Dipole fringing fields

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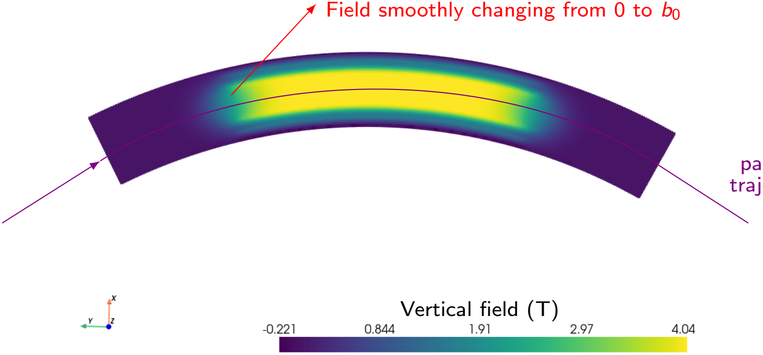
Motivation

The HITRIPlus Canted Cosine Theta magnet



Large *fringe field* region:
Field smoothly changing from 0 to b_0

particle
trajectory

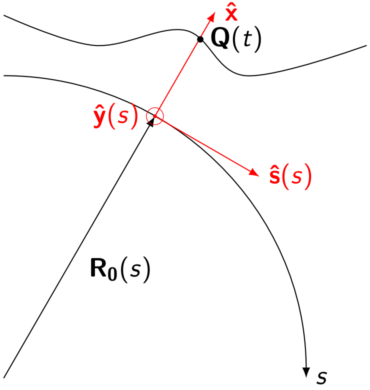
E. De Matteis et al., 2023

Strategy

The fringe field map in PTC

- Describe the field with magnetic field expansion
- Find the appropriate Hamiltonian
- Define a thin fringe field map
- Calculate the effect of the fringe field on p_x and p_y
- Create a symplectic map that has this effect
  (which will induce a displacement in x and y)
- Comparison with SLAC-75 and MAD-8 / MAD-X

Frenet Serret Coordinates



- Unit vectors of the frame

$$\hat{\mathbf{s}}(s) = \mathbf{R_0}'(s)$$
$$\hat{\mathbf{x}}(s) = -\frac{1}{h(s)}\hat{\mathbf{s}}'(s)$$
$$\hat{\mathbf{y}}(s) = -\hat{\mathbf{x}}(s) \times \hat{\mathbf{s}}(s)$$

- Position of a particle

$$\mathbf{Q}(t) = \mathbf{R_0}(s) + x\hat{\mathbf{x}}(s) + y\hat{\mathbf{y}}(s)$$

Magnetic field expansion

The magnetic scalar potential $\phi(x, y, s)$ as general expansion in y

- Magnetic scalar potential

$$\mathbf{B} = -\boldsymbol{\nabla}\phi$$

- General expansion in y

$$\phi(x, y, s) = \sum_{i=0}^{\infty} \phi_i(x, s) \frac{y^i}{i!}$$

$$\Big\downarrow \ \text{Laplace equation } \nabla^2\phi = 0$$

$$\phi_{i+2} = -\frac{1}{1+hx}\left(\partial_x\left((1+hx)\partial_x\phi_i\right) + \partial_s\left(\frac{1}{1+hx}\partial_s\phi_i\right)\right)$$

S. D. Fartoukh, 1997; C. Wang and L. Teng, 2001

Magnetic field expansion

Expansion of the initial functions $\phi_0(x,s)$ and $\phi_1(x,s)$

- Two initial functions $\phi_0(x,s)$ and $\phi_1(x,s)$ can be independently chosen

$$\phi_0(x,s) = -a_0(s) - \sum_{n=1}^{\infty} a_n(s)\frac{x^n}{n!}$$

$$\phi_1(x,s) = -\sum_{n=1}^{\infty} b_n(s)\frac{x^{n-1}}{(n-1)!}$$

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- Apply the formula to determine ϕ_2

$$\phi_2(x,s) = -\frac{1}{1+hx}\left(\partial_x\left((1+hx)\partial_x\phi_0\right) + \partial_s\left(\frac{1}{1+hx}\partial_s\phi_0\right)\right)$$

$$= \tfrac{h}{1+hx}\textstyle\sum_{n=1}^{\infty} a_n \tfrac{x^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty} a_n \tfrac{x^{n-2}}{(n-2)!} - \tfrac{h' x}{(1+hx)^3}\sum_{n=0}^{\infty} a_n'(s)\tfrac{x^n}{n!} + \tfrac{1}{(1+hx)^2}\sum_{n=0}^{\infty} a_n''(s)\tfrac{x^n}{n!}$$

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- Successively determine next ϕ_i and calculate ϕ

S. D. Fartoukh, 1997; C. Wang and L. Teng, 2001

Magnetic field expansion

Relation with the magnetic field $\mathbf{B} = -\boldsymbol{\nabla}\phi$: first terms

	1	x	y	x^2	xy	y^2
B_x	a_1	a_2	b_2	$\frac{a_3}{2}$	b_3	$\frac{-a_3-h(a_2-a_1h-2b_s')+b_sh'-a_1''}{2}$
B_y	b_1	b_2	$-b_s'-a_1h-a_2$	$\frac{b_3}{2}$	$-a_3-h(a_2-a_1h-2b_s')+b_sh'-a_1''$	$-\frac{b_3+hb_2+b_1''}{2}$
B_s	b_s	$-b_sh+a_1'$	b_1'	$b_sh^2-a_1'h+\frac{a_2'}{2}$	$-hb_1'+b_2'$	$-\frac{a_1h'+ha_1'+b_s''+a_2'}{2}$

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B_y	b_1	b_2	$-b_s' - a_1h - a_2$	$\frac{b_3}{2}$	$-a_3 - h(a_2 - a_1h - 2b_s') + b_sh' - a_1''$	$-\frac{b_3+hb_2+b_1''}{2}$
B_s	b_s	$-b_sh + a_1'$	b_1'	$b_sh^2 - a_1'h + \frac{a_2'}{2}$	$-hb_1' + b_2'$	$-\frac{a_1h'+ha_1'+b_s''+a_2'}{2}$

Limit to a straight reference
frame with s-independent fields

$$h = 0,\ \partial_s = 0$$

	1	x	y	x^2	xy	y^2
B_x	a_1	a_2	b_2	$\frac{a_3}{2}$	b_3	$-\frac{a_3}{2}$
B_y	b_1	b_2	$-a_2$	$\frac{b_3}{2}$	$-a_3$	$-\frac{b_3}{2}$
B_s	b_s	0	0	0	0	0

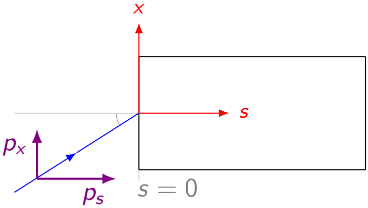
The coefficients a_n, b_n fall back to the usual multipole expansion for straight magnets with transverse-only fields

S. D. Fartoukh, 1997; C. Wang and L. Teng, 2001

Fringe field map: the simplest situation

Straight bend with flat pole face

Simplest situation



- Flat pole face
- Vertical field independent of x
- No horizontal field $B_x = 0$

→ We assume parallel plates with infinite transverse extent

→ Field described by a single coëfficient $b_1(s)$

→ Easiest to describe in straight reference frame

Forest et al., 2006

Straight bend with flat pole face

Simplest situation: fringe field expansion

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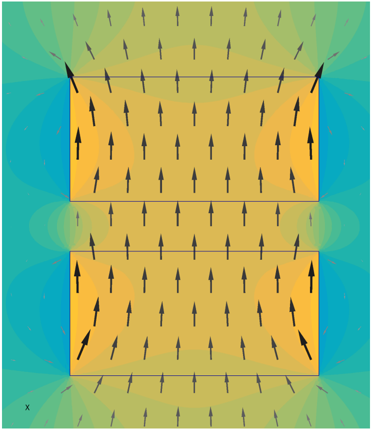
- Magnetic field

$$B_x = 0$$

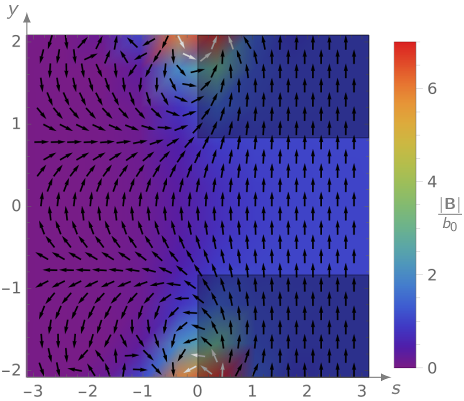
$$B_y = b_1(s) + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} b_1^{[2n]}(s)\, y^{2n}$$

$$B_s = -\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} b_1^{[2n-1]}(s)\, y^{2n-1}\,, \qquad b_1^{[n]}(s) = \frac{d^n}{ds^n} b_1(s)$$

Numerical example

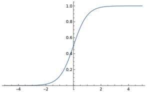


2D field simulation



Field expansion

$$b_1 = \frac{1}{2}\left(\tanh(s) + 1\right)$$



Strategy

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General Hamiltonian

With abuse of notation, we will from now on consider fields scaled by $q/p_0 = 1/B\rho$

General Hamiltonian for particle in an electromagnetic field

$$H = \frac{p_\tau}{\beta_0} - (1 + hx)\left(\sqrt{(1+\delta)^2 - (p_x - a_x)^2 - (p_y - a_y)^2} + a_s\right)$$

- Curvature h
- $\delta = (P - P_0)/P_0$
- $p_\tau = (E - E_0)/(P_0 c)$
- Vector potential $\mathbf{a}$, $\quad \mathbf{b} = \boldsymbol{\nabla} \times \mathbf{a}$

Straight bend with flat pole face

Vector potential and Hamiltonian

- Vector potential **A** $\quad (\mathbf{B} = \nabla \times \mathbf{A})$

$$\begin{aligned} a_x &= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} b_1^{[2n-1]}(s)\, y^{2n} \\ a_y &= 0 \\ a_z &= -b_1(s)x \end{aligned}$$

$$\begin{aligned} b_x &= 0 \\ b_y &= b_1(s) + \textstyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} b_1^{[2n]}(s)\, y^{2n} \\ b_s &= -\textstyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} b_1^{[2n-1]}(s)\, y^{2n-1} \end{aligned}$$

- Transverse Hamiltonian; $h = 0$

$$H_F = -\sqrt{(1+\delta)^2 - (p_x - a_x)^2 - p_y^2} + b_1(s)x$$

with

$$a_x = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} b_1^{[2n-1]}(s)\, y^{2n}\,, \quad b_1^{[n]} = \frac{d^n}{ds^n} b_1(s)$$

Forest et al., 2006

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What are compositional maps and Lie operators?

Definition of compositional maps

- A transfer map of phase space, the type of map used so far, is a map working on the coordinates $\mathbf{F} : (\mathbf{q}, \mathbf{p}) \mapsto \mathbf{F}(\mathbf{q}, \mathbf{p})$

Forest, Beam Dynamics
Dragt, Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics

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- To each of these maps, one can associate a compositional map, denoted with the same name as the map but in curly letters, and defined as

  $$\mathcal{F}g(s) = g(\mathbf{F}(s))$$

  for *any* function (e.g. maps) of phase space $g(\mathbf{q}, \mathbf{p})$

  → A compositional map transforms functions of phase space

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  → A compositional map transforms functions of phase space
- To recover the transfer map, one applies the compositional map to the identity map $g = I$

Forest, Beam Dynamics
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What are compositional maps and Lie operators?

Lie operators are compositional maps

- One can define a Lie operator $:f:$ associated to a function of phase space $f(\mathbf{q}, \mathbf{p})$

$$:f: g = \{f, g\} = \sum_{k=1}^{3} \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right)$$

for *any* function of phase space $g(\mathbf{q}, \mathbf{p})$

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- Lie operators belong to the space of compositional maps

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What are compositional maps and Lie operators?

Time evolution of compositional maps

- The time evolution of a compositional map $\mathcal{F}$ can be described with a differential equation

  $$\frac{d\mathcal{F}}{ds} = \mathcal{F} :-H_F:$$

  with H_F the associated Hamiltonian

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- If the Hamiltonian is independent of z, the solution is

  $$\mathcal{F} = e^{s\,:-H_F:}$$

  which is equivalent to Hamilton equations of motion

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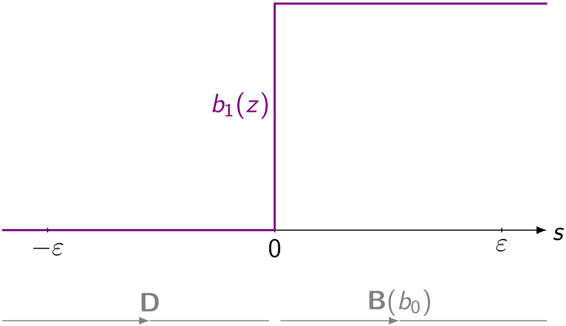
Fringe field map

An approximate thin fringe field map at position $s = 0$



Forest et al., 2006

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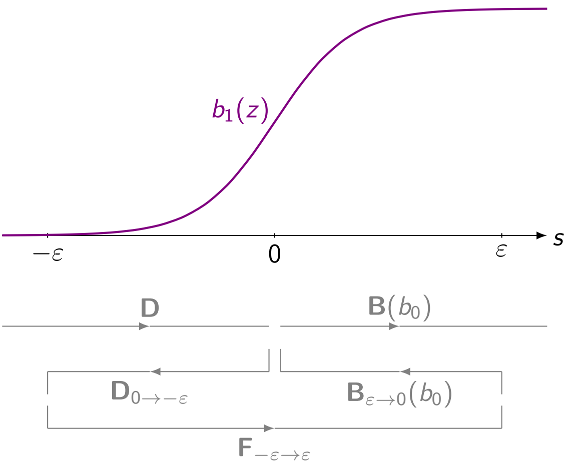

Forest et al., 2006

Lie operator approach

Simplifying the fringe field map **F**

- Split map in drift and remaining contribution $\mathbf{F} = \mathbf{D} \circ \mathbf{P}$

$$\begin{tikzcd}
\mathcal{P} = \mathcal{F}\mathcal{D}^{-1} \arrow[d, "\frac{d\mathcal{F}}{ds} = \mathcal{F} :-H_F: \, , \frac{d\mathcal{D}}{ds} = \mathcal{D} :-H_D:"] \\
\frac{d\mathcal{P}}{ds} = \mathcal{P}\mathcal{D} :-(H_F - H_D): \mathcal{D}^{-1}
\end{tikzcd}$$

Forest et al., 2006

Lie operator approach

Simplifying the fringe field map **F**

- Split map in drift and remaining contribution $\mathbf{F} = \mathbf{D} \circ \mathbf{P}$

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\end{tikzcd}$$

- The solution of this differential equation is a Dyson series or time ordered exponential

$$\mathcal{P}_{-\varepsilon\to s} = \mathcal{P}_{-\varepsilon\to-\varepsilon} + \int_{-\varepsilon}^{s} ds_1 \, \mathcal{P}_{-\varepsilon\to s_1}\mathcal{D}_{-\varepsilon\to s_1} :-(H_F - H_D): \mathcal{D}_{s_1\to-\varepsilon}$$

Forest et al., 2006

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- The solution of this differential equation is a Dyson series or time ordered exponential

$$\mathcal{P}_{-\varepsilon\to s} = \overset{=1}{\boxed{\mathcal{P}_{-\varepsilon\to -\varepsilon}}} + \int_{-\varepsilon}^{s} ds_1 \boxed{\mathcal{P}_{-\varepsilon\to s_1}} \mathcal{D}_{-\varepsilon\to s_1} :-(H_F - H_D): \mathcal{D}_{s_1\to -\varepsilon}$$

$$= 1 + \int_{-\varepsilon}^{s} ds_1\, \mathcal{D}_{-\varepsilon\to s_1} :-(H_F - H_D): \mathcal{D}_{s_1\to -\varepsilon}$$

$$+ \int_{-\varepsilon}^{s} ds_1 \int_{-\varepsilon}^{s_1} ds_2\, \mathcal{D}_{-\varepsilon\to s_2} :-(H_F - H_D): \mathcal{D}_{s_2\to -\varepsilon} \mathcal{D}_{-\varepsilon\to s_1} :-(H_F - H_D): \mathcal{D}_{s_1\to -\varepsilon} + \ldots$$

Forest et al., 2006

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&= 1 + \int_{-\varepsilon}^{s} ds_1\, \mathcal{D}_{-\varepsilon\to s_1} :-(H_F - H_D): \mathcal{D}_{s_1\to-\varepsilon} \\
&\quad + \int_{-\varepsilon}^{s} ds_1 \int_{-\varepsilon}^{s_1} ds_2\, \mathcal{D}_{-\varepsilon\to s_2} :-(H_F - H_D): \mathcal{D}_{s_2\to-\varepsilon} \mathcal{D}_{-\varepsilon\to s_1} :-(H_F - H_D): \mathcal{D}_{s_1\to-\varepsilon} + \ldots \\
&= \mathcal{T}\left(e^{\int_{-\epsilon}^{s} ds\; \mathcal{D} :-(H_F - H_D): \mathcal{D}^{-1}}\right)
\end{aligned}$$

Forest et al., 2006

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Calculation of Δp_x and Δp_y

Order by order in the thin fringe field map expansion

Putting all pieces together...
More details in appendix

- Lowest order $\Delta p_x = 0$
- First order contribution Δp_y

$$\Delta p_{y,1} = -\frac{x'}{1+y'^2} b_0 y$$

- Second order contribution Δp_y

$$\Delta p_{y,2} = \underbrace{\int_{-\infty}^{+\infty} b(s)(b_0 - b(s)) ds}_{g b_0^2 K} \left(\frac{(1+\delta)^2 - p_y^2}{p_s^3} + \frac{p_x^2}{p_s^2} \frac{(1+\delta)^2 - p_x^2}{p_s^3} \right) y$$

$$\boxed{K = \int_{-\infty}^{+\infty} \frac{b(s)(b_0 - b(s))}{g b_0^2} ds}$$

Fringe field integral

Forest et al., 2006

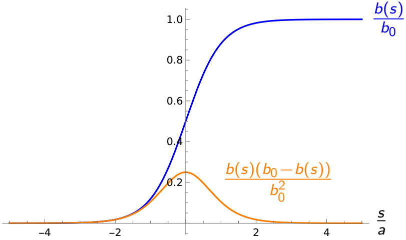
Interpretation of the fringe field integral

Toy model:

$$b(s) = \frac{b_0}{2}\left(\tanh\frac{s}{a} + 1\right)$$

The integral reduces to

$$K = \int_{-\infty}^{+\infty} \frac{b(s)(b_0 - b(s))}{g b_0^2} ds = \frac{a}{2g}$$



Dominated by the region where $b(s)$ and $(b_0 - b(s))$ are not small

- Dimensionless
- Linear in the range of the fringe field
- Independent of the total strength
- Ranges between 0 (hard edge) and ∞

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Generating function: A symplectic map with the correct effect on p_y

The symplectic map for a straight bend with flat pole face

$$F = p_x x_f + p_y y_f - \delta \ell_f - \frac{1}{2}\psi(p_x, p_y, \delta){y_f}^2$$

$$\psi = b_0 \tan\left[\arctan\left(\frac{x'}{1+y'^2}\right) - g b_0 K \left(1 + \frac{p_x^2}{p_s^2}\left(2 + \frac{p_y^2}{p_s^2}\right)\right) p_s\right]$$

$$\boxed{\begin{array}{ll}
x_f = x + \dfrac{1}{2}\dfrac{\partial\psi}{\partial p_x}{y_f}^2 & p_{x,f} = p_x \\[2ex]
y_f = \dfrac{2y}{1+\sqrt{1 - 2\frac{\partial\psi}{\partial p_y} y}} & p_{y,f} = p_y - \psi y_f + c\, y_f^3 \\[2ex]
\ell_f = \ell - \dfrac{1}{2}\dfrac{\partial\psi}{\partial\delta}{y_f}^2 & \delta_f = \delta
\end{array}}$$

Not derived

Forest et al., 2006

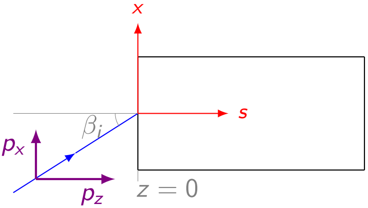
Strategy

The fringe field map in PTC

- Describe the field with magnetic field expansion
- Find the appropriate Hamiltonian
- Define a thin fringe field map
- Calculate the effect of the fringe field on p_x and p_y
- Create a symplectic map that has this effect
  (which will induce a displacement in x and y)
- Comparison with SLAC-75 and MAD-8 / MAD-X

Comparison with SLAC-75 and MAD-8 / MAD-X

Forest with conventions of MAD-X: A symplectic map



- $p_x = \sin\beta_i,\ p_s = \cos\beta_i$ $\rightarrow x' = \frac{p_x}{p_s} = \tan\beta_i$
- $p_y = 0 \rightarrow y' = 0$
- $\delta = 0$

$$\psi = b_0 \tan\left[\arctan\left(\frac{x'}{1+y'^2}\right) - g b_0 K \left(1 + x'^2\left(2 + y'^2\right)\right) p_s\right]$$

$$= b_0 \tan\left[\beta_i - g b_0 K \frac{1}{\cos\beta_i}\left(1 + \sin^2\beta_i\right)\right]$$

$$\boxed{p_{y,f} = p_y - b_0 \tan\left[\beta_i - g b_0 K \frac{1}{\cos\beta_i}\left(1+\sin^2\beta_i\right)\right] y_f}$$

Comparison with SLAC-75 and MAD-8 / MAD-X

MAD-X: A non-symplectic expansion around the origin

$$\overline{\beta_i} = \beta_i - g b_0 K \frac{1}{\cos\beta_i}(1 + \sin^2\beta_i)$$

Pole face rotation

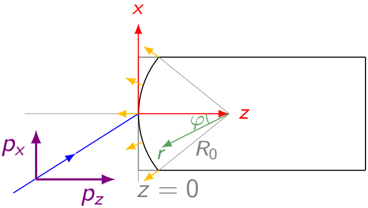
$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \boxed{b_0 \tan\beta_i} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{-b_0 \tan\overline{\beta_i}} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Flat pole face

Fringe field map: Straight bend with curved pole face

Straight bend with curved pole face

A symplectic map with the correct effect on p_x and p_y



- Curved pole face with bending radius R_0
- Straight reference frame
- Cylindrical symmetric pole face
- Pure vertical field for $y = 0$

Vector potential $\mathbf{a} = (a_\varphi, 0, a_r)$

$$a_r = -r\varphi b(r)$$

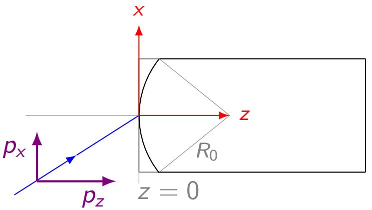
$$a_\varphi = \sum_{n=1}^{\infty} \frac{1}{2n} \frac{da_{n-1}}{dr} y^{2n} , \quad a_n = -\frac{1}{2n(2n+1)} \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} a_{n-1} , \quad q_0 = b(r)$$

$$H = -\sqrt{(1+\delta)^2 - (p_x + \cos\varphi a_\varphi)^2 - p_y^2} + \cos\varphi a_r - \sin\varphi a_\varphi - \frac{\partial}{\partial z} \int^x \sin\varphi a_r dx$$

Forest et al., 2006

Straight bend with curved pole face

A symplectic map with the correct effect on p_x and p_y



- Curved pole face with bending radius R_0
- Straight reference frame
- Cylindrical symmetric pole face
- Pure vertical field for $y = 0$

$$\Xi(p_x, \delta) = \frac{b_0}{2R_0} \frac{1+\delta}{\sqrt{(1+\delta)^2 - p_x^2}}$$

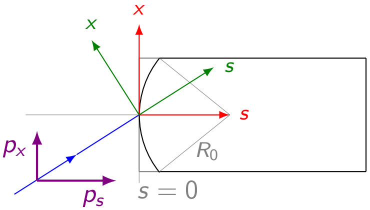
$$\boxed{p_{x,f} = p_x + \frac{b_0}{2R_0}}$$

$$\boxed{\begin{array}{ll} x_f = \dfrac{x}{1 - \frac{\partial \Xi}{\partial p_x} y^2} & p_{x,f} = p_x - \Xi y^2 \\ p_{y,f} = p_y - \Xi x_f y^2 & \ell_f = \ell - \dfrac{\partial \Xi}{\partial \delta} x_f y^2 \end{array}}$$

Forest et al., 2006

Comparison with SLAC-75 and MAD-8 / MAD-X

Forest with conventions of MAD-X: A symplectic map



- $p_x = \sin \beta_i ,\; p_s = \cos \beta_i$ $\rightarrow x' = \frac{p_x}{p_s} = \tan \beta_i$
- $p_y = 0 \rightarrow y' = 0$
- $\delta = 0$
- MAD-X in pipe frame

$$\boxed{p_{x,f} = \frac{1}{\cos^3 \beta_i} \frac{b_0}{2R_0} (x^2 - y^2) \qquad p_{y,f} = -\frac{1}{\cos^3 \beta_i} \frac{b_0}{2R_0} x_f y}$$

Comparison with SLAC-75 and MAD-8 / MAD-X

MAD-X: A non-symplectic expansion around the origin

This means the effect on x of order x^2

$$T_{111} = T_{234} = T_{414} = -\frac{b_0}{2}\tan^2\psi_1$$

$$T_{212} = T_{313} = +\frac{b_0}{2}\tan^2\psi_1$$

$$T_{133} = +\frac{b_0}{2}\sec^2\psi_1$$

$$T_{423} = -\frac{b_0}{2}\sec^2\psi_1$$

$$T_{211} = +\frac{b_0}{2R_0}\sec^3\psi_1 + K_1\tan\psi_1$$

$$T_{233} = -\frac{b_0}{2R_0}\sec^3\psi_1 - K_1\tan\psi_1 + \frac{b_0^2}{2}\tan\psi_1(1+\sec^2\psi_1)$$

$$T_{413} = -\frac{b_0}{2R_0}\sec^3\psi_1 - K_1\tan\psi_1$$

The difference with MAD-X comes from the requirement to be symplectic

Curved pole face

Quadrupole component

(Neglected in our derivation)

Next steps

- Comparison with real magnets (ELENA, HeLICS, canted theta)
- Quadrupole and combined function fringe fields
- Fringe impact on orbit, beta-beating, non-linear RDT, detuning, chromaticity
- Comparison between thin map and realistic fields for ELENA, HeLICS, canted theta for these observables
- Measurement with beam (beta-beating, chromaticity, non-linear RDT)

Appendix

Divergence, gradient and Laplacian in curvilinear coordinates

Divergence and gradient

$$\nabla\phi = \partial_x\phi\hat{\mathbf{x}} + \partial_y\phi\hat{\mathbf{y}} + \frac{1}{1+hx}\partial_s\phi\hat{\mathbf{s}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{1+hx}\partial_x\left((1+hx)A_x\right) + \partial_y A_y + \frac{1}{1+hx}\partial_s A_s$$

Laplacian

$$\nabla\cdot(\nabla\phi) = \frac{1}{1+hx}\partial_x\left((1+hx)\partial_x\phi\right) + \partial_y^2\phi + \frac{1}{1+hx}\partial_s\left(\frac{1}{1+hx}\partial_s\phi\right)$$

Magnetic field expansion

Relation with the magnetic field $\vec{B} = -\vec{\nabla}\phi$ and first terms

$$\boxed{\begin{aligned} a_n(s) &= \partial_x^{n-1} B_x(x,y,s)\Big|_{x=y=0} \\ b_n(s) &= \partial_x^{n-1} B_y(x,y,s)\Big|_{x=y=0} \end{aligned}}$$

	1	x	y	x^2	xy	y^2
B_x	a_1	a_2	b_2	$\frac{a_3}{2}$	b_3	$\frac{-a_3-h(a_2-a_1h-2b_s')+b_sh'-a_1''}{2}$
B_y	b_1	b_2	$-b_s'-a_1h-a_2$	$\frac{b_3}{2}$	$-a_3-h(a_2-a_1h-2b_s')+b_sh'-a_1''$	$-\frac{b_3+hb_2+b_1''}{2}$
B_s	b_s	$-b_sh+a_1'$	b_1'	$b_sh^2-a_1'h+\frac{a_2'}{2}$	$-hb_1'+b_2'$	$-\frac{a_1h'+ha_1'+b_s''+a_2'}{2}$

S. D. Fartoukh, 1997; C. Wang and L. Teng, 2001

Magnetic field expansion

Relation with the magnetic field $\mathbf{B} = -\boldsymbol{\nabla}\phi$

$$B_x(x, y=0, s) = -\partial_x \phi_0(x, s) = \sum_{n=1}^{\infty} a_n(s) \frac{x^{n-1}}{(n-1)!}$$

$$B_y(x, y=0, s) = -\phi_1(x, s) = \sum_{n=1}^{\infty} b_n(s) \frac{x^{n-1}}{(n-1)!}$$

$$B_s(x=0, y=0, s) = -\frac{1}{1+hx} \partial_s \phi_0(x, s)\Big|_{x=0} = b_s(s)$$

$$\boxed{\begin{aligned} a_n(s) &= \partial_x^{n-1} B_x(x, y, s)\Big|_{x=y=0} \\ b_n(s) &= \partial_x^{n-1} B_y(x, y, s)\Big|_{x=y=0} \end{aligned}}$$

S. D. Fartoukh, 1997; C. Wang and L. Teng, 2001

Hamiltonian splitting

Simplifying the fringe field map **F**

- Expand the Hamiltonian in y

$$H_F = \underbrace{-\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}}_{\text{Independent of position} \rightarrow \text{no kick}} + b(s)x$$

$$- \frac{p_x}{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}} y \partial_y a_x + O(y^2)$$

- Split map in drift and remaining contribution, $\mathbf{F} = \mathbf{D} \circ \mathbf{P}$

Forest et al., 2006

Hamiltonian splitting

Simplifying the fringe field map **F**

- Expand the Hamiltonian in y

$$H_F = \underbrace{-\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}}_{\text{Independent of position} \,\rightarrow\, \text{no kick}} + b(s)x$$

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- Split map in drift and remaining contribution, $\mathbf{F} = \mathbf{D} \circ \mathbf{P}$
- Possible approach: determine the Hamiltonian of $\mathbf{P} = \mathbf{D}^{-1} \circ \mathbf{F}$, taking into account the Baker-Campbell-Hausdorf theorem for the associated Lie maps

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]-\frac{1}{24}[Y,[X,[X,Y]]]+\dots}$$

but Hamiltonian depends on s: Magnus series

Forest et al., 2006

Hamiltonian splitting

Simplifying the fringe field map **F**

- Expand the Hamiltonian in y

$$H_F = \underbrace{-\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}}_{\text{Independent of position} \to \text{no kick}} + b(s)x$$

$$- \frac{p_x}{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}} y \partial_y a_x + O(y^2)$$

- Split map in drift and remaining contribution, $\mathbf{F} = \mathbf{D} \circ \mathbf{P}$
- Possible approach: determine the Hamiltonian of $\mathbf{P} = \mathbf{D}^{-1} \circ \mathbf{F}$, taking into account the Baker-Campbell-Hausdorf theorem for the associated Lie maps

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]-\frac{1}{24}[Y,[X,[X,Y]]]+\ldots}$$

  but Hamiltonian depends on s: Magnus series

- Alternative approach: the use of Lie operators

Forest et al., 2006

What are compositional maps and lie operators?

Example: drift

- To give an example, let us have a look at the drift Hamiltonian in a straight reference frame

$$H_D = \frac{p_\tau}{\beta_0} - \sqrt{\underbrace{(1+\delta)^2}_{p_\tau + 2\frac{p_\tau}{\beta_0} + 1} - p_x^2 - p_y^2}$$

- Lie operator

$$\begin{aligned} :H_D: &= \sum_{k=1}^{3} \left(\frac{\partial H_D}{\partial q_k} \frac{\partial}{\partial p_k} - \frac{\partial H_D}{\partial p_k} \frac{\partial}{\partial q_k} \right) \\ &= -\frac{p_x}{p_z} \frac{\partial}{\partial x} - \frac{p_y}{p_s} \frac{\partial}{\partial y} - \left(\frac{1}{\beta_0} - \frac{1}{\beta} \right) \frac{\partial}{\partial \tau} \end{aligned}$$

What are compositional maps and lie operators?

Example: drift

- Compositional map

$$\mathcal{D} = e^{s\,:-H_D:}$$

- Transfer map

$$\begin{aligned}\mathbf{D} &= \mathcal{D} I \\ &= \left(1 + s :-H_D: + \frac{s^2}{2} :-H_D:^{\,2} + ...\right) I\end{aligned}$$

gives the drift map

$$:H_D: \; = -\frac{p_x}{p_s}\frac{\partial}{\partial x} - \frac{p_y}{p_s}\frac{\partial}{\partial y} - \left(\frac{1}{\beta_0} - \frac{1}{\beta}\right)\frac{\partial}{\partial \tau}$$

$$\begin{aligned}x_f &= x_i + \frac{p_x}{p_s} s \\ y_f &= y_i + \frac{p_y}{p_s} s \\ \tau_f &= \tau_i + \left(\frac{1}{\beta_0} - \frac{1}{\beta}\right) s\end{aligned}$$

Alternative approach: The use of Lie operators

Simplifying the fringe field map $\mathbf{F} = \mathbf{D} \circ \mathbf{P} \;\rightarrow\; \mathbf{P} = \mathbf{D}^{-1} \circ \mathbf{F}$

- Calculate the derivative of the associated Lie map $\mathcal{P} = \mathcal{F}\mathcal{D}^{-1}$

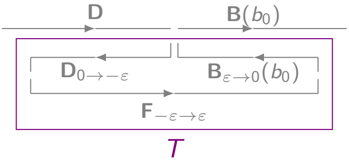
$$\frac{d\mathcal{F}}{ds} = \mathcal{F}:-H_F: \;, \qquad \frac{d\mathcal{D}}{ds} = \mathcal{D}:-H_D: \quad \rightarrow \quad \frac{d\mathcal{D}^{-1}}{ds} = -:-H_D:\mathcal{D}^{-1}$$

$$\Bigg\downarrow \quad \begin{aligned} \frac{d\mathcal{P}}{ds} &= \frac{d\mathcal{F}}{ds}\mathcal{D}^{-1} + \mathcal{F}\frac{d\mathcal{D}^{-1}}{ds} \\ &= \mathcal{F}:-H_F:\mathcal{D}^{-1} - \mathcal{F}:-H_D:\mathcal{D}^{-1} \end{aligned}$$

$$\frac{d\mathcal{P}}{ds} = \mathcal{P}\mathcal{D}:-(H_F - H_D):\mathcal{D}^{-1}$$

Expansion of the thin fringe field map

$$
\begin{aligned}
\mathcal{T} &= \mathcal{D}_{0\to-\varepsilon}\mathcal{F}_{-\varepsilon\to\varepsilon}\mathcal{B}_{\varepsilon\to 0}(b_0) \\
&= \mathcal{D}_{0\to-\varepsilon}\mathcal{P}_{-\varepsilon\to\varepsilon}\mathcal{D}_{-\varepsilon\to 0}\mathcal{D}_{0\to\varepsilon}\mathcal{B}_{\varepsilon\to 0}(b_0)
\end{aligned}
$$



$\mathbf{D}$ $\mathbf{B}(b_0)$

$\mathbf{D}_{0\to-\varepsilon}$ $\mathbf{B}_{\varepsilon\to 0}(b_0)$

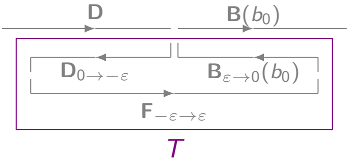
$\mathbf{F}_{-\varepsilon\to\varepsilon}$

T

Forest et al., 2006

Expansion of the thin fringe field map

$$
\begin{aligned}
\mathcal{T} &= \mathcal{D}_{0\to-\varepsilon}\mathcal{F}_{-\varepsilon\to\varepsilon}\mathcal{B}_{\varepsilon\to0}(b_0)\\
&= \mathcal{D}_{0\to-\varepsilon}\mathcal{P}_{-\varepsilon\to\varepsilon}\mathcal{D}_{-\varepsilon\to0}\underbrace{\cancel{\mathcal{D}_{0\to\varepsilon}}\cancel{\mathcal{B}_{\varepsilon\to0}(b_0)}}_{\text{Does not contribute to } \Delta p_y}
\end{aligned}
$$



$\mathbf{D}$

$\mathbf{B}(b_0)$

$\mathbf{D}_{0\to-\varepsilon}$

$\mathbf{B}_{\varepsilon\to0}(b_0)$

$\mathbf{F}_{-\varepsilon\to\varepsilon}$

T

$$
\begin{aligned}
\mathcal{D}_{-\varepsilon}\mathcal{P}_{-\varepsilon\to\varepsilon}\mathcal{D}_{\varepsilon} = 1 &+ \int_{-\varepsilon}^{\varepsilon} ds_1 :-V(x+s_1x', y+s_1y', s_1):\\
&+ \int_{-\varepsilon}^{\varepsilon} ds_1 \int_{-\varepsilon}^{s_1} ds_2 :-V(x+s_2x', y+s_2y', s_2): \ :-V(x+s_1x', y+s_1y', s_1):
\end{aligned}
$$

with $V = H_F - H_D$

Forest et al., 2006

Effect on p_x and p_y

- Lowest order effect on p_x is zero
- Effect on p_y: calculate the Poisson brackets

$$\begin{aligned} p_{y,f} &= \mathcal{D}_{-\varepsilon}\mathcal{P}_{-\varepsilon\to\varepsilon}\mathcal{D}_{\varepsilon}\, p_y \\ &= p_y + \int_{-\varepsilon}^{\varepsilon} ds_1 :-V(x+s_1x', y+s_1y', s_1): p_y \\ &\quad + \int_{-\varepsilon}^{\varepsilon} ds_1 \int_{-\varepsilon}^{s_1} ds_2 :-V(x+s_2x', y+s_2y', z_2): \,:-V(x+s_1x', y+s_1y', s_1): p_y \end{aligned}$$

Forest et al., 2006

Effect on p_x and p_y

- Lowest order effect on p_x is zero
- Effect on p_y: calculate the Poisson brackets

First order contribution $\Delta p_{y,1}$

$$\begin{aligned}
p_{y,f} &= \mathcal{D}_{-\varepsilon}\mathcal{P}_{-\varepsilon\to\varepsilon}\mathcal{D}_{\varepsilon}\, p_y \\
&= \boxed{p_y} + \boxed{\int_{-\varepsilon}^{\varepsilon} ds_1 :-V(x+s_1x', y+s_1y', s_1): p_y} \\
&\quad + \boxed{\int_{-\varepsilon}^{\varepsilon} ds_1 \int_{-\varepsilon}^{s_1} ds_2 :-V(x+s_2x', y+s_2y', z_2): \, :-V(x+s_1x', y+s_1y', s_1): p_y}
\end{aligned}$$

Identity map

Second order contribution $\Delta p_{y,2}$

Forest et al., 2006

Generating function to create a symplectic map

- Typical generating function:

$$F(x^f, p_x, y^f, p_y, -\delta, \ell^f) = \underbrace{p_x x^f + p_y y^f - \delta\ell^f}_{I} + \Lambda(\ldots)$$

- Map determined from

$$\begin{aligned} x &= \frac{\partial F}{\partial p_x} & p_x^f &= \frac{\partial F}{\partial x^f} \\ y &= \frac{\partial F}{\partial p_y} & p_y^f &= \frac{\partial F}{\partial y^f} \\ -\delta^f &= \frac{\partial F}{\partial \ell^f} & \ell &= \frac{\partial F}{\partial(-\delta)} \end{aligned}$$