# Spin density and local equilibrium Wigner function for spin-half particles

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**Based On :** arXiv: 2505.02657

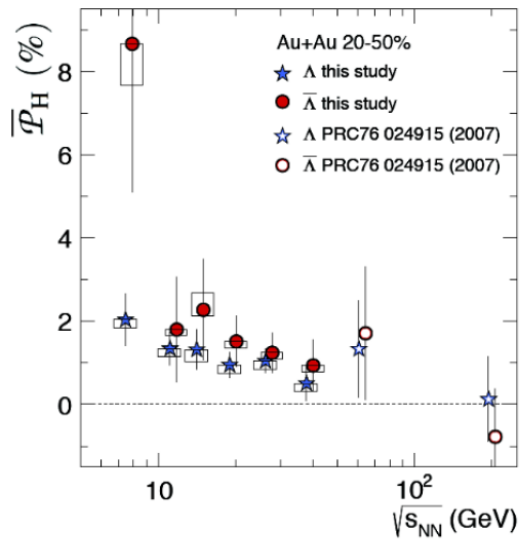
# Section Outline :

# Particle Polarization :

- Spin polarization of hadrons in heavy-ion collisions, predicted in 2004.

  [Z. T. Liang, X. N. Wang, PRL **94**, 102301 (2005), PLB **629**, 20 (2005)]

STAR Collaboration, *Global Lambda hyperon polarization in nuclear collisions*, Nature 548 62-65, 2017





First evidence of a quantum effect in (relativistic) hydrodynamics

Experimental evidence, [STAR Collaboration, Nature 548, 62 (2017), PRL 123, 132301 (2019), PRL 126, 162301 (2021)]

Theoretical models were developed that assume equilibration of spin d.o.f.

[F. Becattini et. al., Annals Phys. **338**, (2013), W. Florkowski et. al. PRC **97**, 041901 (2018)]

# Describing Spin-1/2 particles

- The dynamics of spin can be explained with the help of
  $\rightarrow$ Spin density matrices $\underbrace{f^{\pm}(x,p)}_{2\times 2}$ and, spinor density matrices $\underbrace{X^{\pm}(x,p)}_{4\times 4}$:

$$f^{+}_{rs}(x,p) = \frac{1}{2m}\,\bar{u}_r(p) X^{+} u_s(p), \qquad f^{-}_{rs}(x,p) = -\frac{1}{2m}\,\bar{v}_s(p) X^{-} v_r(p), \qquad r,s = \{1,2\}$$

- Two possible ways to describe relativistic fluid of spin-1/2 particles:
  - *Classical treatment:* kinetic theory + $\int dS$
  - *Quantum treatment:* Wigner function + tr (spinor space)

$$W^{\pm}(x,k) = \frac{1}{4m}\int dP\,\delta^{(4)}(k \mp p)(\not{p} \pm m) X^{\pm} (\not{p} \pm m)$$

[De Groot, Relativistic Kinetic Theory. Principles and Applications (1980)]

# Spinor density matrix $X^{\pm}$

- Commonly used form of $X^{\pm}$ in local equilibrium.

[Becattini et al., Annals Phys. 338 (2013)]

$$X^{\pm} = \exp\left[-\frac{p^{\mu}u_{\mu}}{T} \pm \frac{\mu}{T}\right] \exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right]$$

$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$
$\omega_{\mu\nu} = \Omega_{\mu\nu}/T$ – spin polarization tensor, $\Omega_{\mu\nu}$ – spin chemical potential

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}$$

$\boldsymbol{e} = (e^1, e^2, e^3)$, $\boldsymbol{b} = (b^1, b^2, b^3)$ – electric- and magnetic-like vectors.

# Motivation

- Mean spin polarization of spin-1/2 particles in equilibrium, $\mu = 0$:

[Florkowski et al. PRD 97, 116017 (2018)]

$$\boldsymbol{P} = \frac{1}{2}\frac{\mathrm{tr}_2\left(f^{\pm}\boldsymbol{\sigma}\right)}{\mathrm{tr}_2\left(f^{\pm}\right)} = -\frac{1}{2}\tanh\left[\frac{\sqrt{\boldsymbol{b}_*\cdot\boldsymbol{b}_* - \boldsymbol{e}_*\cdot\boldsymbol{e}_*}}{2}\right]\frac{\boldsymbol{b}_*}{\sqrt{\boldsymbol{b}_*\cdot\boldsymbol{b}_* - \boldsymbol{e}_*\cdot\boldsymbol{e}_*}}$$

$$f^{+}_{rs}(x,p) = \tfrac{1}{2m}\,\bar{u}_r(p)X^{+}u_s(p), \qquad f^{-}_{rs}(x,p) = -\tfrac{1}{2m}\,\bar{v}_s(p)X^{-}v_r(p).$$

- This spinor density matrix,

[Becattini et al., Annals Phys. 338 (2013)]

$$X^{\pm} = \exp\left[-\frac{p^{\mu}u_{\mu}}{T} \pm \frac{\mu}{T}\right]\exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right]$$

violates the normalization of the polarization vector, $0 \leq |\boldsymbol{P}| \leq 1$

(from no polarization to pure state)

# Section Outline :

# Deriving $X^{\pm}$ from the spin density matrices

- Recall:

$$f^{+}_{rs}(x,p) = \frac{1}{2m}\,\bar{u}_r(p) X^{+} u_s(p), \qquad f^{-}_{rs}(x,p) = -\frac{1}{2m}\,\bar{v}_s(p) X^{-} v_r(p)$$

- Decomposition of $2 \times 2$ Hermitian matrices $f^{\pm}_{rs}(x,p)$ :

$$f^{\pm}_{rs}(x,p) = f^{\pm}_{\mathrm{o}}(x,p)\left[\delta_{rs} + \boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p})\cdot\boldsymbol{\sigma}_{rs}\right]$$

[Florkowski et al., PRD97 (2018)]

  - $f^{\pm}_{\mathrm{o}}(x,p)$ – spin-averaged phase-space density.
  - $\zeta^{\pm\mu}_{*} = (0, \boldsymbol{\zeta}^{\pm}_{*})$ – mean polarization vector in PRF, $\quad 0 \leq |\boldsymbol{\zeta}^{\pm}_{*}| \leq 1$.

- Canonical boost to LAB frame, particle moves with $\boldsymbol{v}_p$:

$$\Lambda^{\mu}{}_{\nu}(\boldsymbol{v}_p)\zeta^{\nu}_{\pm *} \equiv \zeta^{\mu}_{\pm}$$

- If we want $f(x,p)$ to have this form, what should $X$ be?

# Deriving $X^{\pm}$ from the spin density matrices

- From Lorentz covariance we get:

$$\bar{u}_r(p)\gamma_5\zeta^+_\mu\gamma^\mu u_s(p) = 2m\,\boldsymbol{\zeta}^+_* \cdot \boldsymbol{\sigma}_{rs}$$
$$\bar{v}_s(p)\gamma_5\zeta^-_\mu\gamma^\mu v_r(p) = -2m\,\boldsymbol{\zeta}^-_* \cdot \boldsymbol{\sigma}_{rs}$$

$$\Longrightarrow \boxed{X^{\pm}_s(x,p) = f^{\pm}_0(x,p)\left[1 + \gamma_5 \not{\zeta}^{\pm}(x,p)\right]}$$

- Derived with no assumptions on equilibrium properties of the spin density matrix, $f^{\pm}(x,p)$.

# New spinor density $X_s^{\pm}$

$$X_s^{\pm}(x,p) = f_0^{\pm}(x,p)\left[1 + \gamma_5 \not{\zeta}^{\pm}(x,p)\right]$$

For general spacelike four-vectors $a_{\pm}^{\mu}$ satisfying $a_{\pm}^2 < 0$.

$$\exp\left(\gamma_5 \not{a}_{\pm}\right) = \cosh\sqrt{-a_{\pm}^2}\left[1 + \frac{\gamma_5 \not{a}_{\pm}}{\sqrt{-a_{\pm}^2}} \tanh\sqrt{-a_{\pm}^2}\right]$$

Square brackets are equal if :

$$\zeta_{\pm}^{\mu} = \frac{a_{\pm}^{\mu}}{\sqrt{-a_{\pm}^2}} \tanh\sqrt{-a_{\pm}^2} \quad \overset{small\ a}{\longrightarrow} \quad \zeta_{\pm}^{\mu} = a_{\pm}^{\mu}$$

Consequently, $\zeta_{\pm}^{\mu}$ is properly normalized.

# New spinor density $X_s^{\pm}$

$$X_s^{\pm}(x,p) = f_{\rm o}^{\pm}(x,p)\exp\left(\gamma_5 \not{a}_{\pm}\right)$$

*Equilibrium spin density for Boltzmann statistics*

$$X_s^{\pm}(x,p) = \exp\left[-\frac{p_\mu u^\mu}{T} \pm \frac{\mu}{T}\right]\exp\left[\gamma_5 \not{a}\right]$$

$$a_\mu(x,p) = -\frac{1}{2m}\widetilde{\omega}_{\mu\nu}(x)p^\nu, \qquad \widetilde{\omega}_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\nu\alpha\beta}\,\omega^{\alpha\beta}$$

# Comparison of spinor densities

Previously known expression:

$$X^{\pm}(x,p) = \exp\left[-\frac{p^\mu u_\mu}{T} \pm \frac{\mu}{T}\right] \exp\left[\pm\frac{i}{8}\omega_{\mu\nu}(x)[\gamma^\mu,\gamma^\nu]\right]$$

[Becattini, Chandra, Del Zanna, Grossi, Annals Phys. 338 (2013)]

Our new finding:

$$X_s^{\pm}(x,p) = \exp\left[-\frac{p^\mu u_\mu}{T} \pm \frac{\mu}{T}\right] \exp\left[-\frac{1}{4m}\gamma_5\,\epsilon_{\mu\nu\alpha\beta}\,\omega^{\alpha\beta}(x)\,p^\nu\,\gamma^\mu\right]$$

[SB, Z. Drogosz, W. Florkowski, S. K. Kar, V. Mykhaylova, arXiv:2505.02657]

$W^{\pm}(x,k) = \frac{1}{4m}\int \mathrm{d}P\,\delta^{(4)}(k\mp p)(\not{p}\pm m)X_s^{\pm}(\not{p}\pm m) \quad \rightarrow \quad$ thermodynamics

# Thermodynamics from Wigner function

$$N^{\mu}(x) = \sum_{r=1}^{2} \int \mathrm{d}P\, p^{\mu} \left[ f_{rr}^{+}(x,p) - f_{rr}^{-}(x,p) \right] - \text{baryon current}$$

$$\mathcal{N}^{\mu}(x) = \sum_{r=1}^{2} \int \mathrm{d}P\, p^{\mu} \left[ f_{rr}^{+}(x,p) + f_{rr}^{-}(x,p) \right] - \text{particle current}$$

$$T^{\mu\nu}(x) = \sum_{r=1}^{2} \int \mathrm{d}P\, p^{\mu} p^{\nu} \left[ f_{rr}^{+}(x,p) + f_{rr}^{-}(x,p) \right] - \text{energy-momentum tensor}$$

$$S^{\lambda,\mu\nu}(x) = \frac{1}{2} \sum_{r,s=1}^{2} \int \mathrm{d}P\, p^{\lambda} \left[ \sigma_{sr}^{+\mu\nu}(p) f_{rs}^{+}(x,p) + \sigma_{sr}^{-\mu\nu}(p) f_{rs}^{-}(x,p) \right] - \text{spin tensor}$$

[De Groot, Relativistic Kinetic Theory. Principles and Applications (1980)]

$$S^{\mu}(x) = -\frac{1}{2} \int \mathrm{d}P\, p^{\mu} \left\{ \mathrm{tr}_4 \left[ X_s^{+} \left( \ln X_s^{+} - 1 \right) \right] + \mathrm{tr}_4 \left[ X_s^{-} \left( \ln X_s^{-} - 1 \right) \right] \right\} - \text{entropy}$$

[Florkowski et al. PRD 97, 116017 (2018)]

# Thermodynamics for classical and quantum spin

**Generalized thermodynamic relations for perfect spin hydrodynamics:**

$$\beta_\alpha = \frac{u_\alpha}{T}, \qquad \xi = \frac{\mu}{T}$$

$$\text{Entropy}: \qquad S^\mu = T^{\mu\alpha}\beta_\alpha - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta} - \xi N^\mu + \mathcal{N}^\mu$$

$$\text{Change of particle current}: \qquad d\mathcal{N}^\mu = N^\mu d\xi - T^{\lambda\mu}d\beta_\lambda + \tfrac{1}{2}S^{\mu,\alpha\beta}d\omega_{\alpha\beta}$$

$$\text{Change of entropy}: \qquad dS^\mu = -\xi dN^\mu + \beta_\lambda dT^{\lambda\mu} - \tfrac{1}{2}\omega_{\alpha\beta}dS^{\mu,\alpha\beta}$$

[Florkowski, Hontarenko, PRL 134 (2025)]

# Quantum vs. classical spin treatments

**Quantum**

$$X_s^{\pm} = [\cdots + \gamma_5 a_\mu \gamma^\mu]$$

**Classical**

$$f^{\pm}(x,p,s) = \exp\left[\cdots + \frac{1}{2}\omega_{\mu\nu} s^{\mu\nu}\right]$$

$$\boxed{\frac{1}{2}\gamma_5\gamma^\mu \quad \longleftrightarrow \quad s^\mu}$$

$N^\mu$ and $T^{\mu\nu}$ modified by $\omega_{\mu\nu}$ in classical and quantum descriptions

**Quadratic corrections:**

$$\cosh\sqrt{-a^2} = 1 - \frac{1}{2}a^2$$

$$1 + \frac{1}{16}\int \mathrm{d}S\, \omega_{\mu\nu} s^{\mu\nu}\, \omega_{\rho\sigma} s^{\rho\sigma}$$

# Applicability range

$$X_s^{\pm}(x,p,s) = \exp\left[-\frac{p_\mu u^\mu}{T} \pm \frac{\mu}{T} + \gamma_5 a_\mu(x,p)\gamma^\mu\right]$$

$$a_\mu(x,p) = -\tfrac{1}{2m}\widetilde{\omega}_{\mu\nu}(x)p^\nu$$

Convergence criterion in $\mu = 0$ limit[1]:

$$\boxed{\frac{1}{2}\sqrt{\boldsymbol{b}'^2 + \boldsymbol{e}'^2 + 2|\boldsymbol{e}' \times \boldsymbol{b}'|} < \frac{m}{T}}$$

[Z. Drogosz, W. Florkowski, V. Mykhaylova, arXiv:2506.01537]

*Our results are applicable within the range of hydrodynamic parameters used in current models of HIC*

[1] The applicability range of classical spin approach was explored recently:
[N. Abboud, L. Gavassino, R. Singh, E. Speranza, arXiv: 2506.19786]

# Section Outline :

# Conclusions :

- Local equilibrium Wigner function for spin-1/2 particles with proper normalization of the mean polarization vector was obtained.

- We found an agreement of the generalized thermodynamics in quantum and classical spin formalisms. $\rightarrow$ Employed in perfect spin hydrodynamics.

- The applicability range of the spinor density $X_s^{\pm}$ in Wigner-function formalism and of the distribution function $f(x, p, s)$ in kinetic theory with spin was found to be reasonable.

***Thank you***