

# On the local thermodynamic relations in relativistic spin hydrodynamics

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## Abstract

• **Key Result:** We show through two examples (massless and massive free fermions in global equilibrium with rotation and acceleration) that the commonly assumed local thermodynamic relations in spin hydrodynamics *are incomplete*, even when the entropy current definition is unambiguous.

• **Method:** Using a rigorous quantum statistical approach, we find that

$$\left. \frac{\partial p}{\partial \omega_{\lambda\nu}} \right|_{T,\mu} \neq S^{\mu\nu}$$

the pressure derivative with respect to spin potential differs from the spin density by a *finite correction of the same order*.

• **Implication:** Thermodynamic differential relations cannot be relied upon to derive constitutive equations in relativistic spin hydrodynamics. A first-principles approach is essential [1].

## Equilibrium density operator

For a relativistic fluid, by applying the quantum-statistical method, one obtains the **local equilibrium density operator**,  $\hat{\rho}_{LE}$ , by **maximizing entropy** [2]

$$S = -\text{Tr}(\hat{\rho} \log \hat{\rho}), \quad (1)$$

over some space-like hypersurface  $\Sigma$ . This constrains the average values of the energy density, momentum density, charge density, and spin density to be same as their original values

$$\hat{\rho}_{LE} = \frac{1}{Z_{LE}} \exp[-\hat{Y}], \quad (2)$$

with  $Z_{LE}$  as the partition function and

$$\hat{Y} = \int_{\Sigma} \left( \hat{T}^{\mu\nu} \beta_{\nu} - \zeta \hat{j}^{\mu} - \frac{\Omega_{\lambda\nu}}{2} \hat{S}^{\mu\lambda\nu} \right) d\Sigma_{\mu}, \quad (3)$$

where  $d\Sigma_{\mu} \equiv n_{\mu} d\Sigma$  with  $n_{\mu}$  as the unit vector orthogonal to  $\Sigma$ . In Eq. (2),  $\hat{T}^{\mu\nu}$ ,  $\hat{S}^{\mu\lambda\nu}$  are the energy-momentum and spin tensor operators and  $\hat{j}^{\mu}$  is the vector-current operator.

Lagrange multipliers are defined as

$$\beta^{\mu} = \frac{u^{\mu}}{T}, \quad \zeta = \frac{\mu}{T}, \quad \Omega^{\lambda\nu} = \frac{\omega^{\lambda\nu}}{T}. \quad (4)$$

In the global thermodynamic equilibrium,  $\beta_{\mu}$  is the Killing vector,  $\Omega^{\mu\nu}$  equals thermal vorticity ( $\varpi^{\mu\nu}$ ) and is constant, and  $\zeta$  is constant

$$\beta_{\mu} = b_{\mu} + \varpi_{\mu\nu} x^{\nu}, \quad \text{with } b_{\mu} = \text{const.} \quad (5)$$

$$\Omega^{\mu\nu} = \varpi^{\mu\nu} = \text{const.}, \quad (6)$$

$$\zeta = \text{const.}, \quad (7)$$

## Primary references

[1] F. Becattini and R. Singh, [arXiv:2506.20681].

[2] F. Becattini, A. Daher and X. L. Sheng, PLB **850** (2024), 138533 [arXiv:2309.05789].

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## Overview of entropy current

To derive the entropy current using quantum statistical approach, we need to **prove that the log Z is extensive**, meaning it can be written as an integral of some four-vector field over a hypersurface

$$\log Z_{LE} = \int_{\Sigma} \left( \phi^{\mu} - \langle 0 | \hat{Y} | 0 \rangle \right) d\Sigma_{\mu}, \quad (8)$$

where we subtract the vacuum expectation value of the operator (3). Thus, if  $\hat{Y}$  in Eq. (2) is bounded from below and  $|0\rangle$  is non-degenerate, then  $\log Z_{LE}$  is

$$\log Z_{LE} = \int_{\Sigma} \left[ \phi^{\mu} - \left\langle 0 \left| \left( \hat{T}^{\mu\nu} \beta_{\nu} - \zeta \hat{j}^{\mu} - \frac{1}{2} \Omega_{\lambda\nu} \hat{S}^{\mu\lambda\nu} \right) \right| 0 \right\rangle \right] d\Sigma_{\mu}, \quad (9)$$

where  $\phi_{\mu}$  is **thermodynamic potential current**,

$$\phi^{\mu} = \int_1^{\infty} \left( T_{LE}^{\mu\nu}(\lambda) \beta_{\nu} - \zeta j_{LE}^{\mu}(\lambda) - \frac{\Omega_{\lambda\nu} S_{LE}^{\mu\lambda\nu}(\lambda)}{2} \right) d\lambda. \quad (10)$$

Redefining integration parameter leads to

$$\phi^{\mu}(x) = \int_0^{T(x)} \left( u_{\nu}(x) T_{LE}^{\mu\nu}(x)[T', \mu, \omega] - \mu(x) j_{LE}^{\mu}(x)[T', \mu, \omega] - \frac{1}{2} \omega_{\lambda\nu}(x) S_{LE}^{\mu\lambda\nu}(x)[T', \mu, \omega] \right) \frac{dT'}{T'^2}. \quad (11)$$

After defining  $\phi_{\mu}$ , we **compute the entropy current** (focusing on global thermodynamic equilibrium) as

$$s^{\mu} = \phi^{\mu} + T^{\mu\nu} \beta_{\nu} - \zeta j^{\mu} - \frac{1}{2} \Omega_{\lambda\nu} S^{\mu\lambda\nu}, \quad (12)$$

where we incorporated the actual values of the currents and  $\phi^{\mu}$  is

$$\phi^{\mu} = \int_0^T \left( T^{\mu\nu}[T'] u_{\nu} - \mu j^{\mu}[T'] - \frac{1}{2} \omega_{\lambda\nu} S^{\mu\lambda\nu}[T'] \right) \frac{dT'}{T'^2}. \quad (13)$$

## Local thermodynamic relations for massless free fermions

Local thermodynamic relations are obtained by **contracting the entropy current (12) with the fluid four-velocity**, giving

$$\checkmark \quad \boxed{Ts = \rho + p - \mu n - \frac{1}{2} \omega_{\mu\nu} S^{\mu\nu}}, \quad \boxed{dp = s dT + n d\mu + \frac{1}{2} S^{\mu\nu} d\omega_{\mu\nu}}? \quad (14)$$

We work with the **canonical currents for massless free fermions**

$$\text{(Energy-momentum tensor)} \quad \Rightarrow \quad T_{Can}^{\mu\nu} = T_B^{\mu\nu} - \frac{1}{2} \partial_{\lambda} S_{Can}^{\lambda\mu\nu}, \quad (15)$$

$$\text{(vector-current)} \quad \Rightarrow \quad j^{\mu} = \frac{\zeta}{\sqrt{\beta^2}} \left( \frac{\pi^2 + \zeta^2}{3\pi^2 \beta^2} - \frac{\alpha^2}{4\pi^2 \beta^2} + \frac{w^2}{4\pi^2 \beta^2} \right) u^{\mu} - \frac{\zeta l^{\mu}}{6\pi^2 \beta^2 \sqrt{\beta^2}},$$

$$\text{(axial-current)} \quad \Rightarrow \quad j_A^{\mu} = \frac{1}{\beta^2} \left( \frac{1}{6} + \frac{\zeta^2}{2\pi^2} - \frac{w^2}{24\pi^2} - \frac{\alpha^2}{8\pi^2} \right) \frac{w^{\mu}}{\sqrt{\beta^2}},$$

$$\text{with, } u^{\mu} = \frac{\beta^{\mu}}{\sqrt{\beta^2}}, \quad l^{\mu} = \epsilon^{\mu\nu\rho\sigma} \omega_{\nu} \alpha_{\rho} u_{\sigma}, \quad \alpha^{\mu} = \frac{a^{\mu}}{T} = \omega^{\mu\nu} \frac{u_{\nu}}{T}, \quad w^{\mu} = \frac{\omega^{\mu}}{T} = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \omega_{\nu\rho} \frac{u_{\sigma}}{T}. \quad (16)$$

Differential thermodynamic relation is obtained by defining pressure as

$$p = T \phi^{\mu} u_{\mu} = T \int_0^T \left( T_{Can}^{\mu\nu}[T'] u_{\nu} u_{\mu} - \mu u_{\mu} j^{\mu}[T'] - \frac{1}{2} u_{\mu} \omega_{\lambda\nu} S_{Can}^{\mu\lambda\nu}[T'] \right) \frac{dT'}{T'^2}, \quad (17)$$

which using the canonical relations and discarding the vacuum terms leads to

$$\text{(pressure)} \quad \Rightarrow \quad p = \frac{7\pi^2 T^4}{180} + \frac{1}{6} \mu^2 T^2 - (a^2 + w^2) \frac{T^2}{24}. \quad (18)$$

$$\text{We observe } \left. \frac{\partial p}{\partial T} \right|_{\omega,\mu} = s \text{ (entropy-density)}, \quad \& \quad \left. \frac{\partial p}{\partial \mu} \right|_{T,\omega} = n \text{ (charge-density)}, \quad (19)$$

but pressure derivative with respect to  $\omega_{\lambda\nu}$  is not equal to spin density

$$\left. \frac{\partial p}{\partial \omega_{\lambda\nu}} \right|_{T,\mu} = \frac{T^2}{12} (a^{\nu} u^{\lambda} - a^{\lambda} u^{\nu}) + S^{\lambda\nu}, \quad \text{where } S^{\lambda\nu} = u_{\mu} S_{Can}^{\mu\lambda\nu}, \quad (20)$$

and acquire correction terms. Thus, local differential thermodynamic relations need to be modified.