LONGITUDINAL BEAM DYNAMICS

JUAS 2025

COURSE 1: THE SCIENCE OF PARTICLE ACCELERATORS

A. Lasheen

CERN

juas
Joint Universities Accelerator School

INTRODUCTION

ACKNOWLEDGEMENTS

JUAS FORMER LECTURERS AND THEIR LEGACY

ELIAS, BENOIT, BIRK, DAVID, AND LEANDRO FOR THEIR SUPPORT

THE CERN ACCELERATOR SCHOOL AND ITS NUMEROUS REFERENCES

COLLEAGUES FROM THE RF AND ABP GROUPS, AND THE BR SECTION AT CERN

AND YOU!

RESOURCES

WEB

- E. Metral website, JUAS courses, exercises, exams and corrections

COURSES

- G. Dôme, Theory of RF Acceleration
- L. Rinolfi, Longitudinal Beam Dynamics Application to synchrotron
- F. Tecker, Longitudinal Beam Dynamics in Circular Accelerators
- B. Holzer, Introduction to Longitudinal Beam Dynamics
- H. Damerau, Introduction to Non-linear Longitudinal Beam Dynamics
- R. Garoby, RF Gymnastics in Synchrotrons
- B. W. Montague, Single particle dynamics : Hamiltonian formulation
- W. Pirkl, Longitudinal beam dynamics
- J. Le Duff, Longitudinal beam dynamics in circular accelerators
- E. Jensen, RF Cavity Design

RESOURCES

NOTES

- H. G. Hereward, What are the equations for the phase oscillations in a synchrotron?
- J. A. MacLachlan, Difference Equations for Longitudinal Motion in a Synchrotron
- J. A. MacLachlan, Differential Equations for Longitudinal Motion in a Synchrotron
- C. Bovet, R. Gouiran, I. Gumowski, K. H. Reich, A selection of formulae and data useful for the design of A.G. synchrotrons

BOOKS

- A. A. Kolomensky, A. N. Lebedev, Theory of Cyclic Accelerators
- H. Bruck, Accelerateurs Circulaires De Particules
- S. Y. Lee, Accelerator Physics
- S. Humphries, Principles of Charged Particle Acceleration
- T. P. Wangler, RF Linear Accelerators
- H. Wiedemann, Particle Accelerator Physics
- M. Reiser, Theory and Design of Charged Particle Beams

SCHEDULE

COURSE CONTENT

- 1 Introductory session
- 10 Teaching modules including
  - Lecture
  - Derivations
  - Computational exercises
  - Quizz
  - Interleaving exercises with lecture. The last slot of each afternoon dedicated to tutorials/questions.
- Exam preparation
- PyHEADTAIL workshop

SCHEDULE

WEEK 1

	13 Jan. **Monday**	14 Jan. **Tuesday**	15 Jan. **Wednesday**	16 Jan. **Thursday**	17 Jan. **Friday**
09:00 - 09:15		**Transverse Beam Dynamics** B. Holzer	**Transverse Beam Dynamics** B. Holzer	**Transverse Beam Dynamics** B. Holzer	**Transverse Beam Dynamics** B. Holzer
09:15 - 09:30					
09:30 - 09:45	**OFFICIAL OPENING: Presentation of JUAS & Introduction of students** *E. Métral, F. Mouthon, D. Baizhanova*				
09:45 - 10:00					
10:00 - 10:15		**Transverse Beam Dynamics** B. Holzer	**Transverse Beam Dynamics** B. Holzer	**Transverse Beam Dynamics** B. Holzer	**Transverse Beam Dynamics** B. Holzer
10:15 - 10:30					
10:30 - 10:45					
10:45 - 11:00					
11:00 - 11:15		**Transverse Beam Dynamics** B. Holzer	**Transverse Beam Dynamics** B. Holzer	**Transverse Beam Dynamics** B. Holzer	**Longitudinal Beam Dynamics** A. Lasheen
11:15 - 11:30					
11:30 - 11:45					
11:45 - 12:00					
12:00 - 12:15	**LUNCH (TBC)**	LUNCH	LUNCH	LUNCH	LUNCH
12:15 - 12:30					
12:30 - 12:45					
12:45 - 13:00					
13:00 - 13:15					
13:15 - 13:30	**Special relativity, electromagnetism, classical & quantum mechanics: What to remember for particle accelerators** *E. Métral*	**Longitudinal Beam Dynamics** A. Lasheen	**Longitudinal Beam Dynamics** A. Lasheen	**Longitudinal Beam Dynamics** A. Lasheen	**Longitudinal Beam Dynamics** A. Lasheen
13:30 - 13:45					
13:45 - 14:00					
14:00 - 14:15					
14:15 - 14:30		**Longitudinal Beam Dynamics** A. Lasheen	**Longitudinal Beam Dynamics** A. Lasheen	**Longitudinal Beam Dynamics** A. Lasheen	**Longitudinal Beam Dynamics** A. Lasheen
14:30 - 14:45					
14:45 - 15:00					
15:00 - 15:15					
15:15 - 15:30	**Particle Accelerators in the 21st century** (Seminar) *M. Vretenar*	**Longitudinal Beam Dynamics** A. Lasheen	**Introduction to CERN & its Accelerator Complex** (Seminar) *R. Steerenberg*	**Longitudinal Beam Dynamics** A. Lasheen	**Electron-positron circular colliders** (Seminar) *J. Keintzel*
15:30 - 15:45					
15:45 - 16:00					
16:00 - 16:15					
16:15 - 16:30					**Evaluation #1 (weekend)**

SCHEDULE

WEEK 2

	20 Jan. **Monday**	21 Jan. **Tuesday**	22 Jan. **Wednesday**	23 Jan. **Thursday**	24 Jan. **Friday**
09:00 - 09:15 09:15 - 09:30 09:30 - 09:45 09:45 - 10:00	**Transverse Beam Dynamics** B. Holzer	**Longitudinal Beam Dynamics** A. Lasheen	**Linacs** D. Alesini	**Linacs** D. Alesini	**Linacs** D. Alesini
10:00 - 10:15 10:15 - 10:30 10:30 - 10:45 10:45 - 11:00	**Transverse Beam Dynamics** B. Holzer	**Longitudinal Beam Dynamics** A. Lasheen	**Linacs** D. Alesini	**Linacs** D. Alesini	**Linacs** D. Alesini
11:00 - 11:15 11:15 - 11:30 11:30 - 11:45 11:45 - 12:00	**Introduction to MAD-X** N. Fuster Martinez	**Introduction to PyHeadTail** B. Salvant	**Linacs** D. Alesini	**Linacs** D. Alesini	**Transverse linear imperfections** D. Gamba
12:00 - 12:15 12:15 - 12:30 12:30 - 12:45 12:45 - 13:00 13:00 - 13:15	LUNCH	LUNCH	LUNCH	LUNCH	LUNCH
13:15 - 13:30 13:30 - 13:45 13:45 - 14:00 14:00 - 14:15	**MAD-X workshop** N. Fuster Martinez	**Introduction to PyHeadTail** B. Salvant	**Transverse linear imperfections** D. Gamba	**Transverse linear imperfections** D. Gamba	**Transverse linear imperfections** D. Gamba
14:15 - 14:30 14:30 - 14:45 14:45 - 15:00 15:00 - 15:15	**MAD-X workshop** N. Fuster Martinez	**Introduction to PyHeadTail** B. Salvant	**Transverse linear imperfections** D. Gamba	**Transverse linear imperfections** D. Gamba	**Transverse linear imperfections** D. Gamba
15:15 - 15:30 15:30 - 15:45 15:45 - 16:00 16:00 - 16:15	**MAD-X workshop** N. Fuster Martinez	**Introduction to PyHeadTail** B. Salvant	**The US Electron-Ion collider** *- Zoom PPT -* (Seminar) T. Satogata	**Transverse linear imperfections** D. Gamba	**Future high-energy linear colliders** (Seminar) P. Burrows
16:15 - 16:30					**Evaluation #2 (weekend)**

SCHEDULE

WEEK 3

	27 Jan. Monday	28 Jan. Tuesday	29 Jan. Wednesday	30 Jan. Thursday	31 Jan. Friday
09:00 - 09:15	**WRITTEN EXAMINATION (09:00 - 11:00) Transverse beam dynamics**	**Cyclotrons & FFAs** *A. Gerbershagen*	**Synchrotron Radiation** *R. Ischebeck*	**Synchrotron Radiation** *R. Ischebeck*	**Synchrotron Radiation** *R. Ischebeck*
09:15 - 09:30					
09:30 - 09:45					
09:45 - 10:00					
10:00 - 10:15		**Cyclotrons & FFAs** *A. Gerbershagen*	**Synchrotron Radiation** *R. Ischebeck*	**Synchrotron Radiation** *R. Ischebeck*	**Synchrotron Radiation** *R. Ischebeck*
10:15 - 10:30					
10:30 - 10:45					
10:45 - 11:00					
11:00 - 11:15		**Cyclotrons & FFAs** *A. Gerbershagen*	**Synchrotron Radiation** *R. Ischebeck*	**Synchrotron Radiation** *R. Ischebeck*	**Synchrotron Radiation** *R. Ischebeck*
11:15 - 11:30					
11:30 - 11:45					
11:45 - 12:00					
12:00 - 12:15		LUNCH	LUNCH	LUNCH	LUNCH
12:15 - 12:30					
12:30 - 12:45					
12:45 - 13:00					
13:00 - 13:15					
13:15 - 13:30		**Fixed-target beamlines** (Seminar) A. Gerbershagen	**Synchrotron Radiation** *R. Ischebeck*	**Synchrotron Radiation** *R. Ischebeck*	**Synchrotron Radiation** *R. Ischebeck*
13:30 - 13:45					
13:45 - 14:00					
14:00 - 14:15					
14:15 - 14:30	**WRITTEN EXAMINATION (14:15 - 16:15) Longitudinal beam dynamics**		**Transverse nonlinear effects** *S. Kostoglou*	**Transverse nonlinear effects** *S. Kostoglou*	**Transverse nonlinear effects** *S. Kostoglou*
14:30 - 14:45		**Visit CERN: ALICE (J. Jowett)** **Introductory presentation Exhibition & Experiment at CERN LHC**			
14:45 - 15:00					
15:00 - 15:15					
15:15 - 15:30			**Transverse nonlinear effects** *S. Kostoglou*	**Transverse nonlinear effects** *S. Kostoglou*	**Transverse nonlinear manipulations** (Seminar) *M. Giovannozzi*
15:30 - 15:45					
15:45 - 16:00					
16:00 - 16:15					
16:15 - 16:30					**Evaluation #3 (weekend)**
16:30 - 16:45					
16:45 - 17:00					
17:00 - 17:15		Bus transfer CERN (Prévessin)			
17:15 - 17:30		**VISIT: CERN Control Center** (P. Zisopoulos)			
17:30 - 17:45					
17:45 - 18:00		Bus transfer CERN (Meyrin)			
18:00 - 18:15		**VISIT CERN LEIR Accelerator** (M. Slupecki)			
18:15 - 18:30					
18:30 - 18:45					
18:45 - 19:00					
19:00 - 19:15		**Dinner at CERN (TBC)**			
19:15 - 19:30					
19:30 - 19:45					
19:45 - 20:00					
20:00 - 20:15					
20:15 - 20:30					

COURSE LAYOUT

INTRODUCTORY SESSION

- What is longitudinal beam dynamics?
- How does this lecture relates to the others?

LESSON 1 - FUNDAMENTALS OF PARTICLE ACCELERATION

- Fields, forces
- Accelerator designs
- Relativistic relationships

COURSE LAYOUT

LESSON 2 - SYNCHROTRON DESIGN

- Equations for the synchronous particle
- One word on betatronic acceleration, synchrotron radiation, self induced fields
- Momentum compaction, differential relationships

LESSON 3 - LONGITUDINAL EQUATIONS OF MOTION

- Equations for non synchronous particles
- Introduction to tracking

COURSE LAYOUT

LESSON 4 - SYNCHROTRON MOTION

- Linearized synchrotron motion
- Phase stability and synchrotron frequency/tune
- Non-linear synchrotron motion
- RF bucket, longitudinal emittance, non-linear synchrotron frequency

LESSON 5 - REAL LIFE APPLICATIONS

- Longitudinal bunch profile measurements
- Examples of RF operation
- Introduction to RF manipulations ("gymnastics")

TEACHING AGREEMENT

WHAT YOU SHOULD KNOW AT THE END OF THE COURSE

- Understand how a beam is effectively accelerated in a particle accelerator.
- Understand fundamental concepts of longitudinal beam dynamics (i.e. synchrotron motion, the RF bucket and its parameters).
- How main equations/formulas are derived and underlying assumptions.

WHAT YOU SHOULD BE ABLE TO DO AT THE END OF THE COURSE

- Compute RF parameters and basic design parameters of a synchrotron.
- Interpret the longitudinal motion of a measured bunch of particles.

KEY ASPECTS OF LONGITUDINAL BEAM DYNAMICS

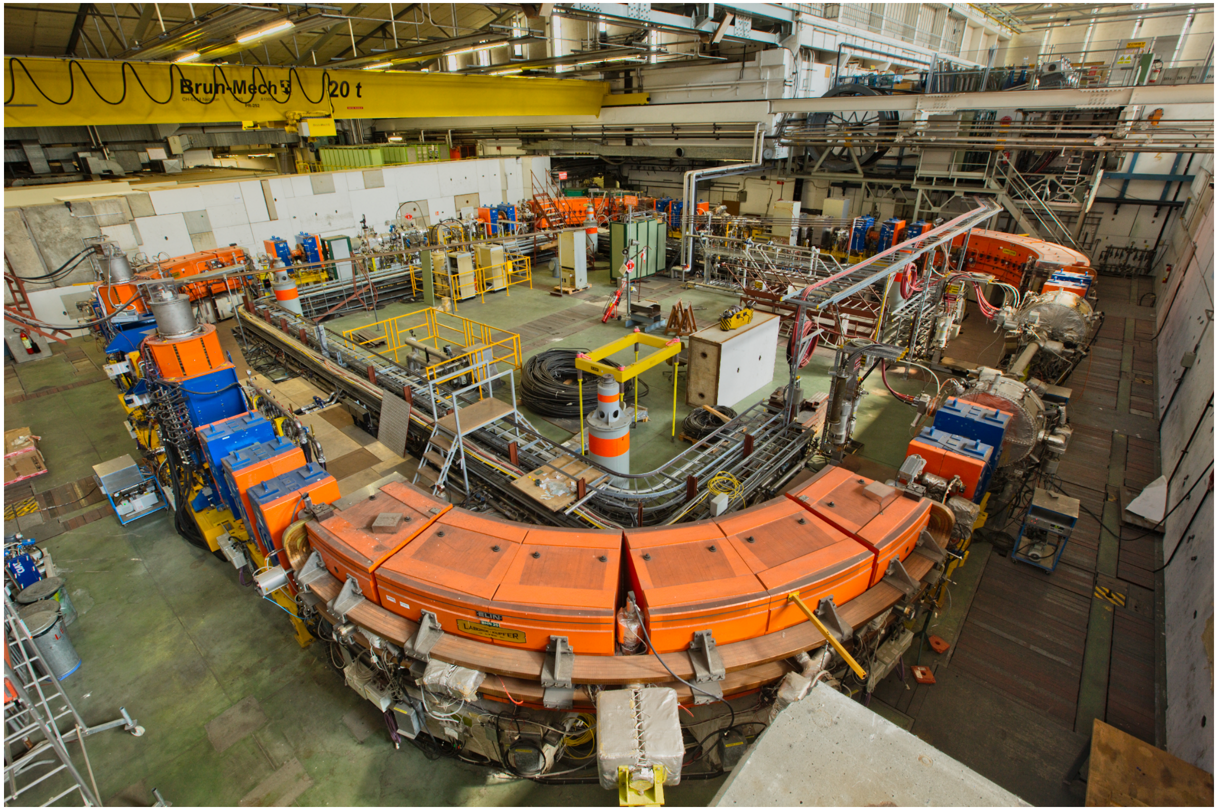
→ **Particle acceleration**

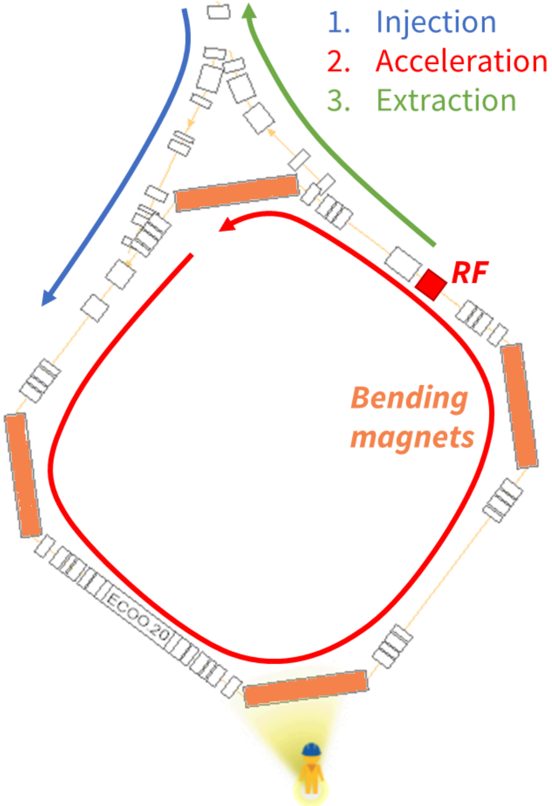
→ **Focusing of particles in the longitudinal direction (bunching)**

→ **Synchrotron motion**

LAYOUT OF A REAL ACCELERATOR

THE LOW ENERGY ION RING (LEIR) AT CERN

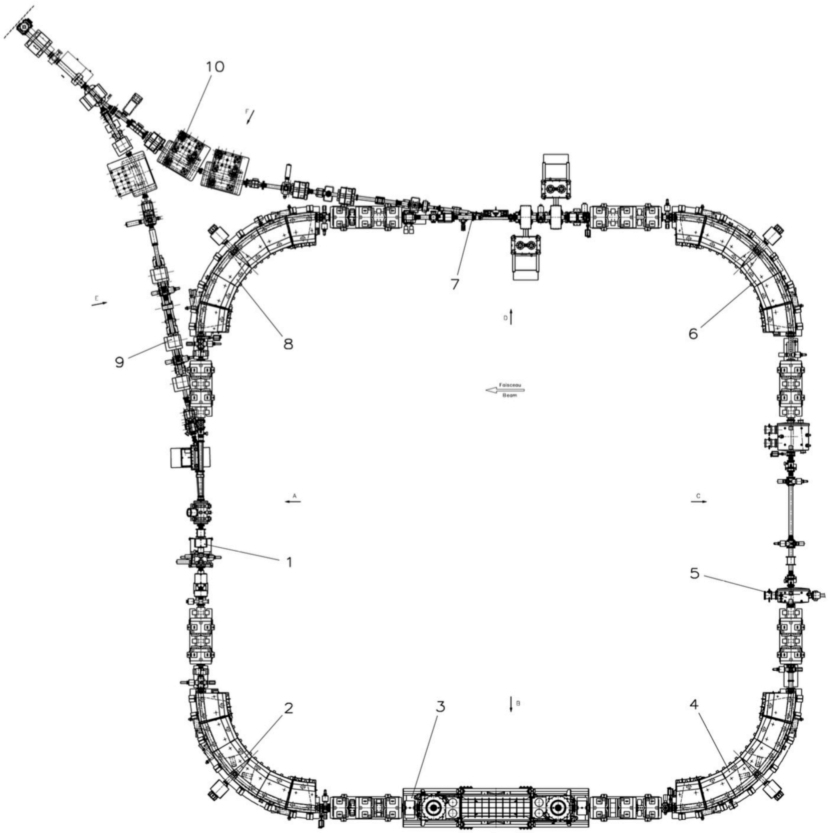


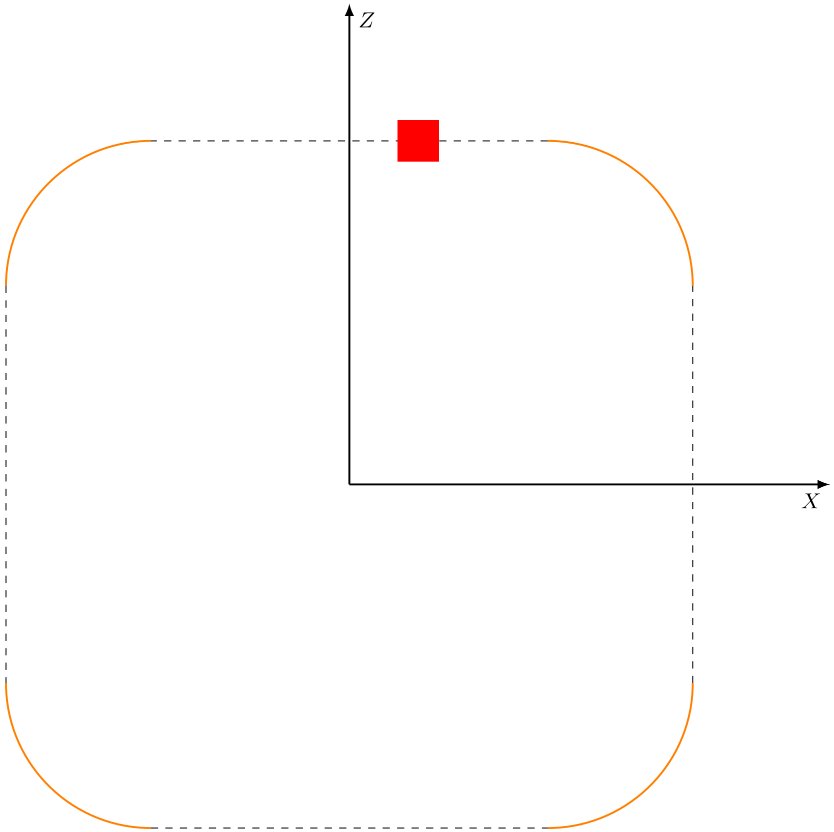


- Virtual walk around LEIR... (visit on the 29/01!)
- To see other accelerators at CERN...

COORDINATE SYSTEM(S)

Accelerator seen from above...

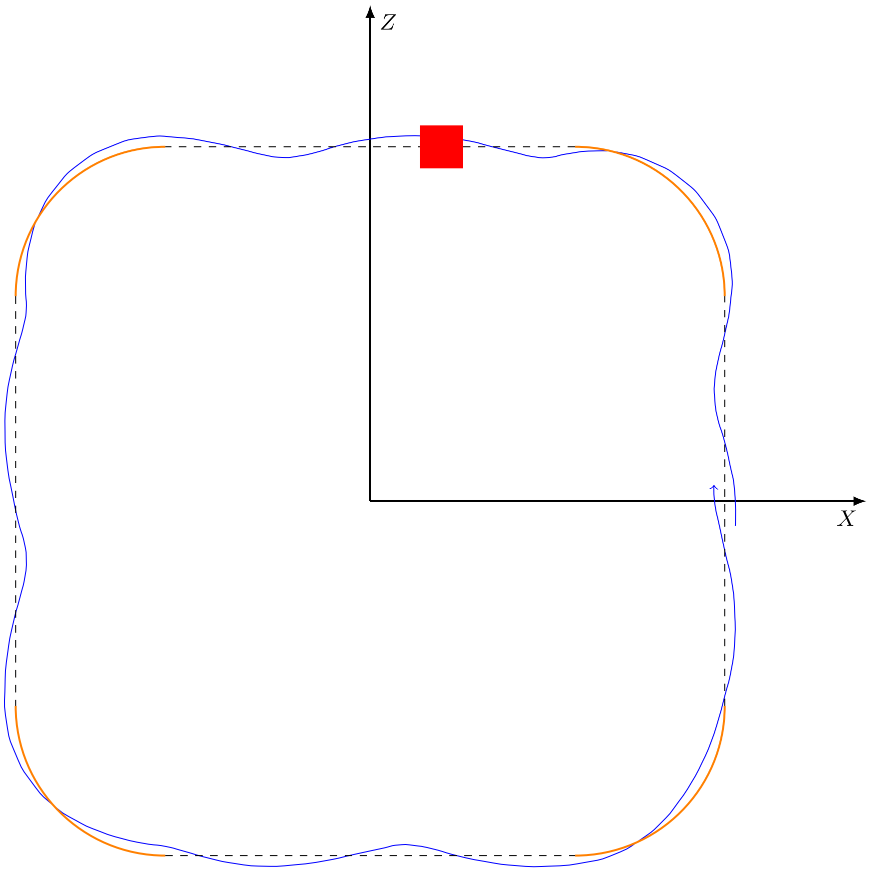




Bending magnets   Accelerating RF cavities

COORDINATE SYSTEM(S)

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



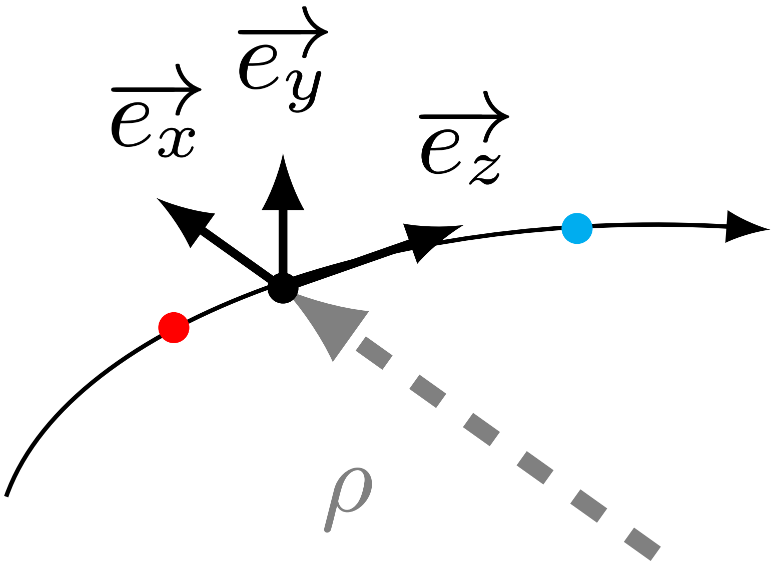
- The black line represents the (ideal) design trajectory of the beam around which a particle oscillate (blue).

- The accelerator layout can be described in fixed cartesian coordinates $\left(\vec{X}, \vec{Z}, \vec{Y}\right)$ where the $\vec{Y}$ direction is the vertical direction.

- However, this coordinate system is not suited to describe particle motion in circular accelerators.

COORDINATE SYSTEM(S)

FRENET-SERRET COORDINATE SYSTEM



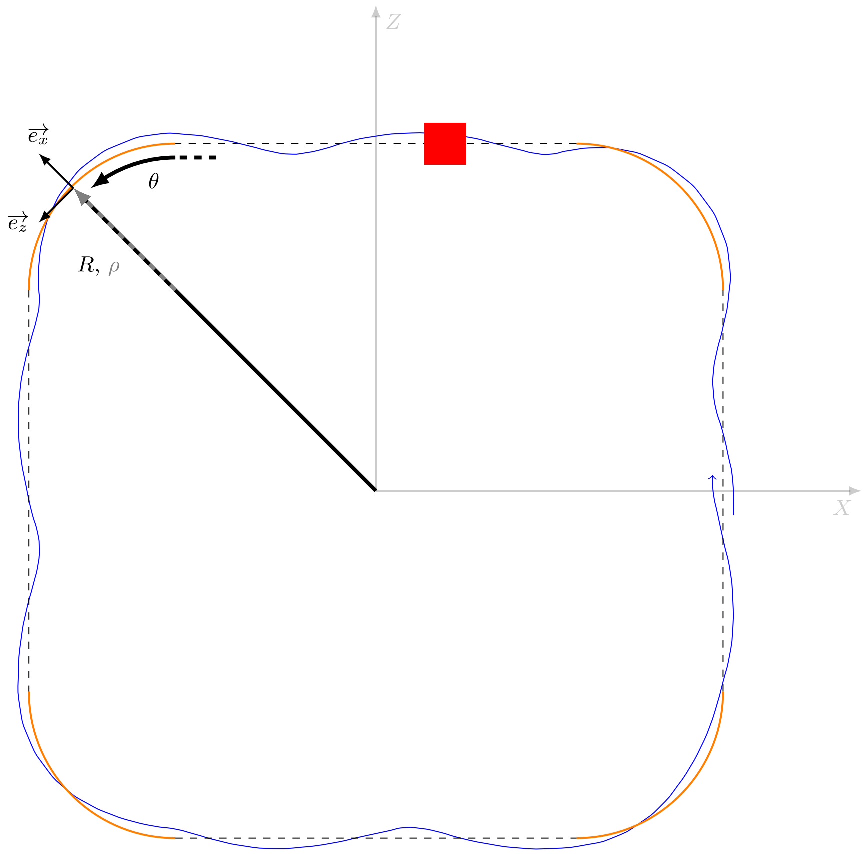
- A particle trajectory follows a curved path, which can be described in the Frenet-Serret coordinate system.
- The particle coordinates are given as offsets with respect to the design trajectory with

$$\begin{array}{ll} x & \text{Horizontal} \\ y & \text{Vertical} \\ z & \text{Longitudinal} \end{array}$$

- The curvature of the trajectory has a local bending radius ρ.

COORDINATE SYSTEM(S)

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



- We use the Frenet-Serret coordinate system $(\vec{x}, \vec{z}, \vec{y})$ as reference to describe the motion of particles.

- We introduce the mean radius

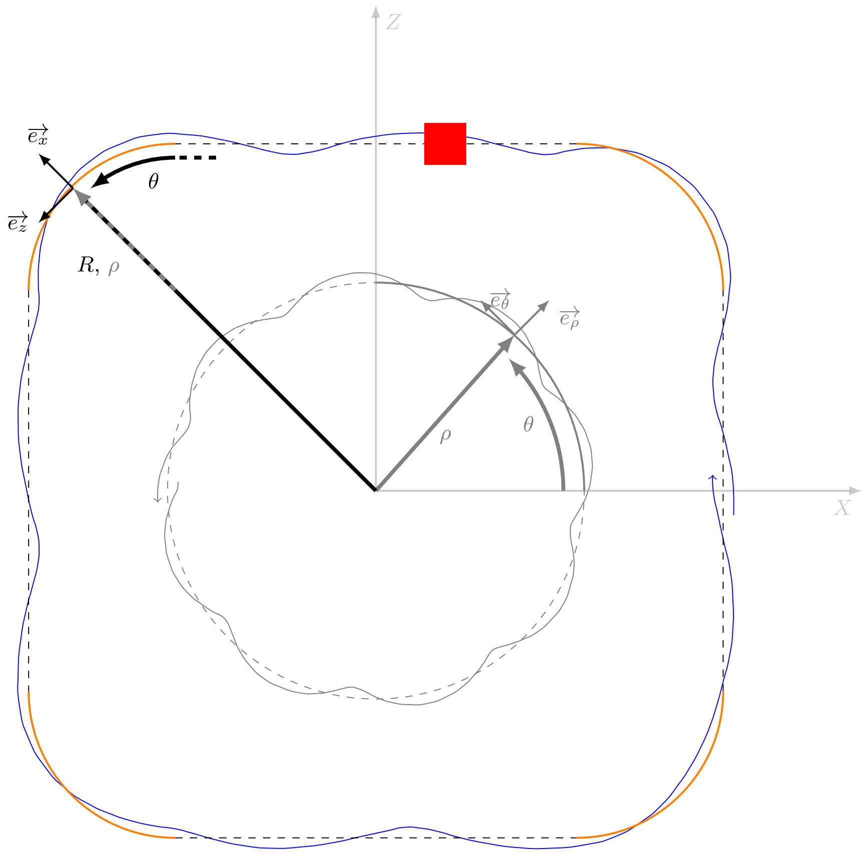
$$R = \frac{C}{2\pi}$$

where C is the path circumference and the generalized azimuth

$$\theta \in [0, 2\pi]$$

COORDINATE SYSTEM(S)

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



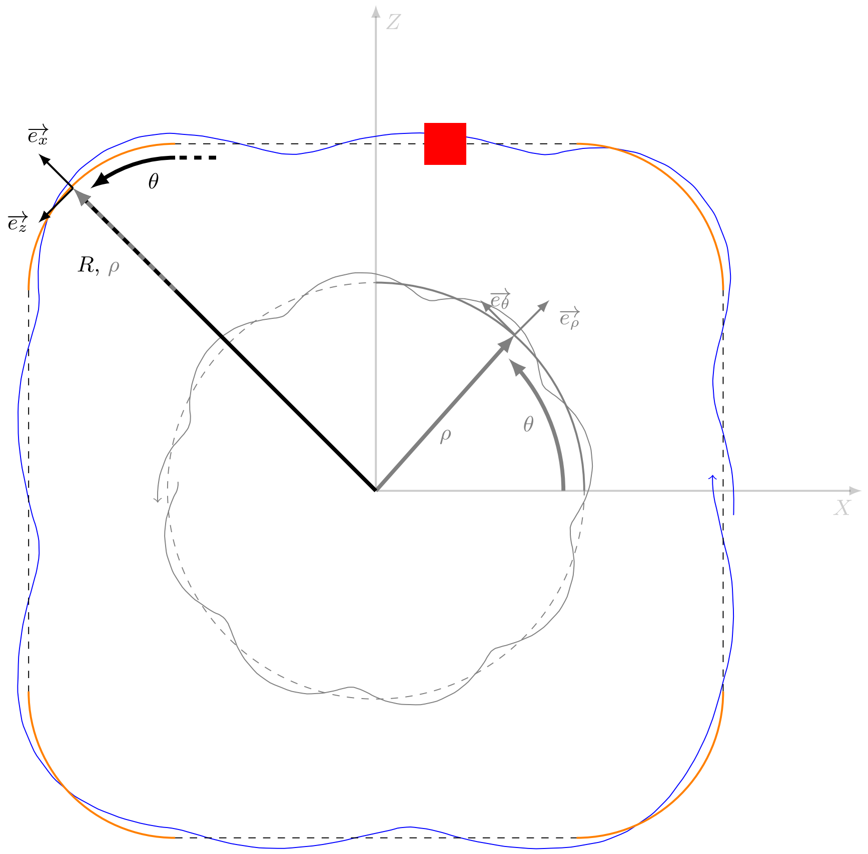
- For a circular accelerator, this coordinate system is comparable to the cylindrical coordinate system $\left(\vec{\rho}, \vec{\theta}, \vec{y}\right)$

- A particle orbit and horzitonal positions are equivalent, as well as the longitudinal position and azimuth.

- Beware, definitions can be interchanged!

COORDINATE SYSTEM(S)

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



- It is also important to disembiguate ρ which is the bending radius and R which is the particle orbit including straight sections of total length L. We have

$$C = 2\pi R = L + 2\pi\rho$$

PARTICLE ACCELERATION

- The primary purpose of a particle accelerator is to produce a beam of particles with a precise energy E.

- The energy can be provided to the particles applying the Lorentz force to charged particles

$$\frac{d\vec{p}}{dt} = \vec{F} = q\left(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}\right)$$

where

- $\vec{p} = m\vec{v}$ is the particle momentum
- q is the particle charge
- m is the particle (relativistic) mass
- $\vec{v}$ is the particle velocity

- $\vec{F}$ is a force
- $\vec{\mathcal{E}}$ is an electric field
- $\vec{\mathcal{B}}$ is a magnetic field

PARTICLE ACCELERATION

ELECTRIC FIELD CONTRIBUTION

$$\vec{F}_{\mathcal{E}} = q\,\vec{\mathcal{E}}$$

- An electric field can effectively **accelerate (or decelerate) particles**.
- Electric fields can also be used to **deflect particles** if applied transversally to the particle trajectory.

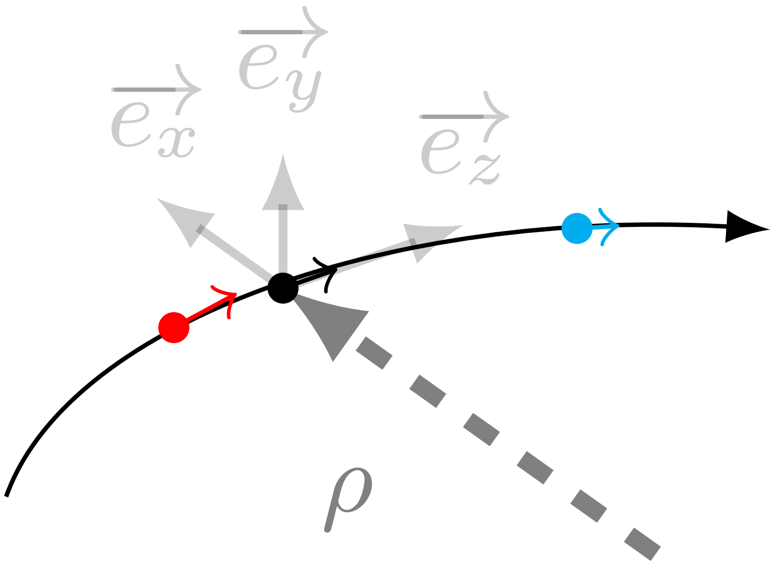
MAGNETIC FIELD CONTRIBUTION

$$\vec{F}_{\mathcal{B}} = q\left(\vec{v} \times \vec{\mathcal{B}}\right)$$

- The force applied by a magnetic field is always orthogonal to the particle trajectory and therefore **cannot accelerate the beam**.
- Magnetic fields are used to **steer the beam**.

PARTICLE TRAJECTORIES

ACCELERATION ALONG THE LONGITUDINAL DIRECTION



- The acceleration is done by applying an electric field tangential to the beam trajectory with

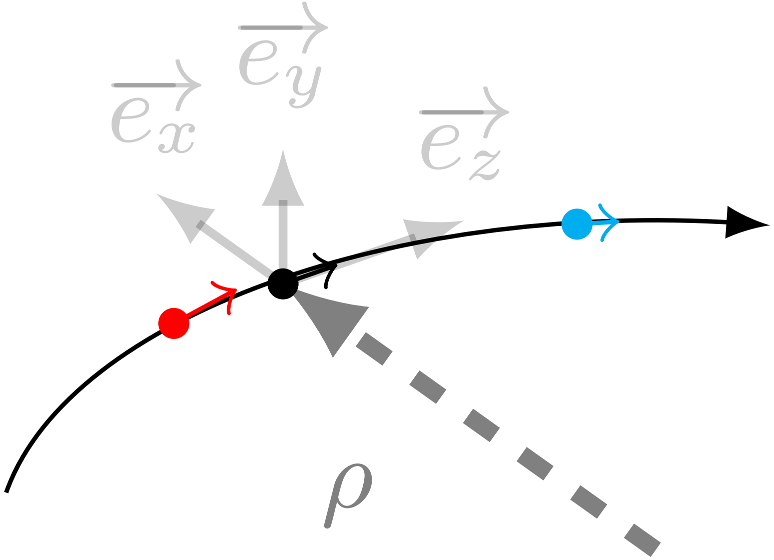
$$\vec{\mathcal{E}} = \mathcal{E}_z \vec{e_z}$$

- Except at extremely low energies (e.g. particle sources), the momentum of a particle is almost exclusively directed towards the longitudinal direction z with small angles in the transverse x and y directions.

- **Assumptions:** $p_z \gg p_{x,y}$ **and** $p \approx p_z$

PARTICLE TRAJECTORIES

STEERING THE DESIGN TRAJECTORY



- The beam trajectory is steered horizontally by applying a vertical magnetic field with

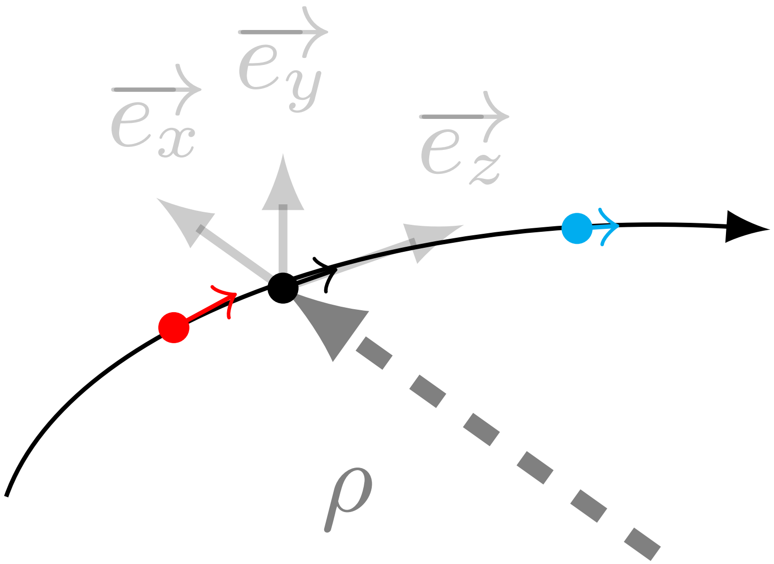
$$\vec{\mathcal{B}} = \mathcal{B}_y \vec{e_y}$$

- The applied force depends on the particle velocity v_z. For particles with different momenta, the steering and trajectories will be different than the design one.

- This effect is called dispersion and will be covered in both transverse and longitudinal beam dynamics lectures.

PARTICLE TRAJECTORIES

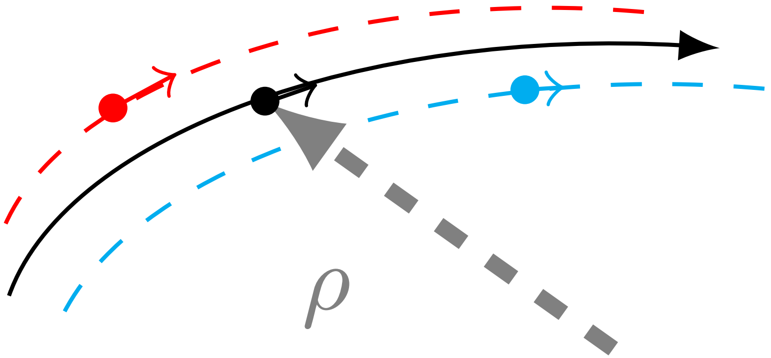
EVOLUTION OF RELATIVE PARTICLE POSITIONS



- In the longitudinal direction, a particle can be in front (in advance), or behind (late) with respect to the **ideal particle (on time)**.
- The relative distance between particles can change
  - Because a particle can also have a smaller/larger velocity v_z (and momentum p_z).

PARTICLE TRAJECTORIES

EVOLUTION OF RELATIVE PARTICLE POSITIONS



Red is faster but at larger orbit, while blue is slower but inner orbit.

How do we accelerate all three particles evenly? How do we keep these particles together?

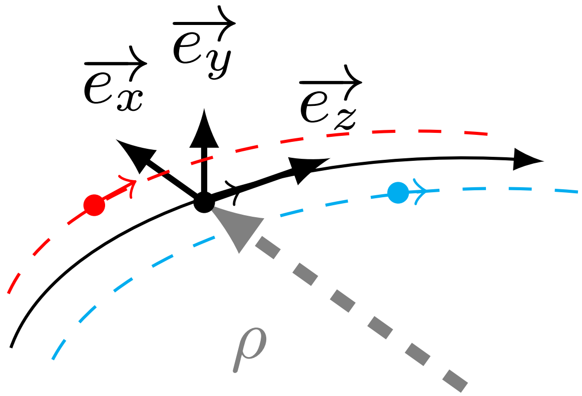
- In the longitudinal direction, a particle can be in front (in advance), or behind (late) with respect to the **ideal particle (on time)**.

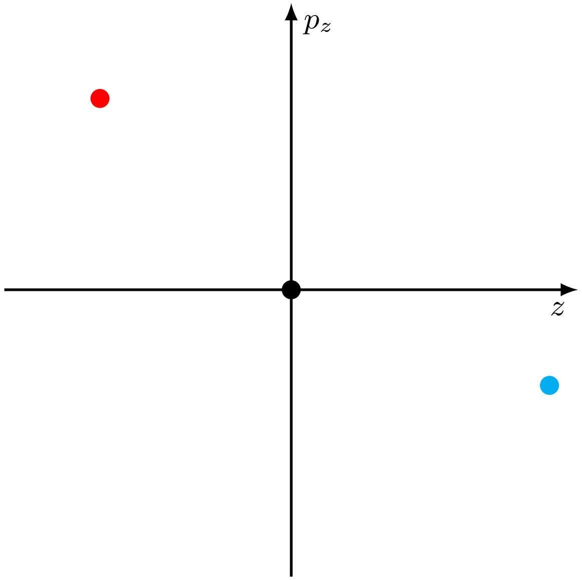
- The relative distance between particles can change

  - Because a particle can also have a smaller/larger velocity v_z (and momentum p_z).

  - Because of a shorter/longer path length in a bending (i.e. smaller/larger orbit), which depends on the particle momentum.

LONGITUDINAL PHASE SPACE





- We will introduce the notion of longitudinal phase space.

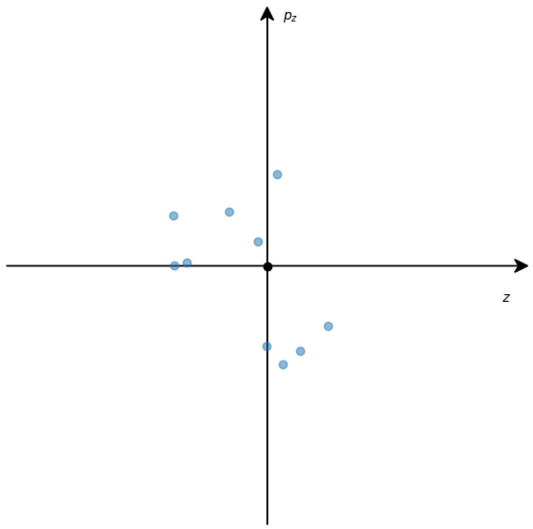
- The particle motion can be described in the (z, p_z) phase space, relative to the **ideal particle** following the design orbit and energy.

- As described before other particles can be
  - In front, or in advance in time (right)
  - In the back, or delayed in time (left)
  - Have higher momentum/velocity (top)
  - Have lower momentum/velocity (bottom)

- The motion of the particles in the longitudinal phase space is called **synchrotron motion**.

SYNCHROTRON OSCILLATIONS

WITH A FEW PARTICLES

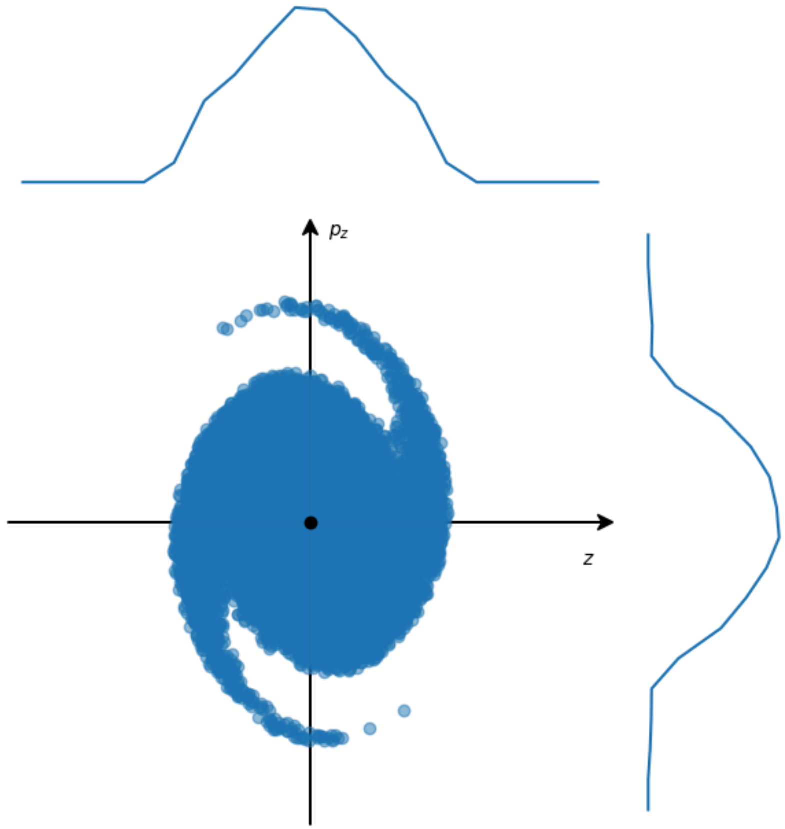


In a bunch, particles rotate around the ideal particle in black used a reference.

These are called **synchrotron oscillations**.

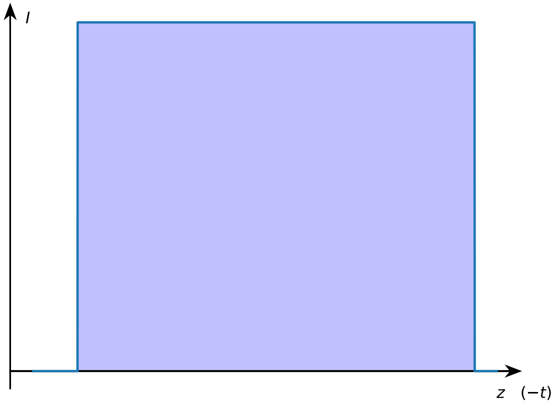
SYNCHROTRON OSCILLATIONS

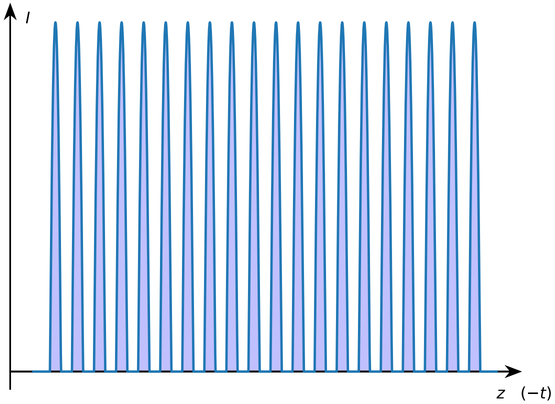
WITH MANY PARTICLES



- A bunch is usually composed of a very large number of particles, typically $\mathcal{O}\left(10^{10} - 10^{12}\right)$ at CERN.
- In a real machine, the coherent motion of a bunch can be measured and analyzed from the longitudinal bunch density (top line, projection along the p_z axis, instantaneous beam current).
- You can notice the non-linear synchrotron motion in phase space at large amplitude.

TEMPORAL DEFINITION OF A BEAM





- Controlling the synchrotron motion allows to define the temporal structure of a pulse of particles.

- The beam current is

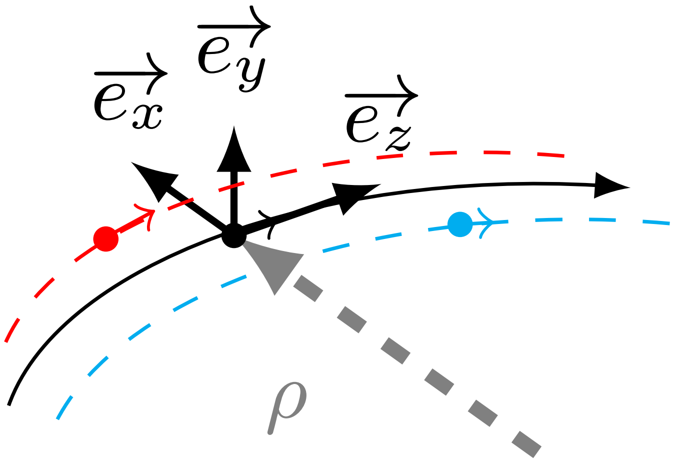
$$I = \frac{dQ}{dt}$$

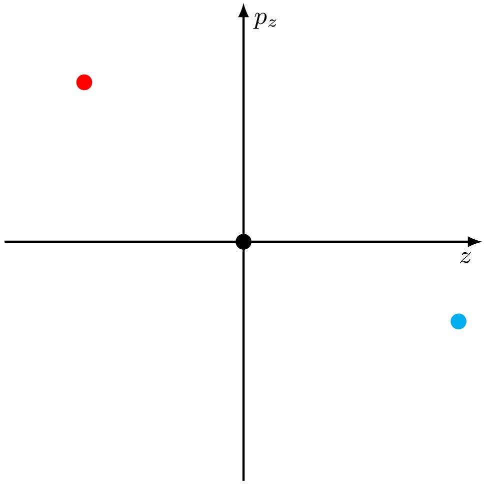
where dQ is the charge passing in a time dt.

- Depending on the destination (experiment or next machine in a chain), parameters defining the synchrotron motion can be adjusted to deliver a continuous or bunched beam.

WHAT IS LONGITUDINAL BEAM DYNAMICS?

- Longitudinal beam dynamics is the description of the acceleration and motion of particles along the forward path of the beam.
- Since the orbit of a particle also plays a role, we will see that the horizontal/radial position of a particle is an important parameter.
- We will derive the equations to describe synchrotron oscillations in longitudinal phase space.





RELATIONSHIP WITH OTHER COURSES

JUAS COURSE 1

- How do we focus the beam in the horizontal and vertical directions, how do we transport the beam to a target?

→ **Transverse Beam Dynamics**

- Can we use the beam in another way than colliding on a target, what is the principle behind light sources ?

→ **Synchrotron radiation**

- Do charged particles interact with each other, can we accelerate an infinite amount of particles?

→ **Collective Effects - Space Charge and Instabilities**

RELATIONSHIP WITH OTHER COURSES

JUAS COURSE 1

- This course is devoted to describe fundamentals of longitudinal beam dynamics with specifities linked to the design of **Synchrotrons**.

- Dedicated courses are devoted to the specificities of **Linacs** and **Cyclotrons**.
- You will find similar concepts between the courses. Nonetheless, beware of definitions, conventions and assumptions used to derive formulas!

RELATIONSHIP WITH OTHER COURSES

JUAS COURSE 2

- What systems do we use to provide the beam with an electric field, how are they designed ?

→ **RF Engineering and Superconducting RF Cavities**

- How do we measure a bunch, specificially in the longitudinal plane ?

→ **Beam Instrumentation**

TAKE AWAY MESSAGE

- Lorentz force

$$\frac{d\vec{p}}{dt} = \vec{F} = q\left(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}\right)$$

$\vec{\mathcal{E}}$ to accelerate and deflect

$\vec{\mathcal{B}}$ to bend trajectories

- Definition of coordinates

x horizontal position

y vertical position

z longitudinal position

ρ local bending radius

R mean radius / orbit

θ azimuth

- Assumptions made so far: $p_z \gg p_{x,y}$ and $p \approx p_z$