Spin hydrodynamics: the hybrid approach

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XQCD 2025, Wrocław

We propose a hybrid approach to help establish consistency between different formulations of spin hydrodynamics. The perfect fluid is described in the framework of kinetic theory of particles with spin 1/2 following Fermi-Dirac statistics. The baryon number, energy, linear momentum, the spin part of angular momentum, and entropy are conserved, although the baryon current, the energy-momentum tensor, and the spin tensor contain terms whose mathematical form is seemingly typical of dissipation. Genuinely dissipative terms are introduced via the Israel-Stewart method, using the condition of positive entropy production, which allows for transfer between the spin and orbital parts of angular momentum.

Tensor thermodynamic relations

The standard scalar thermodynamical relations consist of the fundamental thermodynamic relation, the first law of thermodynamics and the Gibbs-Duhem relation

$$\varepsilon + P = T\sigma + \mu n + \frac{1}{2}\Omega_{\alpha\beta}S^{\alpha\beta}, \tag{1}$$

$$\mathrm{d}\varepsilon = T\mathrm{d}\sigma + \mu \mathrm{d}n + \frac{1}{2}\Omega_{\alpha\beta}\mathrm{d}S^{\alpha\beta}, \tag{2}$$

$$\mathrm{d}P = \sigma \mathrm{d}T + n\mathrm{d}\mu + \frac{1}{2}S^{\alpha\beta}\mathrm{d}\Omega_{\alpha\beta}, \tag{3}$$

with the spin chemical potential $\Omega_{\alpha\beta} = T\omega_{\alpha\beta}$. The spin polarization tensor $\omega_{\alpha\beta}$ admits a general parametrization

$$\omega_{\alpha\beta} = k_\alpha u_\beta - k_\beta u_\alpha + t_{\alpha\beta}, \tag{4}$$

with $k\cdot u = \omega \cdot u = 0, t_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta}u^\gamma\omega^\delta$, $t^\mu = t^{\mu\nu}k_\nu = \epsilon^{\mu\nu\alpha\beta}k_\nu u_\alpha \omega_\beta$.

Multiplication of Eqs. (1)–(3) by the hydrodynamic flow u^μ leads to tensor equations

$$S^\mu = P\beta^\mu - \xi N^\mu + \beta_\lambda T^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}, \tag{5}$$

$$\mathrm{d}S^\mu = -\xi \mathrm{d}N^\mu + \beta_\lambda \mathrm{d}T^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta}\mathrm{d}S^{\mu,\alpha\beta}, \tag{6}$$

$$\mathrm{d}(P\beta^\mu) = N^\mu \mathrm{d}\xi - T^{\lambda\mu}\mathrm{d}\beta_\lambda + \frac{1}{2}S^{\mu,\alpha\beta}\mathrm{d}\omega_{\alpha\beta}, \tag{7}$$

with the spin tensor $S^{\mu,\alpha\beta}$ defined by the expression

$$S^{\mu,\alpha\beta} = u^\mu S^{\alpha\beta}. \tag{8}$$

This form of the spin tensor has a long history [J. Weyssenhoff and A. Raabe, Acta Phys. Polon. 9, 7 (1947)]. However, the spin tensor obtained from kinetic theory has a more complex structure and contains terms orthogonal to u^μ.

Because of this discrepancy, we argue that relations (1)–(3) have to be revised, and consistent and nontrivial thermodynamic relations of spin hydrodynamics in local equilibrium should have the following form

$$S^\mu_{\rm eq} = T^{\mu\alpha}_{\rm eq}\beta_\alpha - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}_{\rm eq} - \xi N^\mu_{\rm eq} + \mathcal{N}^\mu_{\rm eq}, \tag{9}$$

$$dS^\mu_{\rm eq} = -\xi dN^\mu_{\rm eq} + \beta_\lambda dT^{\lambda\mu}_{\rm eq} - \frac{1}{2}\omega_{\alpha\beta}dS^{\mu,\alpha\beta}_{\rm eq}, \tag{10}$$

$$d\mathcal{N}^\mu_{\rm eq} = N^\mu_{\rm eq}d\xi - T^{\lambda\mu}_{\rm eq}d\beta_\lambda + \frac{1}{2}S^{\mu,\alpha\beta}_{\rm eq}d\omega_{\alpha\beta}, \tag{11}$$

where

$$\mathcal{N}^\mu = \mathrm{ctgh}\,\xi\ N^\mu. \tag{12}$$

Moreover, local equilibrium for particles with spin should be defined as the case where the spin part of the total angular momentum is conserved.

$$\partial_\mu N^\mu_{\rm eq} = 0, \quad \partial_\mu T^{\mu\lambda}_{\rm eq} = 0, \quad \partial_\mu S^{\mu,\alpha\beta}_{\rm eq} = 0. \tag{13}$$

Perfect spin hydrodynamics

In kinetic theory with classical treatment of spin, one introduces the internal angular momentum [M. Mathisson, Acta Phys. Polon. 6, 163 (1937)]

$$s^{\alpha\beta} = \frac{1}{m}\epsilon^{\alpha\beta\gamma\delta}p_\gamma s_\delta, \tag{14}$$

where the spin 4-vector s^μ is orthogonal to the 4-momentum $s\cdot p = 0$, $s^\alpha = \frac{1}{2m}\epsilon^{\alpha\beta\gamma\delta}p_\beta s_{\gamma\delta}$. In the particle rest frame, $p^\mu = (m,0,0,0)$, $s^\alpha = (0,\mathbf{s}_*)$, $|\mathbf{s}_*| = \mathfrak{s}$, and $\mathfrak{s}^2 = 1/2\,(1+1/2) = 3/4$. The local equilibrium distribution functions for particles (+) and for antiparticles (−) have the Fermi-Dirac form

$$f^\pm_{\rm eq}(x,p,s) = \left[\exp\left(\mp\xi(x) + p\cdot\beta(x) - \frac{1}{2}\omega(x):s\right) + 1\right]^{-1}. \tag{15}$$

The local equilibrium baryon current $N^\mu_{\rm eq}$, the energy-momentum tensor $T^{\mu\nu}_{\rm eq}$, and the spin tensor $S^{\lambda,\mu\nu}_{\rm eq}$ can be obtained as

$$N^\mu_{\rm eq} = \int \mathrm{d}P\,\mathrm{d}S\,p^\mu\ \left[f^+_{\rm eq}(x,p,s) - f^-_{\rm eq}(x,p,s)\right], \tag{16}$$

$$T^{\mu\nu}_{\rm eq} = \int \mathrm{d}P\,\mathrm{d}S\,p^\mu p^\nu\ \left[f^+_{\rm eq}(x,p,s) + f^-_{\rm eq}(x,p,s)\right], \tag{17}$$

$$S^{\lambda,\mu\nu}_{\rm eq} = \int \mathrm{d}P\,\mathrm{d}S\ p^\lambda\, s^{\mu\nu}\left[f^+_{\rm eq}(x,p,s) + f^-_{\rm eq}(x,p,s)\right]. \tag{18}$$

We additionally introduce the current $\mathcal{N}^\mu$ and the entropy current $S^\mu_{\rm eq}$

$$\mathcal{N}^\mu = -\int \mathrm{d}P\,\mathrm{d}S\,p^\mu\ \left[\ln(1-f^+_{\rm eq}) + \ln(1-f^-_{\rm eq})\right], \tag{19}$$

$$S^\mu_{\rm eq} = -\int \mathrm{d}P\,\mathrm{d}S\,p^\mu\left[f^+_{\rm eq}\left(\ln f^+_{\rm eq} - 1\right) + f^-_{\rm eq}\left(\ln f^-_{\rm eq} - 1\right)\right], \tag{20}$$

Starting directly from the definition (19), the entropy current can be expressed by other tensors, Eq. (9). Those can be calculated by expanding (15) to the quadratic order in ω and performing the integrals (16)–(18)

$$N^\mu_{\rm eq} = (n_0 + n_2^k + n_2^\omega)u^\mu + n_t t^\mu, \tag{21}$$

$$T^{\mu\nu}_{\rm eq} = (\varepsilon_0 + \varepsilon_2^k + \varepsilon_2^\omega)u^\mu u^\nu - (P_0 + P_2^k + P_2^\omega)\Delta^{\mu\nu} + P_{k\omega}(k^\mu k^\nu + \omega^\mu\omega^\nu) + P_t(t^\mu u^\nu + t^\nu u^\mu), \tag{22}$$

$$S^{\lambda,\mu\nu}_{\rm eq} = u^\lambda\left[A\,(k^\mu u^\nu - k^\nu u^\mu) + A_1 t^{\mu\nu}\right] + \frac{A}{2}\left(t^{\lambda\mu}u^\nu - t^{\lambda\nu}u^\mu + \Delta^{\lambda\mu}k^\nu - \Delta^{\lambda\nu}k^\mu\right), \tag{23}$$

where $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ is the projector on the space orthogonal to the flow.

The seemingly dissipative new terms marked in blue appear on the perfect-fluid level. The term $\frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}$ in the entropy current contains transverse parts

$$\frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta} = u^\mu(Ak^2 - A_1\omega^2) + At^\mu \equiv \bar{s}u^\mu + s_t t^\mu. \tag{24}$$

All the coefficients can be expressed by the integral J_{mn} and its derivatives [3]

$$J_{mn}(\xi,z) = \int_0^\infty \frac{\sinh^m y\cosh^n y}{\exp(-\xi + z\cosh y) + 1}\mathrm{d}y, \quad m,n = 0,1,2,\ldots \tag{25}$$

The Boltzmann approximation uses the Bessel function of the second kind.

Dissipative spin hydrodynamics

Following the Israel-Stewart method [W. Israel and J. M. Stewart, Annals Phys. 118, 341 (1979)], we write down general nonequilibrium expressions as a sum of the equilibrium terms and dissipative corrections

$$N^\mu = N^\mu_{\rm eq} + \delta N^\mu, \quad T^{\mu\alpha} = T^{\mu\alpha}_{\rm eq} + \delta T^{\mu\alpha}, \quad S^{\mu,\alpha\beta} = S^{\mu,\alpha\beta}_{\rm eq} + \delta S^{\mu,\alpha\beta}. \tag{26}$$

The spin-orbit interaction is introduced: $T^{\mu\nu}$ can now contain nonsymmetric parts, and the spin tensor is no longer seperately conserved

$$\partial_\mu J^{\mu,\alpha\beta} = 0, \quad J^{\mu,\alpha\beta} = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} + S^{\mu,\alpha\beta}, \tag{27}$$

$$\partial_\mu S^{\mu,\alpha\beta} = T^{\beta\alpha} - T^{\alpha\beta}, \quad \partial_\mu N^\mu = 0, \quad \partial_\mu T^{\mu\nu} = 0. \tag{28}$$

We start from the most general hydrodynamic tensors with correct symmetries. From the condition of nonnegativity of the divergence of the entropy current(9)

$$\partial_\mu S^\mu = -\delta N^\mu \partial_\mu \xi + \delta T^{\mu\lambda}_s \partial_\mu \beta_\lambda + \delta T^{\mu\lambda}_a\left(\partial_\mu\beta_\lambda - \omega_{\lambda\mu}\right) - \frac{1}{2}\delta S^{\mu,\alpha\beta}\partial_\mu\omega_{\alpha\beta}, \tag{29}$$

we obtain the corrections δN^μ, $\delta T^{\mu\alpha}$, $\delta S^{\mu,\alpha\beta}$ as combinations of gradients of hydrodynamic variables, multiplied by certain kinetic coefficients.

The *global equilibrium* is defined by the conditions $\partial_\mu\xi = 0$, $\partial_{(\mu}\beta_{\lambda)} = 0$, $\omega_{\lambda\mu} = \partial_{[\mu}\beta_{\lambda]}$. Nevertheless, in *local equilibrium*, $\omega_{\lambda\mu}$ and the thermal vorticity $\partial_{[\mu}\beta_{\lambda]}$ are not directly related and need not be equal. Analogically, in spinless relativistic hydrodynamics, $\xi = \mu/T = $ const. in global equilibrium but generally not in local equilibrium.

The spin polarization tensor $\omega_{\mu\nu}$, which in natural units is a dimensionless quantity, is not considered to be a gradient term. Therefore, in an expansion linear in gradients, terms such as $\Delta^{\alpha\gamma}\partial_\gamma\omega_{\alpha\beta}$ survive and are not neglected as higher-order contributions.

Summary

- The hybrid framework is a synthesis of kinetic theory for perfect spin hydrodynamics and the Israel-Stewart method for inclusion of dissipation.
- Simple definition of local thermodynamic equilibrium of particles with spin: conservation of the spin part of the total angular momentum.
- Expansion up to the quadratic order in the spin polarization tensor $\omega_{\mu\nu}$ leads to consistent and nontrivial spin thermodynamic relations.
- Dissipative effects are added without resorting to nonlocal collisions.
- No assumption that $\omega_{\mu\nu}$ has the character of a gradient term.
- Two-fold expansion: in $\omega_{\mu\nu}$ up to the quadratic order for perfect fluid and independently in gradients of thermodynamic quantities for dissipation.
- The framework will be straightforward to implement and test in numerical simulations.

References

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Acknowledgment

The work was supported in part by the Polish National Science Centre (NCN) Grant No. 2022/47/B/ST2/01372.

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