Introduction to Transverse Beam Dynamics

Bernhard Holzer
CERN

A Few General Statements

The Main Parts of Beam Dynamics in JUAS

* Lectures (Bernhard): "Transverse Beam Dynamics"
—> listen and ask intelligent (?) questions

* Tutorials (Vera):
—> think about interesting (!) questions from real life
… and from typical exams ;-)

* Accelerator Design (Bastian)
—> learn how to build a real accelerator

* Mini-Workshop (Adrian & Friends)
—> and actually do it !!

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Luminosity Run of a typical storage ring:

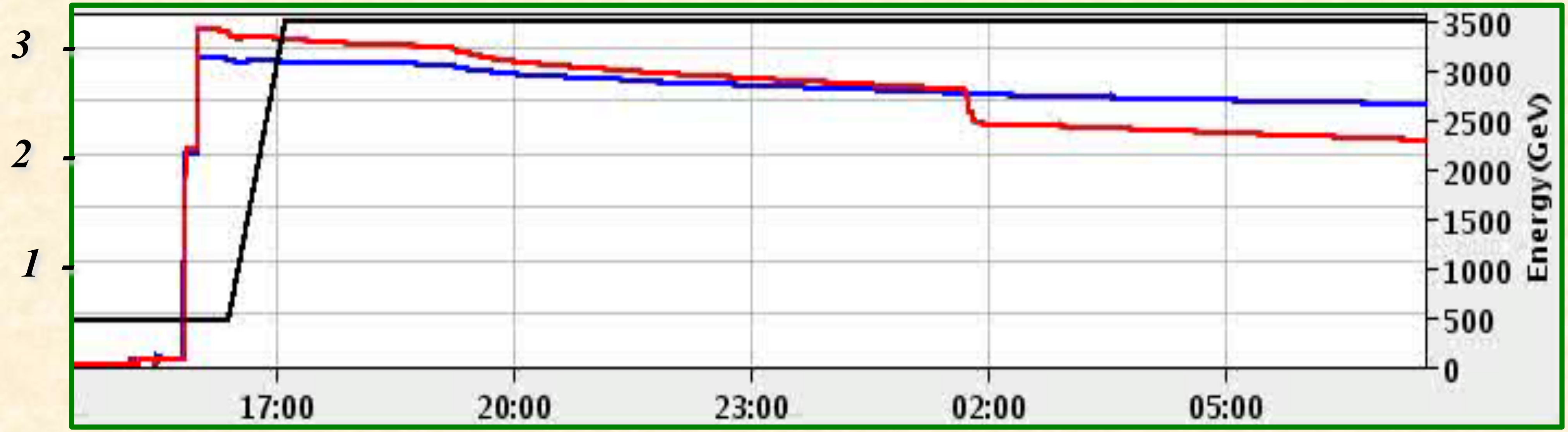
LHC Storage Ring: Protons accelerated and stored for 12 hours

distance of particles travelling at about $v \approx c$

$L = 10^{10}$-10^{11} km

... several times Sun - Pluto and back

intensity (10^{11})



time of the day

→ *guide the particles* on a well defined orbit („design orbit“)

→ *focus the particles* to keep each single particle trajectory within the vacuum chamber of the storage ring, i.e. close to the design orbit.

I.) Introduction and Basic Ideas: The Bending Fields

„ ... in the end and after all it should be a kind of circular machine“
—> need transverse deflecting force

Lorentz force $\vec{F} = q * (\cancel{\vec{E}} + \vec{v} \times \vec{B})$

typical velocity in high energy machines: $v \approx c \approx 3 * 10^8 \, m/s$

Example:

$$B = 1\,T \quad \rightarrow \quad F = q * 3 * 10^8 \frac{m}{s} * 1 \frac{Vs}{m^2}$$

$$F = q * \underbrace{300 \frac{MV}{m}}_{E}$$

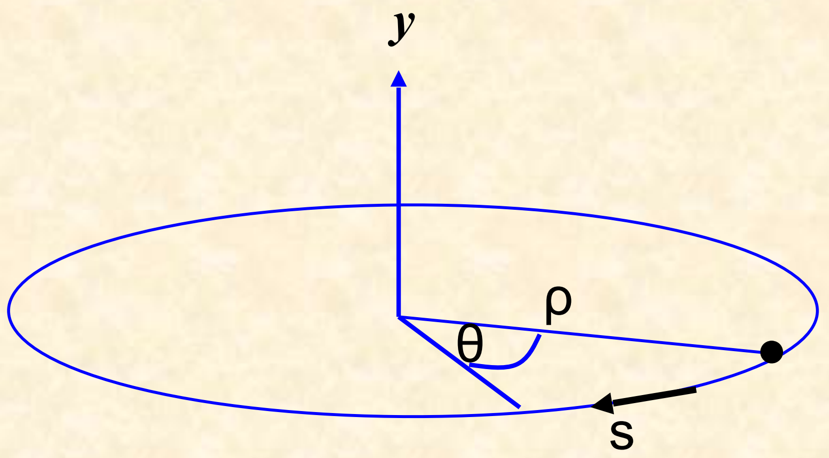
equivalent el. field ... E

technical limit for el. field

$$E \leq 1 \frac{MV}{m}$$

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.

The ideal circular orbit



circular coordinate system

condition for circular orbit:

Lorentz force $F_L = e\,v\,B$

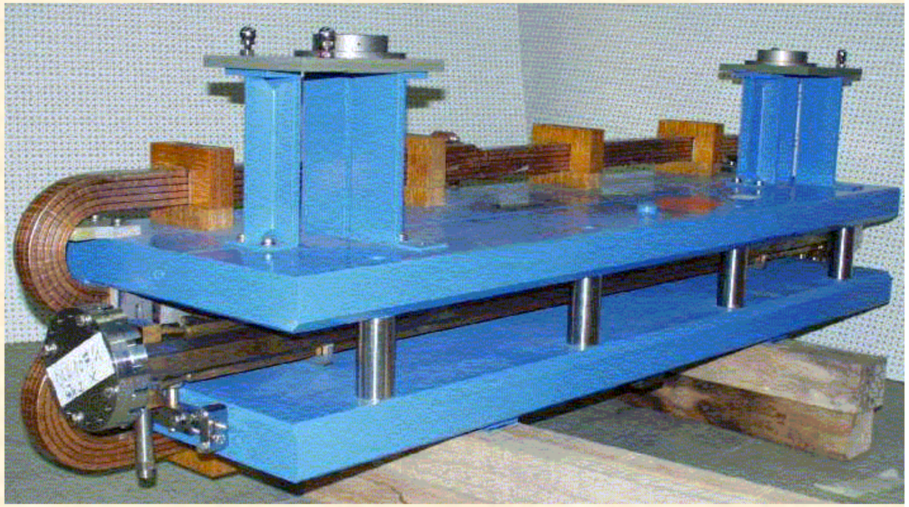
centrifugal force $F_{centr} = \dfrac{\gamma\, m_0\, v^2}{\rho}$

$$\frac{\gamma\, m_0\, v^{\cancel{2}}}{\rho} = e\,\cancel{v}\,B$$

$$\boxed{\frac{p}{e} = B\,\rho}$$

$B\rho$ = "beam rigidity"

2.) The Magnetic Guide Field



Dipole Magnets:

define the ideal orbit
homogeneous field created
by two flat pole shoes

$$B = \frac{\mu_0 \, n \, I}{h}$$

Normalise magnetic field to momentum:

$$\frac{p}{e} = B \cdot \rho \longrightarrow \frac{1}{\rho} = \frac{B}{p/e}$$

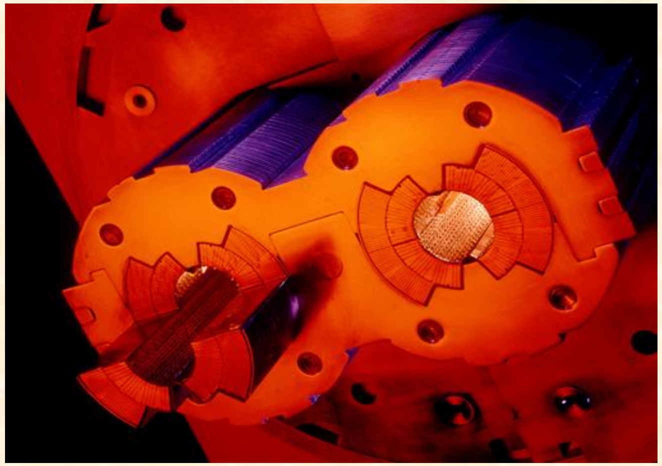
convenient units:

$$\boldsymbol{B} = [\boldsymbol{T}] = \left[\frac{\boldsymbol{Vs}}{\boldsymbol{m}^2}\right] \qquad \boldsymbol{p} = \left[\frac{\boldsymbol{GeV}}{\boldsymbol{c}}\right]$$

Example LHC:

$$\left.\begin{array}{l} B = 8.33 \; T \\ \boldsymbol{p} = 7000 \dfrac{\boldsymbol{GeV}}{\boldsymbol{c}} \end{array}\right\} \quad \rho = \frac{p}{e\,B} = \frac{7000 \cdot 10^9 \; eV}{3 \cdot 10^8 m/s * 8Vs/m^2}$$

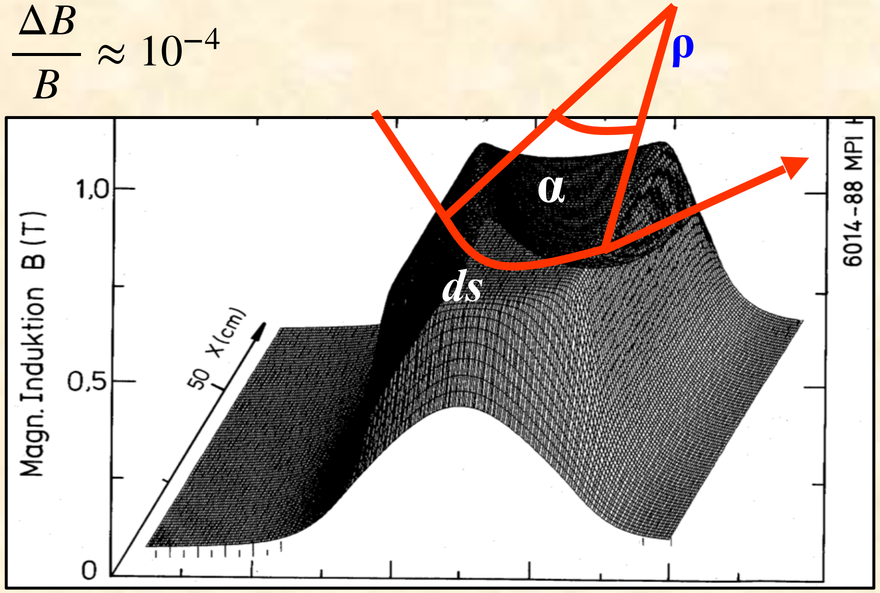
$$\rho = 2.83 \; km$$



nota bene: for ultra relativistic particles we get $p \approx \dfrac{E}{c}$

The Magnetic Guide Field

$$\frac{\Delta B}{B} \approx 10^{-4}$$



field map of a storage ring dipole magnet

Bending Angle

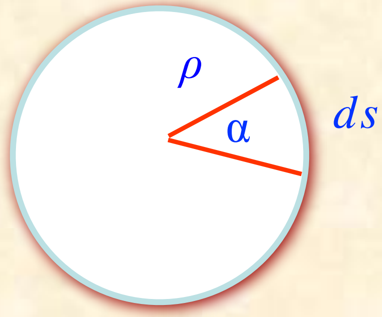
„how many dipoles do we need ???

Dipoles produce a constant (!) magnet field

Angle of one Dipole $\alpha_{dipol} = \frac{ds}{\rho} = \frac{\int B\, ds}{B\,\rho} \approx \frac{B \cdot l_{dipol}}{B\,\rho}$

All Dipoles $\Sigma\,(\alpha_{dipoles}) = \frac{\int_{dipoles} B\, dl}{B\,\rho} \approx \frac{n_{dipoles} \cdot B \cdot l_{dipol}}{B\,\rho} = 2\pi$

number of Dipole Magnets: $N_{dipole} = \frac{2\pi}{\alpha_{dipol}}$

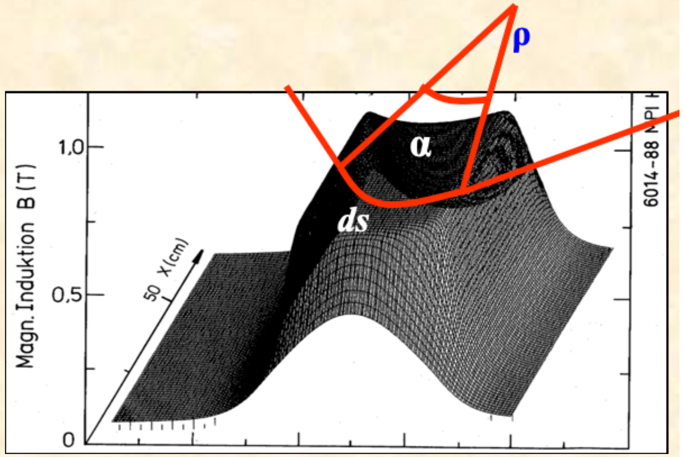


Circumference of the storage ring: $C_0 = 2\pi \cdot \rho \quad = 2\pi \cdot 2.83\ km \approx 18km$

?

The Magnetic Guide Field

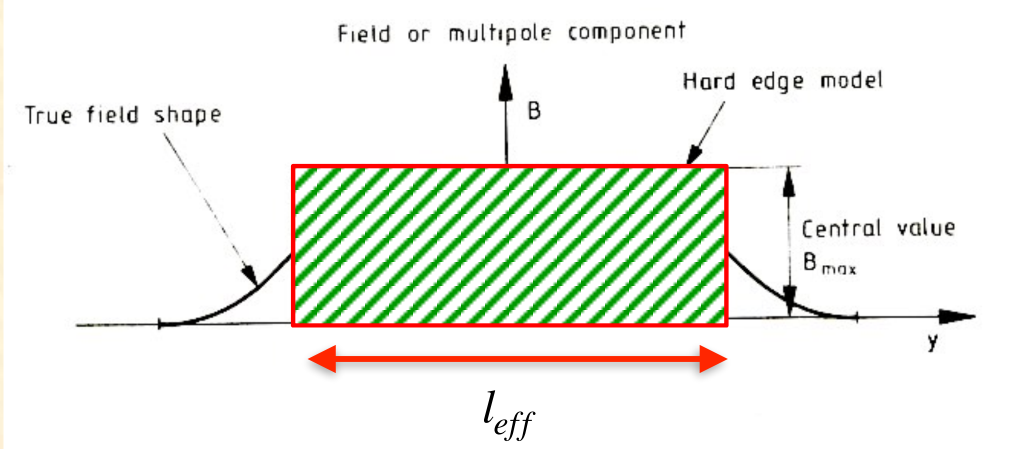
"effective magnet length"



Bending angle $$\alpha_{dipol} = \frac{ds}{\rho} = \frac{\int B\, ds}{B\rho} \approx \frac{B \cdot l_{dipol}}{B\rho}$$

Replace the integral by an effective length with:

$$B \cdot l_{eff} := \int_0^{l_{mag}} B\, ds$$

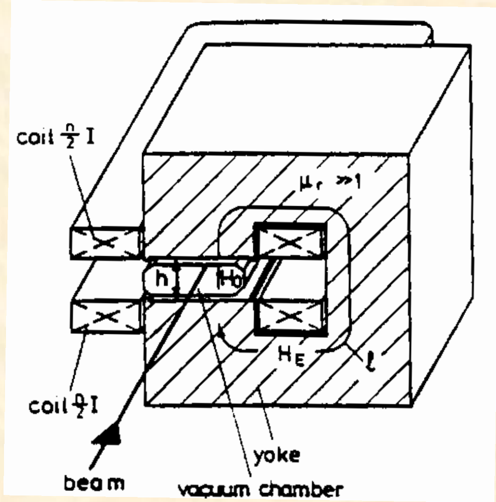


Given that definition, the strength and number of dipoles
define the maximum particle momentum!!

$$n \cdot B \cdot l_{dipol} = 2\pi \cdot \frac{p}{q}$$

Dipole Magnets:

***homogeneous field** created by two flat pole shoes*



Field Calculation:

3rd Maxwell equation for a static field: $\vec{\nabla} \times \vec{H} = \vec{j}$

according to Stokes theorem:

$$\int_S (\vec{\nabla} \times \vec{H})\vec{n}\, da = \oint \vec{H}\, d\vec{l} = \int_S \vec{j} \cdot \vec{n}\, da = N \cdot I$$

$$\oint \vec{H}\, d\vec{l} = H_0 * h + H_{Fe} * l_{Fe}$$

in matter we get with $\mu_r \approx 1000$

$$\oint \vec{H}\, d\vec{l} = H_0 * h + \cancel{\frac{H_0}{\mu_r} * l_{Fe}} \approx H_0 * h$$

Magnetic field of a dipole magnet:

$$H_0 = B_0 / \mu_0 \quad \rightarrow \quad B_0 = \frac{\mu_0 N I}{h}$$

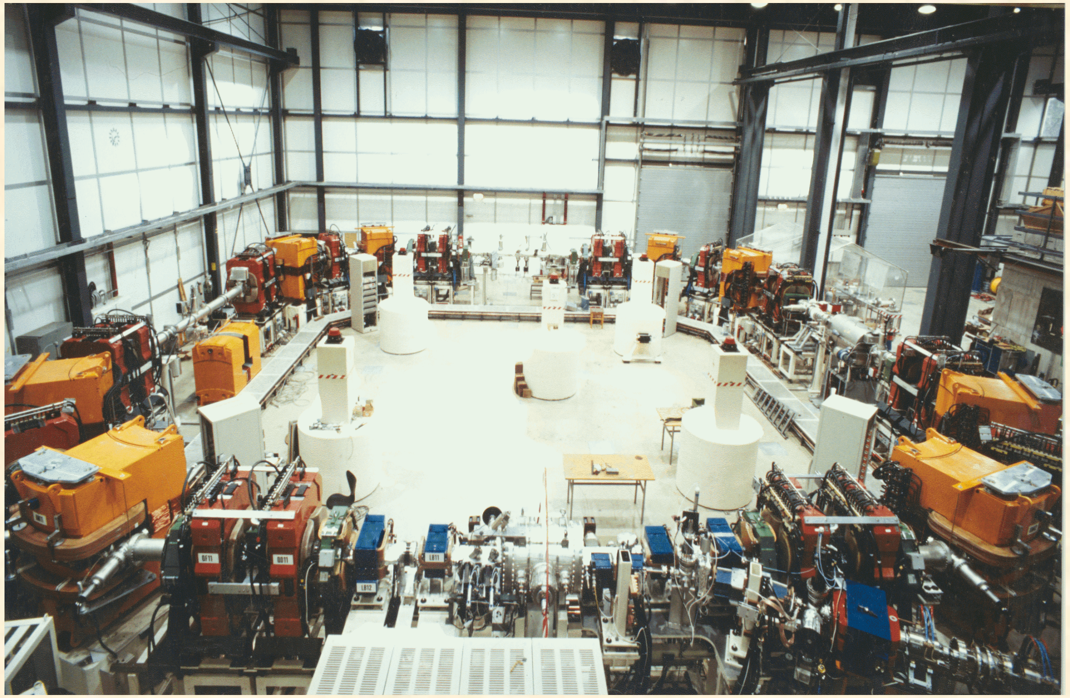
h = gap height

The dipole strength depends on the gap height h ,
aka aperture "r_0" of the magnet.

$$B_0 = \frac{\mu_0 N I}{h}$$

—> keep the beam dimensions small !!

Example:

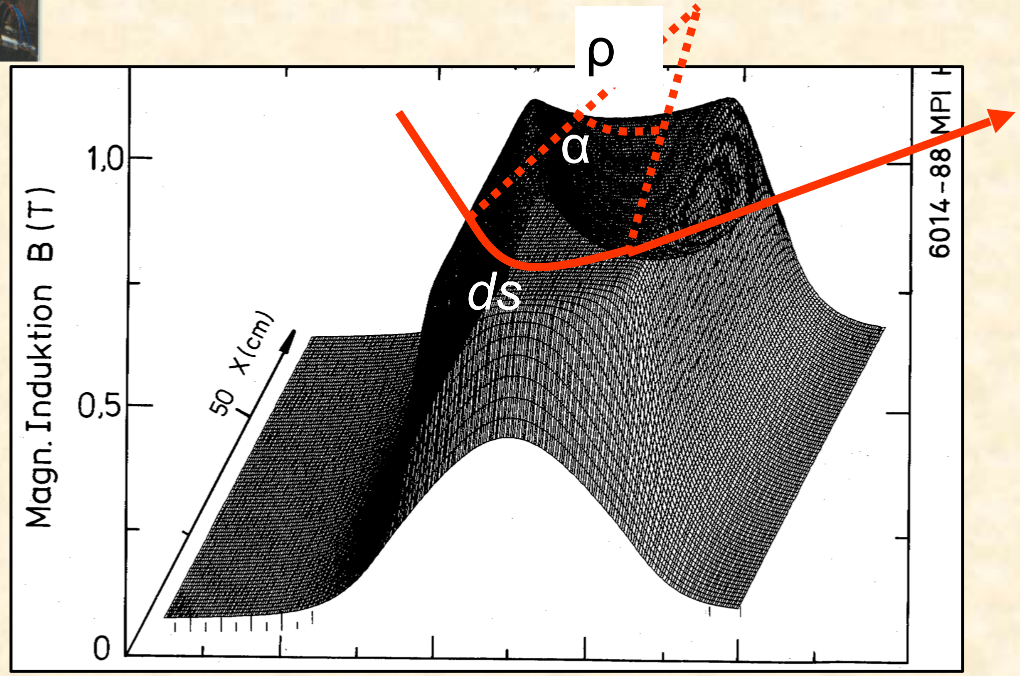


Heavy ion storage ring TSR
8 dipole magnets
of equal bending strength

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} \qquad \alpha = \frac{B * dl}{B * \rho}$$

The B fields integrated over the path-length of the beam through the eight dipole magnets has to add up to give an overall angle of 2π

$$\alpha_{dipole} = \frac{2\pi}{8}$$



field map of a storage ring dipole magnet

The integrated dipole strength (along "s") defines the momentum of the particle beam.

$$\alpha = \frac{\int B dl}{B * \rho} = 2\pi \qquad \rightarrow \qquad \int B dl = 2\pi * \frac{p}{q}$$

Attention: LHC

$$\left.\begin{array}{l} \rho = 2.83\ km \rightarrow 2\pi \cdot \rho \approx 18\ km \\ C_0 = 27\ km \end{array}\right\}$$

… the ring defined by the diples covers typically 66 % of the circumference

… there seems to be more than dipoles

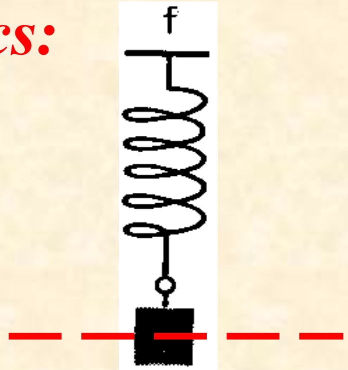
A Tandem "Van de Graaf" Accelerator

12 MV Voltage DC over 25 m
linear accelerating structure, no dipoles, no focusing,
just straight onto the target.

4.)Particles in Quadrupole Fields:

Focusing Properties of a magnet lattice

Classical Mechanics:
pendulum



there is a restoring force, proportional to the elongation x:

$$F = m \cdot \frac{d^2x}{dt^2} = -k \cdot x$$

Ansatz $$x(t) = A \cdot cos(\omega t + \varphi)$$

general solution: free harmonic oscillation

$$\dot{x}(t) = -A\ \omega \cdot sin(\omega t + \varphi)$$

$$\ddot{x}(t) = -A\ \omega^2 \cdot cos(\omega t + \varphi)$$

Solution $\omega = \sqrt{k/m}$ $\qquad x(t) = x_0 \cdot cos(\sqrt{\frac{k}{m}}\ t + \varphi)$

Apply this concept to magnetic forces: we need a Lorentz force that rises as a function of the distance to .... the design orbit

$$\boxed{F(x) = q * v * B(x)}$$

Quadrupole Magnets:

required: ***focusing forces to keep trajectories in vicinity of the ideal orbit***
linear increasing Lorentz force
linear increasing magnetic field $\boldsymbol{B_y = g\,x}$ $\boldsymbol{B_x = g\,y}$

normalised quadrupole field:

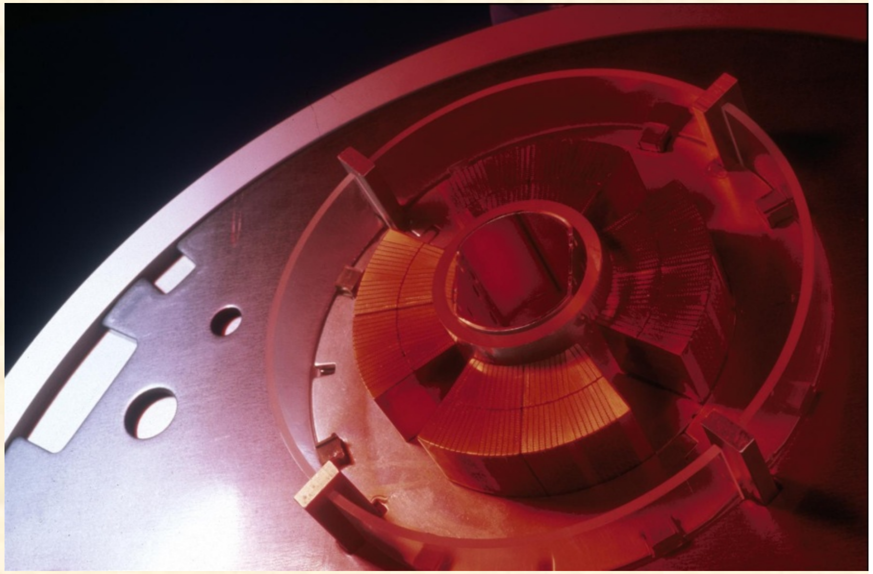
gradient of a quadrupole magnet:

$$\boldsymbol{g} = \frac{2\mu_0 \boldsymbol{nI}}{\boldsymbol{r}^2}$$

$$\longrightarrow \quad k = \frac{g}{p/e} = \frac{g}{B \cdot \rho}$$

simple rule:

$$\boxed{k \approx 0.3 \cdot \frac{g(T/m)}{p(GeV/c)}}$$



LHC main quadrupole magnet

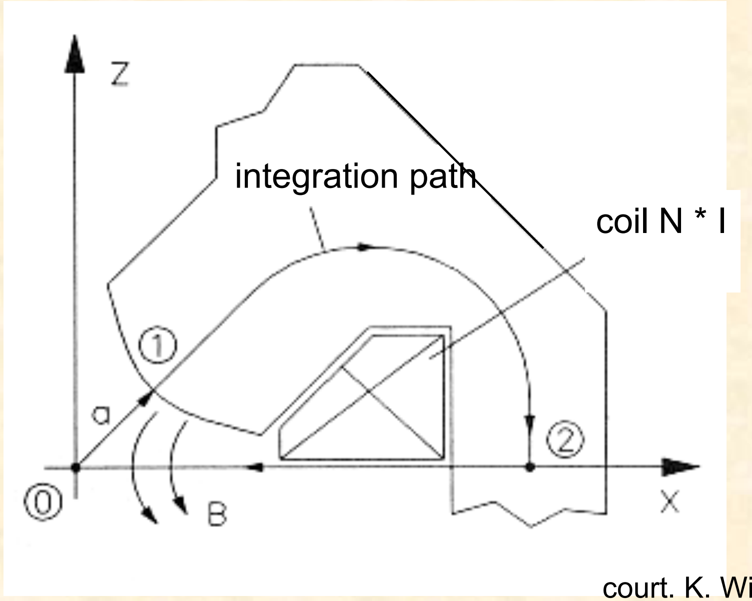
$$g \approx 25 \ldots 225 \; T/m$$

what about the vertical plane:
... Maxwell

$$\vec{\nabla} \times \vec{B} = \cancel{\vec{j}} + \cancel{\frac{\partial \vec{E}}{\partial t}} = 0 \qquad \longrightarrow g = \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

Quadrupole Magnets:

Calculation of the Quadrupole Field:



court. K. Wille

$$\oint \boldsymbol{H ds} = \boldsymbol{N * I}$$

$$\oint \boldsymbol{H ds} = \int_0^1 \boldsymbol{H}_0 \boldsymbol{ds} + \underbrace{\cancel{\int_1^2 \boldsymbol{H}_{Fe} \boldsymbol{ds}}}_{H_{Fe}=H_0/\mu_{Fe}} + \underbrace{\cancel{\int_2^0 \boldsymbol{H ds}}}_{H\perp ds} = \boldsymbol{N * I}$$

$\mu_{Fe} \approx 1000$

now we know that $\boldsymbol{H} = \boldsymbol{B}/\mu_0$

and we require $\boldsymbol{B(r) = -g * r}$

$$\longrightarrow \int_0^1 H_0 \, ds = \int_0^a \frac{B_0}{\mu_0} dr = \int_0^a \frac{g \cdot r}{\mu_0} dr = g \cdot \frac{r^2}{2\,\mu_0} = N \cdot I$$

gradient of a quadrupole field:

$$\boldsymbol{g} = \frac{2\mu_0 * \boldsymbol{N * I}}{\boldsymbol{r}^2}$$

Linear Transverse Beam Dynamics

Dipoles:

Create a constant field $B_y = const$

Quadrupoles:

Create a linear increasing magnetic field $B_y = g \cdot x, \quad B_x = g \cdot y$

A linear increasing restoring force leads always (!) to a harmonic oscillation.

==> quadrupoles do that for us

And dipoles define the particle momentum

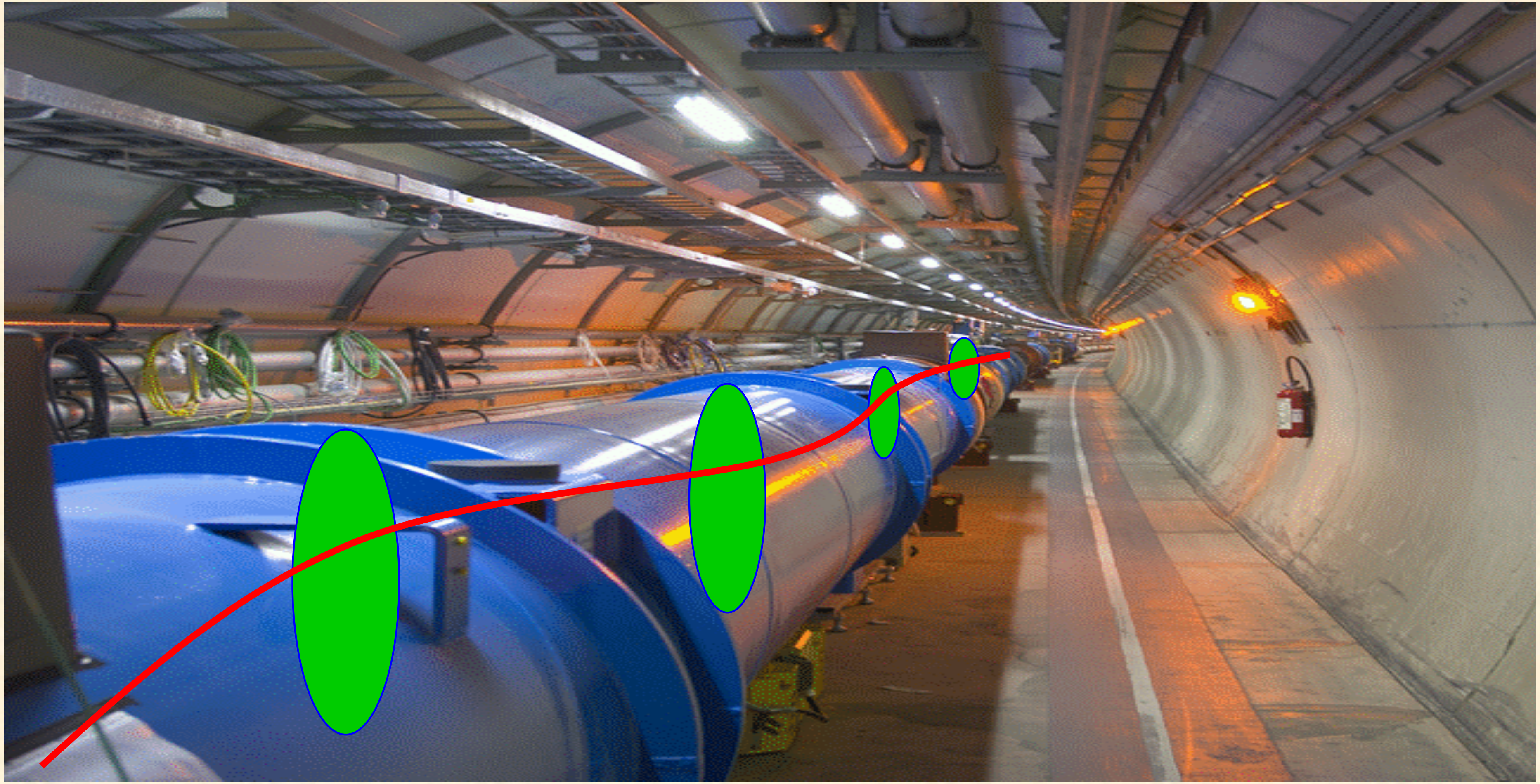
$$B_{quadrupole} = g * x \qquad B_{dipole} = const$$

Linear Lattice:

Dipoles & Quadrupoles

… and Drifts in between

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k\,x + \cancel{\frac{1}{2!} m\,x^2} + \cancel{\frac{1}{3!} n\,x^3} + \ldots$$



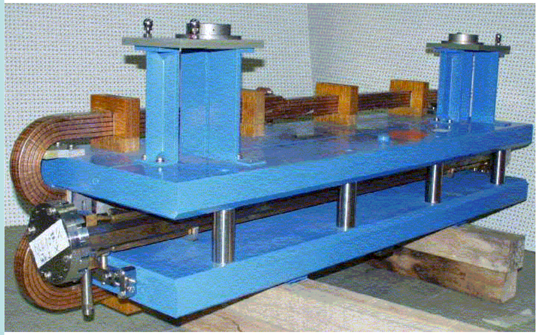
Magnetic fields used in an accelerator:

dipole

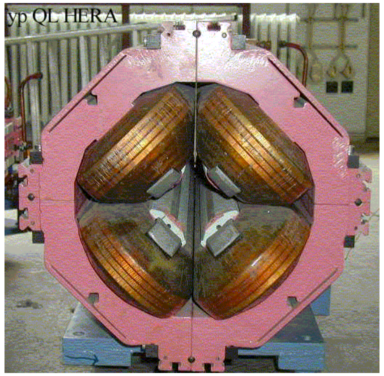
quadrupole

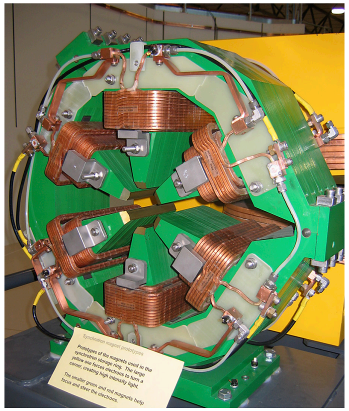
sextupole

octupole

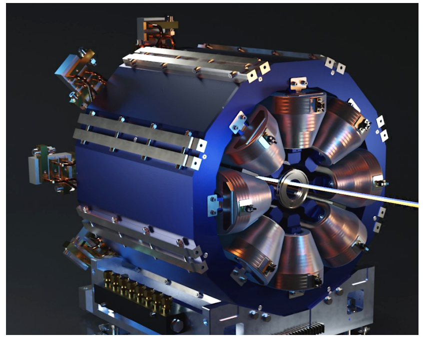


court. HERA collider





court. Australian Synchrotron



court. Pyramid Inc.

Linear Lattice ***Chromatic Correction*** ***Landau Damping***

5.) Nothing is perfect, not even magnetic fields …

non-linear fields

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k\,x + \cancel{\frac{1}{2!} m\,x^2} + \cancel{\frac{1}{3!} n\,x^3} + \ldots$$

only terms linear in x, y taken into account *dipole fields*
quadrupole fields

Separate Function Machines:

Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR

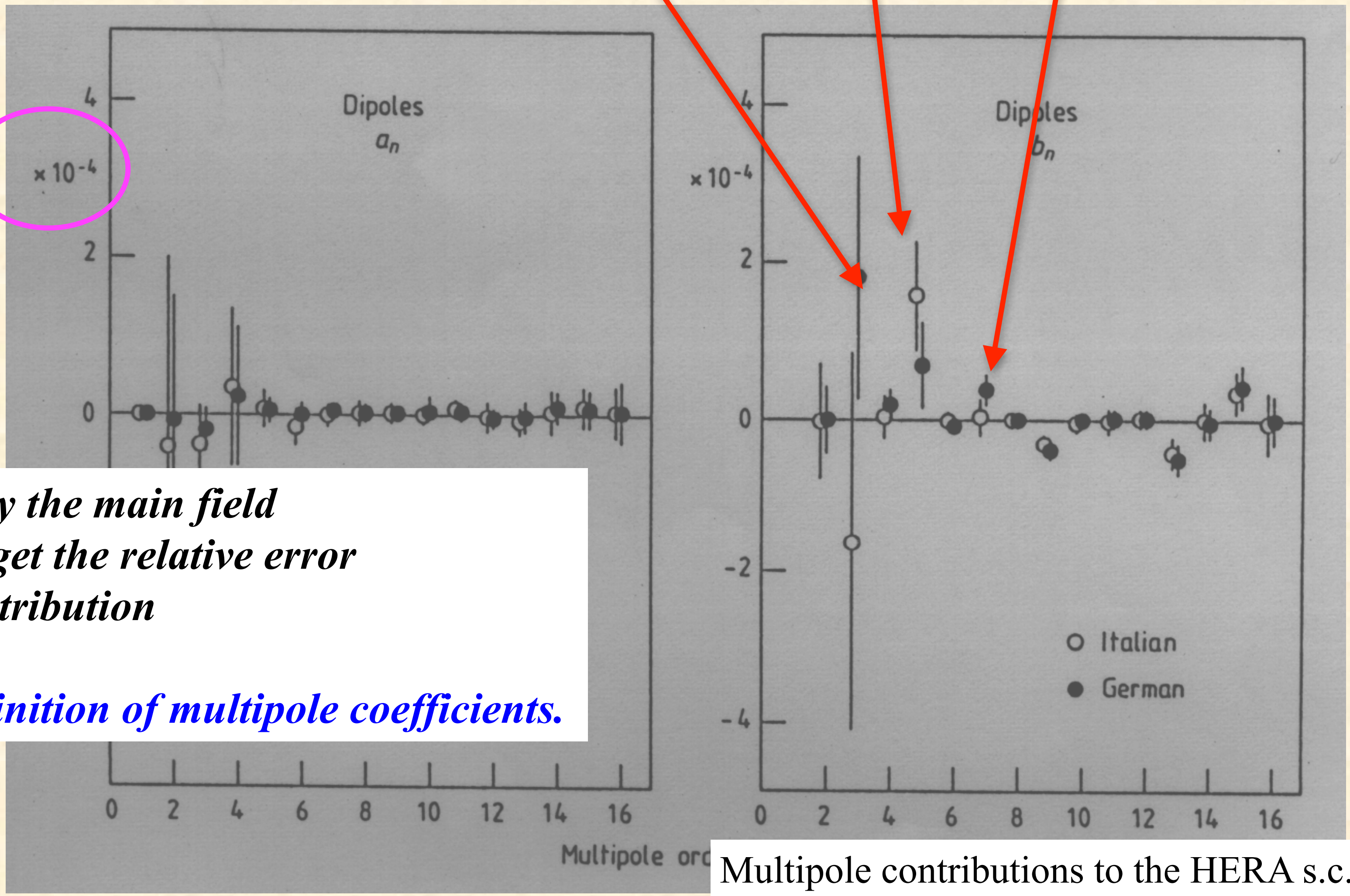
* *man sieht nur*
dipole und quads —> linear

s 1

*** ***Linear Lattice = Dipoles & Quadrupoles.***
However linear also means no prominent multipole contributions !!!

Taylor Expansion of the B field:

$$B_y(x) = B_{y0} + \frac{dB_y}{dx}x + \frac{1}{2!}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{d^3B_y}{dx^3}x^3 + \ldots$$

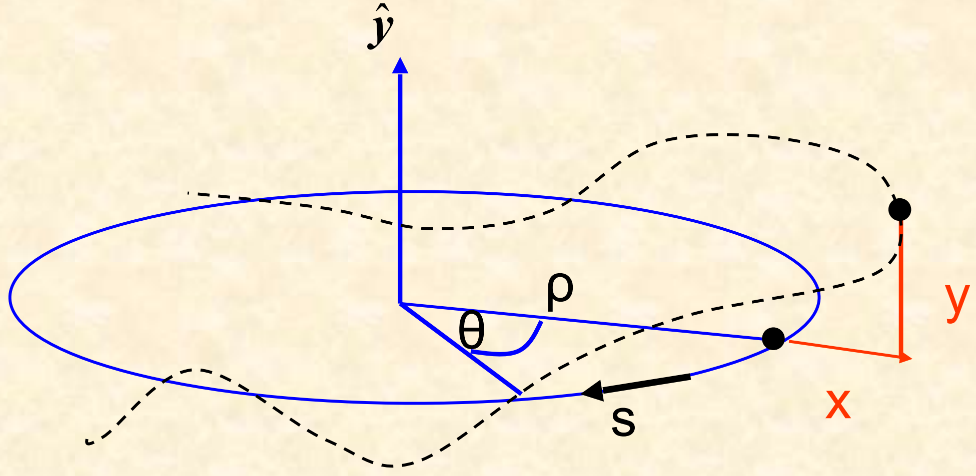


divide by the main field
to get the relative error
contribution

—> definition of multipole coefficients.

Multipole contributions to the HERA s.c. dipole field

6.) The equation of motion:



Linear approximation:

** x = hor. amplitude with respect to the design orbit*
y = vert. amplitude with " "
s = position along the design orbit, moving with the particle considered

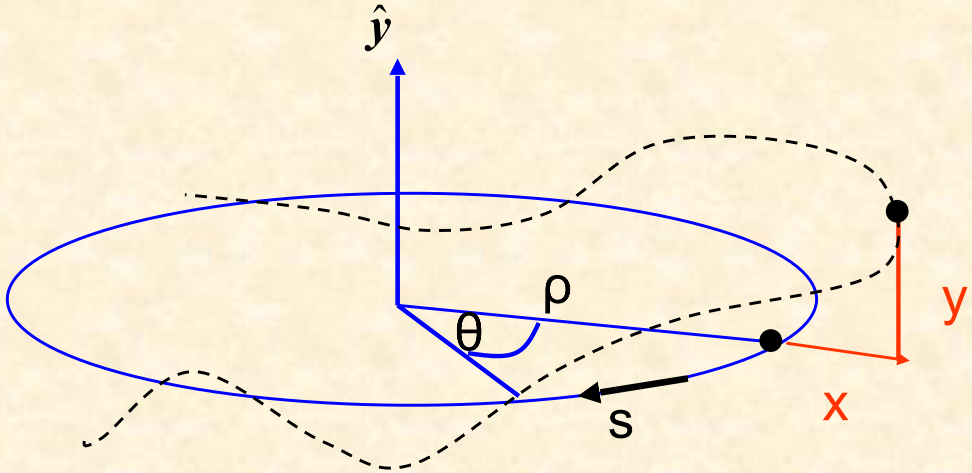
** ideal particle: x = y = 0, —> design orbit*

** any other particle —> coordinates x, y small quantities*

$x\,,\, y \ll \rho$

** magnetic guide field: only linear terms in x & y of B*
have to be taken into account

Equation of Motion:



Consider local segment of a particle trajectory
... and remember the old days:
(Goldstein page 27)

radial acceleration:

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

Ideal orbit: $\rho = const, \quad \frac{d\rho}{dt} = 0$

centrifugal Force:

$$F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2 = mv^2/\rho$$

general trajectory: $\rho \longrightarrow \rho + x$

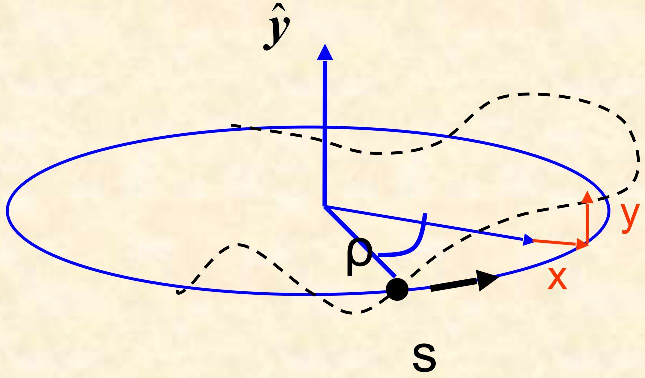
condition for circular orbit:

$$F_{centrifugal} + F_{Lorentz} = 0$$

$$\boxed{F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = -\ eB_y v}$$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = - \ eB_y v$$

1 2



1 $$\frac{d^2}{dt^2}(x+\rho) = \frac{d^2}{dt^2}x$$ ***… as ρ = const***

2 ***remember: x ≈ mm , ρ ≈ m … —> develop for small x***

$$\frac{1}{x+\rho} \approx \frac{1}{\rho}(1-\frac{x}{\rho})$$

Taylor Expansion

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \ldots$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = - \ eB_y v$$

guide field in linear approx.

$$\boxed{\begin{aligned} m\frac{d^2x}{dt^2}-\frac{mv^2}{\rho}(1-\frac{x}{\rho}) &= -ev\{B_0+x\frac{\partial B_y}{\partial x}\} \\ \frac{d^2x}{dt^2}-\frac{v^2}{\rho}(1-\frac{x}{\rho}) &= -\frac{evB_0}{m}+\frac{evxg}{m} \end{aligned}}$$

$$B=B_0+x\cdot\frac{\partial B_y}{\partial x} \quad , \qquad : m$$

$$g=-\frac{\partial B_y}{\partial x}$$

independent variable: t → s

$$\frac{dx}{dt}=\frac{dx}{ds}\frac{ds}{dt}$$

$$x'=\frac{dx}{ds} \qquad = \textit{angle of the particle trajectory}$$

$$\frac{d^2x}{dt^2}=\frac{d}{dt}\left(\frac{dx}{ds}\frac{ds}{dt}\right)=\frac{d}{ds}\left(\underbrace{\frac{dx}{ds}}_{x'}\underbrace{\frac{ds}{dt}}_{v}\right)\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2}=x''v^2+\cancel{\frac{dx}{ds}\frac{dv}{ds}v}$$

$$\boxed{x''v^2-\frac{v^2}{\rho}(1-\frac{x}{\rho})=-\frac{evB_0}{m}+\frac{evxg}{m}}$$

$$: v^2$$

$$x'' - \frac{1}{\rho}(1 - \frac{x}{\rho}) = -\frac{eB_0}{mv} + \frac{exg}{mv}$$

m v = p

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = -\frac{B_0}{p/e} + \frac{xg}{p/e}$$

normalize to momentum of particle

$$x'' \cancel{-\frac{1}{\rho}} + \frac{x}{\rho^2} = \cancel{-\frac{1}{\rho}} + kx$$

$$\frac{1}{\rho} = \frac{B_0}{p/e} \quad , \quad \frac{g}{p/e} = k$$

$$\boxed{x'' + x(\frac{1}{\rho^2} - k) = 0}$$

Equation for the horizontal plane:

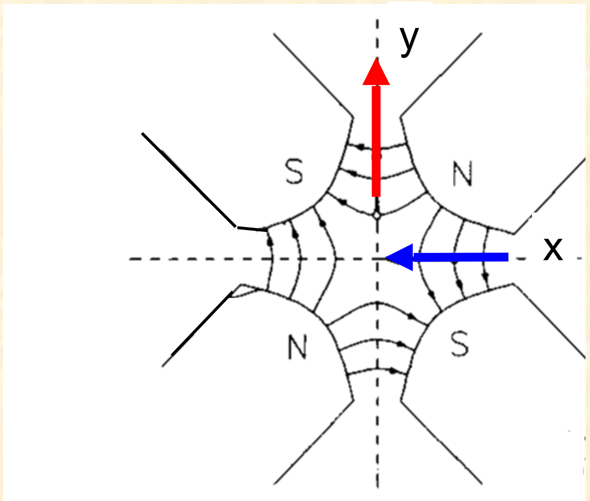
(k < 0 —> foc quad)

* ***Equation for the vertical motion:***

$$\frac{1}{\rho^2} = 0$$ ***no dipoles … in general …***

$$k \leftrightarrow -k$$ ***quadrupole field changes sign***

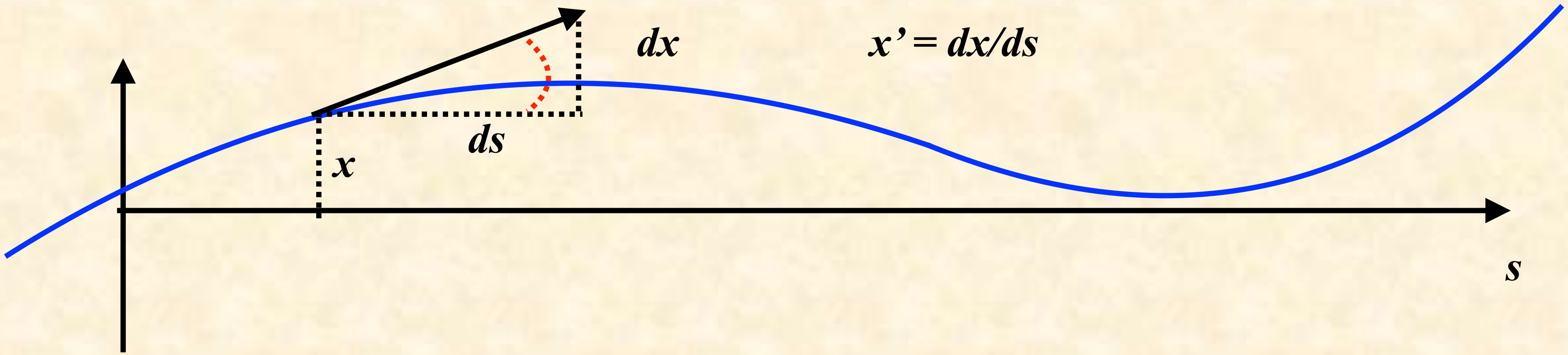
$$\boxed{y'' + k \cdot y = 0}$$



!

Nota bene:

Our coordinates are Amplitude and Angle



hor. Amplitude $\quad x \qquad \dots \quad \dots \quad \dots \qquad [mm]$

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{v_x}{v_s} = \frac{p_x}{p_s} \approx \frac{p_x}{p_0} \qquad [m\ rad]$$

vert. Amplitude $\quad y \qquad [mm]$

$$y' = \frac{dy}{ds} = \frac{dy}{dt}\frac{dt}{ds} = \frac{v_y}{v_s} = \frac{p_y}{p_s} \approx \frac{p_y}{p_0} \qquad [m\ rad]$$

Remark:

* $$x'' + x(\frac{1}{\rho^2} - k) = 0$$

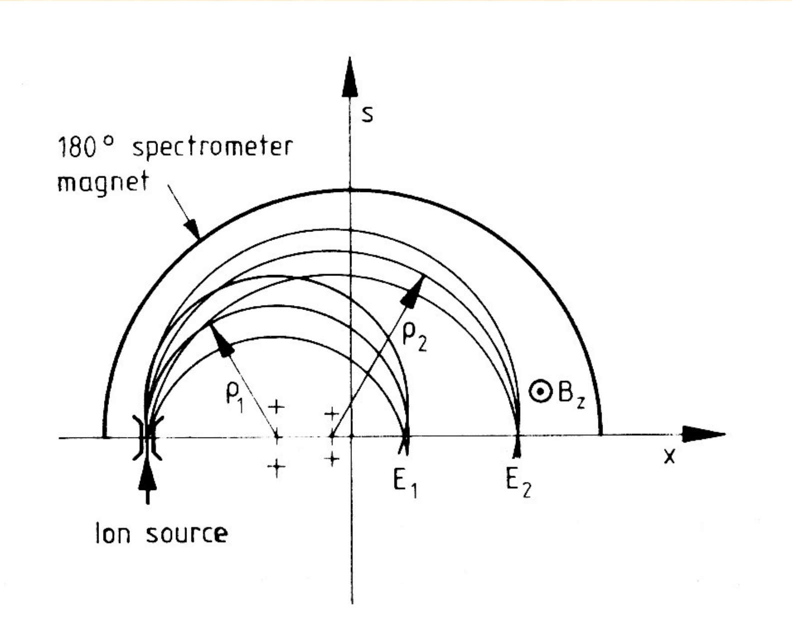
… there seems to be a focusing even without a quadrupole gradient

„weak focusing of dipole magnets“

$$\boldsymbol{k} = 0 \quad \Rightarrow \quad \boldsymbol{x}'' = -\frac{1}{\rho^2}\,\boldsymbol{x}$$

even without quadrupoles there is a retriving force (i.e. focusing) in the bending plane of the dipoles

… in large machines it is weak. *(!)*



Mass spectrometer: particles are separated according to their energy and focused due to the 1/ρ effect of the dipole

7.) Solution of Trajectory Equations

Define … hor. plane: $K = (\frac{1}{\rho^2} - k)$

… vert. Plane: $K = k$

$$\boldsymbol{x'' + K\ x = 0}$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz: $x(s) = a_1 \cdot \cos(\omega\, s) + a_2 \cdot \sin(\omega\, s)$

general solution: linear combination of two independent solutions

$$x'(s) = -a_1\omega \sin(\omega\, s) + a_2\omega \cos(\omega\, s)$$

$$x''(s) = -a_1\omega^2 \cos(\omega\, s) - a_2\omega^2 \sin(\omega\, s) = -\omega^2 x(s) \longrightarrow \omega = \sqrt{K}$$

general solution:

$$x(s) = a_1 \cos(\sqrt{K}s) + a_2 \sin(\sqrt{K}s)$$

determine a_1 , a_2 by boundary conditions:

$$s = 0 \longrightarrow \begin{cases} x(0) = x_0 \quad , \quad a_1 = x_0 \\ x'(0) = x'_0 \quad , \quad a_2 = \dfrac{x'_0}{\sqrt{K}} \end{cases}$$

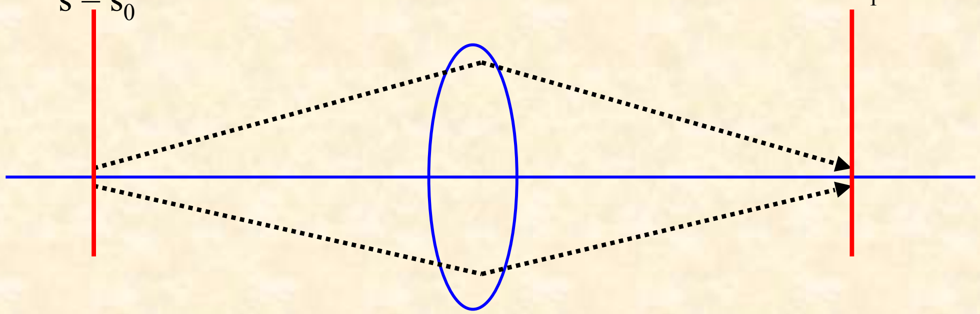
Hor. Focusing Quadrupole K > 0:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

For convenience expressed in matrix formalism:

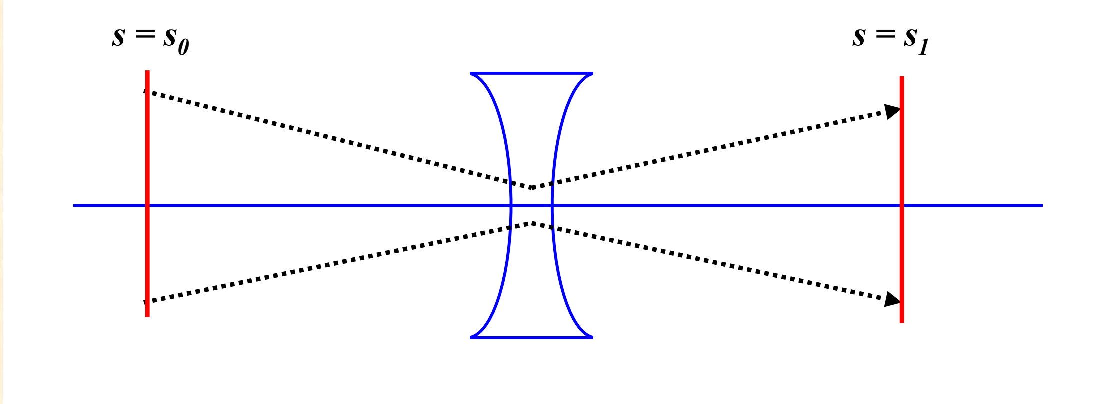
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$\boldsymbol{M_{foc}} = \begin{pmatrix} \cos\left(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}\right) & \dfrac{1}{\sqrt{|\boldsymbol{K}|}} \sin\left(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}\right) \\ -\sqrt{|\boldsymbol{K}|} \sin\left(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}\right) & \cos\left(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}\right) \end{pmatrix}$$

hor. defocusing quadrupole:

$$x'' - K\,x = 0$$



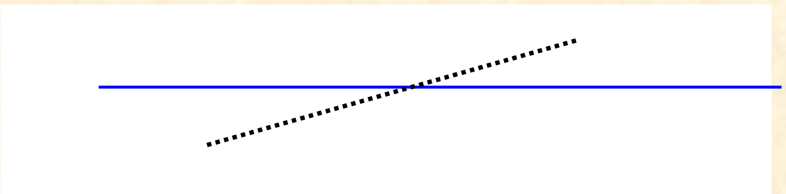
Remember from school:

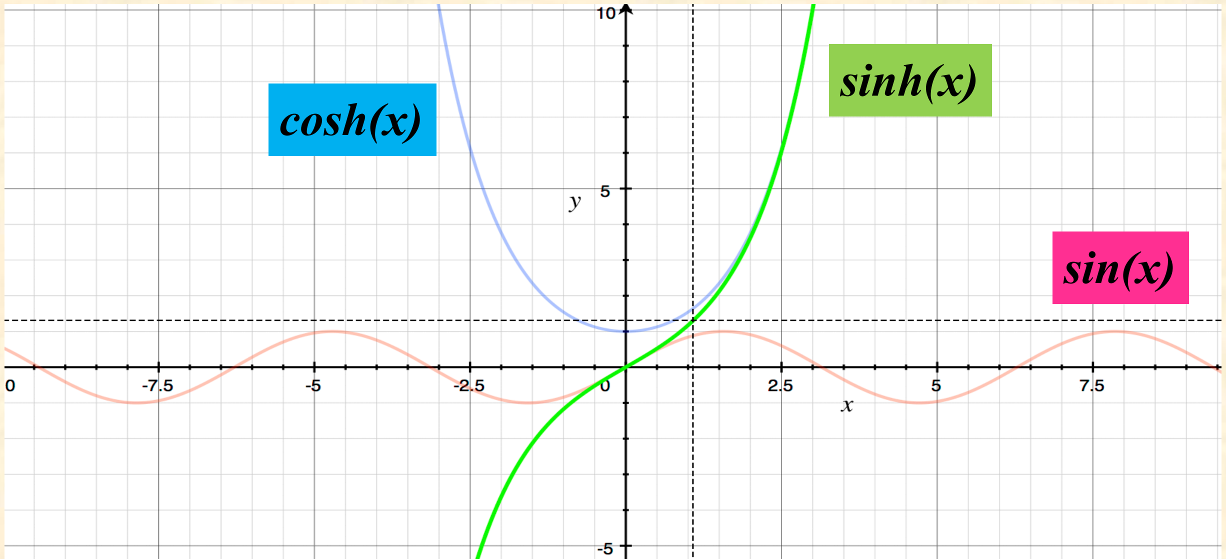
$$f(s) = \cosh(s) \quad , \quad f'(s) = \sinh(s)$$

Ansatz: $$x(s) = a_1 \cdot \cosh(\omega\, s) + a_2 \cdot \sinh(\omega\, s)$$

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\,l \\ \sqrt{|K|}\sinh\sqrt{|K|}\,l & \cosh\sqrt{|K|}\,l \end{pmatrix}$$

drift space: ***K = 0***





$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

! ***with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled“***

One word for the Math Lovers

We talk about a differential equation of second order.
... which has two independent solutions.

$$\boxed{x'' + K\, x = 0}$$

e.g. hor. Focusing Quadrupole $K > 0$:

$$x(s) = \underbrace{x_0 \cdot \cos(\sqrt{|K|}s)}_{C} + \underbrace{x'_0 \cdot \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s)}_{S}$$

$$x'(s) = \underbrace{-x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s)}_{C'} + \underbrace{x'_0 \cdot \cos(\sqrt{|K|}s)}_{S'}$$

Wronski tells us:
The two solutions are independent of each other if the Wronski determinant $\neq 0$.

Each of the two solutions fulfils

$$C'' + K(s)C = 0 \qquad S'' + K(s)S = 0$$

$$W = \left|\begin{pmatrix} C & S \\ C' & S' \end{pmatrix}\right| \quad \longrightarrow \quad \frac{d}{ds}W = CS'' - SC'' = -K(CS - SC) = 0$$

So, W= const.
We can choose the initial values at s=0

$$\left.\begin{matrix} C_0 = 1 & S_0 = 0 \\ C'_0 = 0 & S'_0 = 1 \end{matrix}\right\} \longrightarrow$$

W=1 for all linear accelerator element matrices

$$W = det\begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = 1$$

8.) Thin Lens Approximation:

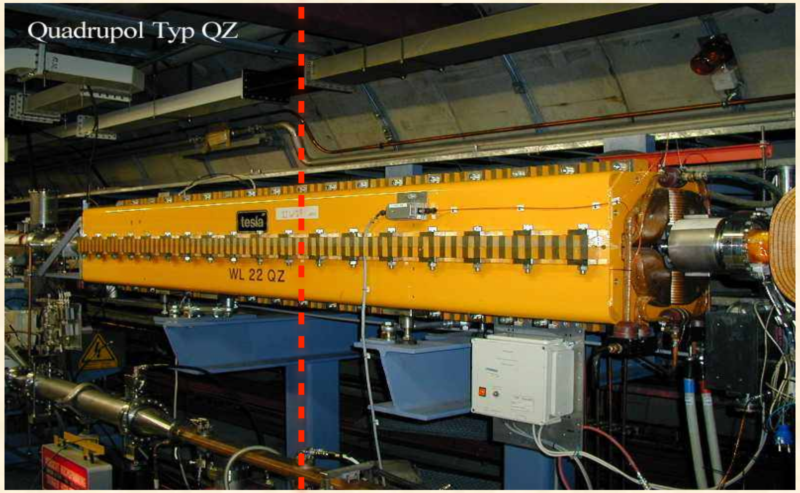
matrix of a quadrupole lens

$$M = \begin{pmatrix} \cos\sqrt{|k|}\,l & \frac{1}{\sqrt{|k|}}\sin\sqrt{|k|}\,l \\ -\sqrt{|k|}\sin\sqrt{|k|}\,l & \cos\sqrt{|k|}\,l \end{pmatrix}$$

in many practical cases we have the situation:

$f = \frac{1}{k l_q} >> l_q$ ... *focal length of the lens is much bigger than the length of the magnet*

limes: $l_q \rightarrow 0$ *while keeping* $k\, l_q = const$



$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}$$

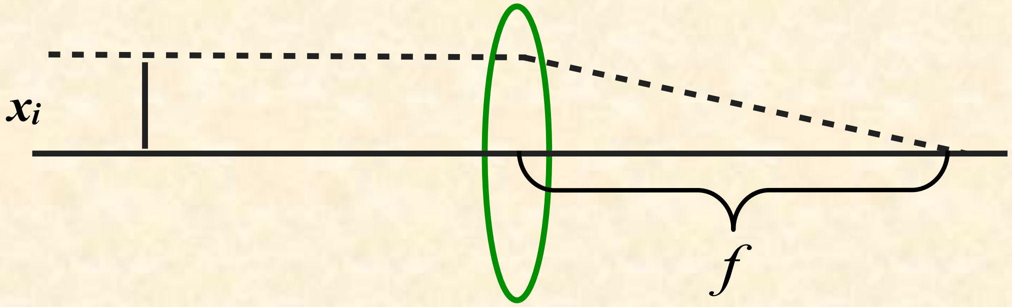
$$M_y = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

... useful for fast (and in large machines still quite accurate) „back on the envelope calculations“ ... and for the guided studies !

Focal Length of a Quadrupole:

matrix of a (thin) quadrupole lens

$$\boldsymbol{M_x} = \begin{pmatrix} 1 & 0 \\ \dfrac{-1}{\boldsymbol{f}} & 1 \end{pmatrix} \qquad f = \frac{1}{kl_q}$$



Definition of focal length:

a trajectory with amplitude x_0 parallel to "s" will be focussed to $x_i = 0$ within the length f

$$x_i - x_f' \cdot f = 0 \qquad x_f' = \frac{l}{\rho} = \frac{Bl}{B\rho} \qquad \textit{angle in a circle}$$

$$x_f' = \frac{x_i\, g\, l_q}{B\rho} = x_i k l_q$$

$$x_i - x_i \cdot kl_q \cdot f = 0 \qquad 1 - kl_q \cdot f = 0 \longrightarrow \boxed{f = \frac{1}{kl_q}}$$

„veni vidi vici ...“ .... or in english .... „we got it !“

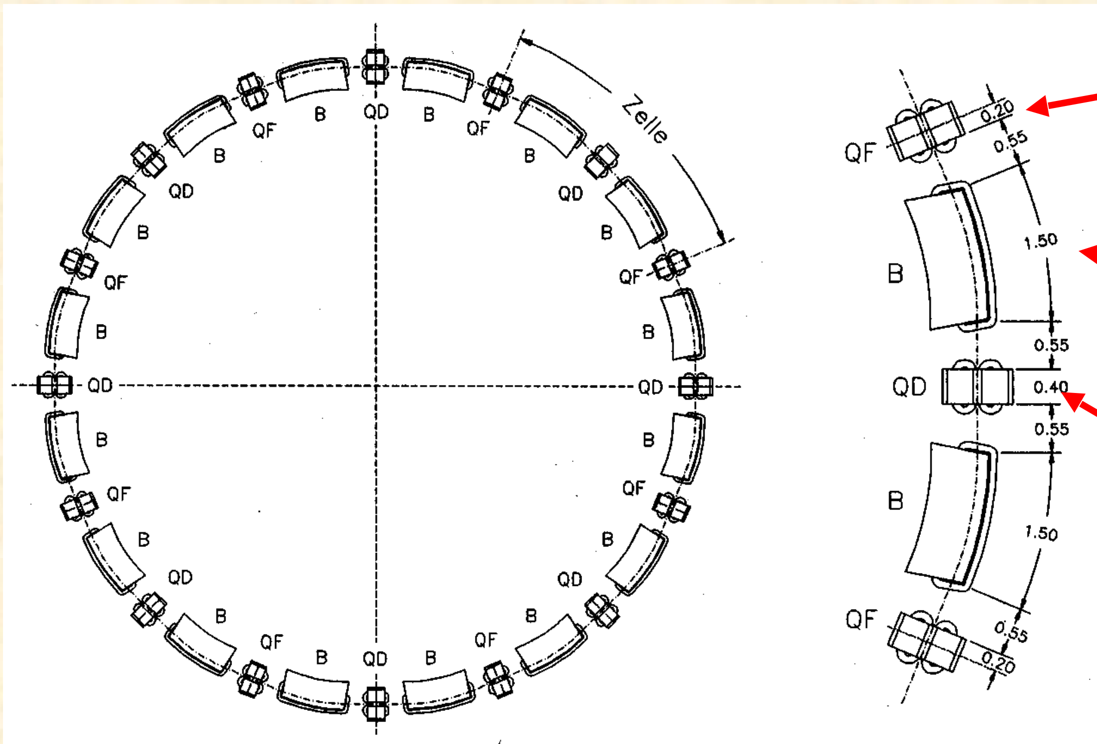
** we can calculate the trajectory of a single particle, inside a storage ring magnet (lattice element)*

** for arbitrary initial conditions x_0, x'_0*

** we can combine these trajectory parts (also mathematically) and so get the complete transverse trajectory around the storage ring*

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D^*} .....$$

Example:
Toy storage ring
for the kids
(court. K.Wille)



horizontal focussing quadrupole lens

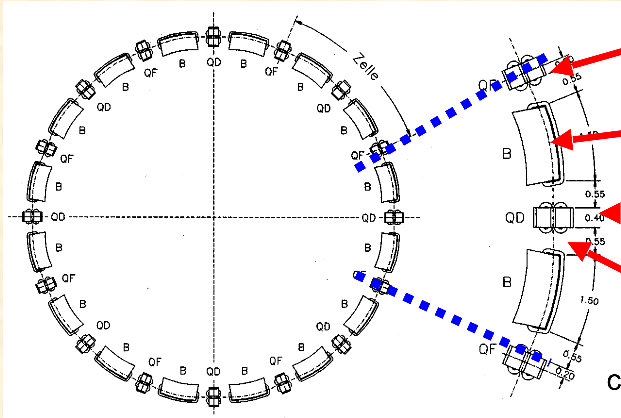
dipole magnet

horizontal defocussing quadrupole lens

Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D^*.....}$$



court. K. Wille

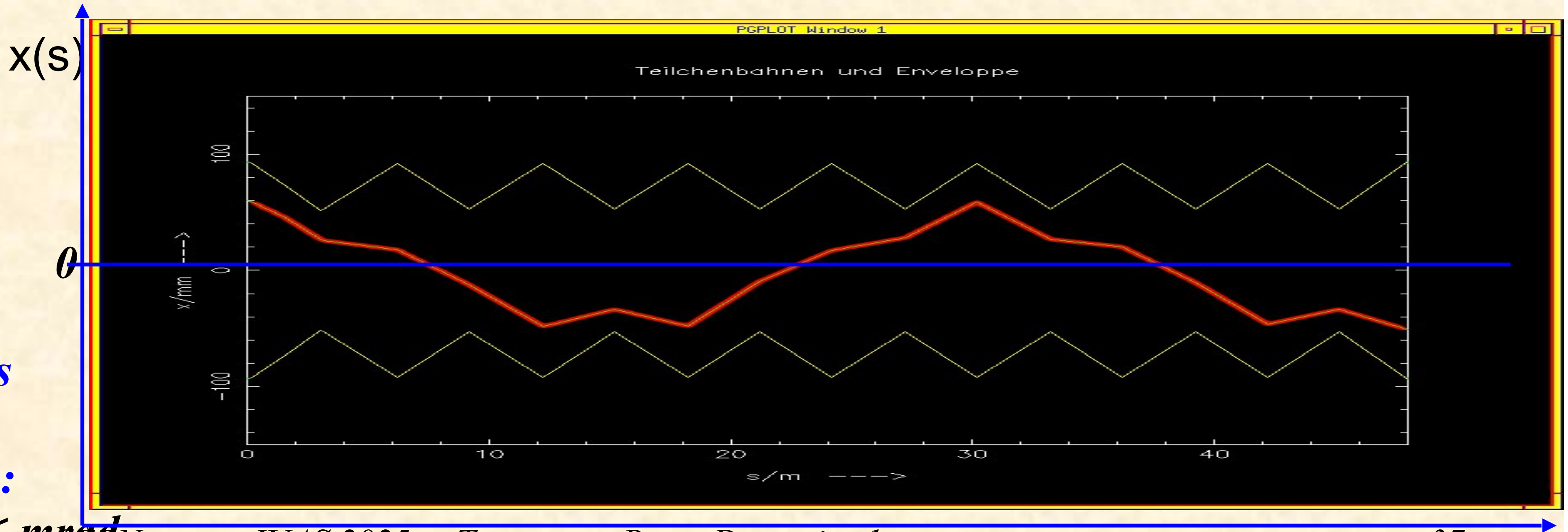
focusing lens

dipole magnet

defocusing lens

drift

in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „



typical values in a strong foc. machine:

$x \approx$ mm, $x' \leq$ mrad

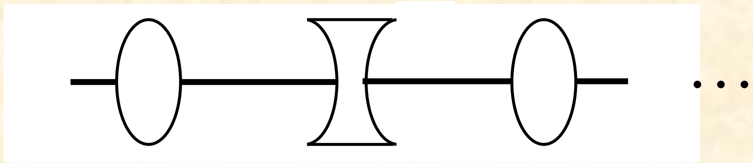
Example: Product Matrix of many Accelerator Elements:

matrices

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l_q) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l_q) \\ -\sqrt{K} \sin(\sqrt{K} * l_q) & \cos(\sqrt{K} * l_q) \end{pmatrix}, \qquad M_{QD} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}} \sinh\sqrt{|K|}l \\ \sqrt{|K|} \sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix} \qquad M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1_d \end{pmatrix}$$

put in some "reasonable" numbers *K = +/- 0.54102 m*$^{-2}$
lq = 0.5 m
ld = 2.5 m

The matrix for a certain "sequence" is obtained by multiplication of the element matrices



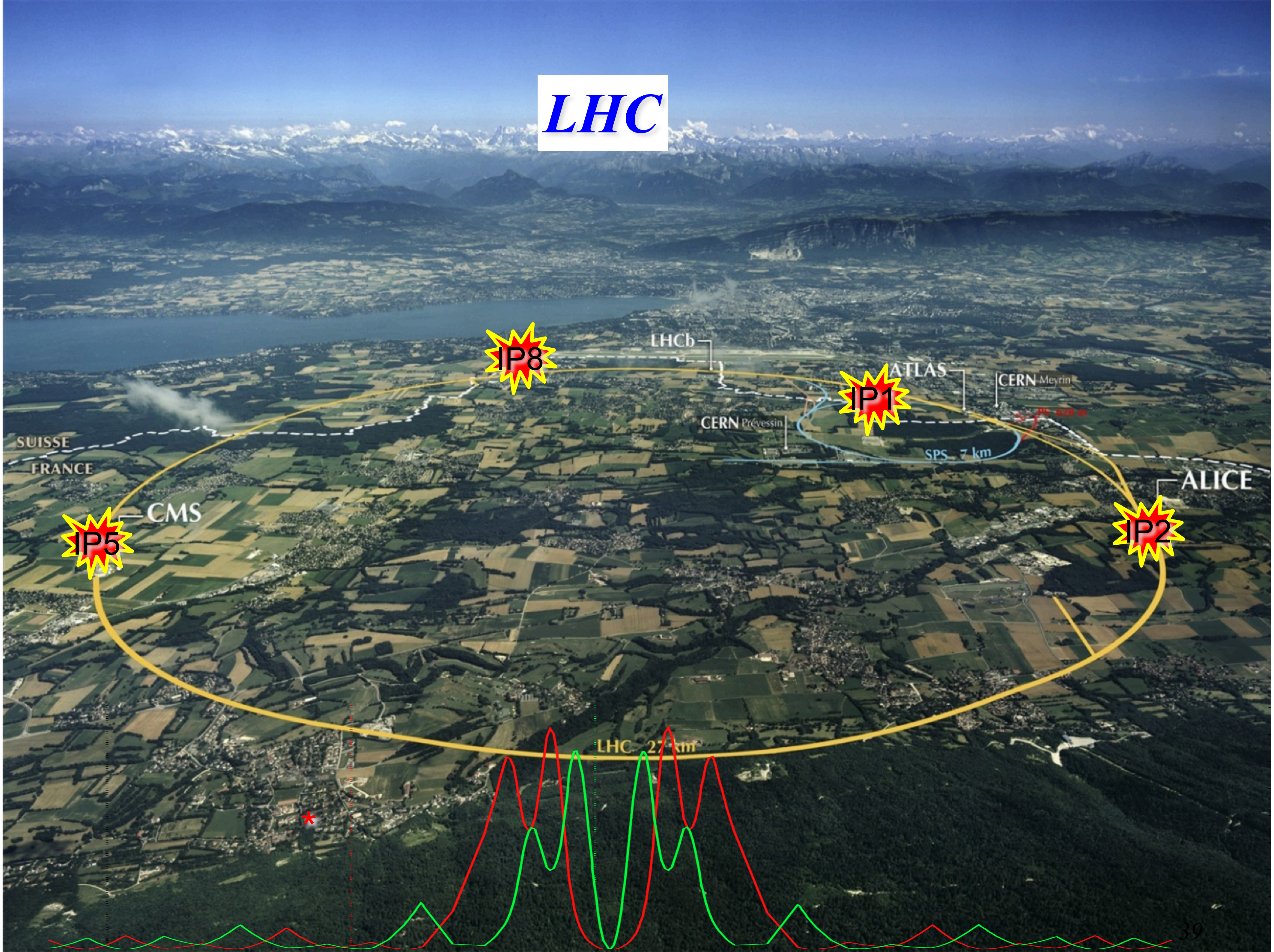
$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qfh} \;\ldots$$

Putting the numbers in and multiplying out we get e.g. …

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

… and so we can calculate position and angle (x,x') anywhere in the ring by using the product matrix in between two points.

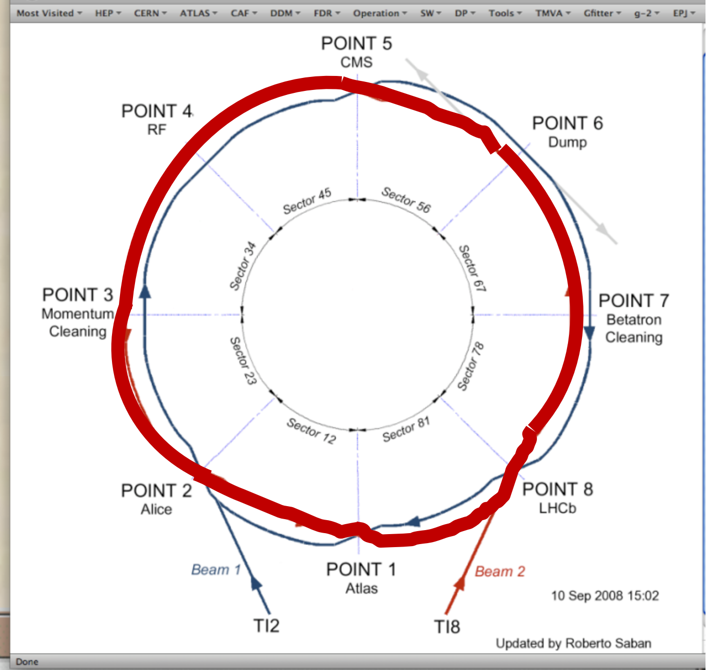
LHC

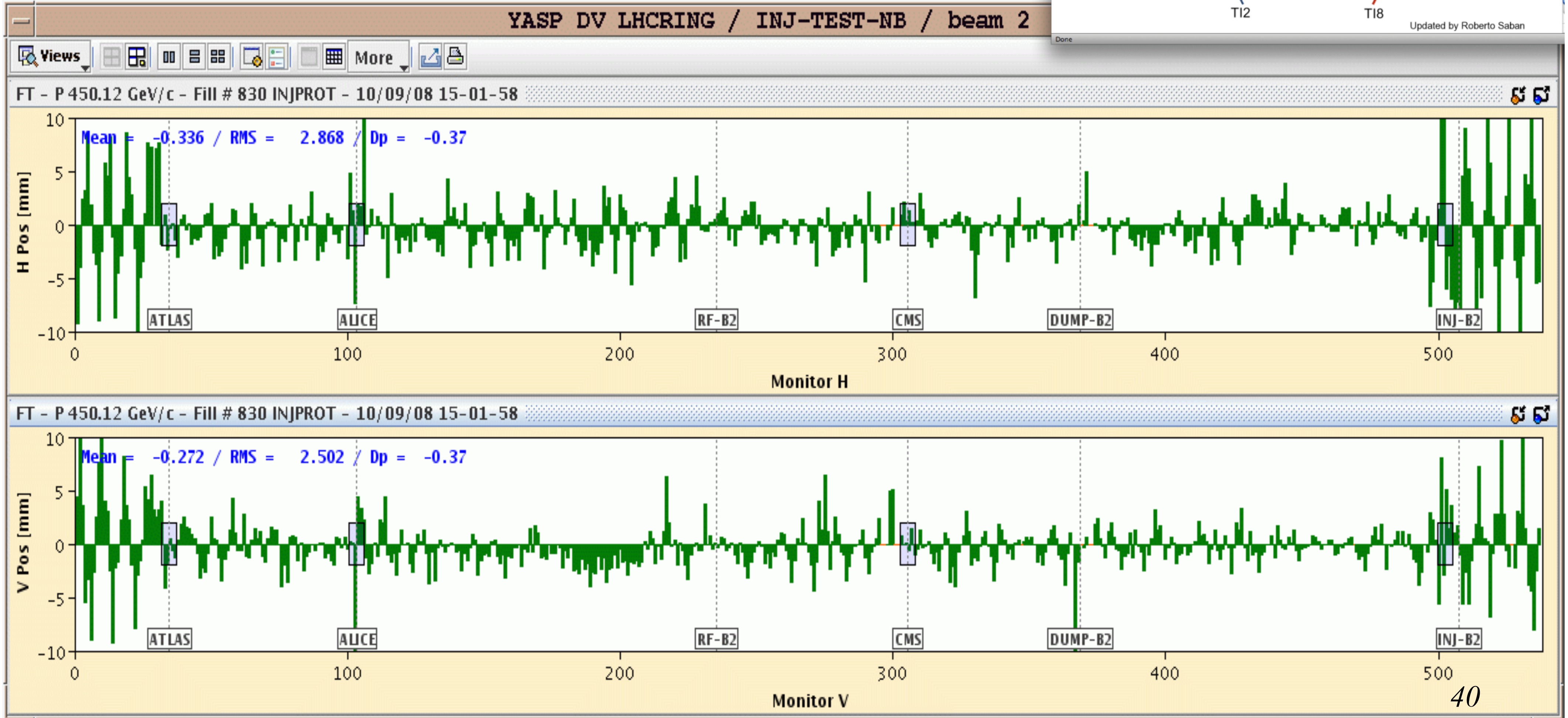

LHCb
IP8
ATLAS
IP1
CERN Meyrin
CERN Prévessin
SPS 7 km
SUISSE
FRANCE
CMS
IP5
ALICE
IP2
LHC 27 km

LHC Operation: Beam Commissioning

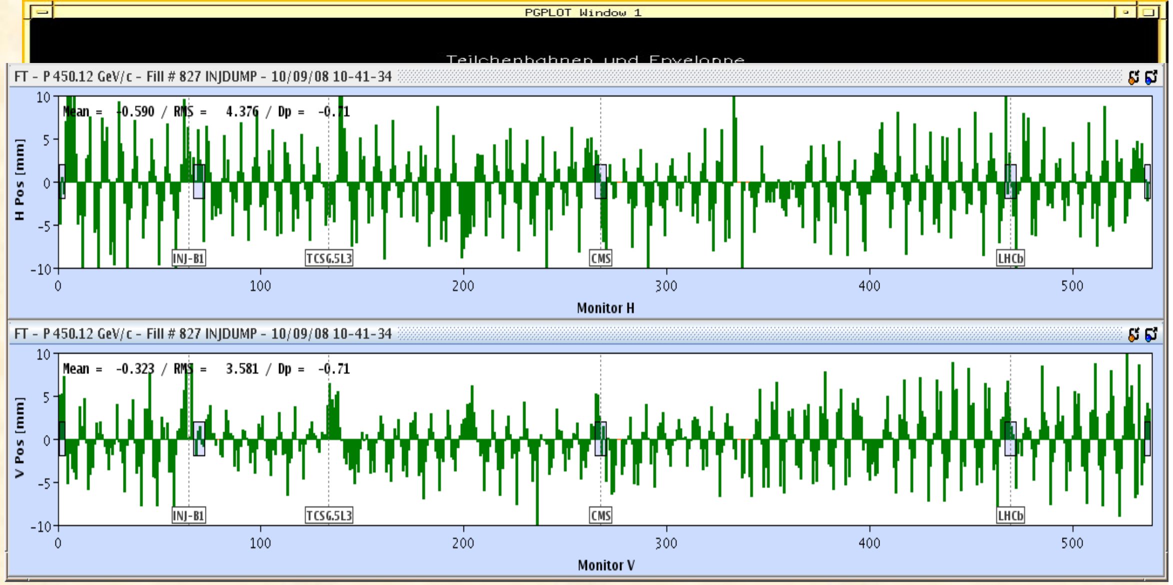
First turn steering "by sector:"

- One beam at the time
- Beam through 1 sector (1/8 ring),
correct trajectory, open collimator and move on.





9.) Orbit & Tune:

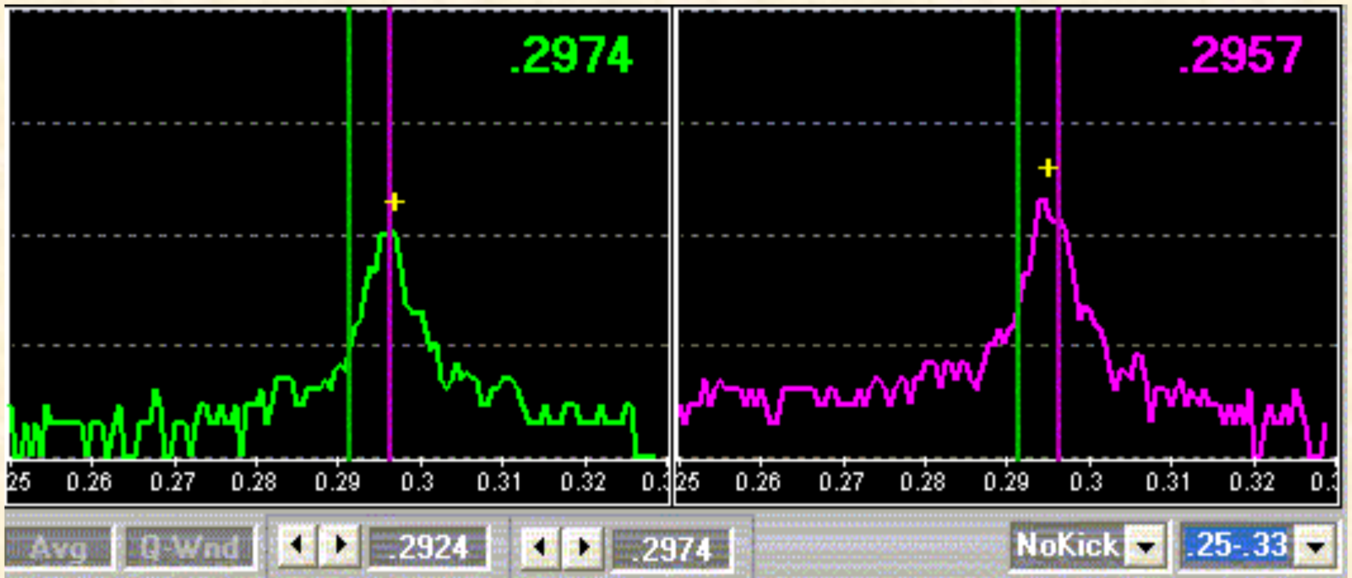


Tune: number of oscillations per turn

64.31
59.32

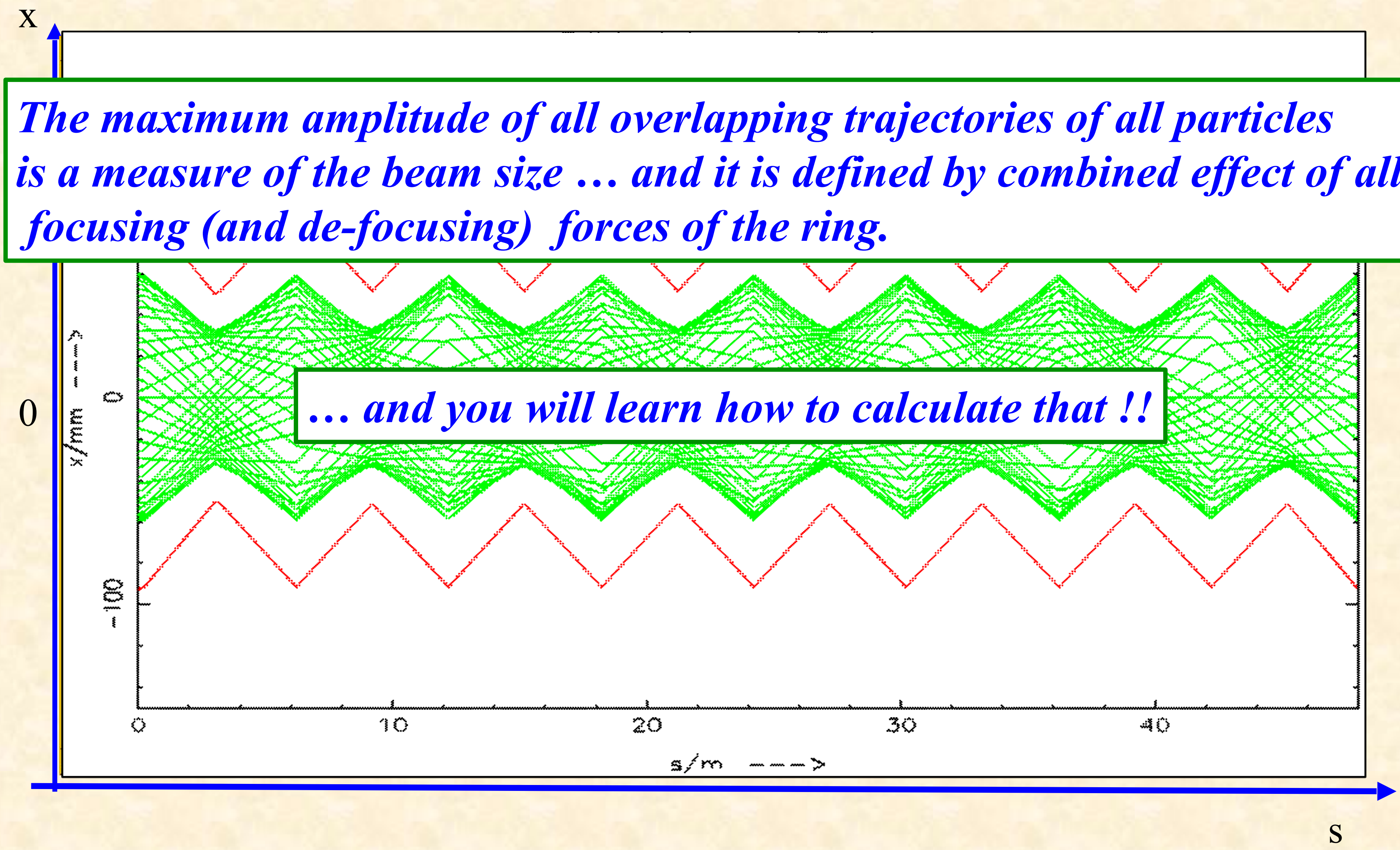
Relevant for beam stability:
non integer part

LHC revolution frequency: 11.3 kHz



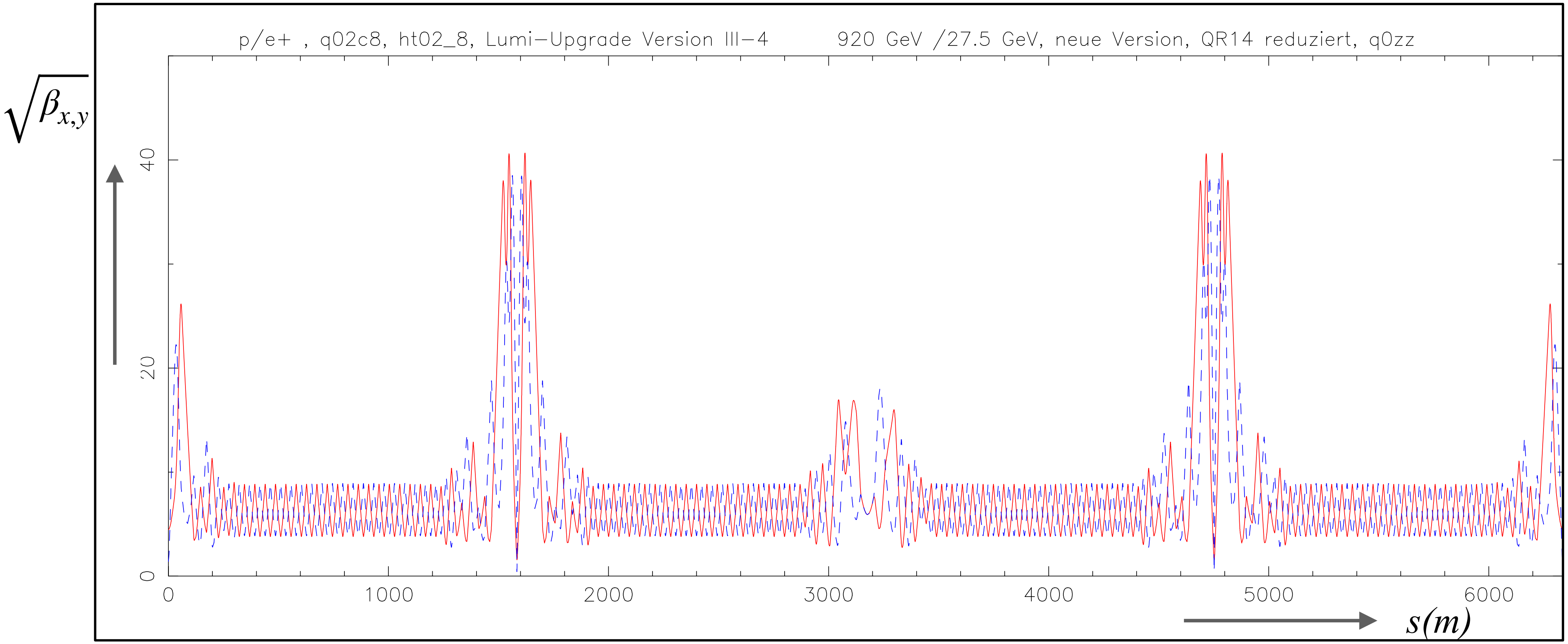
Question: ***what will happen, if the particle performs a second turn ?***

... or a third one or ... 10^{10} turns

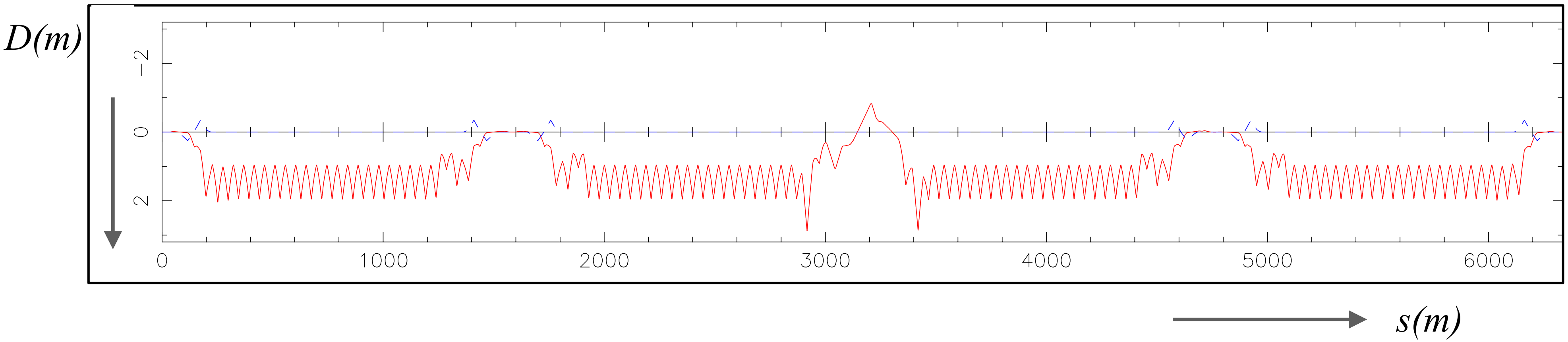


The maximum amplitude of all overlapping trajectories of all particles is a measure of the beam size … and it is defined by combined effect of all focusing (and de-focusing) forces of the ring.

… and you will learn how to calculate that !!



$\sqrt{\beta_{x,y}}$
p/e+ , q02c8, ht02_8, Lumi–Upgrade Version III–4
920 GeV /27.5 GeV, neue Version, QR14 reduziert, q0zz
0
20
40
0
1000
2000
3000
4000
5000
6000
s(m)



D(m)
−2
0
2
0
1000
2000
3000
4000
5000
6000
s(m)

Résumé:

beam rigidity: $B \cdot \rho = p/q$

bending strength of a dipole: $\frac{1}{\rho}\left[m^{-1}\right] = \frac{0.2998 \cdot B_0(T)}{p(GeV/c)}$

focusing strength of a quadrupole: $k\left[m^{-2}\right] = \frac{0.2998 \cdot g}{p(GeV/c)}$

focal length of a quadrupole: $f = \frac{1}{k \cdot l_q}$

equation of motion: $x'' + Kx = \frac{1}{\rho}\frac{\Delta p}{p}$

matrix of a foc. quadrupole: $x_{s2} = M \cdot x_{s1}$

$$M = \begin{pmatrix} \cos\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sin\sqrt{|K|}l \\ -\sqrt{|K|}\sin\sqrt{|K|}l & \cos\sqrt{|K|}l \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

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