Equation of State of QCD with $N_f = 3$ flavours up to the electroweak scale

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Phys. Rev. Lett. 134 (2025) 201904 [2501.11603]

Overview

- **Motivation**
- **Strategy**
- **Lattice computation**
- **Equation of State of QCD**
- **Conclusions**

Motivation

- Equation of State (EoS) encodes equilibrium properties of QCD plasma
- Pressure $p(T)$, energy density $e(T)$, entropy density $s(T)$
- Relevant for Heavy Ion Collision ($T \gtrsim 100$ MeV) and Cosmology ($T \gg 100$ MeV)

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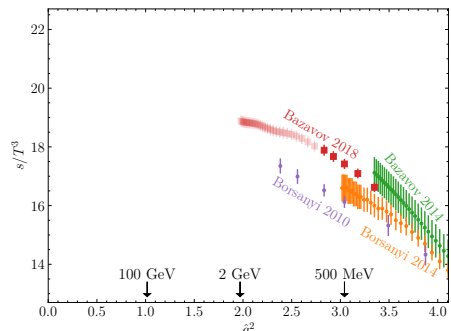
  EoS known non-perturbatively (NP)
  for $N_f = 2 + 1$ QCD



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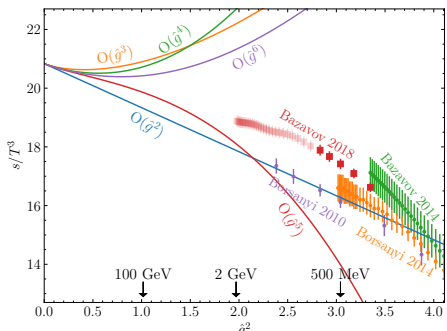

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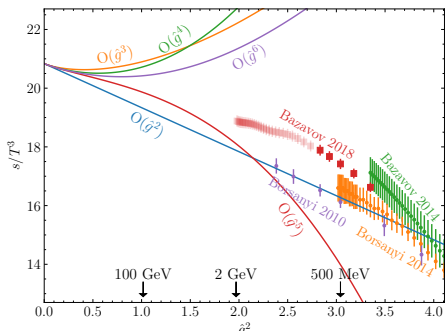

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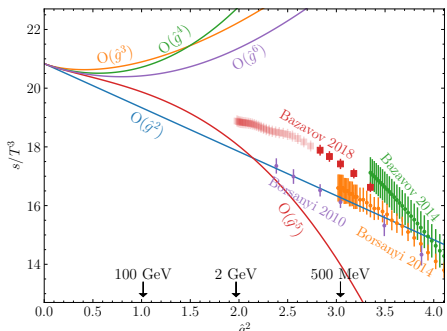


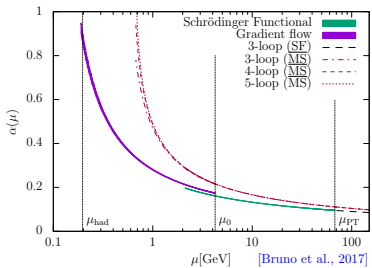
Our goal: NP determination of the QCD EoS up to the electroweak scale

Strategy: renormalization

Hadronic scheme not viable for $T \gg \mathcal{M}_{\rm had} \sim 100$ MeV
$\Rightarrow$ way to go is a NP definition of the QCD coupling

- NP defined coupling $\bar{g}^2$ (e.g. SF scheme)
- Relate $\bar{g}^2(\mu_{\rm had})$ to $\mathcal{M}_{\rm had}$
- Compute $\bar{g}^2(\mu)$ up to the electroweak scale using step-scaling techniques

[Lüscher et al. 91; Dalla Brida et al. 2016]



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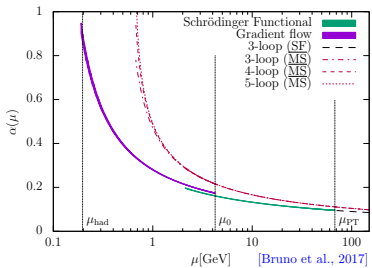


[Bruno et al., 2017]

For each value of T, renormalize thermal QCD by imposing

[Giusti, Pepe 2015]
[Dalla Brida et al. 2021]

$$\bar{g}^2_{\rm SF}(g_0, a\mu) = \bar{g}^2_{\rm SF}(\mu)\,, \quad a\mu \ll 1\,, \quad \mu = T\sqrt{2}$$

$\Rightarrow$ Pietro Rescigno's poster

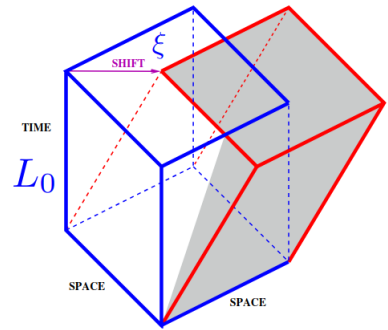
$\Rightarrow$ $g_0^2(a)$ determined for values of a at which μ and T can be easily simulated

Strategy: shifted boundary conditions

Euclidean path integral with shifted boundary conditions:

$$U_\mu(x_0 + L_0, \boldsymbol{x}) = U_\mu(x_0, \boldsymbol{x} - L_0\boldsymbol{\xi})$$
$$\psi(x_0 + L_0, \boldsymbol{x}) = -\psi(x_0, \boldsymbol{x} - L_0\boldsymbol{\xi})$$
$$\overline{\psi}(x_0 + L_0, \boldsymbol{x}) = -\overline{\psi}(x_0, \boldsymbol{x} - L_0\boldsymbol{\xi})$$

$\boldsymbol{\xi} \rightarrow$ spatial shift parameter



Computation of the entropy density:

[Giusti, Meyer 2010-2013]

$$\frac{s}{T^3} = \frac{(1+\boldsymbol{\xi}^2)}{\xi_k}\frac{1}{T^4}\frac{\partial f_{\boldsymbol{\xi}}}{\partial \xi_k}\,, \quad T = \frac{1}{L_0\sqrt{1+\boldsymbol{\xi}^2}}\,, \quad f_{\boldsymbol{\xi}} = -\frac{1}{L_0 L^3}\ln \mathcal{Z}(L_0, L, \boldsymbol{\xi})$$

$\Rightarrow$ Zero-temperature subtraction in the EoS avoided

Lattice computation

Lattice parameters:

- $N_f = 3$ massless $O(a)$-improved Wilson fermions
- 9 temperatures simulated (see table)
- Continuum limit with $L_0/a = 4, 6, 8, 10$ at fixed temperature [Dalla Brida et al. 2022]
- Spatial sizes such that $10 \lesssim LT \lesssim 25$
- Chosen shift $\boldsymbol{\xi} = (1, 0, 0)$

	$\bar{g}^2_{\mathrm{SF}}(\mu = T\sqrt{2})$	T (GeV)
T_0	1.01636	164.6(5.6)
T_1	1.11000	82.3(2.8)
T_2	1.18450	51.4(1.7)
T_3	1.26570	32.8(1.0)
T_4	1.36270	20.63(63)
T_5	1.48080	12.77(37)
T_6	1.61730	8.03(22)
T_7	1.79430	4.91(13)
T_8	2.01200	3.040(78)

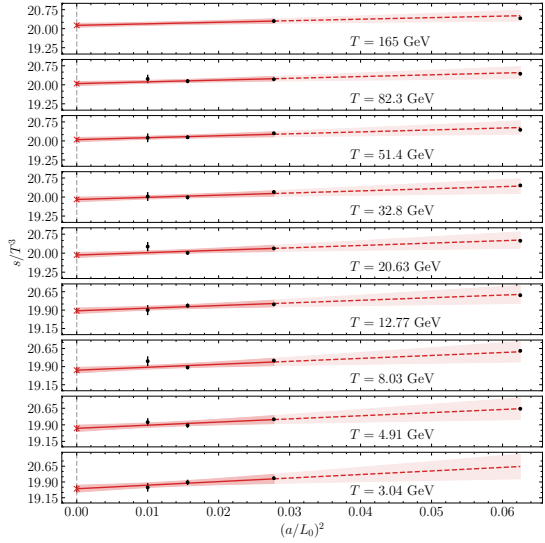
At a given T and L_0/a:

$$\frac{s}{T^3} = \frac{1+\boldsymbol{\xi}^2}{\xi_k}\frac{1}{T^4}\frac{\Delta f_{\boldsymbol{\xi}}}{\Delta \xi_k}\,, \qquad \frac{\Delta f_{\boldsymbol{\xi}}}{\Delta \xi_k} = \frac{L_0}{4a}\left(f_{\boldsymbol{\xi}+\frac{2a}{L_0}\hat{k}} - f_{\boldsymbol{\xi}-\frac{2a}{L_0}\hat{k}}\right)$$

Lattice computation

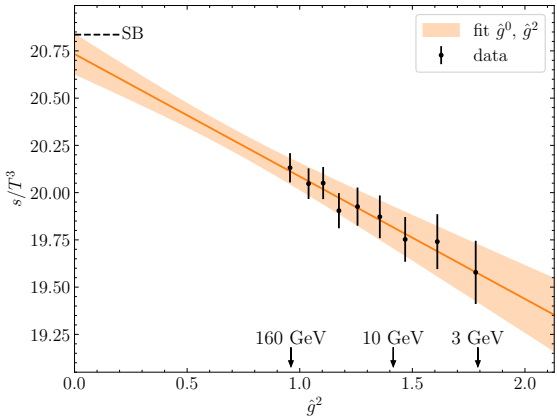
- Extrapolate lattice data to continuum $a/L_0 \to 0$
- Global fit to all temperatures
- Fit $L_0/a > 4$, cutoff effects $\propto \bar{g}^3_{\mathrm{SF}}$
- 0.5-1.0% relative errors of continuum results, $\chi^2/\mathrm{dof} = 0.82$

$$s(\bar{g}_{\mathrm{SF}}, a/L_0)/T^3 = s(\bar{g}_{\mathrm{SF}})/T^3 + d_2\, \bar{g}^3_{\mathrm{SF}} \left(\frac{a}{L_0}\right)^2$$



Equation of State of QCD

- $\hat{g}^2(T)$ = renormalized coupling, logarithmic function of T. Convenient for comparing to PT
- Linear fit in $\hat{g}^2$: $s_0 = 2.954(15)$ compatible with Stefan-Boltzmann (SB) $s_0^{\mathrm{SB}} = 2.969$



$$\frac{s}{T^3} = \frac{32\pi^2}{45}\left[s_0 + s_2\left(\frac{\hat{g}}{2\pi}\right)^2\right]$$

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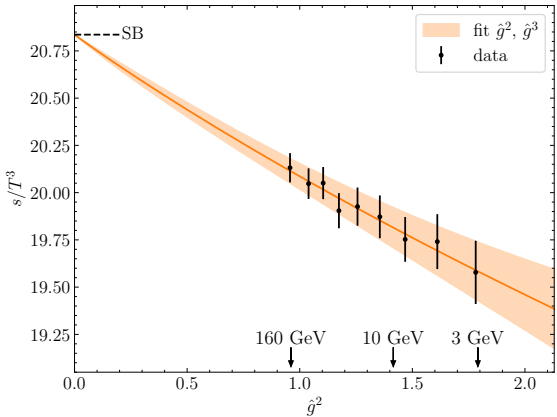
- Enforce s_0^{SB}, add $\sim \hat{g}^3$: $s_2 = -5.1(9)$, $s_3 = 5(5)$ (in PT $s_2 = -8.438$)



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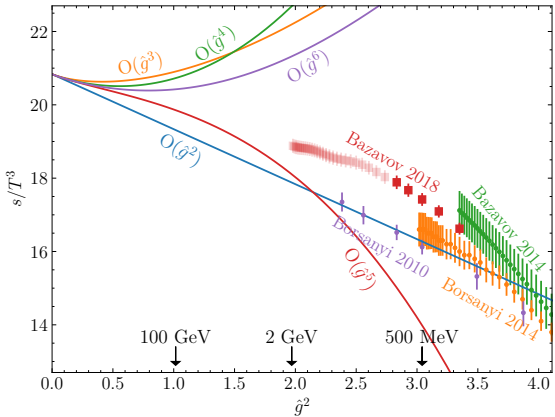

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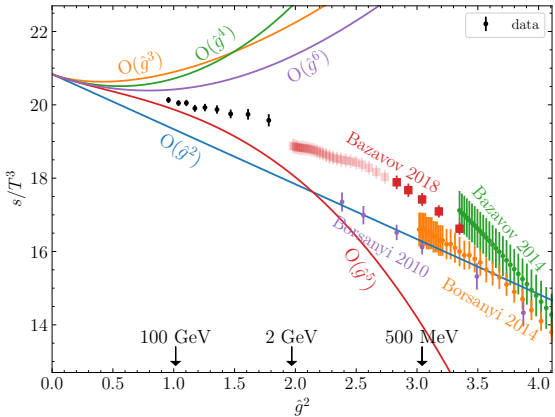


Equation of State of QCD

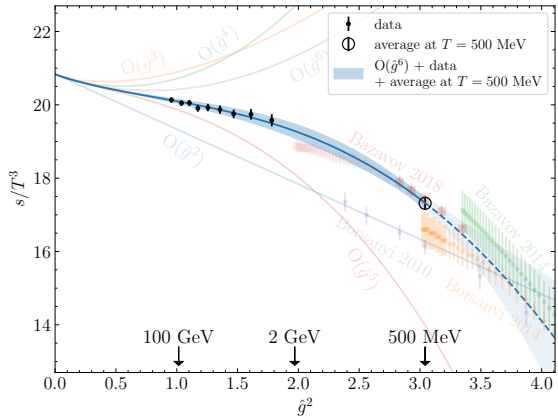
- Fit including $T = 500$ MeV point

  [Borsanyi et al. 2014; Bazavov et al. 2018]

- Enforce all known PT result, fit q_c and s_7

$$q_c = -4.0(1.1) \cdot 10^3$$
$$s_7 = 7(4) \cdot 10^3$$

- Contributions beyond known PT give $\sim 40\%$ of interactions at $T = 165$ GeV



$$\frac{s}{T^3} = \frac{32\pi^2}{45}\left[\sum_{k=0}^{6} s_k \left(\frac{\hat{g}}{2\pi}\right)^k + q_c\left(\frac{\hat{g}}{2\pi}\right)^6 + s_7\left(\frac{\hat{g}}{2\pi}\right)^7\right]$$

Equation of State of QCD

- Write pressure as [Kajantie et al. 2002]

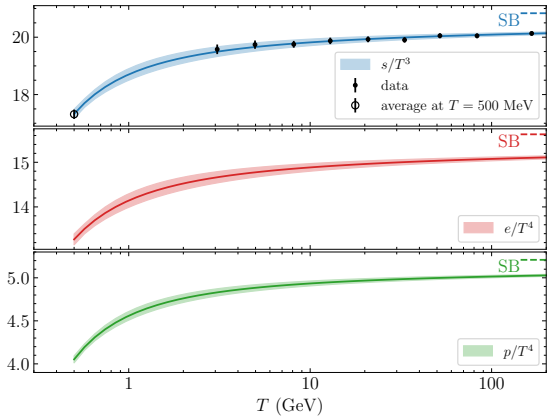
$$\frac{p}{T^4} = \frac{8\pi^2}{45}\left[\sum_{k=0}^{6} p_k \left(\frac{\hat{g}}{2\pi}\right)^k + q_c\left(\frac{\hat{g}}{2\pi}\right)^6 + p_7\left(\frac{\hat{g}}{2\pi}\right)^7\right]$$

  where $p_1, ..., p_6$ are known in PT

- Match to entropy using $s = \frac{dp}{dT}$

$$p_7 = s_7 + \frac{45}{16}\,p_5 + 3\,p_3$$

- Energy density $e = Ts - p$



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- Terms beyond known PT, containing NP effects, still relevant at $T \gtrsim 100$ GeV
  $\Rightarrow$ Strong interactions at EW scale are stronger than what predicted by PT
  $\Rightarrow$ QCD EoS is non-perturbative up to the EW scale
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- Results obtained thanks to a new strategy based on
  - Renormalization through running of NP defined coupling
  - Thermal QCD with shifted boundary conditions

  $\Rightarrow$ Pietro Rescigno's poster
- Generalizable to QCD with 4 or 5 (massive) flavours

BACKUP

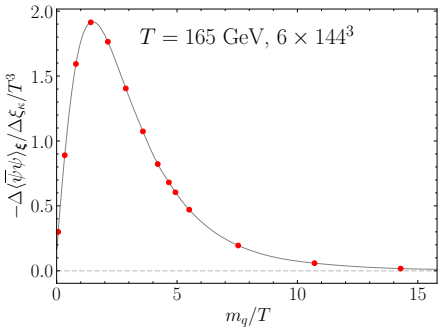
Lattice computation

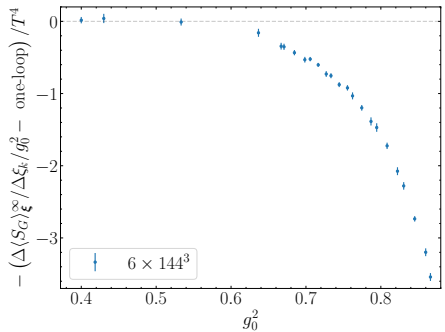
At a given T and L_0/a:

$$\frac{s}{T^3} = \frac{1+\boldsymbol{\xi}^2}{\xi_k}\frac{1}{T^4}\frac{\Delta f_{\boldsymbol{\xi}}}{\Delta\xi_k} = \frac{1+\boldsymbol{\xi}^2}{\xi_k}\frac{1}{T^4}\left\{\frac{\Delta}{\Delta\xi_k}(f_{\boldsymbol{\xi}} - f^{\infty}_{\boldsymbol{\xi}}) + \frac{\Delta f^{\infty}_{\boldsymbol{\xi}}}{\Delta\xi_k}\right\}$$

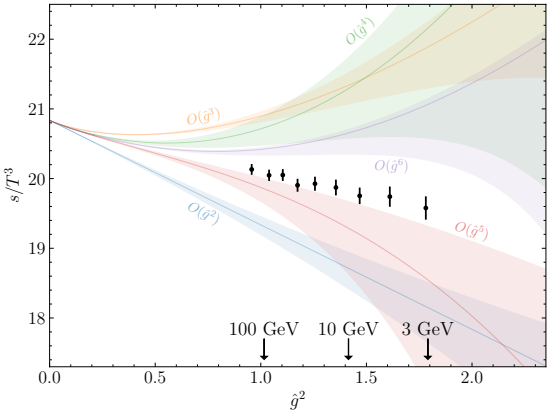
$$\frac{\Delta}{\Delta\xi_k}(f_{\boldsymbol{\xi}} - f^{\infty}_{\boldsymbol{\xi}}) = -\int_0^{\infty} dm_q \frac{\Delta}{\Delta\xi_k}\langle\overline{\psi}\psi\rangle^{m_q}_{\boldsymbol{\xi}}$$

$$\frac{\Delta f^{\infty}_{\boldsymbol{\xi}}}{\Delta\xi_k} = \frac{\Delta f^{\infty}_{\boldsymbol{\xi}}}{\Delta\xi_k}\Big|_{1\text{loop}} - \int_0^{g_0^2} du \left[\frac{1}{u}\frac{\Delta}{\Delta\xi_k}\langle S_G\rangle^{\infty}_{\boldsymbol{\xi}} - 1\text{loop}\right]_u$$

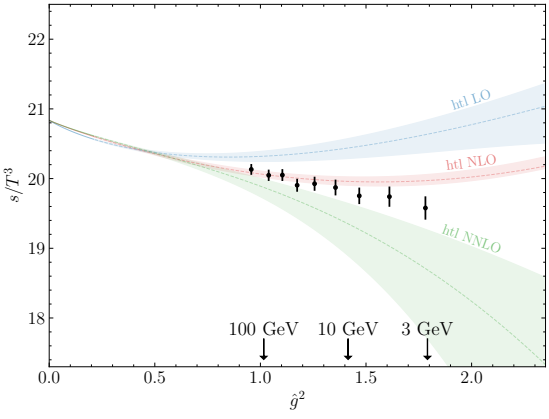




Comparison with PT and HTL



[Kajantie et al., 2002]



[Andersen et al., 2011]

Thermal effective theories of QCD

- High-T expansion of QCD
- Three relevant scales: (g = QCD ren. coupling)

$M = \pi T + \ldots$	Fermions and non-zero Matsubara gluon modes (ptQCD)	hard scale
$m_E \propto gT + \ldots$	A_0 zero Matsubara gluon modes (3D EQCD)	soft scale
$g_E^2 = g^2 T + \ldots$	A_i zero Matsubara gluon modes (3D MQCD)	ultrasoft scale

- Contributions from ultrasoft scale of NP nature ($g_E^2 \sim T/\log^2 T$)
- Asymptotic freedom $\Rightarrow$ hierarchy of scales at asymptotically high T

$$\frac{g_E^2}{\pi} \ll m_E \ll \pi T \quad \Leftrightarrow \quad \left(\frac{g}{\pi}\right)^2 \ll \frac{g}{\pi} \ll 1 \quad (g/\pi \sim 0.3 \text{ at } T = 100\,\text{GeV})$$

Schrödinger functional coupling

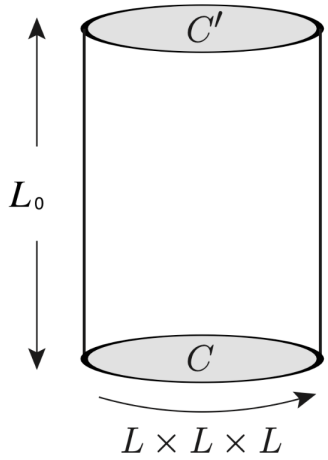
- Schrödinger functional = path integral with Dirichlet boundary conditions C, C' in time
- $\mathcal{Z}[C, C']$ partition function, free-energy

$$\Gamma = -\ln \mathcal{Z}$$

- NP definition of the coupling ($L_0 = L$)

$$\frac{1}{\bar{g}^2_{\rm SF}(\mu)} = \left\{ \frac{\partial \Gamma}{\partial \eta} \Big/ \frac{\partial \Gamma_0}{\partial \eta} \right\}_{\eta=0}, \quad \mu = \frac{1}{L_0}$$

where C, C' depend on the dimensionless parameter η



[Lüscher et al. 1991]