Isotropization in Heavy Ion Collisions in the 2PI Formalism

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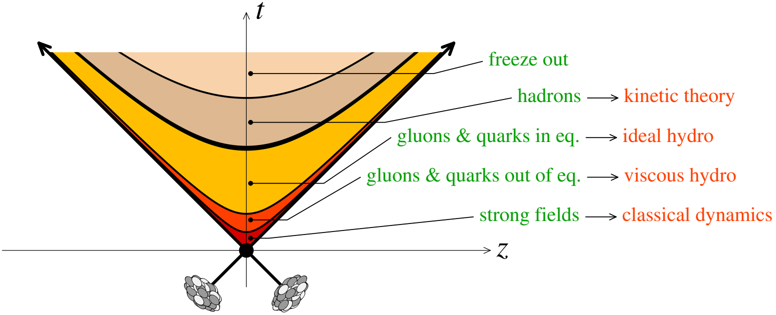
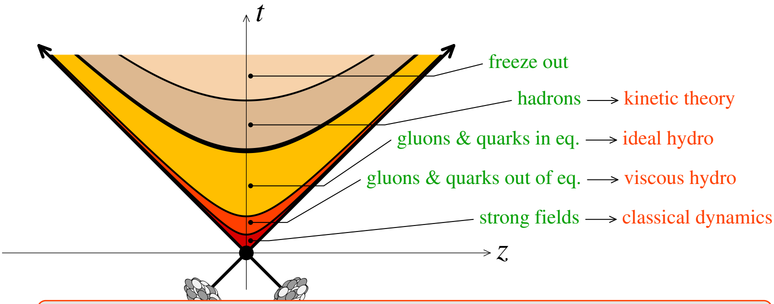
Outline

- 2PI effective action
- Renormalization
- Numerical implementation
- Results

Stages of a nucleus-nucleus collision



Stages of a nucleus-nucleus collision



- Well described as a fluid expanding into vacuum according to relativistic hydrodynamics

Color Glass Condensate at leading order

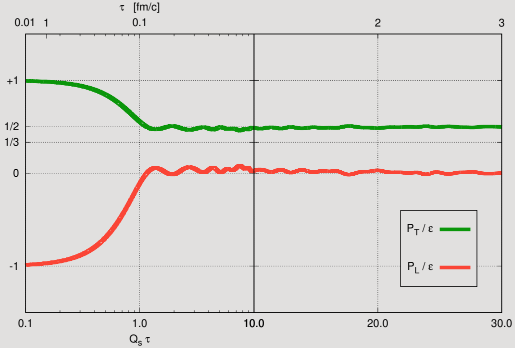
- CGC at leading order $\Longleftrightarrow$ <u>classical</u> color fields:

$$T^{00}_{\text{LO}} = \frac{1}{2}\big[\underbrace{\mathbf{E}^2 + \mathbf{B}^2}_{\text{class. fields}}\big], \qquad T^{0i}_{\text{LO}} = \big[\mathbf{E}\times\mathbf{B}\big]^i,$$

$$T^{ij}_{\text{LO}} = \frac{\delta^{ij}}{2}\big[\mathbf{E}^2 + \mathbf{B}^2\big] - \big[\mathbf{E}^i\mathbf{E}^j + \mathbf{B}^i\mathbf{B}^j\big],$$

• CGC a ,

No isotropization at leading order :



Alternative: Kinetic Theory (but has many limitations...)

- Kinetic theory requires:
  - long-lived quasiparticles
  - moderate occupation numbers
  - small gradients

- For times $Q_s\tau \sim 1$, the occupation number is $\sim \alpha_s^{-1}$, the expansion rate is $\sim \tau^{-1}$ (i.e., gradients are large), and presumably there are no well-defined quasiparticles yet

- We need a framework that can handle large occupation numbers, large gradients and where we deal with fields instead of on-shell particles

2PI effective action

Full QFT	$\longrightarrow$	Dyson-Schwinger $G = G_0 + G_0 \Sigma G$	$\longrightarrow$	Boltzmann $\partial_\tau f =$ Collision Term

- The Dyson-Schwinger equation is exact, as long as the self-energy Σ is exact
- To get Boltzmann:
  - Truncate Σ at 2-loops (in terms of G)
  - Assume spatial gradients are small
  - Assume $G(p) = (\frac{1}{2} + f_p)\delta(p^2 - m^2)$ (quasi-particle approximation)

- **2PI formalism:** *solve the Dyson-Schwinger equations without the extra approximations needed to get Boltzmann*

2PI EFFECTIVE ACTION

- The usual effective action $\Gamma[\varphi]$ is a generalization of the classical action that includes all quantum corrections, under the assumption that we know the field expectation value φ

- The 2PI effective action $\Gamma[\varphi, G]$ is an extension of this concept, where we pretend that we know the field expectation value φ and the <u>exact propagator G</u>

- The diagrammatic expansion based on $\Gamma[\varphi, G]$ contains fewer graphs, because a lot of the complexity is hidden in G

2PI EFFECTIVE ACTION (SCALAR FIELD THEORY)

- For a scalar field theory with $g^2\phi^4$ interaction:

$$\Gamma[\varphi, G] = S[\varphi] - \frac{i}{2}\mathrm{tr}\left(\log G\right) + \frac{i}{2}\mathrm{tr}\left(G_0^{-1}\, G\right) + \Phi[\varphi, G]$$

  G_0 is the free propagator in the background field φ:

$$G_0^{-1} = i\left(\square_x + m^2 + \tfrac{g^2}{2}\varphi^2(x)\right)\delta(x-y)$$

  G = exact propagator
  $S[\varphi]$ = classical action
  $\Phi[\varphi, G]$ = sum of all **2PI** vacuum graphs

- Equations of motion:

$$\frac{\delta\Gamma}{\delta\varphi(x)} = 0, \qquad \frac{\delta\Gamma}{\delta G(x,y)} = 0$$

EQUATIONS OF MOTION

- Explicit form of the equations of motion:

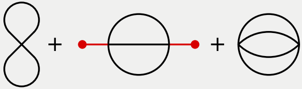
$$(\square_x + m^2)\varphi(x) + \frac{g^2}{6}\varphi^3(x) + \frac{g^2}{2}G(x,x)\varphi(x) = \frac{\delta\Phi}{\delta\varphi(x)}$$

$$(\square_x + m^2)G(x,y) + \frac{g^2}{2}\varphi^2(x)G(x,y) = -i\delta(x-y) + \int d^4z\,\frac{\delta\Phi}{\delta G(x,z)}\,G(z,y)$$

- These equations are untractable unless we truncate $\Phi[\varphi, G]$

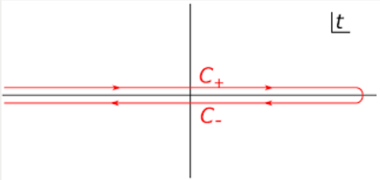
- In this work, truncation at order g^4:

$$i\Phi[\varphi, G] \underset{\text{order } g^4}{=}$$



Equations of motion for ϕ^4 theory

- Times live on the Schwinger-Keldish contour:



- Define "statistical" (F) and "spectral" (ρ) functions:

$$F(x,y) \equiv \frac{1}{2}\Big(G_{-+}(x,y) + G_{+-}(x,y)\Big),$$
$$\rho(x,y) \equiv -i\Big(G_{-+}(x,y) - G_{+-}(x,y)\Big)$$

(both are real valued)

Equations of motion for ϕ^4 theory

- The EoMs at order g^4 read:

$$\Big[\square_x + m^2 + T(x)\Big]\varphi(x) + \tfrac{g^2}{6}\varphi^3(x) = \int_{t_{\rm init}}^{x^0} d^4z\; \Pi_\rho(x,z)\,\varphi(z)$$

$$\Big[\square_x + m^2 + T(x) + \tfrac{g^2}{2}\varphi^2(x)\Big]\,F(x,y) = \int_{t_{\rm init}}^{x^0} d^4z\; \Sigma_\rho(x,z)\,F(z,y) + \int_{t_{\rm init}}^{y^0} d^4z\; \Sigma_F(x,z)\,\rho(z,y)$$

$$\Big[\square_x + m^2 + T(x) + \tfrac{g^2}{2}\varphi^2(x)\Big]\,\rho(x,y) = \int_{y^0}^{x^0} d^4z\; \Sigma_\rho(x,z)\,\rho(z,y)$$

with

$$T(x) \equiv \frac{g^2}{2} F(x,x) \qquad \text{(tadpole)}$$

Renormalization

- The tadpole has a quadratic UV divergence (of order g^2)

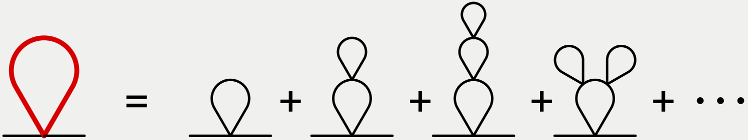
  Since the tadpole does not depend on momentum, no need for field renormalization

- Some of the resummed graphs contain 1-loop corrections to the vertex $\Longrightarrow$ logarithmic UV divergence

- 2-loop self-energies require both mass and field renormalization (at order g^4)

- Because of the 2PI resummation, fixed-order counterterms are not sufficient to remove all UV divergences

- Because the propagator F in the tadpole is the solution of the EoM (that contains the tadpole), this skeleton graph contains an infinite series of graphs built with the bare propagator:



- These graphs contain powers of the bare tadpole, and of 1-loop vertex corrections

- We need both δm^2 and δg^2 counterterms to make it finite
  These counterterms contain all orders in g^2

Renormalization of the tadpole skeleton graph (vacuum)

- Gap equation for renormalized mass:

$$m_R^2 = m_R^2 + \delta m^2 + \frac{g_R^2 + \delta g^2}{2} \int_k F_{vac}(k, m_R^2)$$

gives the counterterm $\delta m^2 = -\frac{g_R^2 + \delta g^2}{2} \int_k F_{vac}(k, m_R^2)$

- Bethe-Salpeter equation for renormalized coupling:

$$g_R^2 = g_R^2 + \delta g^2 - \frac{(g_R^2 + \delta g^2) g_R^2}{2} \int_k F_{vac}(k, m_R^2) \rho_{vac}(k, m_R^2)$$

solution: $g_R^2 + \delta g^2 = \dfrac{g_R^2}{1 - \frac{g_R^2}{2} \int_k F_{vac}(k, m_R^2) \rho_{vac}(k, m_R^2)}$

Renormalization beyond the tadpole

- Renormalization beyond the tadpole possible
- Requires field renormalization
- Much more complicated in practice
- Not implemented in this work
- With $g^4 = 500$:
  δm^2 from the tadpole is the largest subtraction
  δg^2 provides a $(10 - 15)\%$ correction
  δm^2 from 2-loop self-energy is small
  The field renormalization δZ is around $(0.1 - 1)\%$

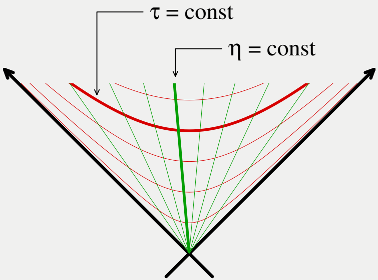
Numerical implementation

Memory requirements

- Memory usage is a major concern because the memory integrals require that we store the past propagators
- Discretize $3D$-space with N^3 points. With fixed times x^0, y^0, a propagator uses N^6 numbers
- Store the last N_t timesteps
- Needed memory: $\frac{1}{2}N_t^2 N^6$ numbers
- With $N_t = 1000$, $N = 500$, simple precision:
  $\sim 28 \times 10^9$ TeraBytes...

  We must assume some symmetries to reduce the memory footprint to practical levels!

Milne coordinates



- Collisions at high energy are approximately invariant under boosts along the collision axis
- This is best expressed by using rapidity and proper time:

$$\eta \equiv \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right) \qquad \tau \equiv \sqrt{t^2 - z^2}$$

- Boost invariance $\Longleftrightarrow$ translation invariance in η at fixed τ

Symmetries assumed in this work

- Translation and rotation invariance in the transverse plane

  F, ρ depend only on $|x_\perp - y_\perp|$
- Boost invariance in the z direction

  F, ρ depend only on $|\eta_x - \eta_y|$

- Discretize $|x_\perp - y_\perp|$ with $N_\perp$ points

  Discretize $\eta_x - \eta_y$ with N_η points

- Needed memory: $\frac{1}{2}N_t^2 N_\perp N_\eta$ numbers
- With $N_t = 1000$, $N_\perp = N_\eta = 500$, simple precision:
  ~ 0.5 TeraBytes
- Note: all this data must be manipulated at each timestep, in a very parallel fashion $\implies$ GPU are well suited for this

Results

Free parameters

- Initial time τ_0
- Initial occupation number $f(\tau_0, p_\perp, \nu)$
  Typical momentum Q for the initial particle distribution
- Field $\varphi(\tau_0)$ and derivative $\dot{\varphi}(\tau_0)$
- Renormalized mass $m_{_R}$
- Renormalized coupling constant $g_{_R}^2$ (we used $g_{_R}^2 \approx 22$)
  This is a fairly weak coupling: [viscosity/entropy] ~ 25

Occupation number

- By solving the 2PI equations of motion, one gets φ, F, ρ

- Assuming a quasi-particle ansatz, one may extract an occupation number from the propagator F

$$\frac{1}{2} + f(p_\perp, \nu; \tau) \underset{\tau=\tau'}{\approx} \tau \sqrt{F(\partial_\tau \partial_{\tau'} F) - (\partial_\tau F)(\partial_{\tau'} F)}$$

(Note: this works even if we do not know the mass of the quasi-particles)

Effective mass

- To extract a thermal mass, we consider

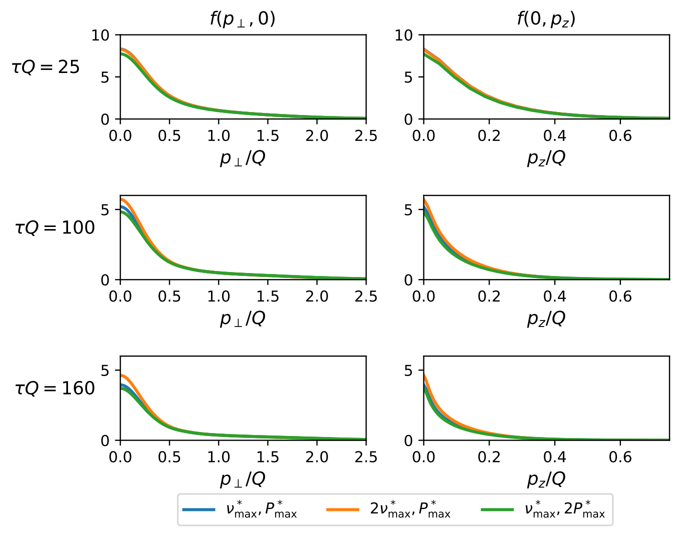
$$\Gamma(\tau,\tau';p_\perp,\nu) \equiv \frac{F(\tau,\tau';p_\perp,\nu)}{\frac{1}{2}+f(p_\perp,\nu)} - i\rho(\tau,\tau';p_\perp,\nu)$$

- With a quasi-particle ansatz, this combination is proportional to

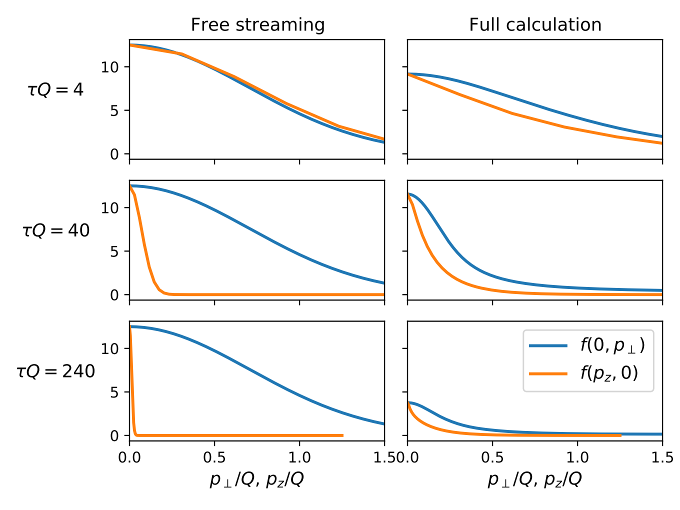
$$H^{(1)}_{i\nu}(m_\perp\tau)H^{(2)}_{i\nu}(m_\perp\tau') \qquad [H^{(1,2)}_{i\nu} \equiv \text{Hankel functions}]$$

- The value of the transverse mass can be deduced from the oscillation period in τ' of $\Gamma(\tau,\tau';p_\perp,\nu)$
(provided we know the oscillation period of $H^{(1)}_{i\nu}(z)$)

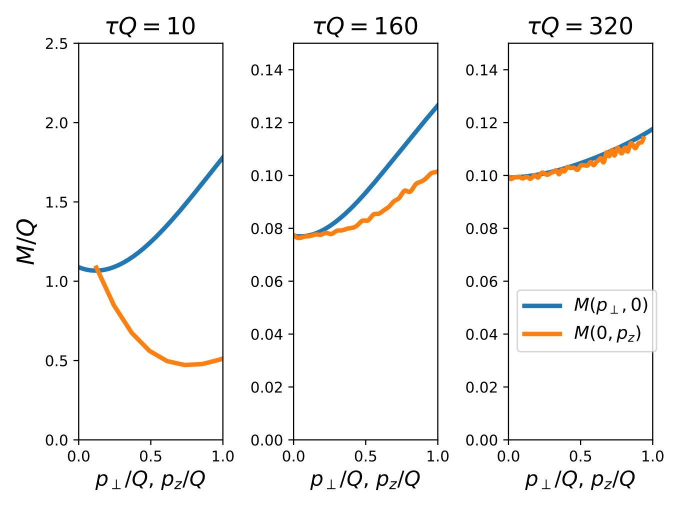
Residual cutoff dependence



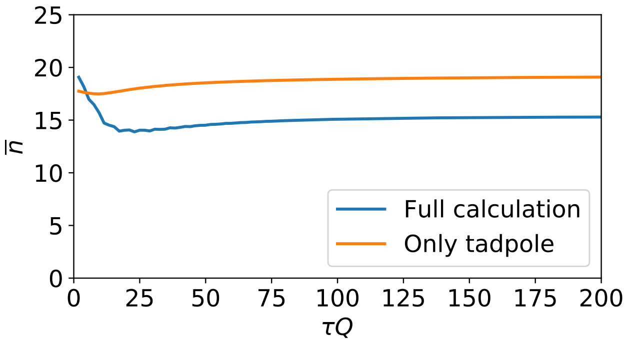
Momentum distribution



Thermal mass



Number density



Summary and Outlook

- 2PI equations of motion, truncated at order g^4
- Implementation in Milne coordinates $(\tau, \boldsymbol{x}_\perp, \eta)$
- Need to assume some symmetries to make it manageable
- Note: memory (amount, bandwidth) as important as pure computational power
- Visible trend towards isotropization
- Extracted thermal mass also shows isotropization
- Variation of the number density ($\neq$ kinetic theory)

- Possible extensions:
  - Renormalization beyond tadpole
  - Include a non-zero φ. Goal: study how a classical field is converted into particles
  - Yang-Mills: in principle possible, but would require considerably larger resources

Extra bits

Self-energies at order g^4

$$\Sigma_F(x,y) \equiv \frac{g^4}{2}\varphi(x)\varphi(y)\Big[F^2(x,y) - \frac{1}{4}\rho^2(x,y)\Big] + \frac{g^4}{6}F(x,y)\Big[F^2(x,y) - \frac{3}{4}\rho^2(x,y)\Big]$$

$$\Sigma_\rho(x,y) \equiv -g^4\varphi(x)\varphi(y)F(x,y)\rho(x,y) - \frac{g^4}{6}\rho(x,y)\Big[3F^2(x,y) - \frac{1}{4}\rho^2(x,y)\Big]$$

$$\Pi_\rho(x,y) \equiv -\frac{g^4}{6}\rho(x,y)\Big[3F^2(x,y) - \frac{1}{4}\rho^2(x,y)\Big]$$

Renormalization of the tadpole skeleton graph (in matter)

- Gap equation:

$$M^2 = m_R^2 + \delta m^2 + \frac{g_R^2 + \delta g^2}{2} \int_k F(k, M^2)$$

- Bethe-Salpeter equation:

$$G^2 = g_R^2 + \delta g^2 - \frac{(g_R^2 + \delta g^2) G^2}{2} \int_k F(k, M^2) \rho(k, M^2)$$

- Solution for G^2 (using the previously determined δg^2):

$$G^2 = \frac{g_R^2}{1 + \frac{g_R^2}{2}\Big[\int_k F(k, M^2)\rho(k, M^2) - \int_k F_{vac}(k, m_R^2)\rho_{vac}(k, m_R^2)\Big]}$$

- The two integrals in the denominator have a logarithmic divergence, but their difference is finite

- G^2 is UV finite with the δg^2 determined in vacuum

Renormalization of the tadpole skeleton graph (in matter)

- The difference of the in-vacuum and in-matter tadpoles still has a logarithmic divergence:

$$\int_k F(k, M^2) - \int_k F_{\text{vac}}(k, m_{\text{R}}^2) = (m_{\text{R}}^2 - M^2) \int_k F_{\text{vac}}(k, m_{\text{R}}^2) \rho_{\text{vac}}(k, m_{\text{R}}^2) + \text{finite terms}$$

- The solution of the gap equation is implicitly given by

$$M^2 = m_{\text{R}}^2 + \frac{g_{\text{R}}^2}{2} \int_{\mathbf{k}} \frac{f(\mathbf{k})}{\sqrt{\mathbf{k}^2 + M^2}} + \text{finite terms}$$

- The integrand is proportional to the occupation number $\implies$ the integral is finite
- M^2 is UV finite with the δm^2 and δg^2 determined in vacuum

Specificities of Milne coordinates

- A fixed volume in the coordinates $(x_\perp, \eta)$ corresponds to a time-dependent volume in Cartesian coordinates $(x_\perp, z)$
  We must check that the physical scales stay on the grid at all times
  The UV cutoff in p_z is time-dependent
  The UV counterterms are time-dependent

- The analogue of plane waves are the Hankel functions $H^{(1,2)}_{i\nu}(m_\perp \tau)$.
  ν is the Fourier conjugate of η. Roughly: $\nu \approx p_z \tau$

- The self-energies involve convolutions in momentum space
  Easier to evaluate in coordinate space, where they are given by ordinary products
  BUT: since they depend only on $|x_\perp - y_\perp|$, the Fourier transform to go back to $p_\perp$ is a Hankel transform:

$$\widetilde{F}(p_\perp) = 2\pi \int_0^{+\infty} dr_\perp \; r_\perp \; J_0(p_\perp r_\perp) \; F(r_\perp)$$

Bare propagators

$$F_0(\tau,\tau';p_\perp,\nu) = \frac{\pi\big(\frac{1}{2}+f(p_\perp,\nu)\big)}{4}\Big\{H^{(1)}_{i\nu}(m_\perp\tau)H^{(2)}_{i\nu}(m_\perp\tau') + H^{(2)}_{i\nu}(m_\perp\tau)H^{(1)}_{i\nu}(m_\perp\tau')\Big\}$$

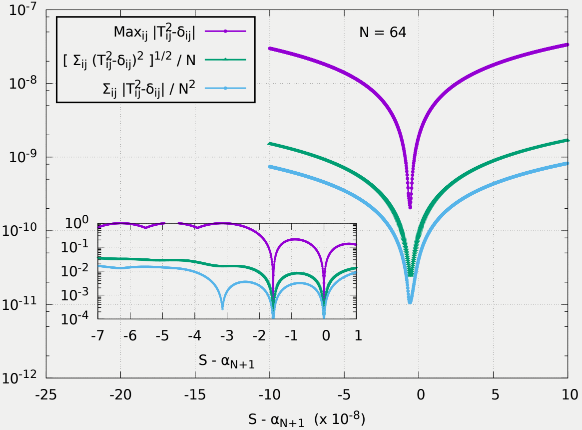
$$\rho_0(\tau,\tau';p_\perp,\nu) = \frac{i\pi}{4}\Big\{H^{(1)}_{i\nu}(m_\perp\tau)H^{(2)}_{i\nu}(m_\perp\tau') - H^{(2)}_{i\nu}(m_\perp\tau)H^{(1)}_{i\nu}(m_\perp\tau')\Big\}$$

- Denote by α_i the zeroes of the Bessel function $J_0(z)$
- Choose a length R and a momentum P
- Choose N discrete radii $r_i \equiv \alpha_i/P$
- Choose N discrete transverse momenta $p_k \equiv \alpha_k/R$
- The Quasi Discrete Hankel Transform reads:

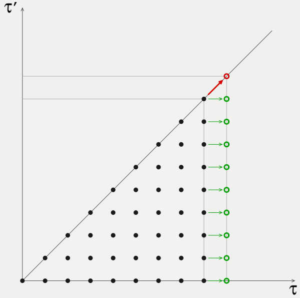
$$F(r_i) = \frac{1}{\pi R^2} \sum_{k=1}^{N} \widetilde{F}(p_k) \, \frac{J_0(\frac{\alpha_k \alpha_i}{RP})}{J_1^2(\alpha_k)}, \quad \widetilde{F}(p_k) = \frac{4\pi}{P^2} \sum_{i=1}^{N} F(r_i) \, \frac{J_0(\frac{\alpha_k \alpha_i}{RP})}{J_1^2(\alpha_i)}$$

- "Quasi": because these two formulas are mutually consistent only when $N \to \infty$. For finite N, the error is minimized by choosing $RP \approx \alpha_{N+1}$. For $N = 512$, the error is $\sim 10^{-14}$

QDHT : VIOLATION OF INVOLUTIVITY



Time stepping



- $F(\tau, \tau', p_\perp, \nu)$ is symmetric, $\rho(\tau, \tau', p_\perp, \nu)$ is antisymmetric
  $\Longrightarrow$ we consider only $\tau \geq \tau'$
- The steps $(\tau, \tau') \to (\tau + \delta\tau, \tau')$ vary only τ at fixed τ'
- For the diagonal propagators, we need $(\tau, \tau) \to (\tau + \delta\tau, \tau + \delta\tau)$

- The EoM for evolution in τ has the following generic form

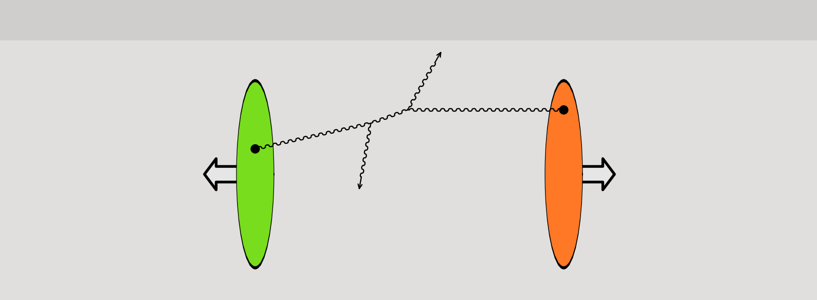
$$\left(\partial_\tau^2 + \tau^{-1}\partial_\tau + p_\perp^2 + m^2 + \frac{\nu^2}{\tau^2}\right) X = \mathrm{RHS}(\tau)$$

- Define $\mathbf{X} \equiv \begin{pmatrix} X \\ \tau\partial_\tau X \end{pmatrix}$. The solution from τ_n to τ_{n+1} reads

$$\mathbf{X}(\tau_{n+1}) = \mathbf{L}(\tau_{n+1},\tau_n)\mathbf{X}(\tau_n) + \int_{\tau_n}^{\tau_{n+1}} d\tau''\,\tau''\,\mathbf{L}(\tau_{n+1},\tau'')\begin{pmatrix} 0 \\ \mathrm{RHS}(\tau'') \end{pmatrix}$$

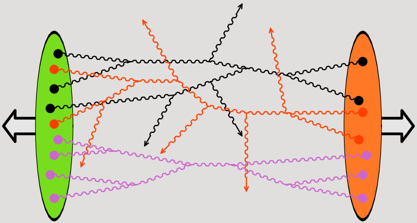
$$\mathbf{L}(\tau,\tau') = \frac{i\pi}{4}\begin{pmatrix} \begin{matrix} m_\perp\tau'\big[H^{(1)}_{i\nu}(m_\perp\tau)\dot{H}^{(2)}_{i\nu}(m_\perp\tau') \\ -H^{(2)}_{i\nu}(m_\perp\tau)\dot{H}^{(1)}_{i\nu}(m_\perp\tau')\big] \end{matrix} & \begin{matrix} H^{(2)}_{i\nu}(m_\perp\tau)H^{(1)}_{i\nu}(m_\perp\tau') \\ -H^{(1)}_{i\nu}(m_\perp\tau)H^{(2)}_{i\nu}(m_\perp\tau') \end{matrix} \\ & \\ \begin{matrix} m_\perp^2\tau\tau'\big[\dot{H}^{(1)}_{i\nu}(m_\perp\tau)\dot{H}^{(2)}_{i\nu}(m_\perp\tau') \\ -\dot{H}^{(2)}_{i\nu}(m_\perp\tau)\dot{H}^{(1)}_{i\nu}(m_\perp\tau')\big] \end{matrix} & \begin{matrix} m_\perp\tau\big[\dot{H}^{(2)}_{i\nu}(m_\perp\tau)H^{(1)}_{i\nu}(m_\perp\tau') \\ -\dot{H}^{(1)}_{i\nu}(m_\perp\tau)H^{(2)}_{i\nu}(m_\perp\tau')\big] \end{matrix} \end{pmatrix}$$

Multiple scatterings and gluon recombination



- Dilute regime : one parton in each projectile interact
  - ▷ single parton distributions, standard perturbation theory

Multiple scatterings and gluon recombination



- Dense regime : multiparton processes become crucial
  - ▷ gluon recombinations are important (saturation)
  - ▷ we need multi-parton distributions

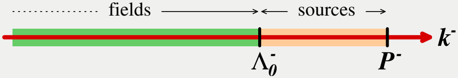
Color Glass Condensate Effective Theory

- The fast partons ($k^- > \Lambda_0$, for a left-moving projectile) are frozen by time dilation
  - ▷ described as static color sources on the light-cone :

$$J^\mu = \delta^{\mu -}\rho(x^+, \vec{x}_\perp) \qquad\qquad (0 < x^+ < 1/\Lambda_0)$$

- The color sources ρ are random, and described by a probability distribution $W_{\Lambda_0}[\rho]$
- Slow partons ($k^- < \Lambda_0$) cannot be considered static over the time-scales of the collision process
  - ▷ must be treated as standard gauge fields
  - ▷ eikonal coupling to the current J^μ : $A_\mu J^\mu$

- CGC effective theory with cutoff at the scale Λ_0 :



$$\mathcal{S} = \underbrace{-\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}}_{\mathcal{S}_{\rm YM}} + \int \underbrace{(J_1^\mu + J_2^\mu)}_{\text{fast partons}} A_\mu$$

- Expansion in g^2 in the saturated regime:

$$T^{\mu\nu} \sim \frac{1}{g^2}\left[c_0 + c_1\, g^2 + c_2\, g^4 + \cdots\right]$$