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R-RATIO OF ELECTRON-POSITRON ANNIHILATION INTO HADRONS AND DETERMINATION OF THE STRONG COUPLING

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INTRODUCTION

The hadronic vacuum polarization (HVP) function $\Pi(q^2)$ plays a central role in various issues of QCD and Standard Model. In particular, the theoretical description of some strong interaction processes and hadronic contributions to electroweak observables is inherently based on the HVP function $\Pi(q^2)$:

- electron–positron annihilation into hadrons
- inclusive τ lepton hadronic decay
- muon anomalous magnetic moment
- running of the electromagnetic coupling

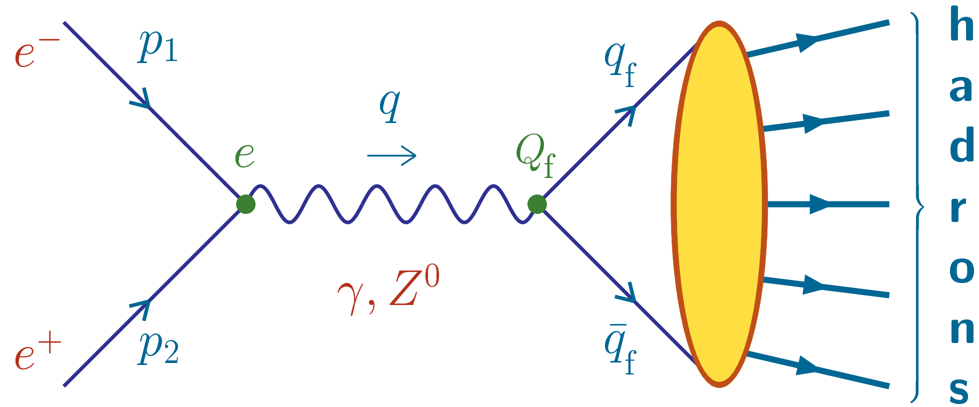
The relevant energy scales span from IR to UV domain.

GENERAL DISPERSION RELATIONS

Cross–section of $e^+e^- \longrightarrow$ hadrons reads

$$\sigma = 4\pi^2 \frac{2\alpha^2}{s^3} L^{\mu\nu} \Delta_{\mu\nu},$$

where $s = q^2 = (p_1 + p_2)^2 > 0$, [timelike]



$$L_{\mu\nu} = \frac{1}{2}\left[q_\mu q_\nu - g_{\mu\nu} q^2 - (p_1 - p_2)_\mu (p_1 - p_2)_\nu\right],$$

$$\Delta_{\mu\nu} = (2\pi)^4 \sum_\Gamma \delta(p_1 + p_2 - p_\Gamma) \langle 0 | J_\mu(-q) | \Gamma \rangle \langle \Gamma | J_\nu(q) | 0 \rangle,$$

and $J_\mu = \sum_f Q_f :\bar{q}\,\gamma_\mu\, q:$ is the electromagnetic quark current.

Kinematic restriction: the hadronic tensor $\Delta_{\mu\nu}(q^2)$ assumes non–zero values only for $q^2 \geq 4m_\pi^2 = m^2$, since otherwise no hadron state Γ could be excited

■ Feynman (1972); Adler (1974).

The hadronic tensor can be represented as $\Delta_{\mu\nu} = 2\,\mathrm{Im}\,\Pi_{\mu\nu}$,

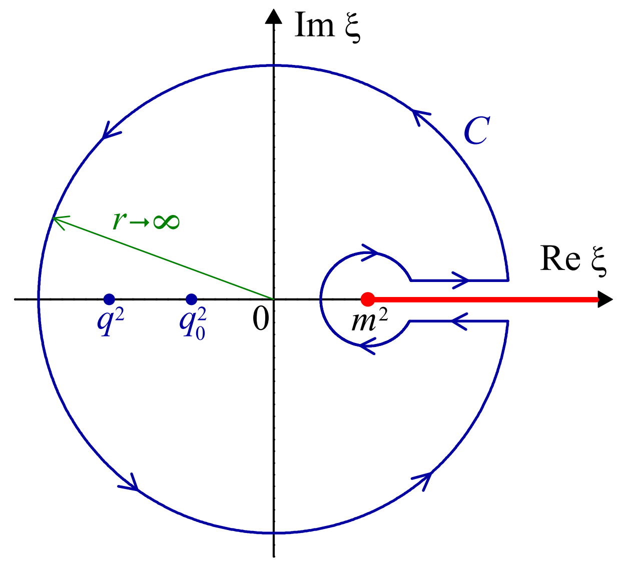
$$\Pi_{\mu\nu}(q^2) = i\int e^{iqx}\langle 0|\,T\{J_\mu(x)\,J_\nu(0)\}|0\rangle\, d^4x = i\left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\frac{\Pi(q^2)}{12\pi^2}.$$

Kinematic restriction: $\Pi(q^2)$ has the only cut $s = q^2 \geq m^2$ [timelike]

Dispersion relation for $\Pi(q^2)$:

$$\Delta\Pi(q^2,\, q_0^2) = \frac{1}{2\pi i}(q^2 - q_0^2)\oint_C \frac{\Pi(\xi)}{(\xi - q^2)(\xi - q_0^2)}\, d\xi$$

$$= (q^2 - q_0^2)\int_{m^2}^{\infty} \frac{R(s)}{(s - q^2)(s - q_0^2)}\, ds,$$



where $\Delta\Pi(q^2,\, q_0^2) = \Pi(q^2) - \Pi(q_0^2)$ and $R(s)$ denotes the measurable ratio of two cross–sections

$$R(s) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\left[\Pi(s + i\varepsilon) - \Pi(s - i\varepsilon)\right] = \frac{\sigma(e^+e^- \to \text{hadrons};\, s)}{\sigma(e^+e^- \to \mu^+\mu^-;\, s)}.$$

Kinematic restriction: $R(s) = 0$ for $s = q^2 < m^2$.

For practical purposes it is convenient to employ the Adler function

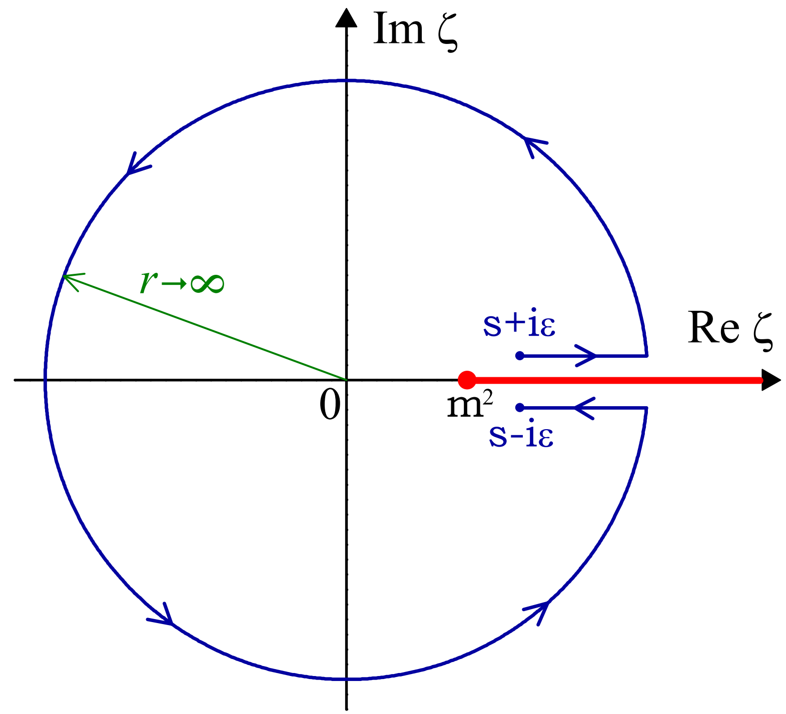
$$D(Q^2) = -\frac{d\,\Pi(-Q^2)}{d\ln Q^2}, \quad D(Q^2) = Q^2 \int_{m^2}^{\infty} \frac{R(s)}{(s+Q^2)^2}\, ds, \quad Q^2 = -q^2 > 0 \ \text{[spacelike]}$$

■ Adler (1974); De Rujula, Georgi (1976); Bjorken (1989).

The Adler function plays a valuable role for the congruous analysis of spacelike/timelike data: the dispersion relation provides a link between experimentally measurable and theoretically computable quantities.

The inverse relations between the functions on hand read

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon\to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\, \frac{d\zeta}{\zeta}$$



■ Radyushkin (1982); Krasnikov, Pivovarov (1982).

$$\Delta\Pi(-Q^2, -Q_0^2) = -\int_{Q_0^2}^{Q^2} D(\xi)\, \frac{d\xi}{\xi}$$

■ Pennington, Ross (1981); Pivovarov (1992).

The massless limit $m = 0$ is assumed hereinafter.

QCD PERTURBATIVE PREDICTIONS

STRONG RUNNING COUPLING

The QCD running coupling $\alpha_s(\mu^2) = g^2(\mu^2)/(4\pi)$ satisfies the RG equation

$$\frac{\partial a_s(\mu^2)}{\partial \ln \mu^2} = \beta(a_s), \qquad \beta^{(\ell)}(a_s) = -\sum_{i=0}^{\ell-1} B_i \left[a_s^{(\ell)}(\mu^2)\right]^{i+2}, \qquad B_i = \frac{\beta_i}{\beta_0^{i+1}},$$

where $\mu^2 > 0$, $a_s(\mu^2) = \alpha_s(\mu^2)\beta_0/(4\pi)$ is the so–called “QCD couplant”, $\beta_0 = 11 - 2n_f/3$, $B_0 = 1$, $\beta_1 = 102 - 38n_f/3$, $B_1 = \beta_1/\beta_0^2$, etc.

Perturbative coefficients β_i are available up to the 5–loop level ($i = 0\ldots4$):

- Baikov, Chetyrkin, Kuhn, Phys. Rev. Lett. 118, 082002 (2017);
  Herzog, Ruijl, Ueda, Vermaseren, Vogt, JHEP02, 090 (2017);
  Luthe, Maier, Marquard, Schroder, JHEP10, 166 (2017);
  Chetyrkin, Falcioni, Herzog, Vermaseren, JHEP10, 179 (2017).

In what follows for the higher–order scheme–dependent perturbative coefficients the $\overline{\text{MS}}$–scheme is assumed.

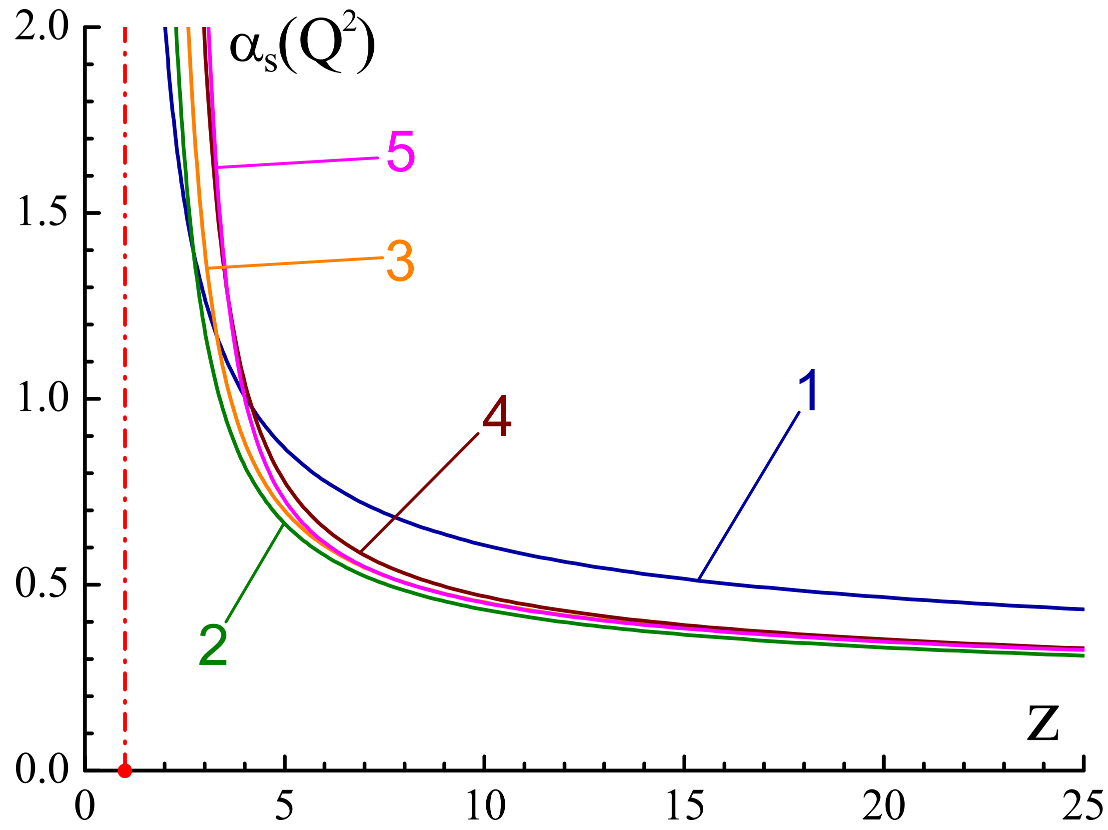
The perturbative QCD running coupling at the ℓ–loop level:

$$\alpha_s^{(\ell)}(Q^2) = \frac{4\pi}{\beta_0} \sum_{n=1}^{\ell} \sum_{m=0}^{n-1} b_n^m \frac{\ln^m(\ln z)}{\ln^n z},$$

where $z = Q^2/\Lambda^2 > 0$, $b_1^0 = 1$, $b_2^0 = 0$, $b_2^1 = -B_1$, etc.

The one–loop expression reads

$$\alpha_s^{(1)}(Q^2) = \frac{4\pi}{\beta_0} a_s^{(1)}(Q^2), \quad a_s^{(1)}(Q^2) = \frac{1}{\ln z}.$$

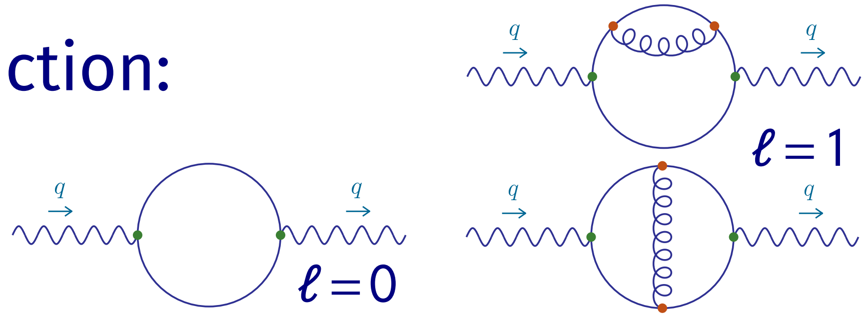


The one–loop perturbative QCD running coupling suffers from the infrared unphysical singularity, namely, the ghost pole, and the inclusion of the higher–loop corrections does not resolve this issue.

■ Prosperi, Raciti, Simolo, Prog. Part. Nucl. Phys. **58**, 387 (2007);
Deur, Brodsky, de Teramond, Prog. Part. Nucl. Phys. **90**, 1 (2016).

Perturbative hadronic vacuum polarization function:

$$\Pi^{(\ell)}(q^2,\mu^2,a_s) = \sum_{j=0}^{\ell}\left[a_s^{(\ell)}(\mu^2)\right]^j \sum_{k=0}^{j+1} \Pi_{j,k} \ln^k\!\left(\frac{\mu^2}{-q^2}\right),$$



where $q^2 < 0$, $\mu^2 > 0$, $a_s(\mu^2) = \alpha_s(\mu^2)\beta_0/(4\pi)$, and the prefactor $N_c \sum_{f=1}^{n_f} Q_f^2$ is omitted throughout. At the one–loop level ($\ell = 1$) it reads

$$\Pi^{(1)}(q^2,\mu^2,a_s) = \frac{5}{3} - \ln\!\left(\frac{-q^2}{\mu^2}\right) + \frac{\alpha_s^{(1)}(\mu^2)}{\pi}\left[\frac{55}{12} - 4\zeta(3) - \ln\!\left(\frac{-q^2}{\mu^2}\right)\right], \quad q^2 \to -\infty.$$

The function $\Pi(q^2,\mu^2,a_s)$ satisfies the inhomogeneous RG equation

$$\left[\frac{\partial}{\partial \ln \mu^2} + \frac{\partial a_s(\mu^2)}{\partial \ln \mu^2}\frac{\partial}{\partial a_s}\right]\Pi(q^2,\mu^2,a_s) = \gamma(a_s), \qquad \gamma^{(\ell)}(a_s) = \sum_{j=0}^{\ell}\gamma_j\left[a_s^{(\ell)}(\mu^2)\right]^j.$$

which binds together the higher–order coefficients $\Pi_{j,k}$ and γ_j. At the first few orders the corresponding RG relations are given in, e.g.,

- Baikov, Chetyrkin, Kuhn, Rittinger (2009), (2012); Nesterenko (2019), (2020).

ADLER FUNCTION

Perturbative expression for the Adler function takes the form

$$D^{(\ell)}(Q^2,\mu^2,a_s) = \sum_{j=0}^{\ell}\left[a_s^{(\ell)}(\mu^2)\right]^j \sum_{k=0}^{j+1} k\,\Pi_{j,k}\,\ln^{k-1}\!\left(\frac{\mu^2}{Q^2}\right), \qquad Q^2=-q^2\to\infty.$$

The Adler function satisfies the homogeneous RG equation

$$\left[\frac{\partial}{\partial\ln\mu^2} + \frac{\partial a_s(\mu^2)}{\partial\ln\mu^2}\frac{\partial}{\partial a_s}\right] D(Q^2,\mu^2,a_s) = 0,$$

which, similarly to the previous case, enables one to express the higher–order perturbative coefficients $\Pi_{j,k}$ ($j \ge 2$; $k = 2,\dots,j+1$) in terms of the lower–order ones $\Pi_{i,1}$ ($i = 1,\dots,j-1$). For example, at the first few orders such RG relations read

$$\Pi_{2,2} = \frac{1}{2}\Pi_{1,1}, \qquad \Pi_{3,2} = \frac{1}{2}\Pi_{1,1}B_1 + \Pi_{2,1}, \qquad \Pi_{3,3} = \frac{1}{3}\Pi_{1,1}$$

■ Beneke, Jamin (2008); Nesterenko (2019), (2020).

Explicit form of the RG relations for the higher–order coefficients $\Pi_{j,k}$ is obtained at an arbitrary loop level in a compact recurrent form:

$$\Pi_{j,2} = \frac{1}{2}\sum_{i=1}^{j-1} i B_{j-i-1}\Pi_{i,1} \quad (j \geq 2), \qquad \Pi_{j,j+1} = 0 \quad (j \geq 1), \qquad \mathfrak{B}_n = \frac{1}{4}\sum_{i=0}^{n} B_i\, B_{n-i},$$

$$\Pi_{j,k} = \frac{1}{T_{k-1}} \sum_{i=k-2}^{j-2} i\,(i+j)\mathfrak{B}_{j-i-2}\Pi_{i,k-2} \quad (j \geq k, \quad k \geq 3), \qquad T_n = \frac{1}{2}n(n+1)$$

■ Nesterenko, J. Phys. G47, 105001 (2020).

The higher–order coefficients $\Pi_{j,k}$ can also be expressed in terms of the coefficients $\Pi_{i,0}$ and γ_i. For this purpose the obtained results should be supplemented by the relations

$$\Pi_{0,1} = \gamma_0, \qquad \Pi_{1,1} = \gamma_1, \qquad \Pi_{j,1} = \gamma_j + \sum_{k=1}^{j-1} k\,\Pi_{k,0}\, B_{j-k-1} \quad (j \geq 2), \qquad B_i = \frac{\beta_i}{\beta_0^{i+1}}$$

■ Nesterenko, J. Phys. G46, 115006 (2019).

The obtained RG relations for the higher–order coefficients $\Pi_{j,k}$ can also be represented in the unfolded explicit form:

$$\Pi_{j,2k+1} = \frac{2^k}{(2k+1)!} \sum_{i_1=2(k-1)+1}^{j-2} \underbrace{\sum_{i_2=2(k-2)+1}^{i_1-2} \ldots \sum_{i_n=2(k-n)+1}^{i_{n-1}-2} \ldots \sum_{i_k=1}^{i_{k-1}-2}}_{(k-1)\text{ sums}} (j+i_1) i_1 \times$$

$$\times \underbrace{(i_1+i_2) i_2 \times \ldots \times (i_{n-1}+i_n) i_n \times \ldots \times (i_{k-1}+i_k) i_k}_{(k-1)\text{ products}} \times$$

$$\times \mathfrak{B}_{j-i_1-2} \underbrace{\mathfrak{B}_{i_1-i_2-2} \ldots \mathfrak{B}_{i_{n-1}-i_n-2} \ldots \mathfrak{B}_{i_{k-1}-i_k-2}}_{(k-1)\text{ terms}} \Pi_{i_k,1}, \qquad j \geq (2k+1), \qquad k \geq 1,$$

$$\Pi_{j,2k} = \frac{2^{k-1}}{(2k)!} \sum_{i_1=2(k-1)}^{j-2} \underbrace{\sum_{i_2=2(k-2)}^{i_1-2} \ldots \sum_{i_n=2(k-n)}^{i_{n-1}-2} \ldots \sum_{i_{k-1}=2}^{i_{k-2}-2}}_{(k-2)\text{ sums}} \sum_{i_k=1}^{i_{k-1}-1} (j+i_1) i_1 \times$$

$$\times \underbrace{(i_1+i_2) i_2 \times \ldots \times (i_{n-1}+i_n) i_n \times \ldots \times (i_{k-2}+i_{k-1}) i_{k-1}}_{(k-2)\text{ products}} \times i_k \times$$

$$\times \mathfrak{B}_{j-i_1-2} \underbrace{\mathfrak{B}_{i_1-i_2-2} \ldots \mathfrak{B}_{i_{n-1}-i_n-2} \ldots \mathfrak{B}_{i_{k-2}-i_{k-1}-2}}_{(k-2)\text{ terms}} B_{i_{k-1}-i_k-1} \Pi_{i_k,1}, \qquad j \geq 2k, \qquad k \geq 2$$

■ Nesterenko, J. Phys. G47, 105001 (2020).

RG SUMMATION IN THE SPACELIKE DOMAIN

For a general choice of renormalization scale μ^2 in the spacelike domain all the coefficients $\Pi_{j,k}$ contribute to the resulting expression for the Adler function, while the native choice $\mu^2 = Q^2$ casts $D^{(\ell)}(Q^2, \mu^2, a_s)$ to ($\Pi_{0,1} = 1$)

$$D^{(\ell)}(Q^2) = \sum_{j=0}^{\ell} \Pi_{j,1} \left[a_s^{(\ell)}(Q^2) \right]^j = 1 + d^{(\ell)}(Q^2), \qquad d^{(\ell)}(Q^2) = \sum_{j=1}^{\ell} d_j \left[a_s^{(\ell)}(Q^2) \right]^j.$$

The coefficients $d_j = \Pi_{j,1}$ are known up to the 4–loop level ($j = 1 \ldots 4$):

- Baikov, Chetyrkin, Kuhn, Phys. Rev. Lett. 101, 012002 (2008);
  Baikov, Chetyrkin, Kuhn, Phys. Rev. Lett. 104, 132004 (2010);
  Baikov, Chetyrkin, Kuhn, Rittinger, Phys. Lett. B714, 62 (2012)

whereas the numerical estimations of d_5 are also available, see, e.g.,

- Kataev, Starshenko, Mod. Phys. Lett. A10, 235 (1995).

The one–loop expression for the Adler function reads

$$D^{(1)}(Q^2) = 1 + d^{(1)}(Q^2), \quad d^{(1)}(Q^2) = d_1 a_s^{(1)}(Q^2), \quad d_1 = \frac{4}{\beta_0}, \quad a_s^{(1)}(Q^2) = \frac{1}{\ln(Q^2/\Lambda^2)}.$$

In the expression for the hadronic vacuum polarization function $\Pi(q^2, \mu^2, a_s)$ the dependence on the renormalization scale μ^2 can be eliminated in the following way:

$$\Pi(q^2,\mu^2,a_s) \;\longrightarrow\; D(Q^2,\mu^2,a_s) \;\xrightarrow[\mu^2 = Q^2]{\text{proper RG summation in spacelike domain}}\; D(Q^2) \;\longrightarrow\; \Delta\Pi(-Q^2,-Q_0^2)$$

$$\left[-\frac{d\,\Pi(-Q^2)}{d\ln Q^2} = D(Q^2)\right] \qquad \left[-\int_{Q_0^2}^{Q^2} D(\xi)\,\frac{d\xi}{\xi} = \Delta\Pi(-Q^2,\,-Q_0^2)\right]$$

The expression for $\Delta\Pi^{(\ell)}(-Q^2,-Q_0^2)$ at the one–loop level $(\ell = 1)$:

$$\Delta\Pi^{(1)}(-Q^2,-Q_0^2) = -\ln\left(\frac{Q^2}{Q_0^2}\right) - d_1\ln\left[\frac{a_s^{(1)}(Q_0^2)}{a_s^{(1)}(Q^2)}\right], \qquad a_s^{(1)}(Q^2) = \frac{1}{\ln z}, \qquad z = \frac{Q^2}{\Lambda^2}$$

- Moorhouse, Pennington, Ross, Nucl. Phys. B124, 285 (1977);
  Pennington, Ross, Phys. Lett. B102, 167 (1981);
  Pennington, Roberts, Ross, Nucl. Phys. B242, 69 (1984);
  Pivovarov, Nuovo Cim. A105, 813 (1992).

The explicit expression for the hadronic vacuum polarization function $\Delta\Pi^{(\ell)}(-Q^2, -Q_0^2)$ is obtained at an arbitrary loop level:

$$\Delta\Pi^{(\ell)}(-Q^2, -Q_0^2) = -\ln\left(\frac{Q^2}{Q_0^2}\right) + \sum_{j=1}^{\ell} d_j \left[p_j^{(\ell)}(Q^2) - p_j^{(\ell)}(Q_0^2)\right],$$

$$p_j^{(\ell)}(Q^2) = \sum_{n_1=1}^{\ell} \dots \sum_{n_j=1}^{\ell} \sum_{m_1=0}^{n_1-1} \dots \sum_{m_j=0}^{n_j-1} \left(\prod_{i=1}^{j} b_{n_i}^{m_i}\right) J\left(Q^2, \sum_{i=1}^{j} n_i, \sum_{i=1}^{j} m_i\right),$$

$$J(Q^2, n, m) = \begin{cases} -\dfrac{\ln^{m+1}(\ln z)}{m+1}, & \text{if } n = 1, \\ \displaystyle\sum_{k=0}^{m} \frac{m!}{k!}(n-1)^{k-m-1} \frac{\ln^k(\ln z)}{\ln^{n-1} z}, & \text{if } n \geq 2, \end{cases}$$

where $z = Q^2/\Lambda^2$, $d_j = \Pi_{j,1}$, and b_n^m denotes the combination of the β function perturbative expansion coefficients

- Nesterenko, J. Phys. G46, 115006 (2019).

R–RATIO OF e^+e^- ANNIHILATION INTO HADRONS

R–RATIO ("METHOD I")

$$\Pi(q^2, \mu^2, a_s) \;\rightarrow\; D(Q^2, \mu^2, a_s) \;\xrightarrow[\mu^2 = Q^2]{\text{proper RG summation in spacelike domain}}\; D(Q^2) \;\rightarrow\; R(s)$$

$$\left[-\frac{d\,\Pi(-Q^2)}{d\ln Q^2} = D(Q^2)\right]$$

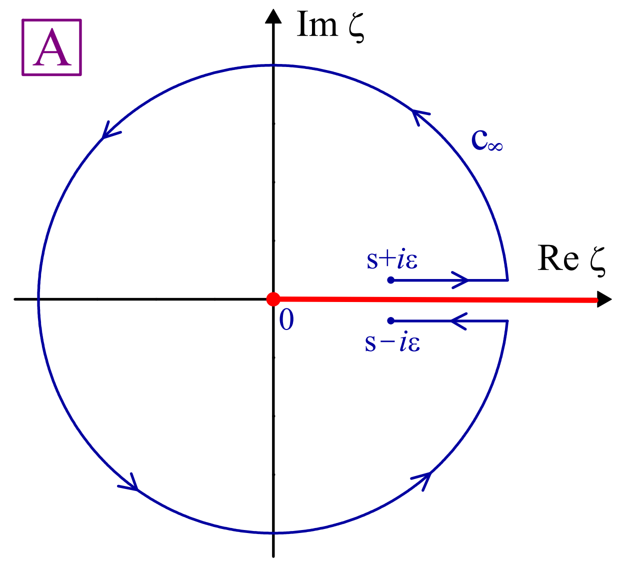
$$\left[\frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\,\frac{d\zeta}{\zeta} = R(s)\right]$$

This method eventually leads to

$$R^{(\ell)}(s) = 1 + r^{(\ell)}(s), \qquad r^{(\ell)}(s) = \int_s^{\infty} \rho^{(\ell)}(\sigma)\,\frac{d\sigma}{\sigma},$$

where the spectral function is

$$\rho^{(\ell)}(\sigma) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\left[d^{(\ell)}(-\sigma - i\varepsilon) - d^{(\ell)}(-\sigma + i\varepsilon)\right].$$



At the one–loop level the calculation of $R^{(1)}(s)$ is quite straightforward.

Strong correction to the Adler function reads

$$d^{(1)}(Q^2) = d_1 a_s^{(1)}(Q^2), \qquad d_1 = \frac{4}{\beta_0}, \qquad a_s^{(1)}(Q^2) = \frac{1}{\ln z}, \qquad z = \frac{Q^2}{\Lambda^2}.$$

One–loop spectral function acquires the form

$$\rho^{(1)}(\sigma) = d_1 \bar{\rho}_1^{(1)}(\sigma), \qquad \bar{\rho}_1^{(1)}(\sigma) = \frac{1}{y^2 + \pi^2}, \qquad y = \ln\left(\frac{\sigma}{\Lambda^2}\right).$$

Resulting expression for the R–ratio at the one–loop level:

$$R^{(1)}(s) = 1 + r^{(1)}(s), \quad r^{(1)}(s) = d_1 A_{\mathrm{TL},1}^{(1)}(s), \quad A_{\mathrm{TL},1}^{(1)}(s) = \frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{\ln w}{\pi}\right), \quad w = \frac{s}{\Lambda^2}.$$

Expression for $A_{\mathrm{TL},1}^{(1)}(s)$ first appeared in: ■ Schrempp, Schrempp, Z. Phys. C6, 7 (1980).

All basic ideas were given in: ■ Radyushkin, report JINR E2–82–159 (1982), hep-ph/9907228.

The function $A_{\mathrm{TL},1}^{(1)}(s)$ was also reported in:

■ Pivovarov, Nuovo Cim. A105, 813 (1992); Jones, Solovtsov, Phys. Lett. B349, 519 (1995).

At the ℓ–loop level the function $R^{(\ell)}(s)$ can be represented as

$$R^{(\ell)}(s) = 1 + r^{(\ell)}(s), \qquad r^{(\ell)}(s) = \int_s^\infty \rho^{(\ell)}(\sigma)\frac{d\sigma}{\sigma}.$$

For the illustrative purposes it is convenient to cast $r^{(\ell)}(s)$ into

$$r^{(\ell)}(s) = \sum_{j=1}^{\ell} d_j \int_s^\infty \bar{\rho}_j^{(\ell)}(\sigma)\frac{d\sigma}{\sigma} = \sum_{j=1}^{\ell} d_j\, A_{\mathrm{TL},\,j}^{(\ell)}(s)$$

■ Milton, Shirkov, Solovtsov (1997), (2007); Howe, Brooks, Maxwell (2002), (2004), (2006).

The perturbative expression for the ℓ–loop spectral function reads

$$\rho^{(\ell)}(\sigma) = \sum_{j=1}^{\ell} d_j\, \bar{\rho}_j^{(\ell)}(\sigma), \qquad \bar{\rho}_j^{(\ell)}(\sigma) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\left\{\left[a_{\mathrm{s}}^{(\ell)}(-\sigma - i\varepsilon)\right]^j - \left[a_{\mathrm{s}}^{(\ell)}(-\sigma + i\varepsilon)\right]^j\right\}.$$

At the first few loop levels the explicit form of the spectral function $\rho^{(\ell)}(\sigma)$ can be calculated directly:

■ Nesterenko (2003); Baldicchi, Nesterenko, Prosperi, Simolo (2008); Nesterenko, Simolo (2010).

At the higher loop levels the function $\rho^{(\ell)}(\sigma)$ becomes rather cumbrous.

Explicit expression for $\rho^{(\ell)}(\sigma)$ is obtained at an arbitrary loop level:

$$\rho^{(\ell)}(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[d^{(\ell)}(-\sigma - i\varepsilon) - d^{(\ell)}(-\sigma + i\varepsilon) \Big] = \sum_{j=1}^{\ell} d_j \, \bar{\rho}_j^{(\ell)}(\sigma) =$$

$$= \sum_{j=1}^{\ell} d_j \sum_{k=0}^{K(j)} \binom{j}{2k+1} (-1)^k \, \pi^{2k} \left[\sum_{n=1}^{\ell} \sum_{m=0}^{n-1} b_n^m \, u_n^m(\sigma) \right]^{j-2k-1} \left[\sum_{n=1}^{\ell} \sum_{m=0}^{n-1} b_n^m \, v_n^m(\sigma) \right]^{2k+1},$$

that makes the (numerical) calculation of the R–ratio at the higher–loop levels easily accessible ■ Nesterenko, Eur. Phys. J. C77, 844 (2017).

In this equation ℓ is the loop level, $d_j = \Pi_{j,1}$,

$$u_n^m(\sigma) = \begin{cases} u_n^0(\sigma), & \text{if } m = 0, \\ u_n^0(\sigma) u_0^m(\sigma) - \pi^2 v_n^0(\sigma) v_0^m(\sigma), & \text{if } m \geq 1, \end{cases}$$

$$v_n^m(\sigma) = \begin{cases} v_n^0(\sigma), & \text{if } m = 0, \\ v_n^0(\sigma) u_0^m(\sigma) + u_n^0(\sigma) v_0^m(\sigma), & \text{if } m \geq 1, \end{cases}$$

$$v_0^m(\sigma) = \sum_{k=0}^{K(m)} \binom{m}{2k+1} (-1)^{k+1} \pi^{2k} \Big[L_1(y)\Big]^{m-2k-1} \Big[L_2(y)\Big]^{2k+1},$$

$$u_0^m(\sigma) = \sum_{k=0}^{K(m+1)} \binom{m}{2k} (-1)^{k} \pi^{2k} \Big[L_1(y)\Big]^{m-2k} \Big[L_2(y)\Big]^{2k},$$

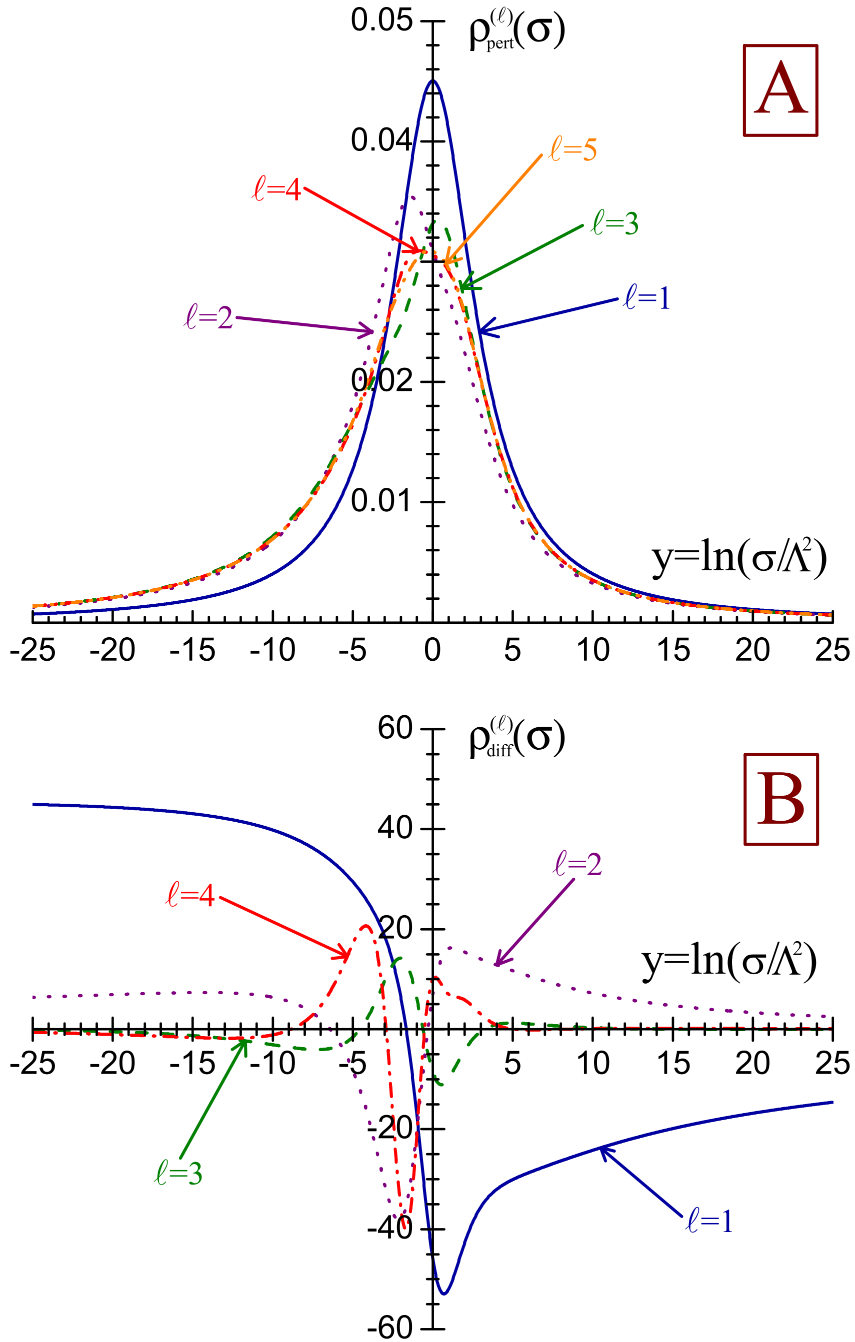
$$v_n^0(\sigma) = \frac{1}{(y^2+\pi^2)^n} \sum_{k=0}^{K(n)} \binom{n}{2k+1} (-1)^k \pi^{2k} y^{n-2k-1}, \qquad L_1(y) = \ln\sqrt{y^2+\pi^2},$$

$$u_n^0(\sigma) = \frac{1}{(y^2+\pi^2)^n} \sum_{k=0}^{K(n+1)} \binom{n}{2k} (-1)^k \pi^{2k} y^{n-2k}, \qquad L_2(y) = \frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{y}{\pi}\right),$$

$$K(n) = \frac{n-2}{2} + \frac{n \bmod 2}{2}, \qquad \binom{n}{m} = \frac{n!}{m!\,(n-m)!}, \qquad y = \ln\left(\frac{\sigma}{\Lambda^2}\right),$$

and Λ is the QCD scale parameter

■ Nesterenko, Eur. Phys. J. C77, 844 (2017).



Function $\rho^{(4)}_{\rm diff}(\sigma)$ is scaled by the factor of 10

The spectral function $\rho^{(\ell)}(\sigma)$ is remarkably stable with respect to the higher–loop corrections. In particular, the range of $y = \ln(\sigma/\Lambda^2)$, where the difference between $\rho^{(\ell)}(\sigma)$ and $\rho^{(\ell+1)}(\sigma)$ is sizable, is located in the vicinity of $y = 0$ and becomes smaller at larger ℓ. This fact eventually leads to an enhanced higher–loop stability of the resulting expression for the R–ratio at moderate and low energies.

plot A: $[\,\rho^{(\ell)}(\sigma),\ \ell = 1\ldots5\,]$

plot B: $[\,\rho^{(\ell)}_{\rm diff}(\sigma),\ \ell = 1\ldots4\,]$

$$\rho^{(\ell)}_{\rm diff}(\sigma) = \left[1 - \frac{\rho^{(\ell)}(\sigma)}{\rho^{(\ell+1)}(\sigma)}\right] \times 100\%$$

■ Nesterenko, Eur. Phys. J. C77, 844 (2017).

R–RATIO ("METHOD II",   EQUIVALENT TO "METHOD I")

$$\Pi(q^2,\mu^2,a_s) \longrightarrow D(Q^2,\mu^2,a_s) \xrightarrow[\mu^2 = Q^2]{\text{proper RG summation in spacelike domain}} D(Q^2) \longrightarrow \Delta\Pi(-Q^2,-Q_0^2) \longrightarrow R(s)$$

$$\left[-\frac{d\,\Pi(-Q^2)}{d\ln Q^2} = D(Q^2)\right] \qquad \left[-\int_{Q_0^2}^{Q^2} D(\xi)\,\frac{d\xi}{\xi} = \Delta\Pi(-Q^2,\,-Q_0^2)\right]\left[\frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\Big[\Pi(s+i\varepsilon)-\Pi(s-i\varepsilon)\Big] = R(s)\right]$$

This method proves to be more appealing than the previous one. Specifically, the derived expression for the hadronic vacuum polarization function $\Delta\Pi^{(\ell)}(-Q^2,-Q_0^2)$ (see page 13) and the technique developed for the calculation of the spectral function $\rho^{(\ell)}(\sigma)$ (see pages 17 and 18) make it possible to obtain the explicit expression for the R–ratio, which properly accounts for all the effects due to continuation of the spacelike perturbative results into the timelike domain and, being valid at an arbitrary loop level, can easily be employed in practical applications.

Explicit expression for $R^{(\ell)}(s)$ is obtained at an arbitrary loop level:

$$R^{(\ell)}(s) = 1 + r^{(\ell)}(s), \qquad r^{(\ell)}(s) = \sum\nolimits_{j=1}^{\ell} d_j\, A^{(\ell)}_{\mathrm{TL},j}(s),$$

$$A^{(\ell)}_{\mathrm{TL},j}(s) = \sum_{n_1=1}^{\ell} \dots \sum_{n_j=1}^{\ell} \sum_{m_1=0}^{n_1-1} \dots \sum_{m_j=0}^{n_j-1} \left(\prod_{i=1}^{j} b_{n_i}^{m_i} \right) T\left(s, \sum_{i=1}^{j} n_i, \sum_{i=1}^{j} m_i \right),$$

$$T(s,n,m) = \begin{cases} -V_0^1(s), & \text{if } n = 1 \text{ and } m = 0, \\ \displaystyle\sum_{k=0}^{m} \frac{m!}{k!} (n-1)^{k-m-1}\, V_{n-1}^{k}(s), & \text{if } n \geq 2, \end{cases}$$

$$V_n^m(s) = \begin{cases} 0, & \text{if } n = 0 \text{ and } m = 0, \\ v_0^m(s), & \text{if } n = 0 \text{ and } m \geq 1, \\ v_n^0(s), & \text{if } n \geq 1 \text{ and } m = 0, \\ v_n^0(s) u_0^m(s) + u_n^0(s) v_0^m(s), & \text{if } n \geq 1 \text{ and } m \geq 1, \end{cases}$$

where the functions $v_n^0(s)$, $u_0^m(s)$, $u_n^0(s)$, $v_0^m(s)$ are defined on page 18

■ Nesterenko, Eur. Phys. J. C77, 844 (2017); J. Phys. G46, 115006 (2019).

R–RATIO ("METHOD III", APPROXIMATE TO "METHOD I" AND "METHOD II")

$$\Pi(q^2, \mu^2, a_s) \;\longrightarrow\; R(s, \mu^2, a_s) \;\xrightarrow[\mu^2 = |s|]{\text{incomplete RG summation in timelike domain}}\; R_{\mathrm{appr}}(s)$$

$$\left[\frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\Big[\Pi(s+i\varepsilon) - \Pi(s-i\varepsilon)\Big] = R(s)\right]$$

However, the assignment of the renormalization scale $\mu^2 = |s|$ factually amounts to an incomplete RG summation in the timelike domain

- Pennington, Ross (1977), (1981), (1984); Pivovarov (1992).

This method eventually yields

$$R^{(\ell)}_{\mathrm{appr}}(s) = 1 + r^{(\ell)}_{\mathrm{appr}}(s), \qquad r^{(\ell)}_{\mathrm{appr}}(s) = \sum_{j=1}^{\ell} r_j \left[a_s^{(\ell)}(|s|)\right]^j, \qquad r_j = d_j - \delta_j,$$

where

$$\delta_1 = 0, \qquad \delta_2 = 0, \qquad \delta_j = \sum_{k=1}^{K(j)} (-1)^{k+1}\pi^{2k}\Pi_{j,2k+1}, \qquad j \geq 3.$$

The coefficients embodying the retained π^2–terms in $R^{(\ell)}_{\text{appr}}(s)$ read

$$\delta_1 = \delta_2 = 0, \quad \delta_3 = \frac{\pi^2}{3} d_1, \quad \delta_4 = \frac{\pi^2}{3}\left(\frac{5}{2} d_1 B_1 + 3 d_2\right), \quad \delta_5 = \frac{\pi^2}{3}\left[\frac{3}{2} d_1 \left(B_1^2 + 2B_2\right) + 7 d_2 B_1 + 6 d_3\right] - \frac{\pi^4}{5} d_1,$$

$$\delta_6 = \frac{\pi^2}{3}\left[\frac{7}{2} d_1 \left(B_1 B_2 + B_3\right) + 4 d_2 \left(B_1^2 + 2B_2\right) + \frac{27}{2} d_3 B_1 + 10 d_4\right] - \frac{\pi^4}{5}\left(\frac{77}{12} d_1 B_1 + 5 d_2\right),$$

$$\delta_7 = \frac{\pi^2}{3}\left[4 d_1\left(B_1 B_3 + \frac{1}{2}B_2^2 + B_4\right) + 9 d_2\left(B_1 B_2 + B_3\right) + \frac{15}{2} d_3\left(B_1^2 + 2B_2\right) + 22 d_4 B_1 + 15 d_5\right] - \frac{\pi^4}{5}\left[\frac{5}{6} d_1\left(17 B_1^2 + 12 B_2\right) + \frac{57}{2} d_2 B_1 + 15 d_3\right] + \frac{\pi^6}{7} d_1$$

- Bjorken (1989); Kataev, Starshenko (1995); Prosperi, Raciti, Simolo (2007); Nesterenko (2017).

Explicit expression for δ_j is obtained at an arbitrary loop level:

$$\delta_j = -\sum_{k=1}^{K(j)} \frac{(-2\pi^2)^k}{(2k+1)!} \sum_{i_1=2(k-1)+1}^{j-2} \underbrace{\sum_{i_2=2(k-2)+1}^{i_1-2} \ldots \sum_{i_n=2(k-n)+1}^{i_{n-1}-2} \ldots \sum_{i_k=1}^{i_{k-1}-2}}_{(k-1)\text{ sums}} (j+i_1) i_1 \times$$

$$\times \underbrace{(i_1 + i_2) i_2 \times \ldots \times (i_{n-1} + i_n) i_n \times \ldots \times (i_{k-1} + i_k) i_k}_{(k-1)\text{ products}} \times$$

$$\times \mathfrak{B}_{j-i_1-2} \underbrace{\mathfrak{B}_{i_1-i_2-2} \ldots \mathfrak{B}_{i_{n-1}-i_n-2} \ldots \mathfrak{B}_{i_{k-1}-i_k-2}}_{(k-1)\text{ terms}} d_{i_k}, \quad d_j = \Pi_{j,1}, \quad \mathfrak{B}_n = \frac{1}{4}\sum_{i=0}^{n} B_i B_{n-i}, \quad j \geq 3$$

- Nesterenko, J. Phys. G47, 105001 (2020).

NOTES ON THE FOREGOING METHODS I, II, AND III

- The methods I and II are equivalent and yield the same result for $R(s)$
- Its re–expansion at high energies takes the form

$$R^{(\ell)}(s) = 1 + \sum_{j=1}^{\ell} d_j \left[a_s^{(\ell)}(|s|)\right]^j - \sum_{j=1}^{\ell} d_j \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\pi^{2n}}{(2n+1)!} \sum_{k_1=0}^{\ell-1} \dots \sum_{k_{2n}=0}^{\ell-1} \left(\prod_{p=1}^{2n} B_{k_p}\right) \times$$

$$\times \left[\prod_{t=0}^{2n-1} \left(j + t + k_1 + k_2 + \ldots + k_t\right)\right] \left[a_s^{(\ell)}(|s|)\right]^{j+2n+k_1+k_2+\ldots+k_{2n}}, \qquad \frac{\sqrt{s}}{\Lambda} > \exp\left(\frac{\pi}{2}\right) \simeq 4.81$$

- The j–th order contribution to $R^{(\ell)}(s)$ on the l.h.s. of this equation appears to be re–distributed over the higher–order terms on its r.h.s.
- This expression can provide a rather accurate approximation of the R–ratio for $\sqrt{s} > \Lambda \exp(\pi/2) \simeq 4.81\,\Lambda$, but only if the number of retained π^2–terms on its right–hand side is large enough
- Its truncation yields the result of method III, i.e., $R^{(\ell)}_{\mathrm{appr}}(s)$

■ Nesterenko, Eur. Phys. J. C77, 844 (2017); J. Phys. G46, 115006 (2019); J. Phys. G47, 105001 (2020).

π^2–TERMS RETAINED IN $R_{\text{appr}}(s)$

- $R_{\text{appr}}(s)$: is only valid for $\sqrt{s} > \Lambda \exp(\pi/2) \simeq 4.81\Lambda$; converges poorly when $\sqrt{s}$ approaches $\Lambda \exp(\pi/2)$; the SL ⟶ TL effects are partially accounted for by coefficients δ_j; contains infrared unphysical singularities
- The coefficients δ_j, which embody the contributions of the retained π^2–terms in $R_{\text{appr}}(s)$, can in no way be regarded as small corrections to the Adler function perturbative expansion coefficients d_j for $j \geq 3$
- On the contrary, the values of coefficients δ_j significantly exceed the values of respective perturbative coefficients d_j, thereby constituting the dominant contribution to the coefficients r_j in $R_{\text{appr}}(s)$
- The values of coefficients δ_j rapidly increase as the order j increases, that amplifies the higher–order terms in $R_{\text{appr}}(s)$, makes its loop convergence worse than that of $R(s)$, and raises resulting uncertainties of α_s and Λ

Adler function perturbative expansion coefficients d_j

n_f	d_1	d_2	d_3	d_4	d_5 (est.)
0	0.3636	0.2626	0.8772	2.3743	*5.40*
1	0.3871	0.2803	0.7946	2.1884	*4.70*
2	0.4138	0.3005	0.7137	2.1466	*3.74*
3	0.4444	0.3239	0.5593	1.9149	*2.52*
4	0.4800	0.3513	0.2868	1.3440	*1.16*
5	0.5217	0.3836	−0.1021	0.6489	*0.0256*
6	0.5714	0.4225	−0.7831	−0.8952	*0.267*

[SL] $D^{(\ell)}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left[a_s^{(\ell)}(Q^2)\right]^j, \qquad a_s^{(\ell)}(Q^2) = \alpha_s^{(\ell)}(Q^2)\,\frac{\beta_0}{4\pi}$

↓ [dispersion relations + re–expansion + truncation]

[TL] $R^{(\ell)}_{\mathrm{appr}}(s) = 1 + \sum_{j=1}^{\ell} r_j \left[a_s^{(\ell)}(|s|)\right]^j, \qquad r_j = d_j - \delta_j, \qquad \frac{\sqrt{s}}{\Lambda} > \exp\left(\frac{\pi}{2}\right) \simeq 4.81$

Coefficients δ_j embodying the contributions of π^2–terms retained in $R^{(\ell)}_{\text{appr}}(s)$

n_f	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7 (est.)
0	0.0000	0.0000	1.1963	5.1127	20.455	69.081	45.7
1	0.0000	0.0000	1.2735	5.4298	18.880	56.819	7.02
2	0.0000	0.0000	1.3613	5.7583	17.118	48.532	−35.7
3	0.0000	0.0000	1.4622	6.0851	13.519	30.365	−82.5
4	0.0000	0.0000	1.5791	6.3850	6.910	−3.843	−115.7
5	0.0000	0.0000	1.7165	6.6090	−3.187	−45.692	−83.0
6	0.0000	0.0000	1.8799	6.6638	−21.168	−120.010	142.5

[SL] $$D^{(\ell)}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left[a_s^{(\ell)}(Q^2)\right]^j, \qquad a_s^{(\ell)}(Q^2) = \alpha_s^{(\ell)}(Q^2)\,\frac{\beta_0}{4\pi}$$

↓ [dispersion relations + re–expansion + truncation]

[TL] $$R^{(\ell)}_{\text{appr}}(s) = 1 + \sum_{j=1}^{\ell} r_j \left[a_s^{(\ell)}(|s|)\right]^j, \qquad r_j = d_j - \delta_j, \qquad \frac{\sqrt{s}}{\Lambda} > \exp\left(\frac{\pi}{2}\right) \simeq 4.81$$

R–ratio perturbative expansion coefficients r_j

n_f	$r_1 = d_1$	$r_2 = d_2$	$r_3 = d_3 - \delta_3$	$r_4 = d_4 - \delta_4$	$r_5 = d_5 - \delta_5$
0	0.3636	0.2626	−0.3191	−2.7383	−15.1
1	0.3871	0.2803	−0.4788	−3.2413	−14.2
2	0.4138	0.3005	−0.6476	−3.6116	−13.4
3	0.4444	0.3239	−0.9028	−4.1703	−11.0
4	0.4800	0.3513	−1.2923	−5.0409	−5.75
5	0.5217	0.3836	−1.8186	−5.9601	3.21
6	0.5714	0.4225	−2.6630	−7.5590	21.4

[SL] $$D^{(\ell)}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left[a_s^{(\ell)}(Q^2)\right]^j, \qquad a_s^{(\ell)}(Q^2) = \alpha_s^{(\ell)}(Q^2)\,\frac{\beta_0}{4\pi}$$

↓ [dispersion relations + re–expansion + truncation]

[TL] $$R_{\text{appr}}^{(\ell)}(s) = 1 + \sum_{j=1}^{\ell} r_j \left[a_s^{(\ell)}(|s|)\right]^j, \qquad r_j = d_j - \delta_j, \qquad \frac{\sqrt{s}}{\Lambda} > \exp\left(\frac{\pi}{2}\right) \simeq 4.81$$

Relative weight of π^2–terms in the coefficients r_j

n_f	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$ (est.)
0	0.00 %	0.00 %	57.7 %	68.3 %	*79.1 %*
1	0.00 %	0.00 %	61.6 %	71.3 %	*80.1 %*
2	0.00 %	0.00 %	65.6 %	72.8 %	*82.1 %*
3	0.00 %	0.00 %	72.3 %	76.1 %	*84.3 %*
4	0.00 %	0.00 %	84.6 %	82.6 %	*85.6 %*
5	0.00 %	0.00 %	94.4 %	91.1 %	*99.2 %*
6	0.00 %	0.00 %	70.6 %	88.2 %	*98.8 %*

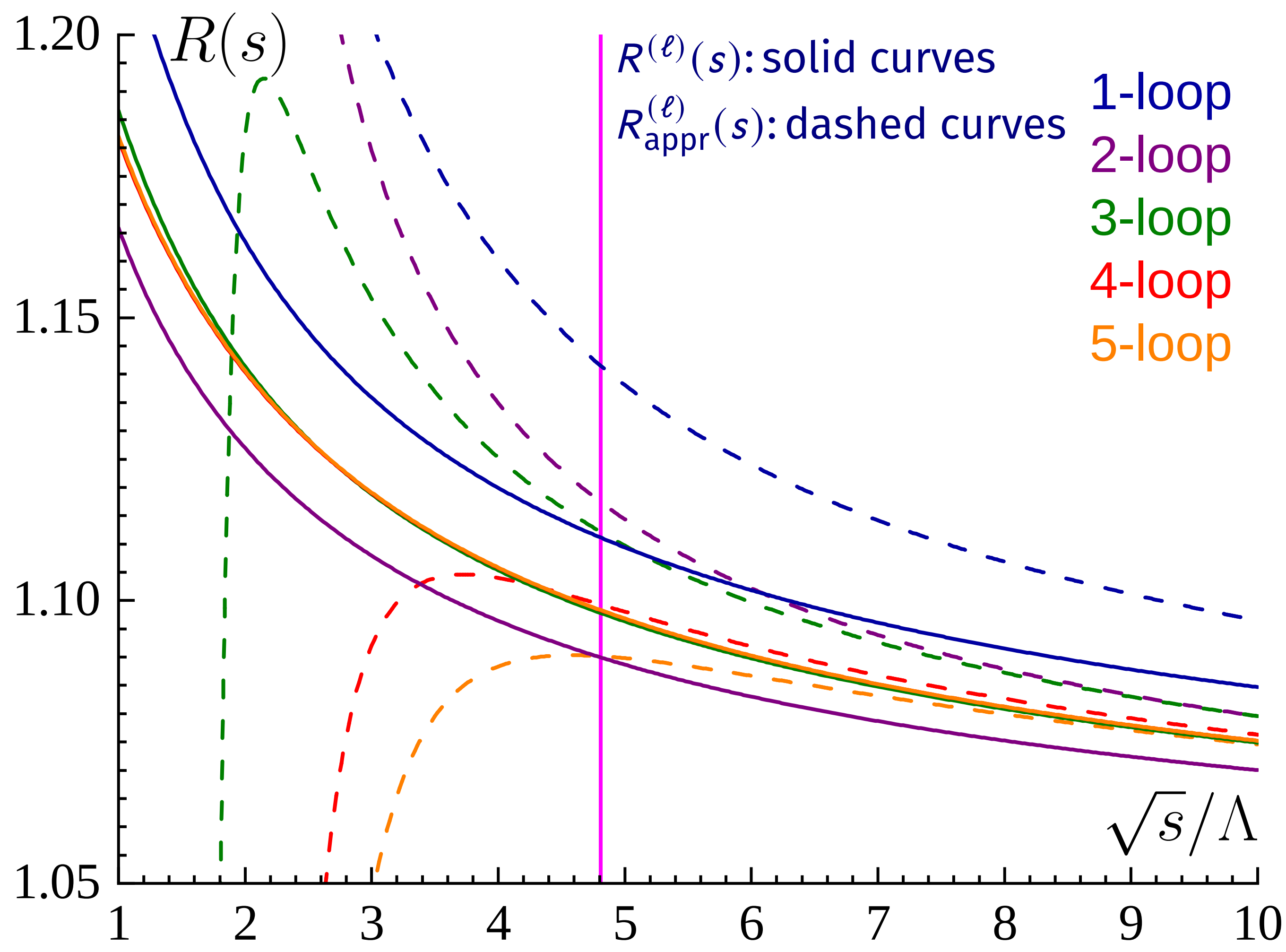
[SL] $D^{(\ell)}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left[a_s^{(\ell)}(Q^2)\right]^j, \qquad a_s^{(\ell)}(Q^2) = \alpha_s^{(\ell)}(Q^2)\,\frac{\beta_0}{4\pi}$

$\downarrow$ [dispersion relations + re–expansion + truncation]

[TL] $R_{\text{appr}}^{(\ell)}(s) = 1 + \sum_{j=1}^{\ell} r_j \left[a_s^{(\ell)}(|s|)\right]^j, \qquad r_j = d_j - \delta_j, \qquad \frac{\sqrt{s}}{\Lambda} > \exp\left(\frac{\pi}{2}\right) \simeq 4.81$

EFFECT OF THE π^2–TERMS OMITTED IN $R^{(\ell)}_{\mathrm{appr}}(s)$

The higher–order terms in $R^{(\ell)}_{\mathrm{appr}}(s)$ appear to be amplified, that makes its loop convergence worse than that of $R^{(\ell)}(s)$. In particular, even at $s = M_Z^2$ at the four–loop level ($\ell = 4$) the 3rd– and 4th–order terms of $R^{(4)}_{\mathrm{appr}}(s)$ comprise 34.2 % and 8.1 % of its 2nd–order term, whereas the 3rd– and 4th–order terms of



$R^{(4)}(s)$ comprise only 1.8 % and 0.8 % of its 2nd–order term

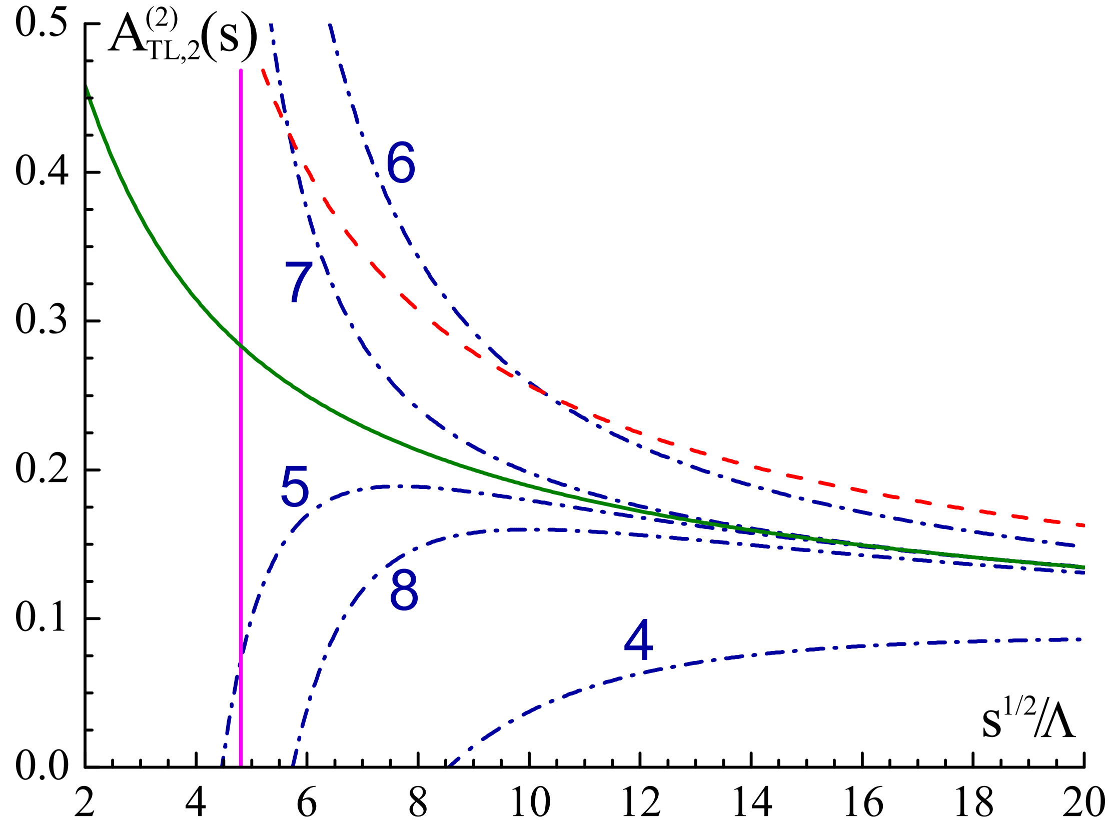
■ Nesterenko, Eur. Phys. J. C77, 844 (2017); J. Phys. G46, 115006 (2019).

Continuation of the two–loop strong running coupling squared into the timelike domain:

$$\text{[SL]} \quad \left[a_s^{(2)}(Q^2)\right]^2 \longrightarrow A_{\mathrm{TL},2}^{(2)}(s) \quad \text{[TL]}$$

Its re–expansion for $\sqrt{s} > \Lambda \exp(\pi/2)$:

$$A_{\mathrm{TL},2}^{(2)}(s) \simeq \left[a_s^{(2)}(|s|)\right]^2 - \frac{\pi^2}{\ln^4 w} + \frac{\pi^2}{\ln^5 w} B_1\left(4\ln\ln w - \frac{7}{3}\right) + O\left(\frac{1}{\ln^6 w}\right).$$



[the functions are scaled by the factor of 10]

However, at the two–loop level all π^2–terms are truncated in $R_{\mathrm{appr}}(s)$, that gives a rather large error even at high energies. For example, $\left[a_s^{(2)}(|s|)\right]^2 \simeq 1.21\, A_{\mathrm{TL},2}^{(2)}(s)$ at $\sqrt{s} = 20\,\Lambda$, and to securely achieve 10 % accuracy one needs to include all the π^2–terms up to $\ln^{-7} w$, $w = s/\Lambda^2$

■ Nesterenko, Eur. Phys. J. C77, 844 (2017).

Extraction of α_s and Λ from experimental data on R–ratio

loop level	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
$R^{(\ell)}(s)$ employed (proper RG sum in SL)					
$\alpha_s^{(\ell)}(\lvert s_0\rvert)$	0.3283	0.3168	0.2955	0.2955	0.2924
$\Lambda^{(\ell)}$ [MeV]	238	417	336	331	331
$R^{(\ell)}_{\text{appr}}(s)$ employed (incomplete RG sum in TL)					
$\bar{\alpha}_s^{(\ell)}(\lvert s_0\rvert)$	0.2827	0.2501	0.2655	0.2881	0.3278
$\bar{\Lambda}^{(\ell)}$ [MeV]	169	263	269	315	408

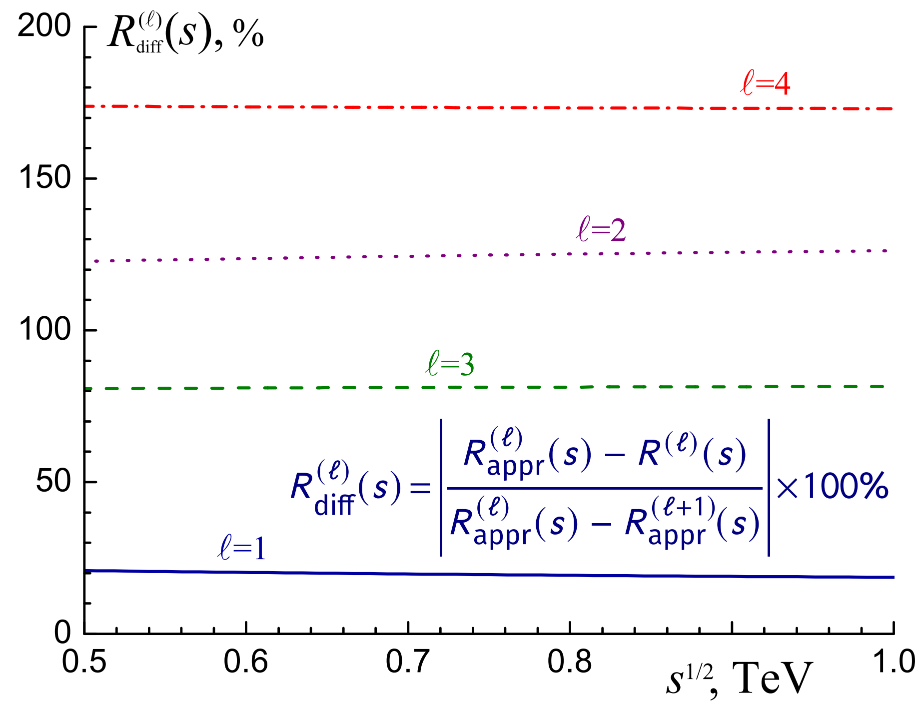
$R(s_0) = 2.18$, $\sqrt{s_0} = 2.0\,\text{GeV}$ ■ BES Collaboration, Phys. Rev. Lett. 88, 101802 (2002).

- 1st and 2nd lines: $R^{(\ell)}(s)$ employed, mild variation of the results for $\ell > 2$
- 3rd and 4th lines: $R^{(\ell)}_{\text{appr}}(s)$ employed, no sign of loop convergence

■ Nesterenko, J. Phys. G46, 115006 (2019).

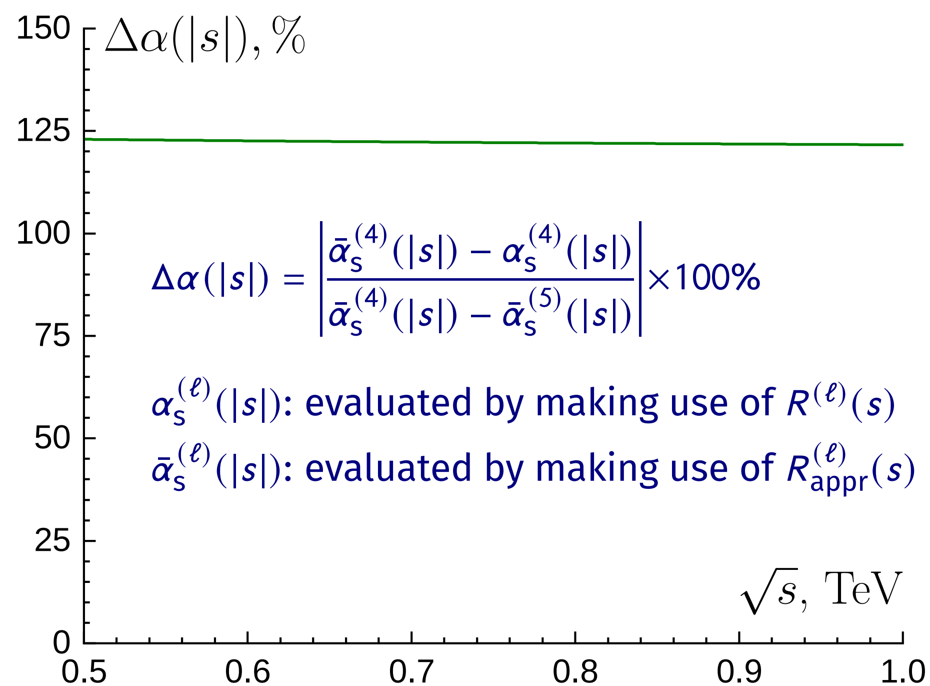
In the energy range planned for the ILC experiment the effect of inclusion of the π^2–terms omitted in $R^{(\ell)}_{\text{appr}}(s)$ appears to be either comparable to or even prevailing over the effect of inclusion of the next–order perturbative correction

- Nesterenko, Eur. Phys. J. C77, 844 (2017).



In this energy range the effect of inclusion of the π^2–terms ignored in $R^{(4)}_{\text{appr}}(s)$ on the resulting value of the strong running coupling is steadily prevailing over the effect of inclusion of the five–loop perturbative correction into $R_{\text{appr}}(s)$

- Nesterenko, J. Phys. G46, 115006 (2019).



SUMMARY

- The calculation of the R–ratio of electron–positron annihilation into hadrons from the hadronic vacuum polarization function is studied within various methods.
- The RG relations for the higher–order hadronic vacuum polarization function perturbative expansion coefficients $\Pi_{j,k}$ are obtained in the folded recurrent and unfolded explicit forms, that can also be employed as an independent crosscheck of the perturbative calculations and in the studies of the renormalization scale setting.
- The explicit expression for the hadronic vacuum polarization function with properly eliminated dependence on the renormalization scale is obtained at an arbitrary loop level.

- The explicit expression for the perturbative spectral function is obtained at an arbitrary loop level, that substantially facilitates the (numerical) calculation of the R–ratio within the method I.

- The explicit expression for the R–ratio, which properly accounts for all the effects due to continuation of the spacelike perturbative results into the timelike domain, is obtained at an arbitrary loop level within the method II.

- It is shown that the methods I and II are equivalent and yield the same result for the R–ratio [i.e., $R(s)$], whereas its truncated re–expansion at high energies is identical to the result of the method III [i.e., $R_{\text{appr}}(s)$].

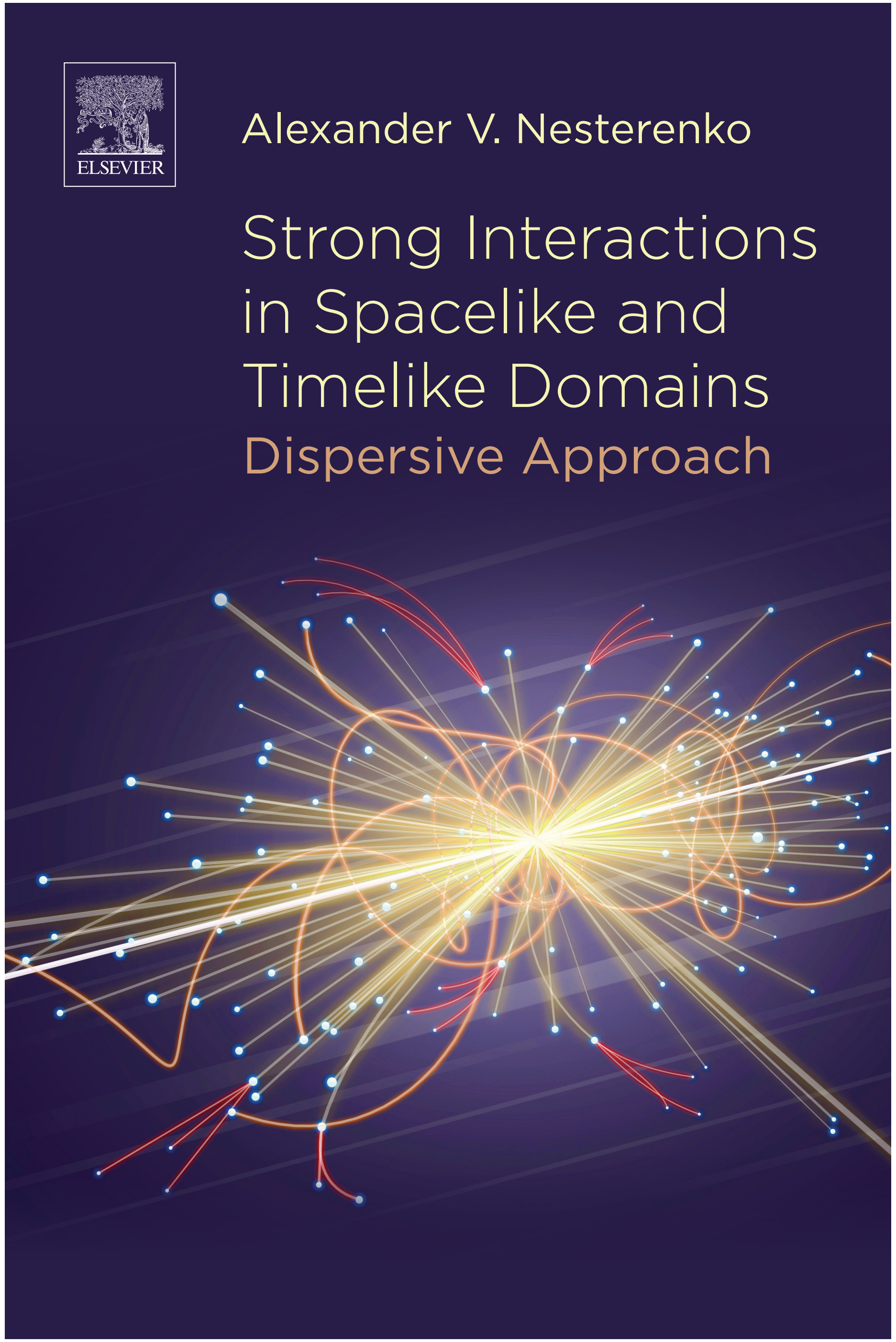
- The explicit expression for the coefficients δ_j, which embody the contributions of the π^2–terms retained in $R_{\text{appr}}(s)$, is derived at an arbitrary loop level.

- It is shown that the j–th order contribution to $R(s)$ is re–distributed over the higher–order terms of $R_{\text{appr}}(s)$, thereby substantially amplifying them. Eventually this makes the loop convergence of $R_{\text{appr}}(s)$ worse than that of $R(s)$ and increases the resulting uncertainty of α_s and Λ associated with the discarded higher–loop perturbative corrections.

- It is shown that the validity range of $R_{\text{appr}}(s)$ is strictly limited to $\sqrt{s} > \Lambda \exp(\pi/2) \simeq 4.81\,\Lambda$ and it converges poorly for $\sqrt{s} \sim \Lambda \exp(\pi/2)$.

- It is shown that the SL $\longrightarrow$ TL effects are only partially accounted for by the coefficients δ_j, which embody the contributions of the retained π^2–terms in $R_{\text{appr}}(s)$.

- It is shown that the higher–order π^2–terms omitted in $R_{\text{appr}}(s)$ can produce a considerable effect on the determination of α_s and Λ from the experimental data on the R–ratio.



The detailed discussion of the interplay between the strong interactions in spacelike and timelike domains, the essentials of the Dispersively improved perturbation theory, and many other closely related topics can be found in the monograph

A.V. Nesterenko

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