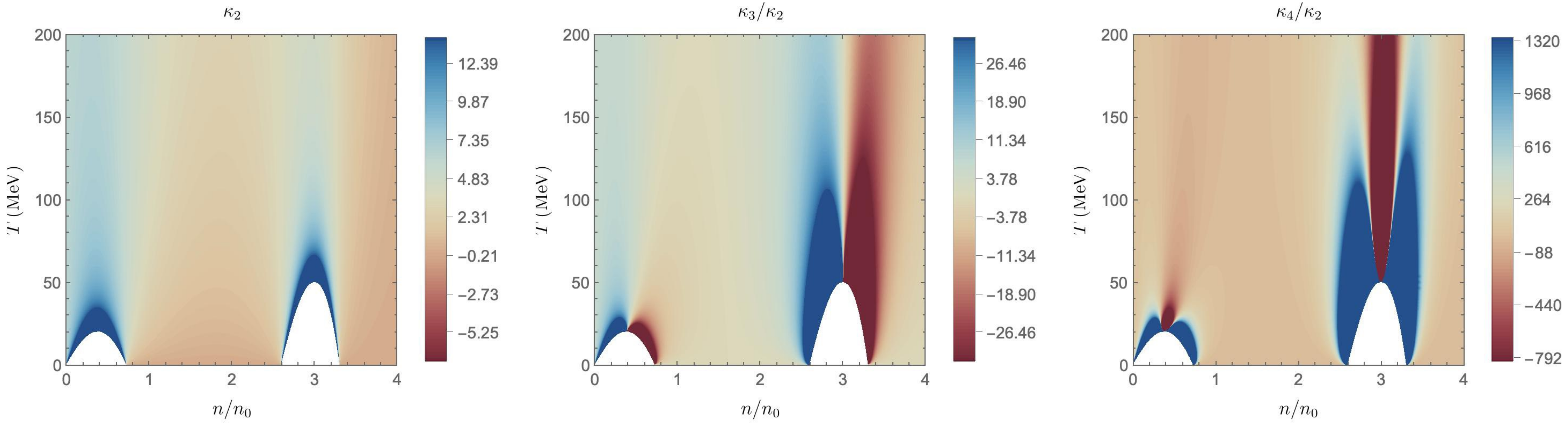
liquid-gas CP:
$T_c = 20$ MeV, $n_c = 0.06$ fm^{-3}
hadron-QGP CP:
$T_c = 100$ MeV, $n_c = 0.48$ fm^{-3}
spinodal boundary:
$T = 0$ MeV, $n_L = 0.42$ fm^{-3}, $n_R = 0.53$ fm^{-3}



Observables

- Observable

$$\langle O \rangle = \frac{1}{Z_N} \int_b^\infty dv\, e^{N\psi(t,x;v)} O(v)$$



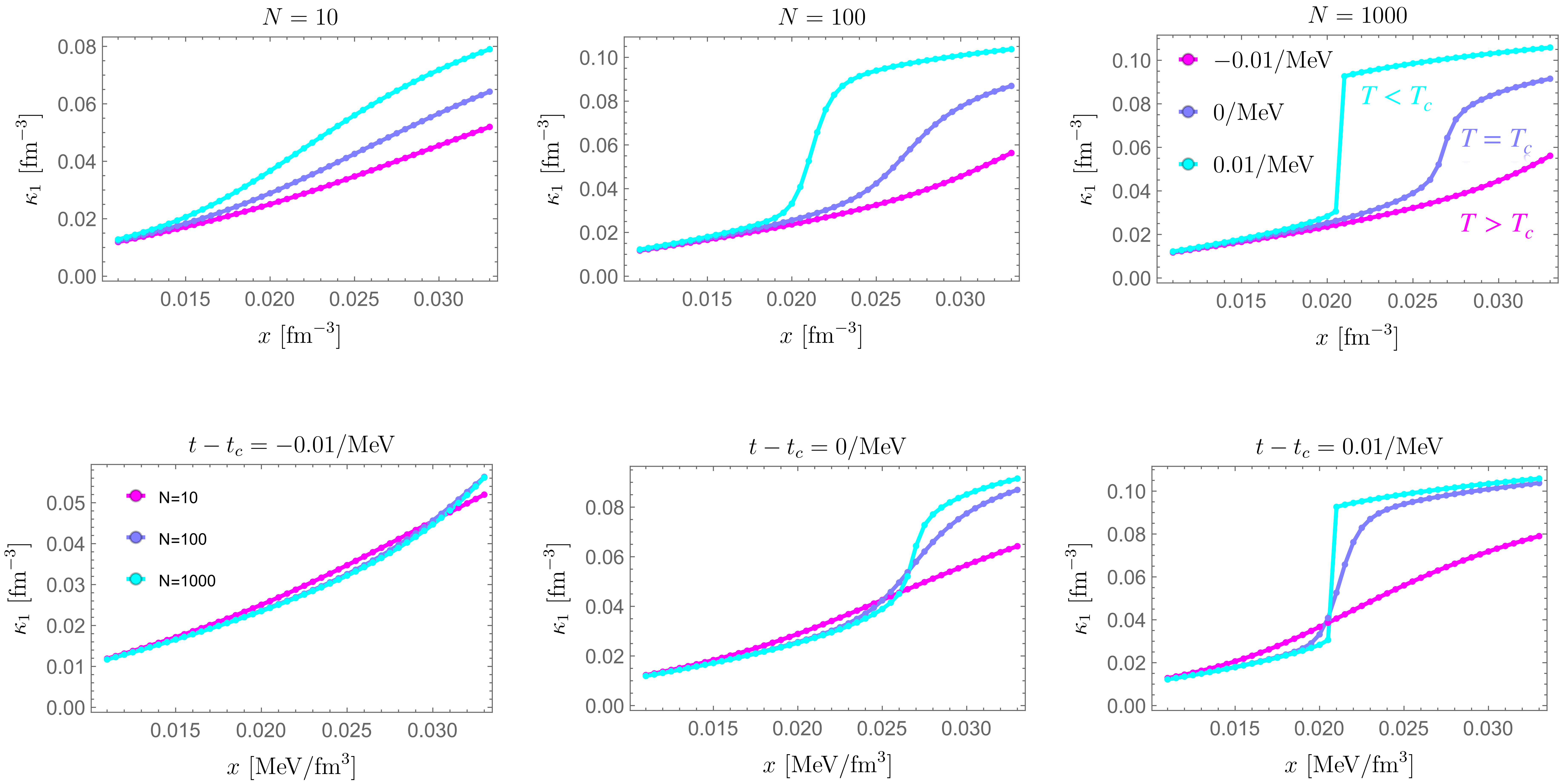
density plot of cumulants $\kappa_m = \langle (n - \langle n \rangle)^m \rangle_c$

Finite size effects on cumulants: 1st

- The 1st cumulant $\kappa_1 \equiv \langle n \rangle = \langle 1/v \rangle$ as function of pressure $x \equiv P/T$ and temperature $t \equiv 1/T$ at different particle number N:

smearing at small N

shift of remnant singularity

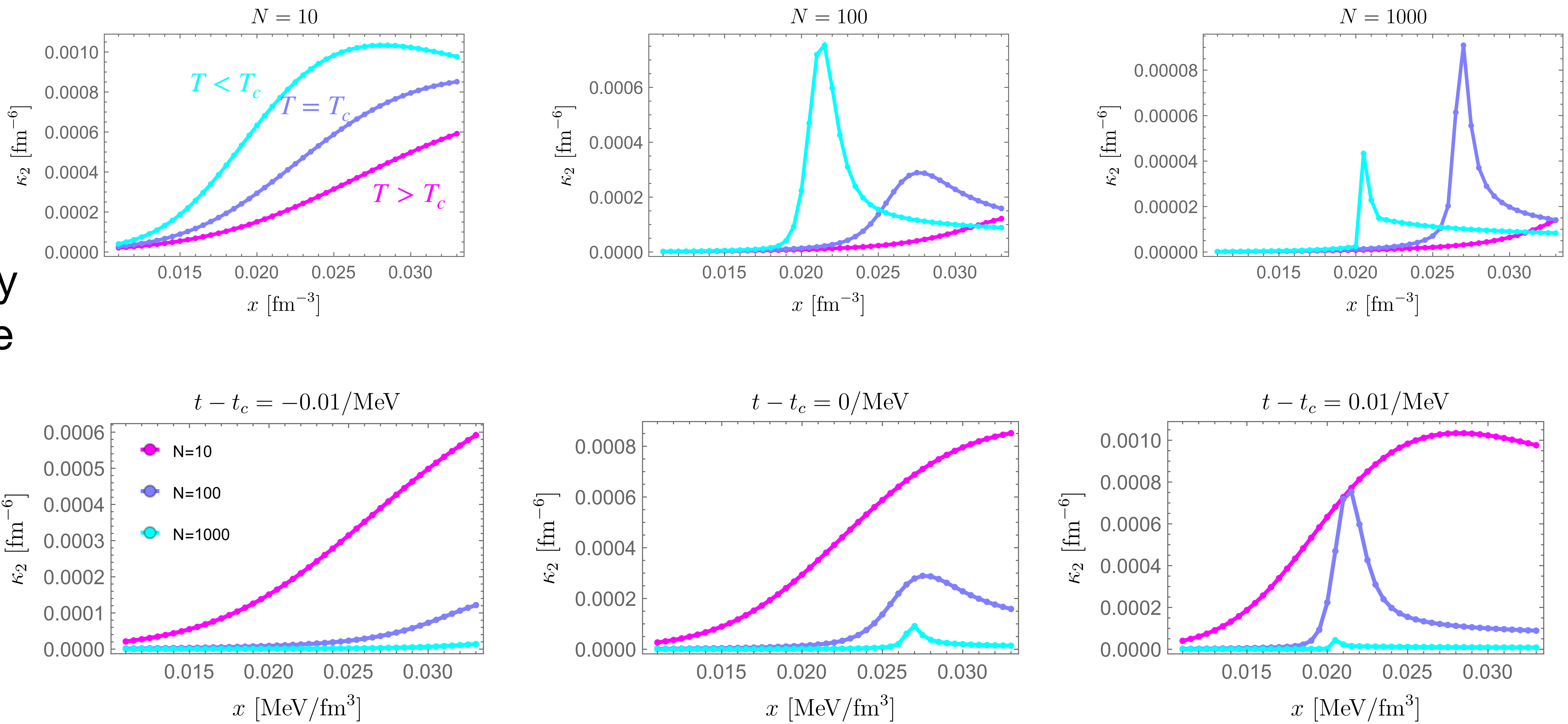


Talk by G. Kovács

Finite size effects on cumulants: 2nd

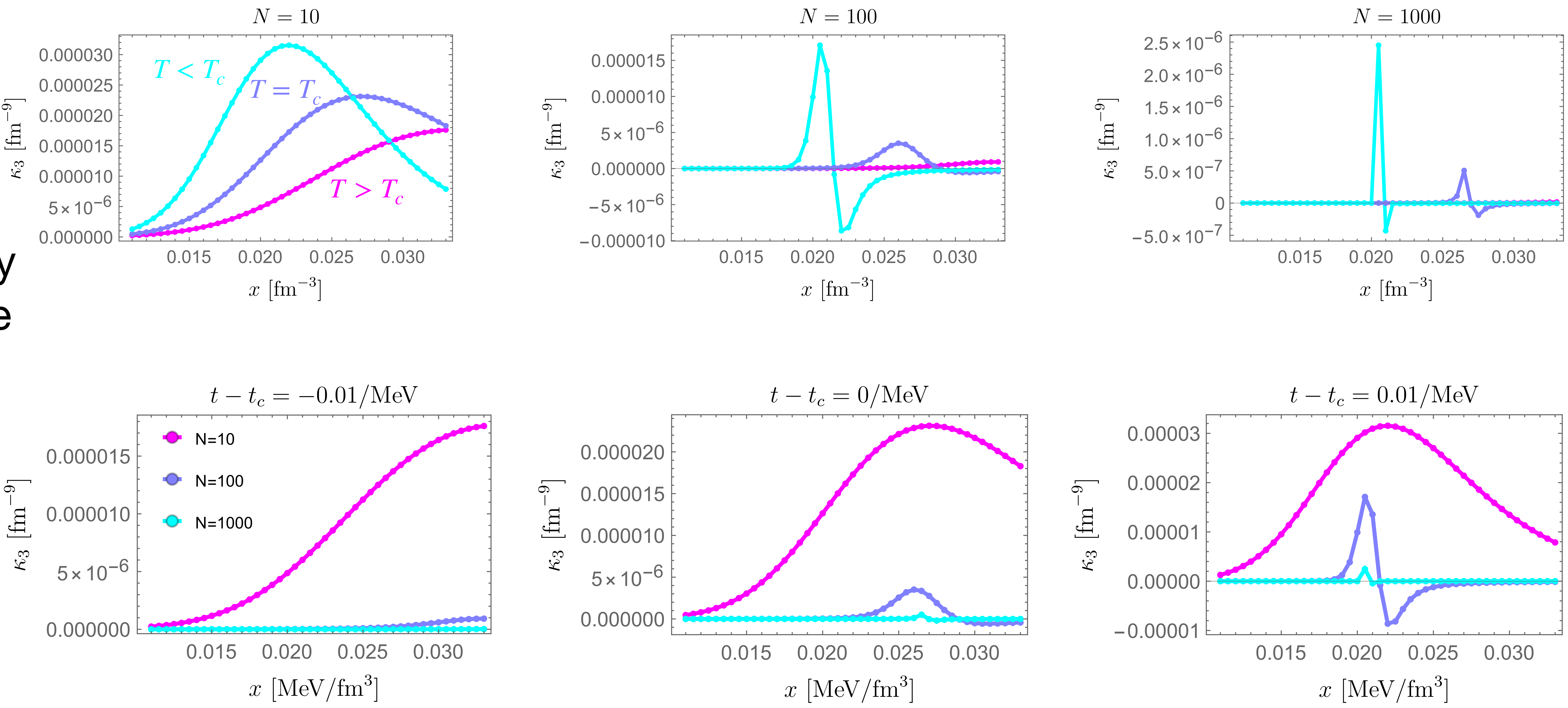
- The 2nd cumulant $\kappa_2 \equiv \langle (n - \langle n \rangle)^2 \rangle_c$ as function of pressure $x \equiv P/T$ and temperature $t \equiv 1/T$ at different particle number N:

smearing singularity
stronger magnitude
shifted location



Finite size effects on cumulants: 3rd

- The 3rd cumulant $\kappa_3 \equiv \langle (n - \langle n \rangle)^3 \rangle_c$ as function of pressure $x \equiv P/T$ and temperature $t \equiv 1/T$ at different particle number N:



smearing singularity
stronger magnitude
shifted location

PDE for partition function

- Partition function satisfies a linear PDE that are integrable (exactly solvable):

$$\left(\partial_x^m \partial_t + \sum_{j=1}^{m} \frac{(-N)^{j+1} a_{j+1}}{j} \partial_x^{m-j}\right) Z_N(t,x) = 0$$

$t \equiv 1/T,$
$x \equiv P/T$

For the conventional vdW model it reduces to

$$\left(\partial_x \partial_t + N^2 a_2\right) Z_N(t,x) = 0$$

(Klein–Gordon equation)

NB: given Cauchy initial conditions at (t_0, x_0), thermodynamics at any (t, x) is inferred.

PDE for free energy and order parameter

- PDE for Gibbs free energy $G_N(t,x) \equiv -\frac{1}{N}\log Z_N(t,x)$

$$B_K \partial_t G_N + \sum_{j=1}^{K} B_{K-j}\left[\binom{K}{j}\partial_x^j \partial_t G_N + \frac{(-N)^j a_{j+1}}{j}\right] = 0, \qquad B_n \equiv B_n(\partial_x G_N, \ldots, \partial_x^n G_N)$$

Bell polynomial

- PDE for order parameter $v_N(t,x) \equiv \partial_x G_N(t,x)$ with $\epsilon_N(t,x) \equiv \partial_t G_N(t,x)$:

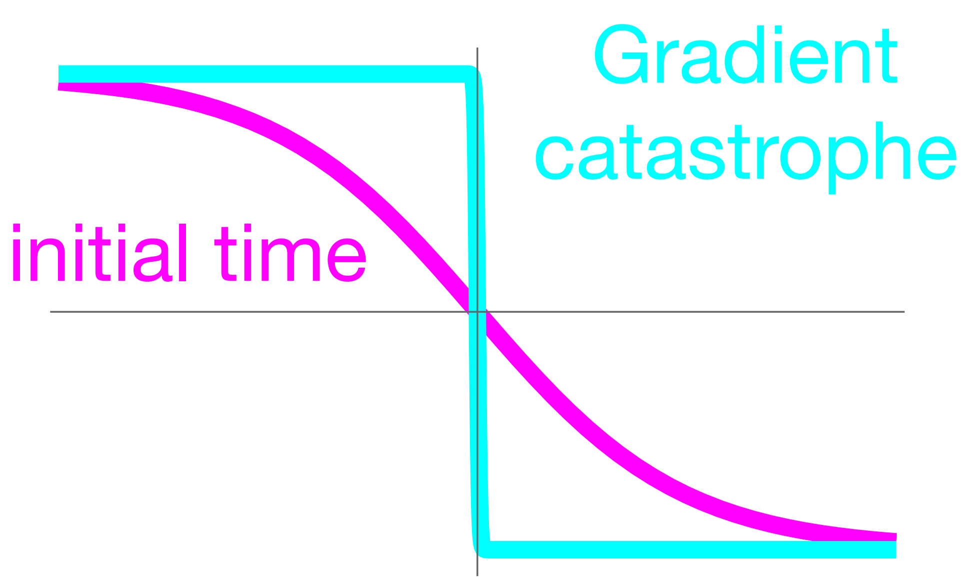
$$\partial_t v_N = \partial_x \left[\epsilon_N(v_N) + \frac{1}{2N}\partial_x \epsilon_N(v_N) + O(1/N^2)\right]$$

(viscous Burgers equation)

For the conventional vdW model $(N \to \infty)$ it reduces to

$$\partial_t v = \partial_x \epsilon(v) = -\partial_x\left(\frac{a_2}{v}\right)$$

(inviscid Burgers equation)



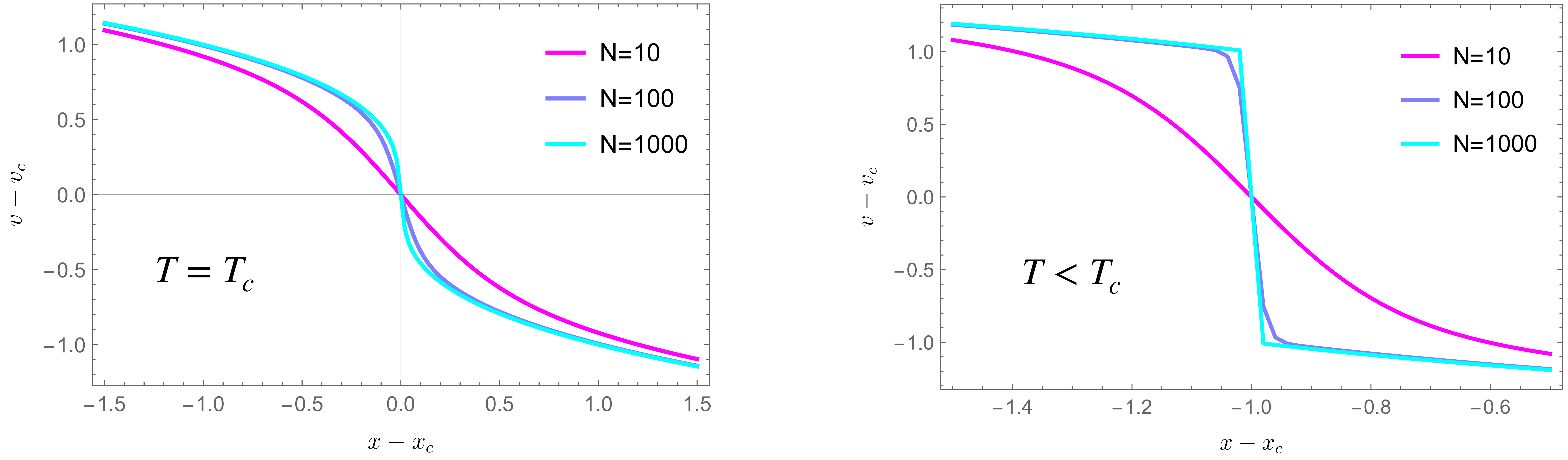
Critical EOS at finite N

- The universal scaling EOS near CP at finite N: Dubrovin et al, 2018

$$v_N(t,x) = v + \frac{\gamma}{N^{1/4}} U\left(\frac{\Delta x - \epsilon'(v)\Delta t}{\alpha N^{-3/4}}, \frac{\Delta t}{\beta N^{-1/4}}\right) + O(1/N^{1/2})$$

where $\Delta t = t - t_c, \quad \Delta x = x - x_c, \quad U(r_1, r_2) = -2\partial_{r_1} \log \int_{-\infty}^{\infty} dr\, e^{-\frac{1}{8}(r^4 - 2r^2 r_2 + 4 r r_1)}$ (Pearcey integral)

α, β, γ : determined by thermodynamic derivatives $\epsilon^{(n)}(v)$ and $s^{(n)}(v)$



Recap

- We developed a family of simple integrable EOS incorporating two critical points in QCD at finite system size.

- The finite size effect plays an important role in heavy-ion collisions.

Outlook

- Finite size corrections for observables at freezeout.

- Different ensembles (e.g., finite volume with particle fluctuations).

Thank You!