Topological observables and θ-dependence in high temperature QCD from lattice simulations

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JÜLICH
Forschungszentrum

[AYuK, M.P. Lombardo, A. Trunin, arXiv: 2502.15407]

XQCD 2025

U(1) axial problem, QCD and topology

Would be Goldstone Bosons	Quark Content	Mass [MeV]
π^+	$u\bar{d}$	140
π^0	$u\bar{u} - d\bar{d}$	135
K^+	$u\bar{s}$	494
K^0	$d\bar{s}$	497
η	$u\bar{u} + d\bar{d} - 2s\bar{s}$	550
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[Witten, 1979][Veneziano, 1979]

Integer: $Q = \int d^4x q(x)$: topological charge

QCD and θ-term

Possible, breaks CP (if $\theta \neq 0$ or $\theta \neq \pi$)

$$\mathscr{L}(\theta) = \mathscr{L}_{QCD} + \theta \frac{1}{32\pi^2} F\tilde{F}$$

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Strong CP, axions: not discussed in this talk

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Topological susceptibility:

$$\chi(T) \equiv c_2(T) = \left. \frac{\partial^2 f(\theta, T)}{\partial \theta^2} \right|_{\theta=0} = \frac{\langle Q^2 \rangle}{V}$$

Kurtosis:

$$b_2(T) = -\frac{c_4(T)}{12 c_2(T)} = -\frac{\langle Q \rangle^4 - 3\langle Q^2 \rangle^2}{12 \langle Q \rangle^2}$$

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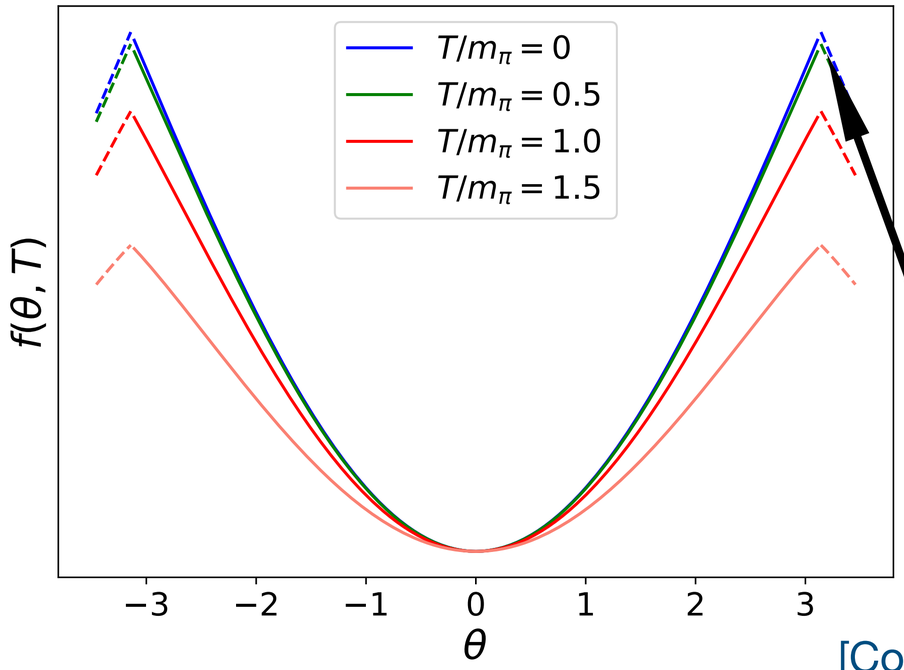
θ-dependence of $Z(\theta, T)$ $\iff$ distribution $Z_Q(T)$ and cumulants $\chi(T), b_2(T), c_{2n}(T)$

Chiral Perturbation Theory (ChPT) and Dilute Instanton Gas Approximation (DIGA)

ChPT @ low $T \simeq 0$



First order P.T. at $\theta = \pi$
Spontaneous breaking of CP
Dashen's phenomenon
[Dashen, 1971]

[Cortona et al., 2016]

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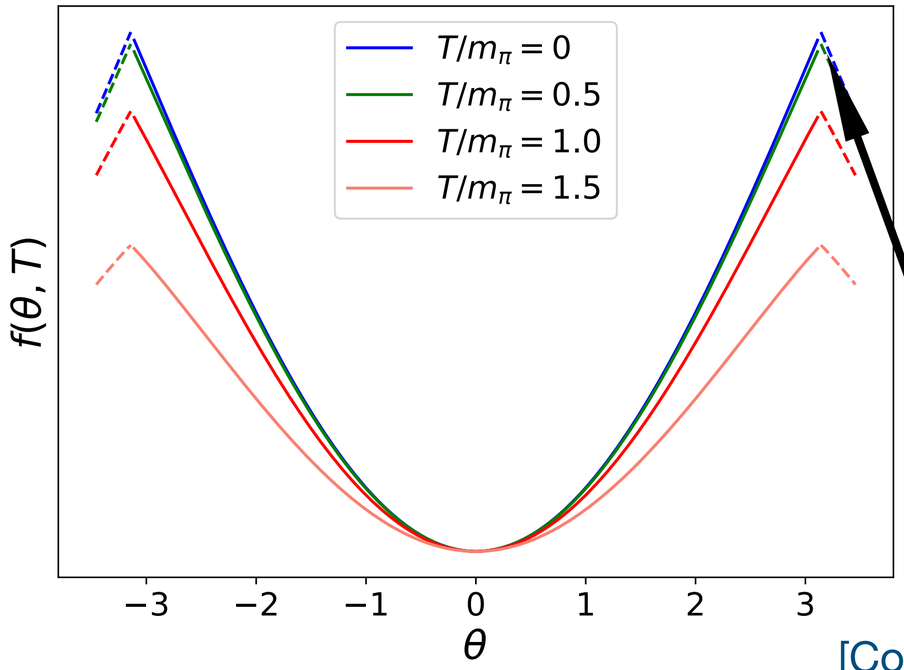

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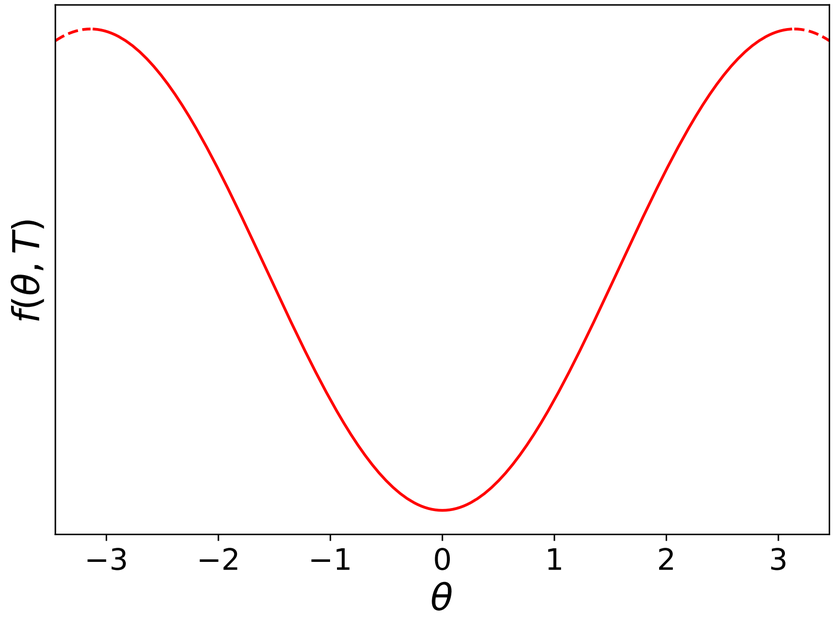
DIGA @ high T: $f(\theta, T) \simeq (1 - \cos\theta) T^{4-\beta_0} \prod \frac{m_i}{T}$

$\chi \sim T^{-b}$ with $b = 7.67/8$ for $N_l = 2/3$, $b_2 = -\frac{1}{12}$

[Gross, Pisarski, Yaffe, 1981]



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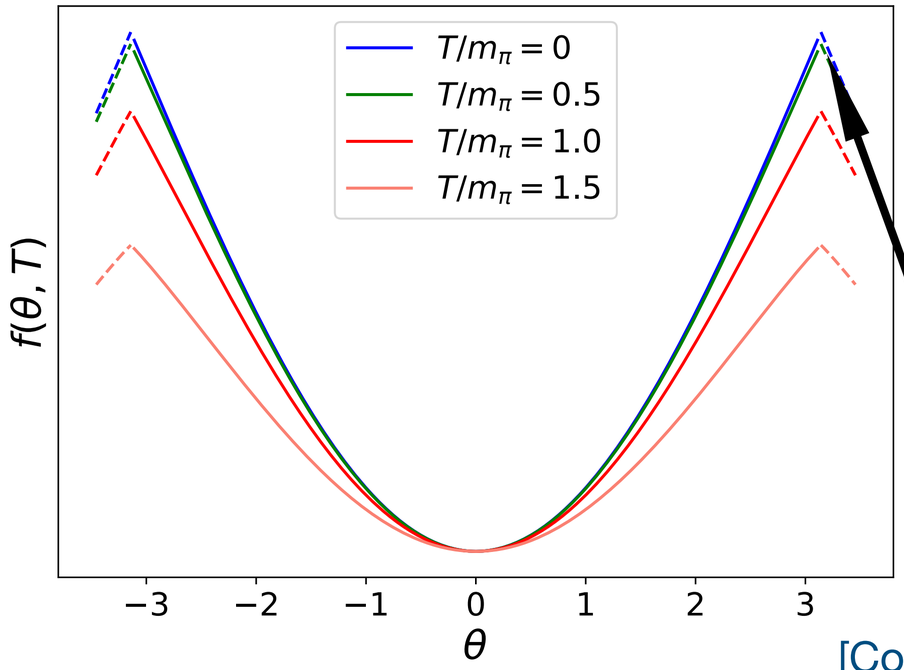

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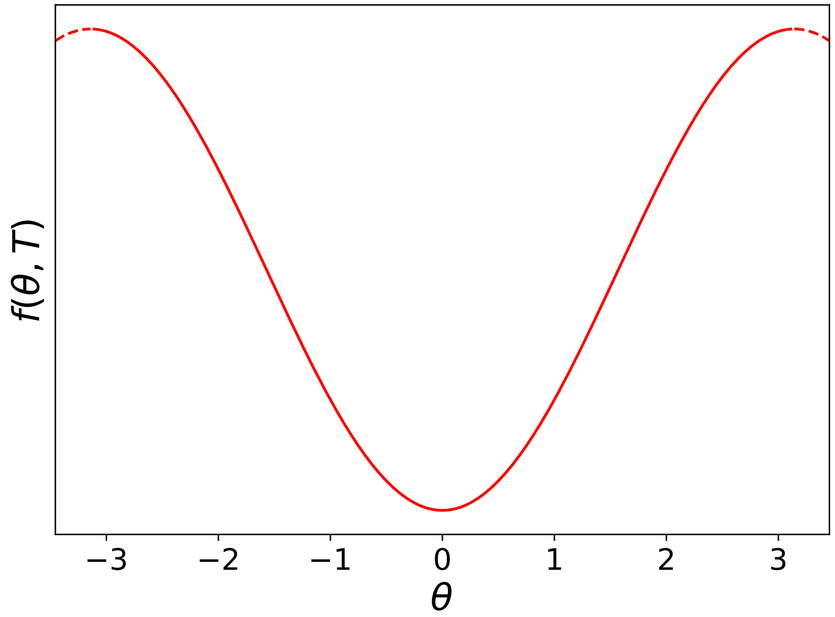

At which T do these approximations work? How do we change from one to another?

TWEXT (Twisted Wilson @ EXTreme) Collaboration

Lattice details

- 2+1+1 Wilson twisted mass fermions at maximal twist

  automatically $O(a)$ improved

  [Frezzotti, Rossi, 2004]

- $m_\pi = m_\pi^{\text{phys}}$; $m_s = m_s^{\text{phys}}$; $m_c = m_c^{\text{phys}}$; isospin symmetric: $m_u = m_d$
- Fixed scale approach: $a =$ fixed, $T \leftrightarrow N_t$

  [Alexandrou et al.,2018][Alexandrou et al.,2021]

- Based on ETMC $T = 0$ parameters & tmLQCD code
- Chiral phase transition (and scaling window): $T_c = 157.8 \pm 1.2$ MeV

  [AYuK, Lombardo, Trunin, 2021]

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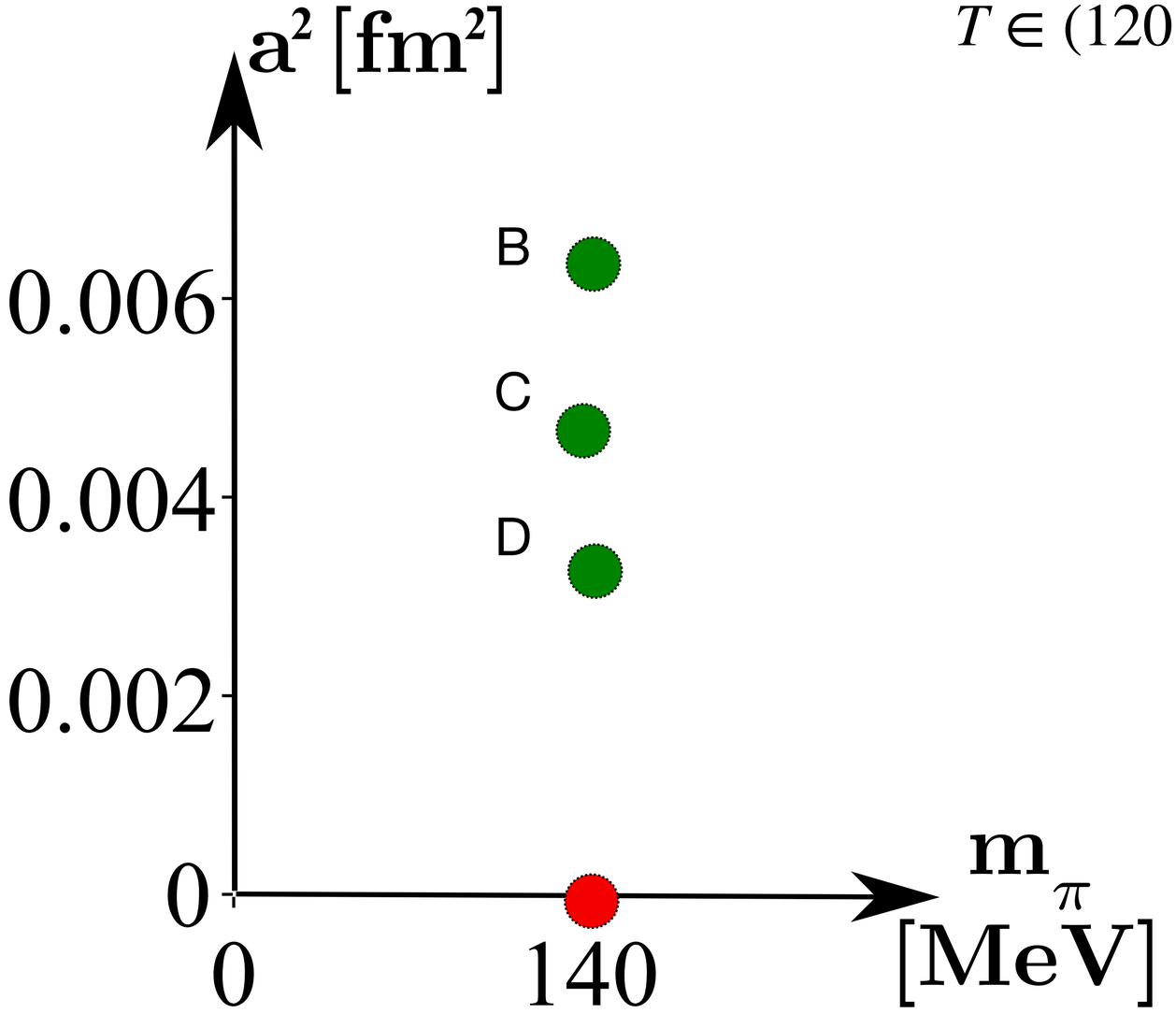
Ensemble summary



$T \in (120 - 900)$ MeV

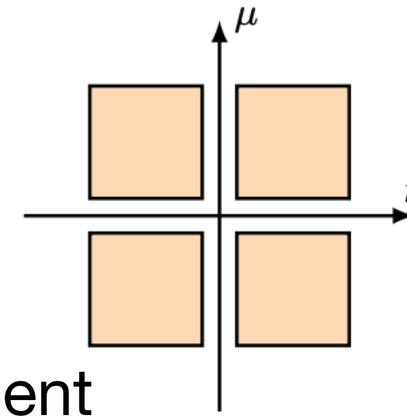
B64: $a = 0.080$ fm, $L = 64a \approx 5.1$ fm
B48: $a = 0.080$ fm, $L = 48a \approx 3.8$ fm
C80: $a = 0.068$ fm, $L = 80a \approx 5.5$ fm
D96: $a = 0.057$ fm, $L = 96a \approx 5.5$ fm

Continuum limit!

Topological susceptibility and other observables

Topological susceptibility and other observables

- Gluonic method:
  - Smearing of gluonic field: Gradient Flow (GF)
  - Topological charge: $Q = \sum_x \frac{1}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a;\mu\nu}(x)$ + 3-loop improvement

$F_{\mu\nu} \sim$ 

[de Forcrand, Perez, Stamatescu, 1997]

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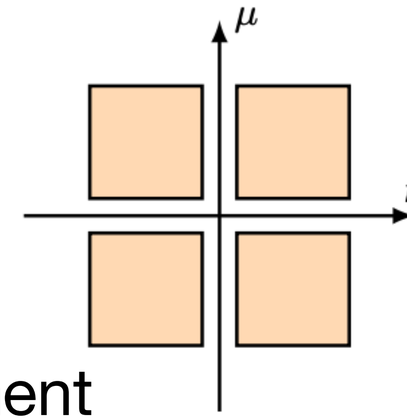
$F_{\mu\nu} \sim$ 

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- Fermionic method (only for $\chi(T)$):

$$m \int d^4x \bar{\psi}\gamma_5\psi = Q_{\text{top}}$$

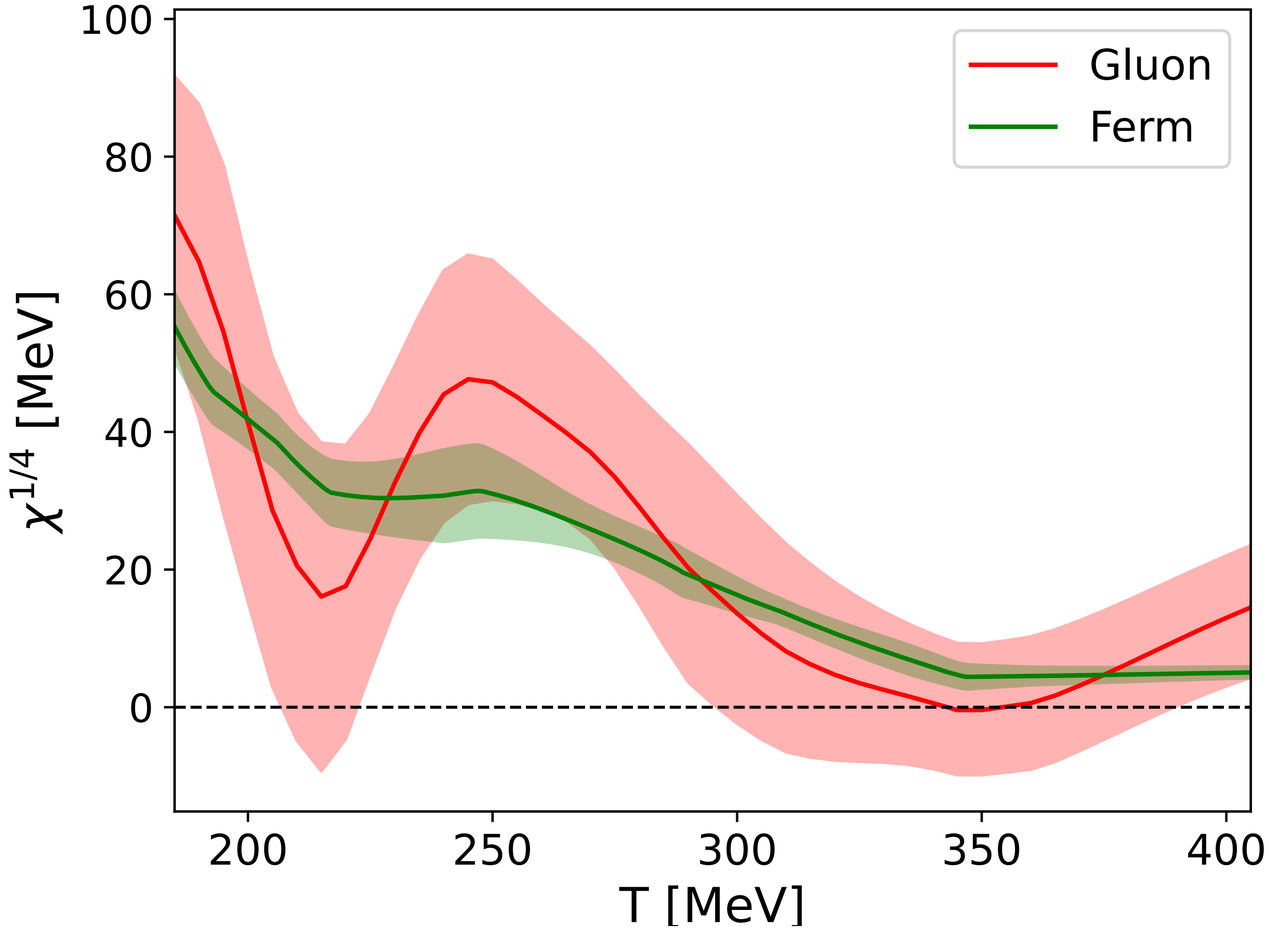
[Kogut, Lagae, Sinclair, 1998]

$$\chi = \langle Q_{\text{top}} \rangle^2 / V = m_l^2 \chi_{5,\text{disc}}$$

$$\chi_{5,\text{disc}} = \chi_{\text{disc}}, \quad \text{for} \quad T \geq T_C, m_l \to 0 \quad \Longrightarrow \chi = \langle Q_{\text{top}} \rangle^2 / V = m_l^2 \chi_{\text{disc}}$$

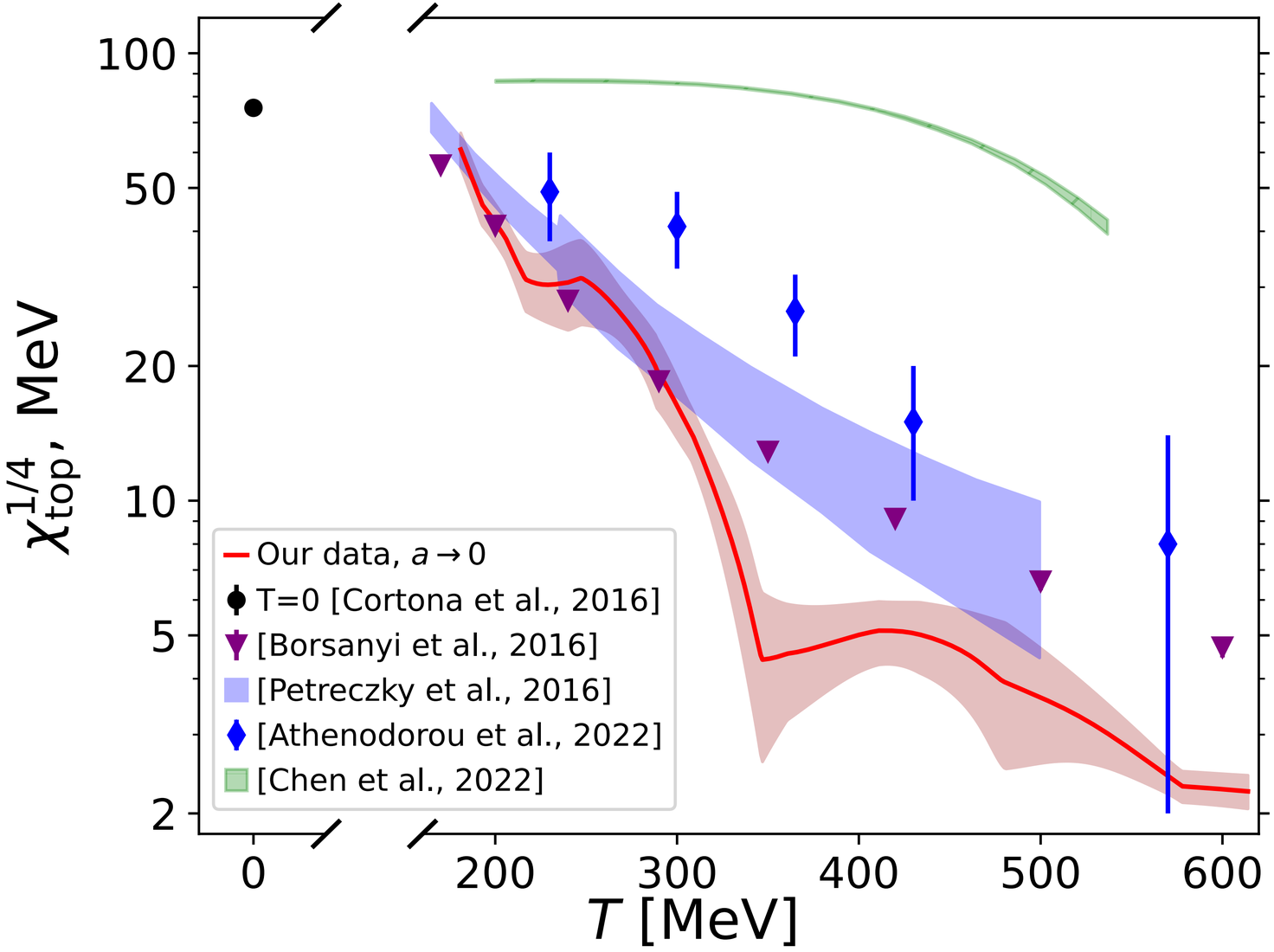
Topological susceptibility, continuum extr'd

Gluonic vs Fermionic definition



Agreement between both definitions!

Topological susceptibility by different groups



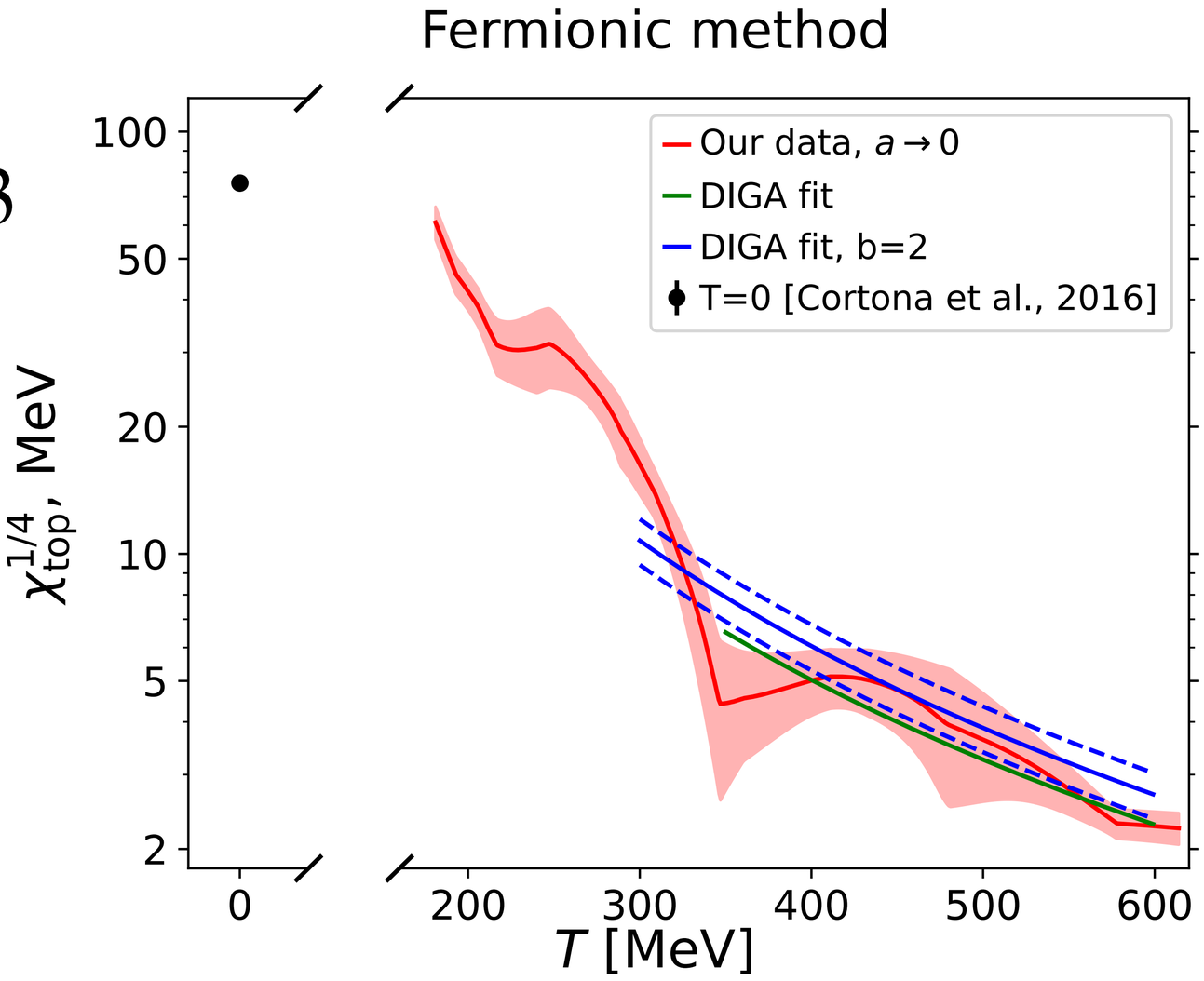
DIGA fit

$$\chi^{1/4} = aT^{-b}$$

with $b = 1.92/2$ **for** $N_l = 2/3$

DIGA: consistent with our data in the range

$300\ \text{MeV} \lesssim T \lesssim 600\ \text{MeV}$



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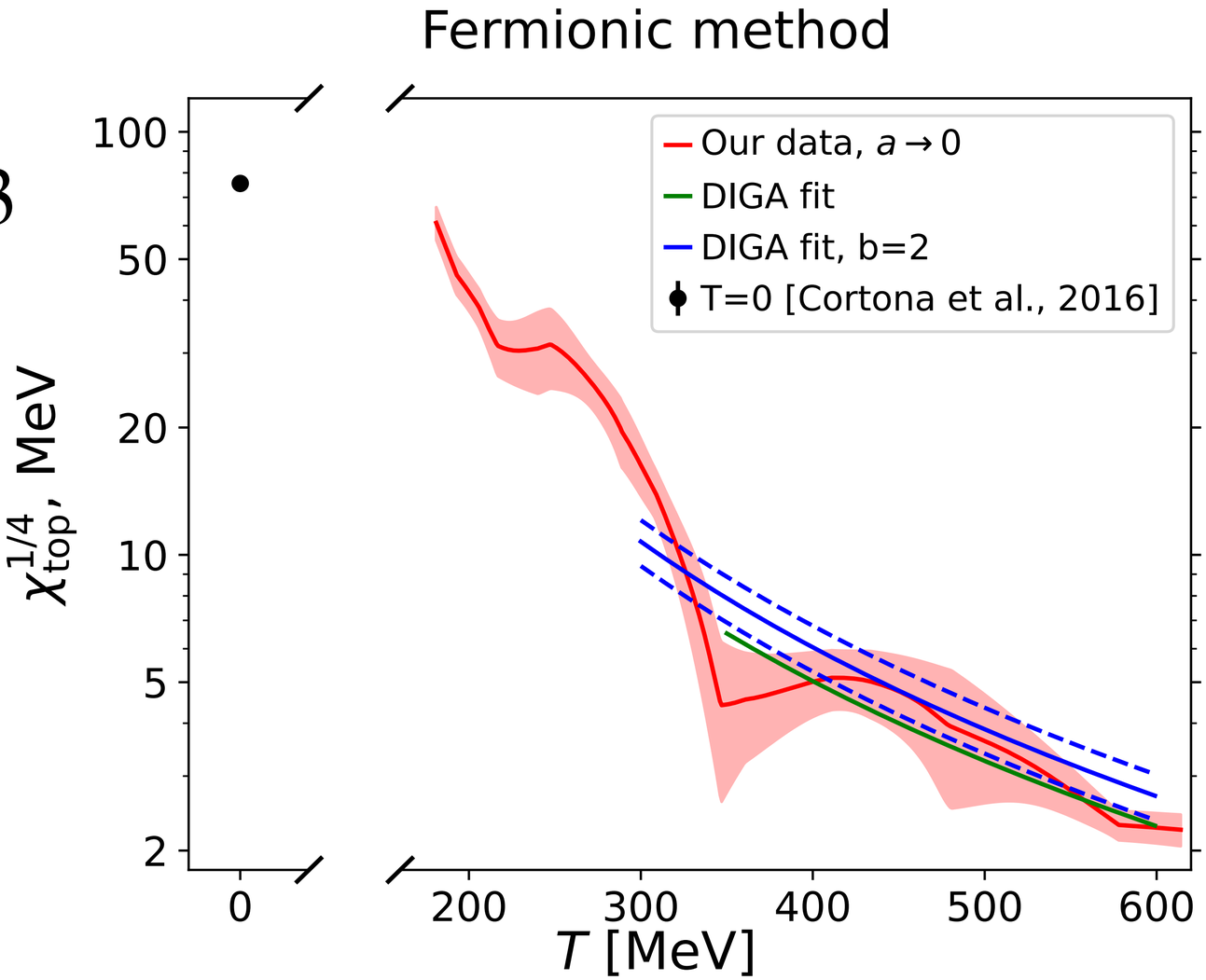
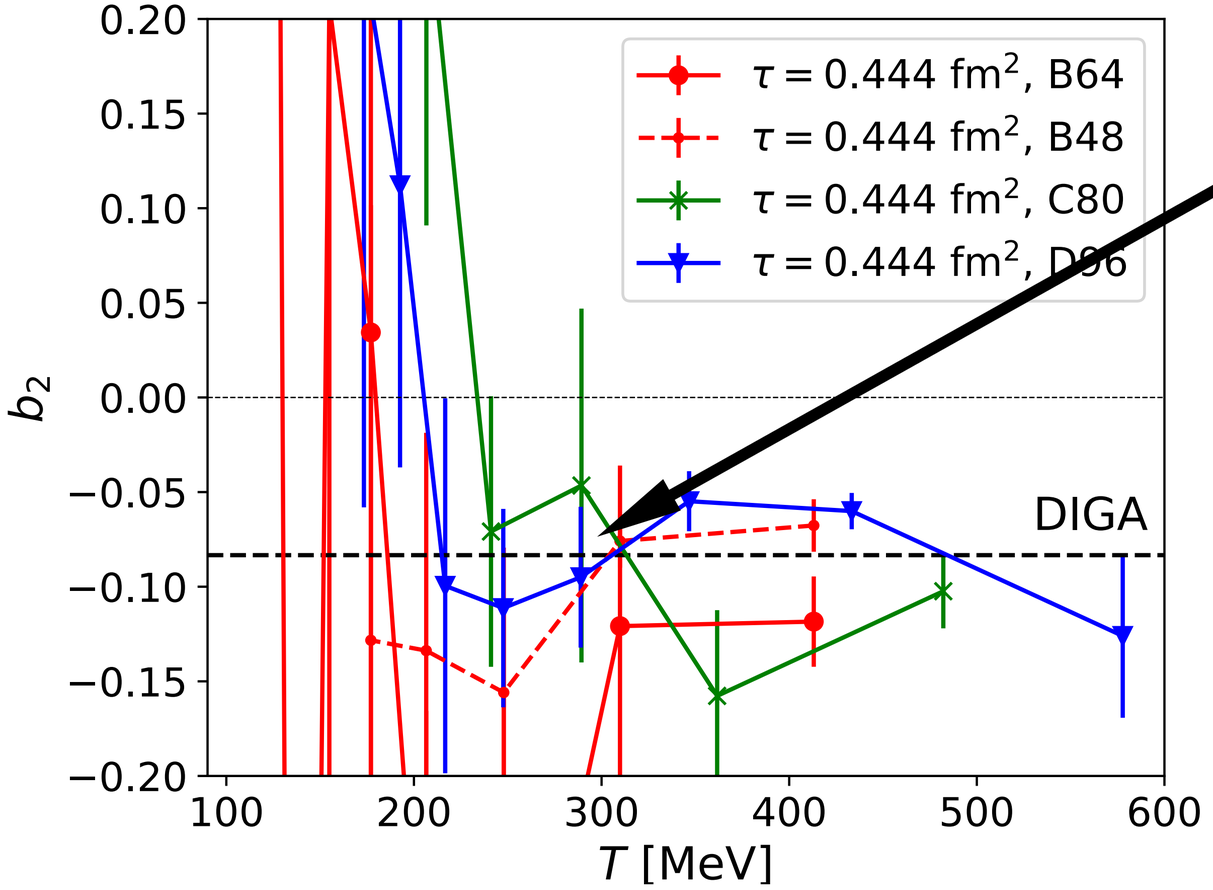
Side remark:
various observables: threshold or transition at $T \sim 300$ MeV,
but full understanding is still missing



[HotQCD, 2016][Mickley et al., 2024][Cardinali et al., 2021]
[Glozman, 2016][Horvath, Alexandru, 2015][TWEXT, 2021]
[Haque et al., 2013]

Kurtosis b_2

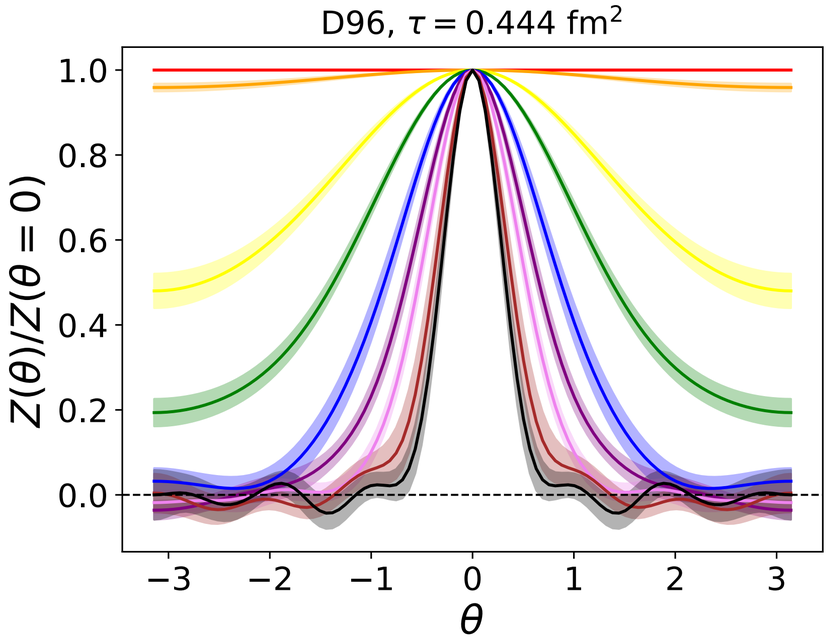
$$b_2 = -\frac{\langle Q^4 \rangle - 3\langle Q^2 \rangle^2}{12\langle Q^2 \rangle} \quad \xrightarrow{\text{DIGA}} \quad -\frac{1}{12}$$



Onset of DIGA behaviour:

$T \sim 300$ MeV

Partition function $Z(\theta)$ and free energy density $f(\theta, T)$



$$Z(\theta, T) = \sum_Q Z_Q(T) e^{i\theta Q}$$

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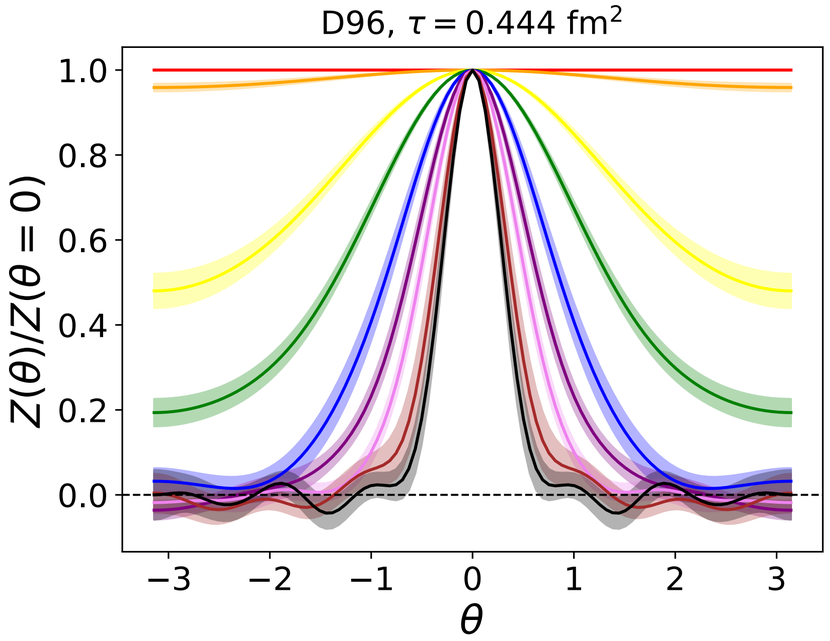

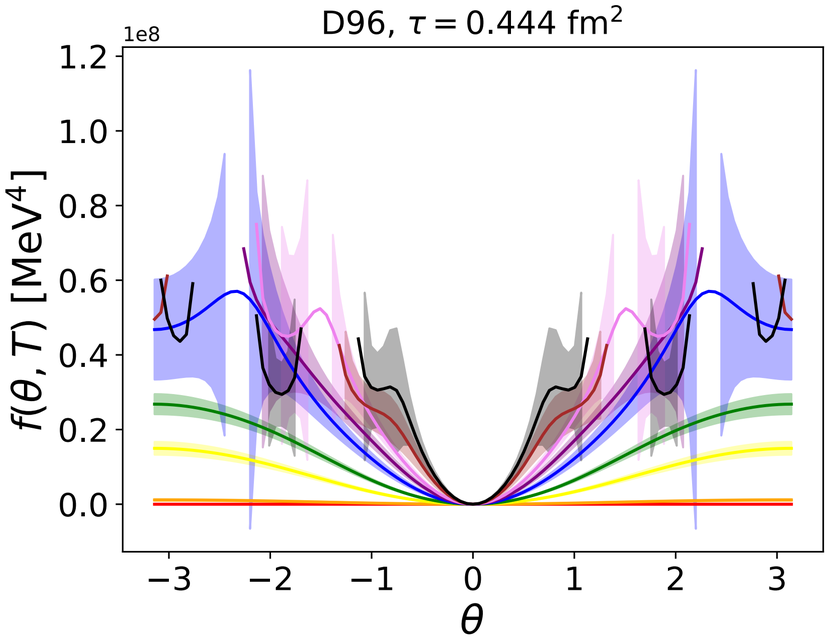

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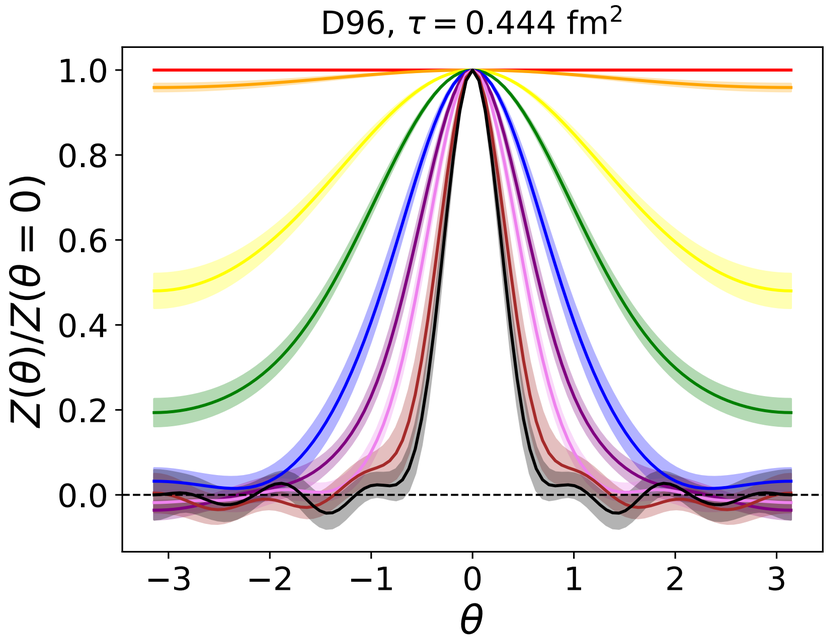

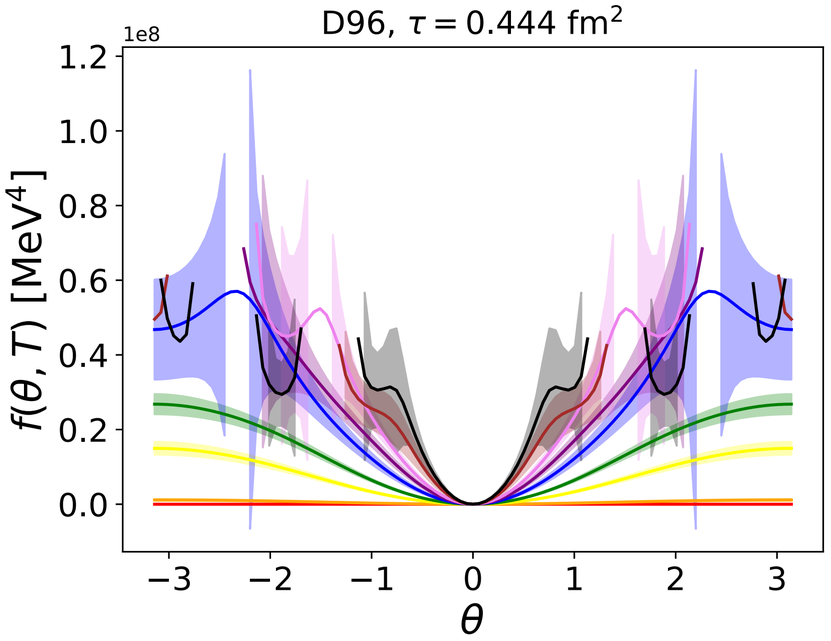

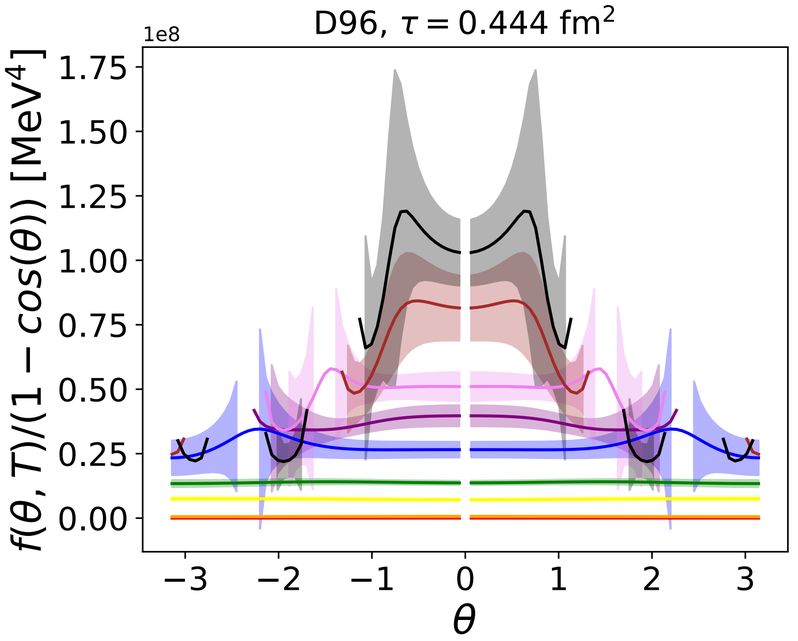

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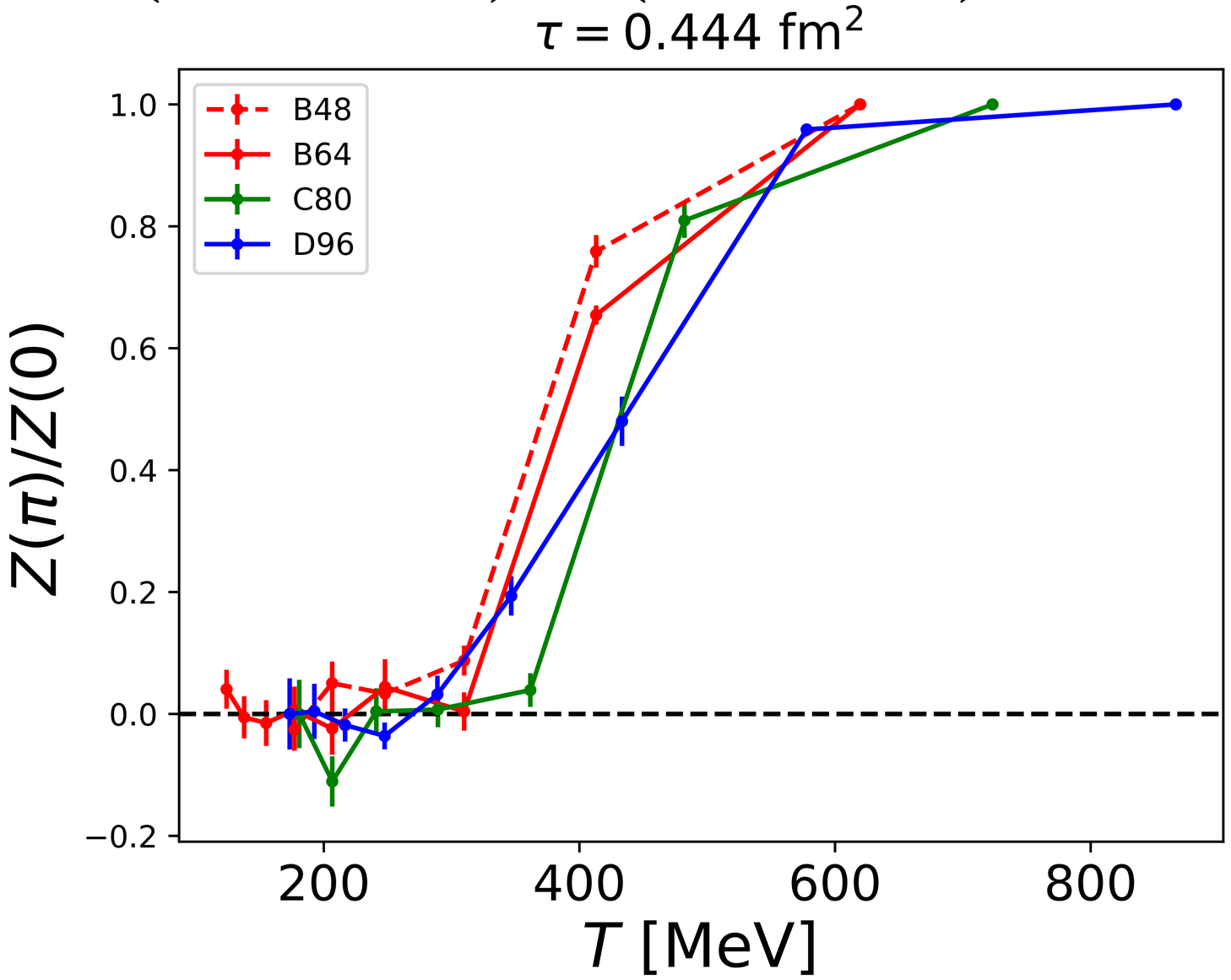
Agreement with DIGA after $T \gtrsim 300$ MeV

$$Z(\theta = \pi, T)/Z(\theta = 0, T)$$

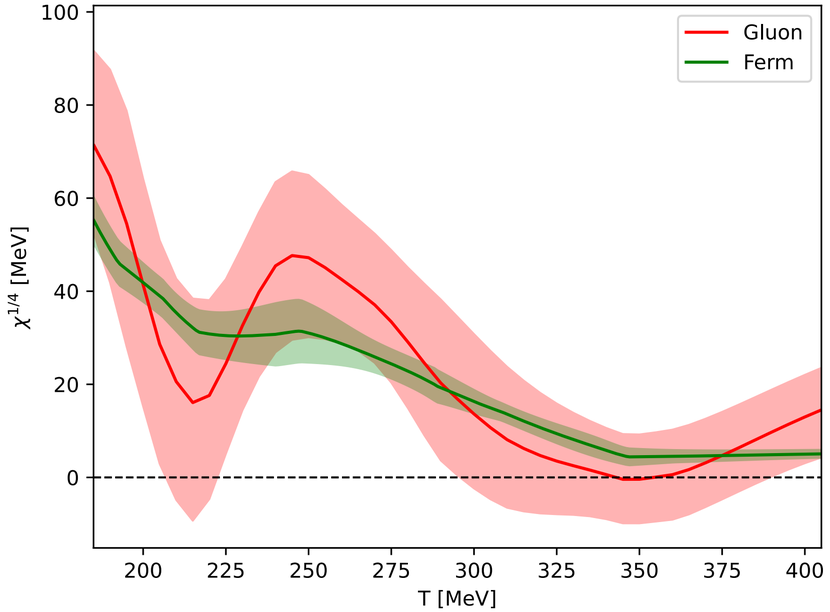


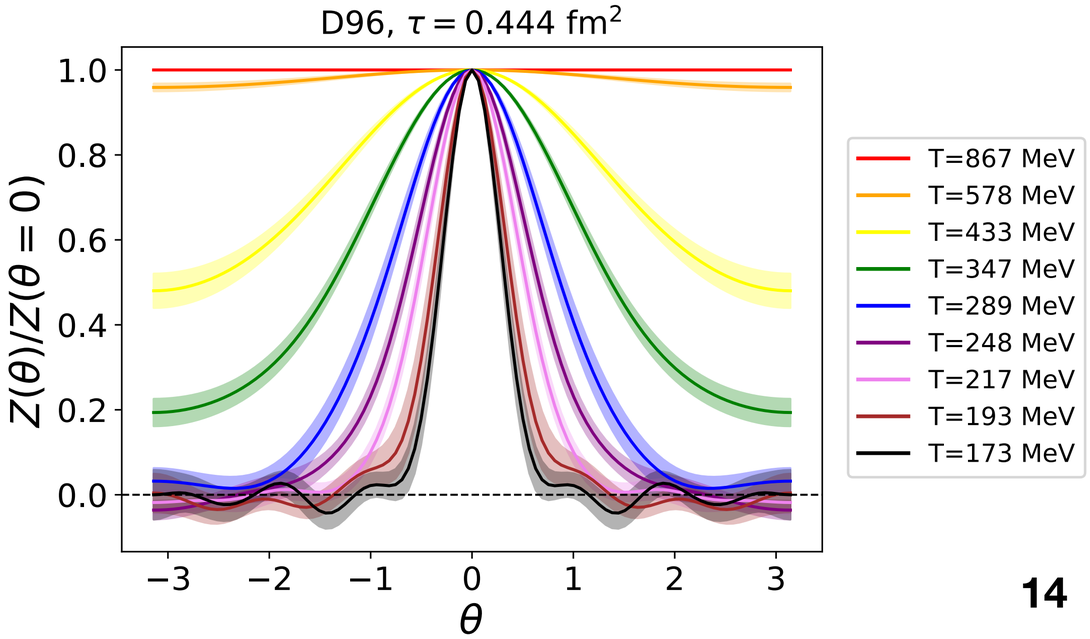
Sharp change of behaviour around $T \sim 300$ MeV

Disappearance of the Dashen's phenomenon ?

Conclusions

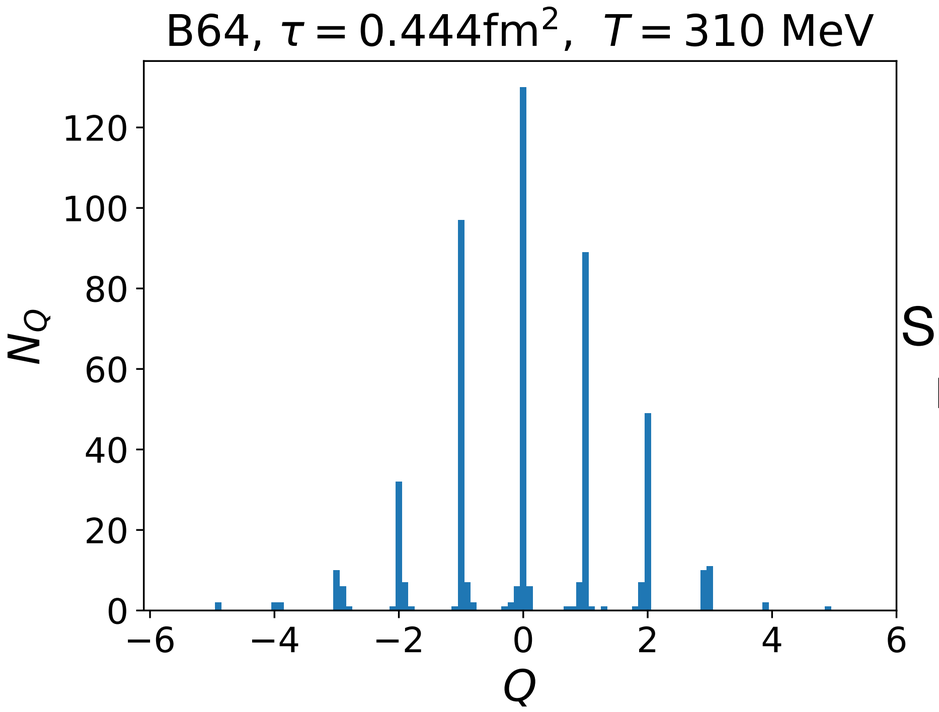
- Topology @ high-T QCD from the lattice
- Results for $\chi(T)$, $b_2(T)$ for $T \gtrsim 180$ MeV
  - Fermionic and gluonic $\chi(T)$: agreement!
- First (up to our knowledge) lattice data for $Z(\theta, T)$, $f(\theta, T)$
- DIGA: $300 \text{ MeV} \lesssim T \lesssim 600 \text{ MeV}$
  - $\chi(T)$, $b_2(T)$, $f(\theta, T)$
- Threshold in QGP for $T \sim 300$ MeV





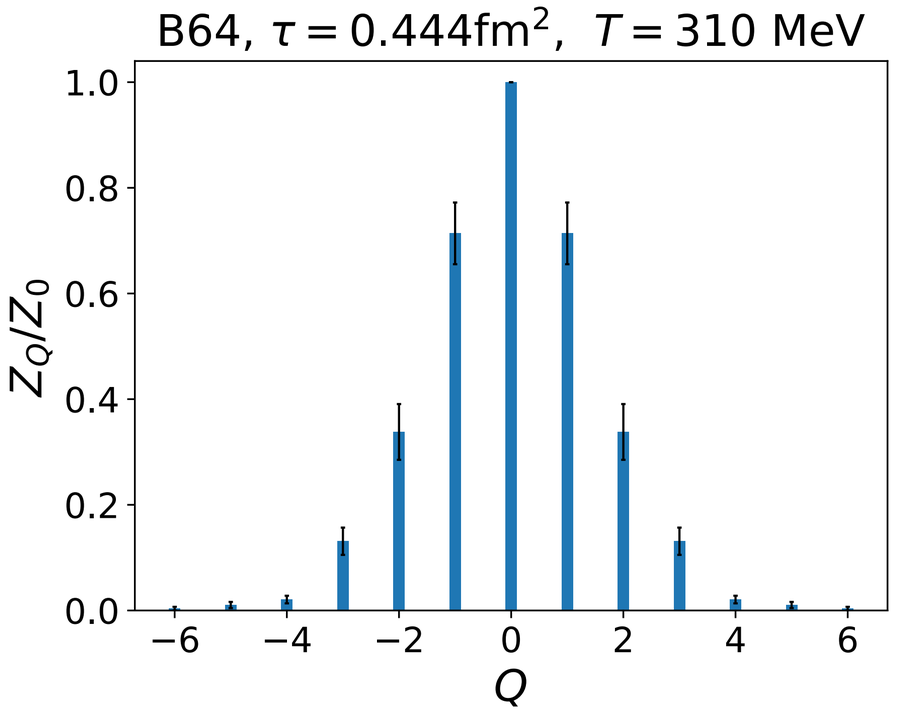
BACKUP

Histogram of the topological charge Q

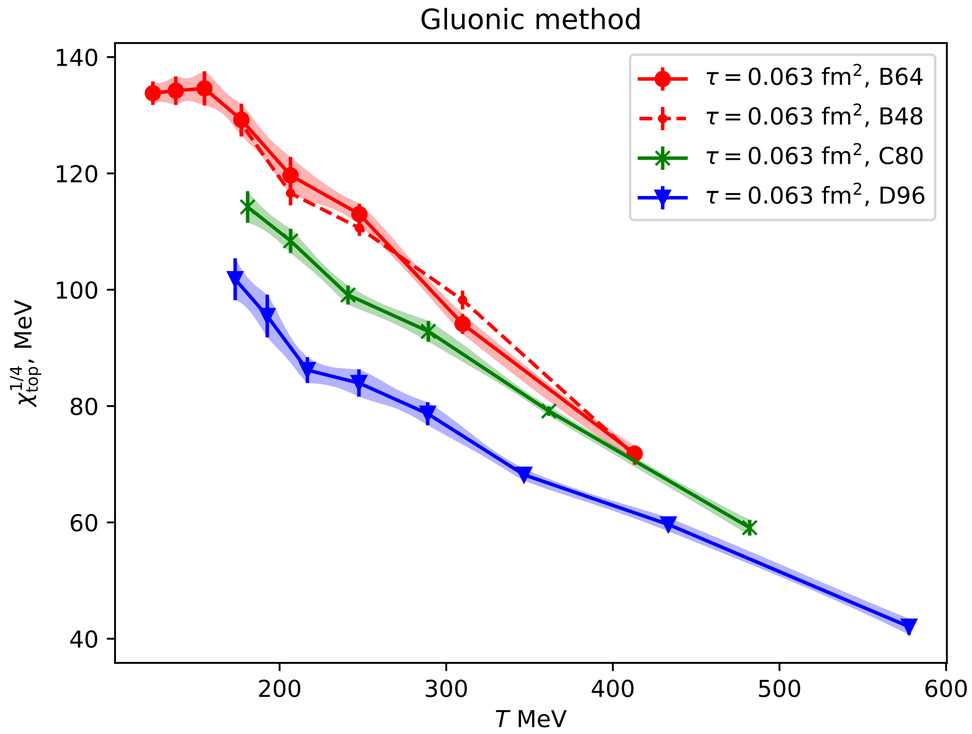


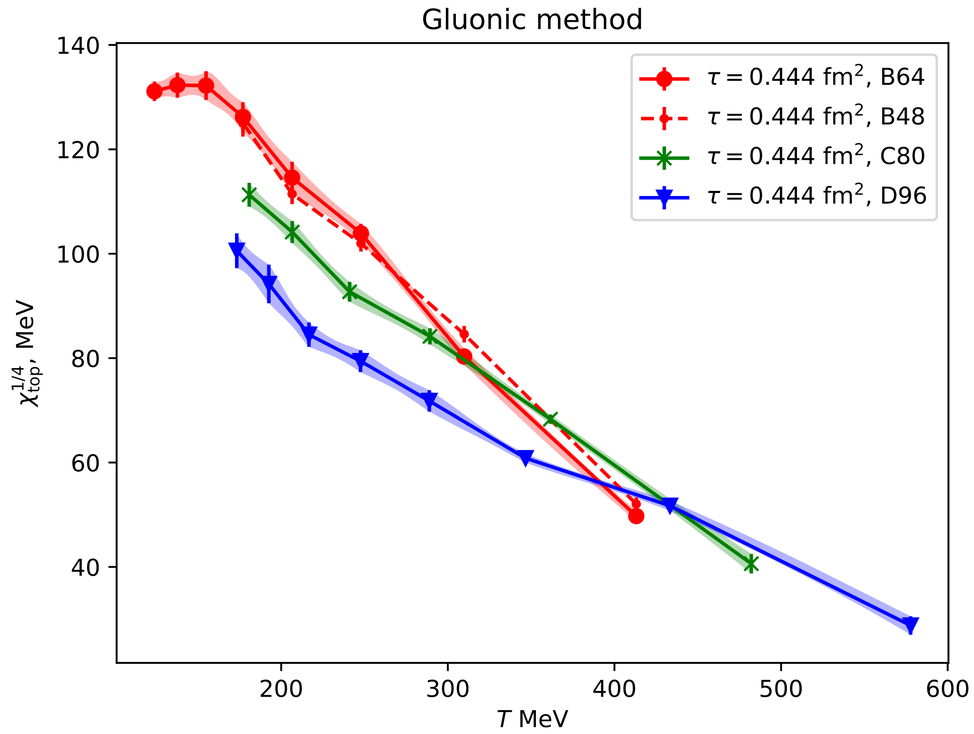
Small rescaling and rounding

[Del Debbio, Panagopoulos, Vicari, 2002]
[Bonati, D'Elia, Scapellato, 2015]

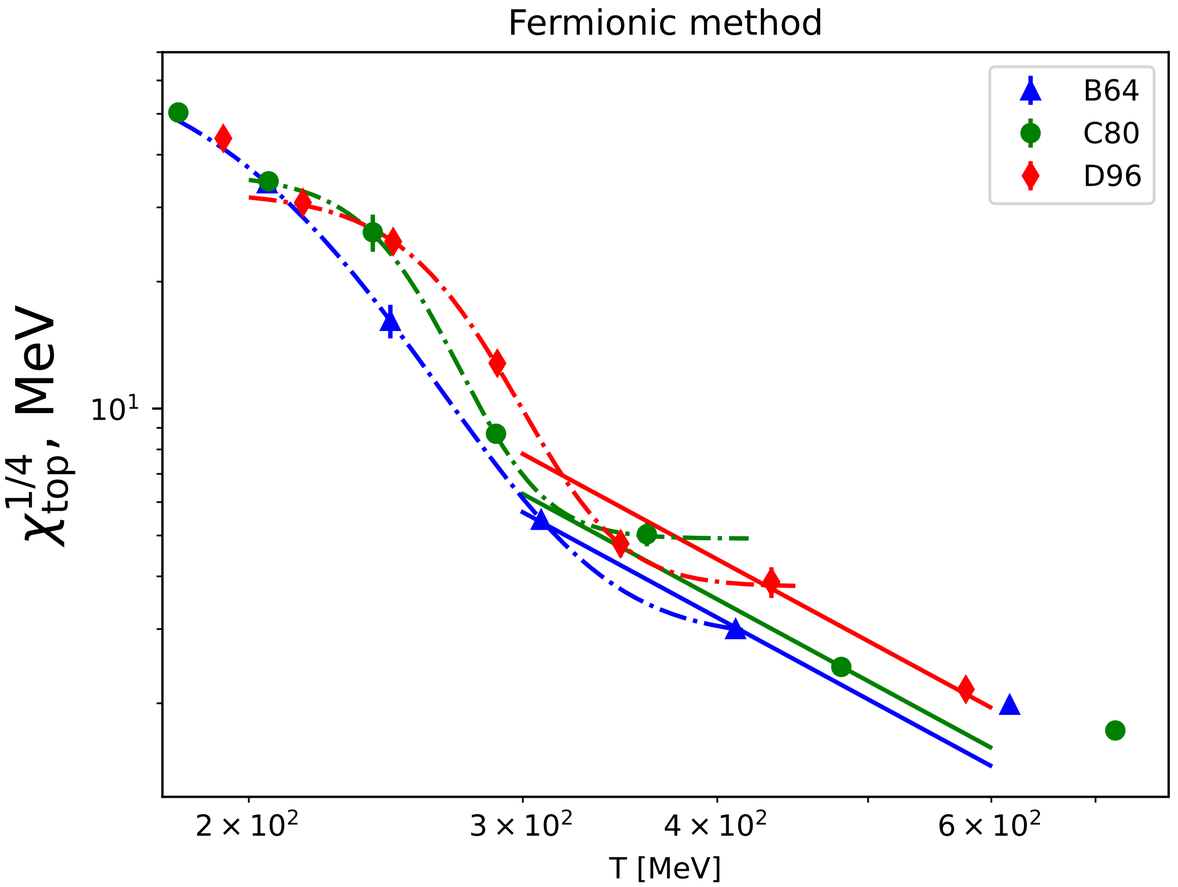


Topological susceptibility from gluonic definition





Topological susceptibility, fermionic method



Topological susceptibility, continuum extrapolation