LONGITUDINAL BEAM DYNAMICS

JUAS 2025

COURSE 1: THE SCIENCE OF PARTICLE ACCELERATORS

A. Lasheen

Joint Universities Accelerator School

LESSON 1: FUNDAMENTALS OF PARTICLE ACCELERATION

MODULE 1: FIELDS AND FORCES

→ **Acceleration in electric fields**

→ **Electrostatic, induction, and RF acceleration**

→ **Circular accelerators and magnetic rigidity**

MAXWELL EQUATIONS

DIFFERENTIAL EQUATIONS IN VACUUM

$$\overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{E}} = \frac{\rho_q}{\epsilon_0} \qquad \text{Gauss' law}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{B}} = 0 \qquad \text{Flux/Thomson's law}$$

$$\overrightarrow{\nabla} \times \overrightarrow{\mathcal{E}} = -\frac{\partial \overrightarrow{\mathcal{B}}}{\partial t} \qquad \text{Faraday's law}$$

$$\overrightarrow{\nabla} \times \overrightarrow{\mathcal{B}} = \mu_0 \left(\overrightarrow{j} + \epsilon_0 \frac{\partial \overrightarrow{\mathcal{E}}}{\partial t} \right) \qquad \text{Ampère's law}$$

ϵ_0 Vacuum permittivity , μ_0 Vacuum permeability

ρ_q Charge density, $\overrightarrow{j}$ Current density

MAXWELL EQUATIONS

INTEGRAL FORM EQUATIONS IN VACUUM

$$\oiint_S \overrightarrow{\mathcal{E}} \cdot \overrightarrow{dS} = \frac{1}{\epsilon_0} \iiint \rho_q dV \qquad \text{Gauss' law}$$

$$\oiint_S \overrightarrow{\mathcal{B}} \cdot \overrightarrow{dS} = 0 \qquad \text{Flux/Thomson's law}$$

$$\oint_C \overrightarrow{\mathcal{E}} \cdot \overrightarrow{dz} = -\frac{d}{dt} \iint_S \overrightarrow{\mathcal{B}} \cdot \overrightarrow{dS} \qquad \text{Faraday's law}$$

$$\oint_C \overrightarrow{\mathcal{B}} \cdot \overrightarrow{dz} = \mu_0 \iint_S \overrightarrow{j} \cdot \overrightarrow{dS} + \mu_0 \epsilon_0 \iint_S \frac{\partial \overrightarrow{\mathcal{E}}}{\partial t} \cdot \overrightarrow{dS} \qquad \text{Ampère's law}$$

dz Line element, dS Surface element, dV Volume element

ACCELERATION IN ELECTROSTATIC FIELDS (DC)

The simplest form of acceleration can be obtained using electrostatic fields along the longitudinal direction

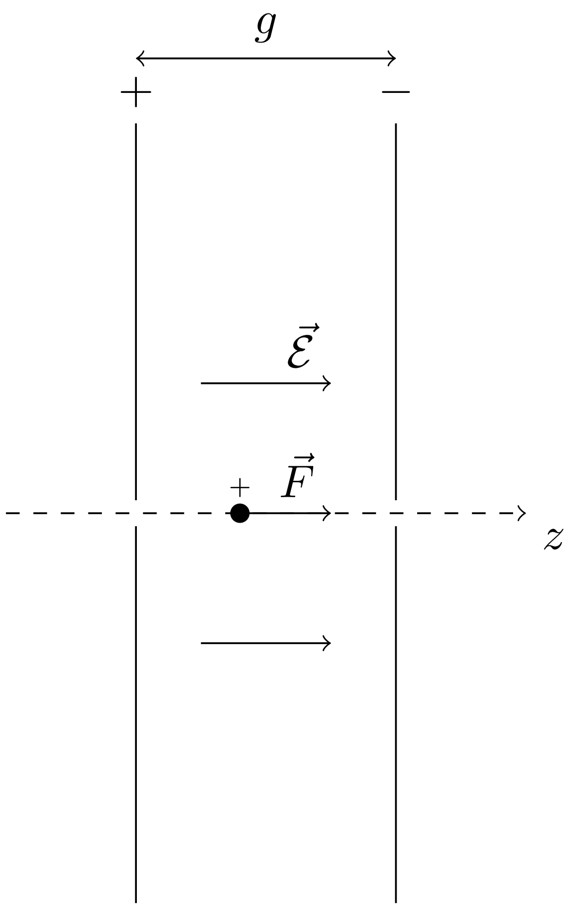
$$\frac{dp}{dt} = \frac{dE}{dz} = q\,\mathcal{E}_z$$

giving an increment in energy

$$\delta E = \int q\mathcal{E}_z dz = q\mathcal{E}_z g = qV_g$$

where the scalar potential V is defined as

$$\overrightarrow{\mathcal{E}} = -\overrightarrow{\nabla} V \implies \mathcal{E}_z = -\frac{\partial V}{\partial z}$$



DEFINITIONS OF ENERGY AND POWER

PARTICLE ENERGY

The energy of particles in accelerators is expressed in electronvolts eV corresponding to the energy gain by a particle with elementary charge e in a potential $V_g = 1V$

$$1 \text{ eV} = 1.602\ 176\ 634 \times 10^{-19} \text{ J}$$

POWER TRANSFERRED TO THE BEAM

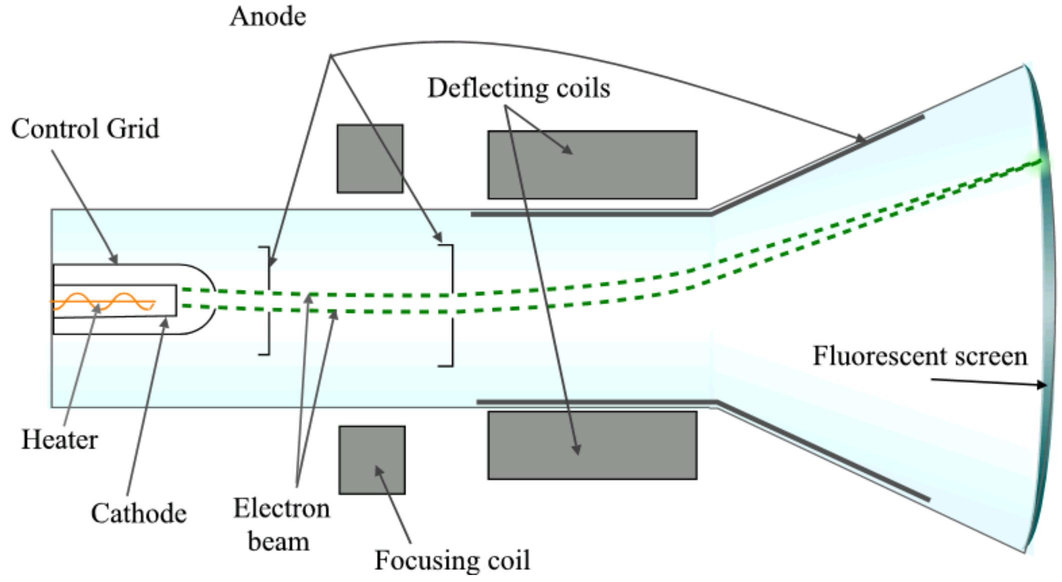
The average power transferred to the beam in W is defined as the total accelerated beam energy $N_p E_{\mathrm{acc}}$ (N_p being the number of particles and E_{acc} expressed in J) delivered in an acceleration time T_{acc}.

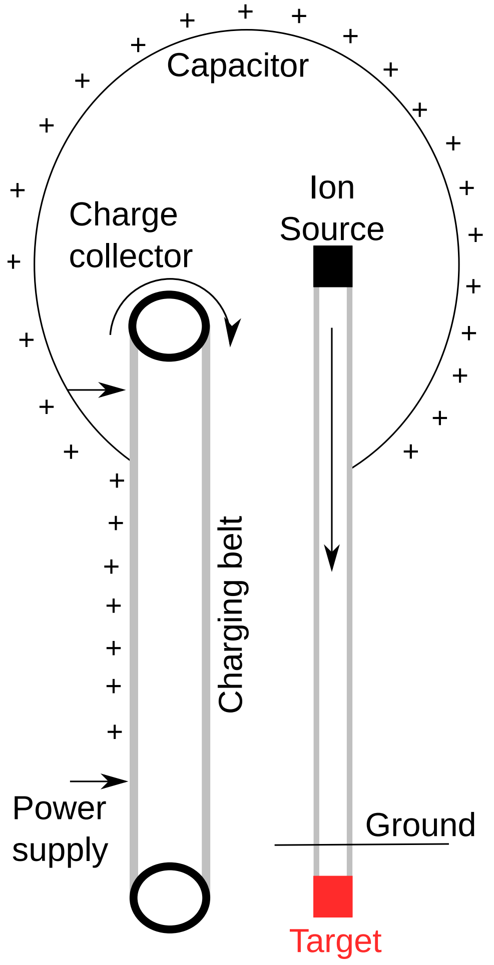
$$\langle P_b \rangle = \frac{N_p E_{\mathrm{acc}}}{T_{\mathrm{acc}}}$$

EXERCISES ON THE EV

- **An accelerator has a potential of 20 MV, what is the corresponding energy gain of a particle in Joules?**
- **What is the total energy of the beam stored in the LHC?** (The beam is composed of 2808 bunches of 1.15×10^{11} protons each at 7 TeV)
- **What is the equivalent speed of a high speed train?** (Assume a 400 tons (200 m long) TGV train)
- **What is the beam power delivered to the LHC beam?** (Consider an acceleration from 450 GeV to 7 TeV in 30 minutes)

EXAMPLES OF ELECTROSTATIC ACCELERATORS





- Various designs exist for extraction from a particle source, high field DC acceleration (e.g. Cockroft-Walton, Van de Graaf, Tandem).
- Various applications exist such as cathode ray tudes for (old) TVs, industrial/medical applications...
- See CAS - Electrostatic accelerators for more details.

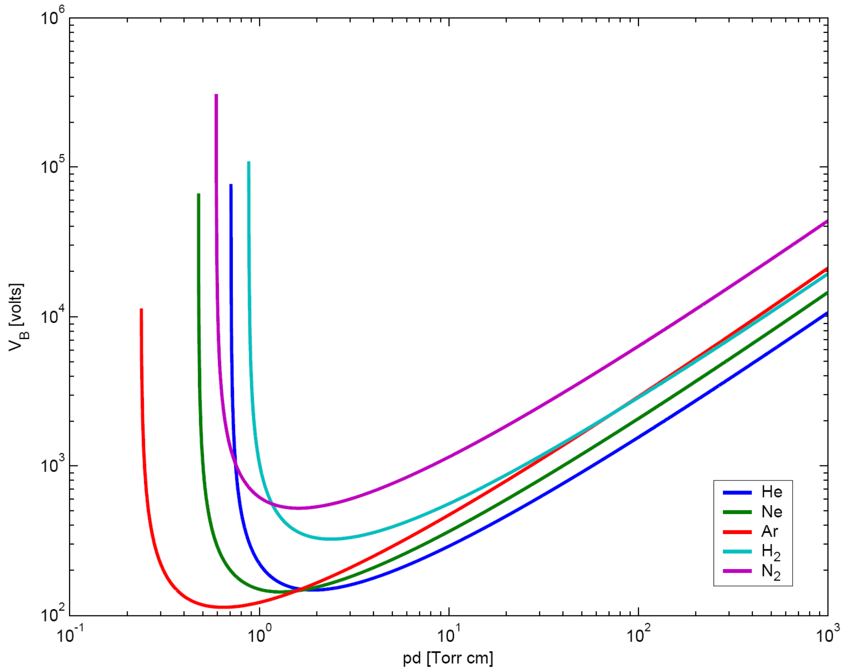
LIMITATIONS OF ELECTROSTATIC ACCELERATORS

- Maximum electric field limited to the MV range due to discharge/arcs.

- The maximum voltage reached depends on the gas nature and pressure and follows the Paschen law.

- Moreover from Faraday's law for static fields implies

$$\oint_C \overrightarrow{\mathcal{E}} \cdot \overrightarrow{dz} = 0$$



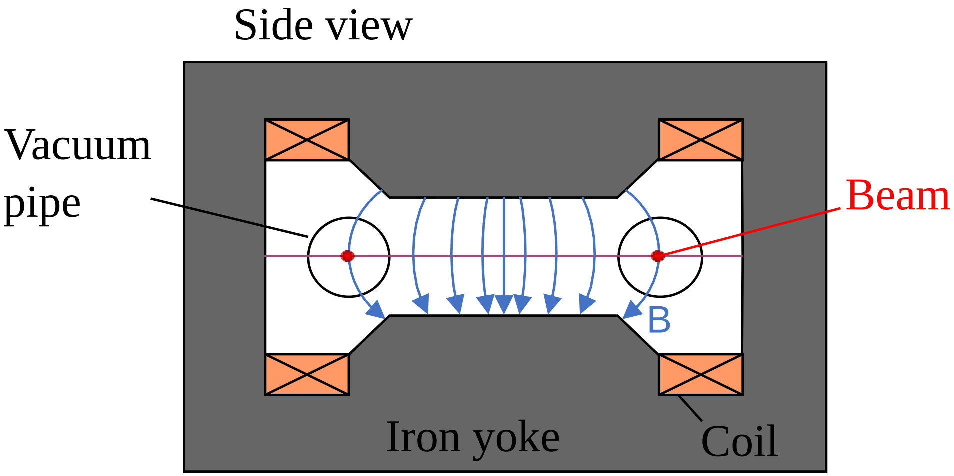
- Single pass accelerator only, cannot reach higher energies than the tens of MeV level (high energy hadron colliders ~TeV!).

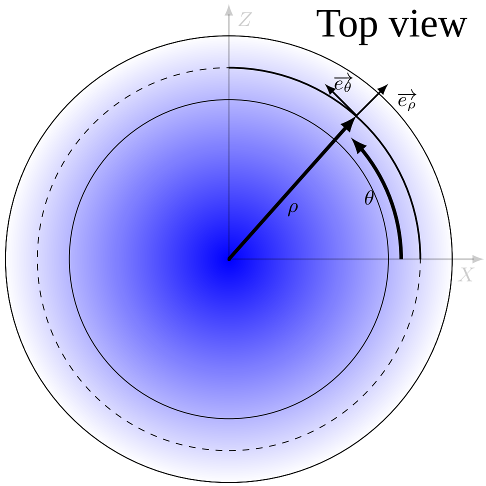
INDUCTION ACCELERATION

An electric field can be obtained with a ramping magnetic field. Again from Faraday's law for induction

$$\oint_C \overrightarrow{\mathcal{E}} \cdot \overrightarrow{dz} = -\frac{d}{dt}\iint_S \overrightarrow{\mathcal{B}} \cdot \overrightarrow{dS}$$

This is the principle behind the betatron accelerator design sketched below, with $B(\rho)$ in blue.





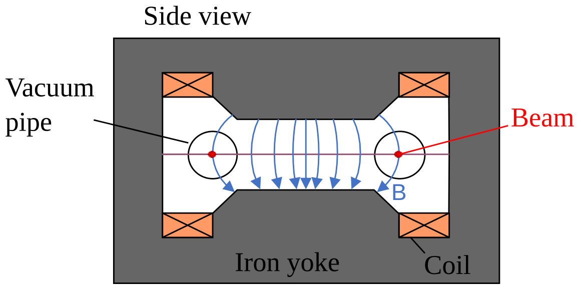
INDUCTION ACCELERATION

BETATRON CONDITION, 2:1 RULE

Assuming azimuthal symmetry, vertical magnetic field $\overrightarrow{\mathcal{B}} = -\mathcal{B}_y(\rho, t)\,\vec{e_y}$ at a constant orbit ρ_0, we get

$$\mathcal{B}_y(\rho_0) = \frac{1}{2}\frac{\Phi_{S,\rho_0}}{\pi\rho_0^2} = \frac{1}{2}\left\langle \mathcal{B}_y \right\rangle_{S,\rho_0}$$

→ **If the particles move in a circular path of orbit ρ_0, the averaged magnetic field (flux) in the surface enclosed in the orbit ρ_0 should be twice the magnetic field on the particle trajectory. This is also stated as the 2:1 rule.**



INDUCTION ACCELERATION

DERIVATION OF THE BETATRON CONDITION

Assuming azimuthal symmetry, vertical magnetic field $\overrightarrow{\mathcal{B}} = -\mathcal{B}_y\left(\rho, t\right)\vec{e_y}$ at a constant orbit ρ_0, can you derive an equation for $\mathcal{E}_\theta$ and the corresponding dp_θ/dt ?

We will introduce the magetic flux Φ_{S,ρ_0} and an averaged magnetic field in the betatron core $\left\langle\mathcal{B}_y\right\rangle_{S,\rho_0}$

$$\Phi_{S,\rho_0} = 2\pi\int_0^{\rho_0}\mathcal{B}_y\left(\rho\right)\rho\, d\rho = \pi\rho_0^2\left\langle\mathcal{B}_y\right\rangle_{S,\rho_0}$$

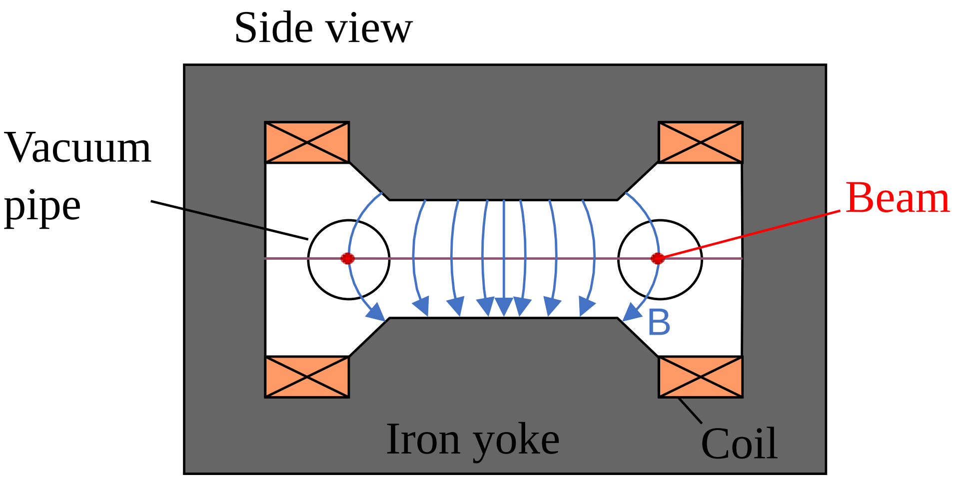
What is the equilibrium condition for a constant p_θ if

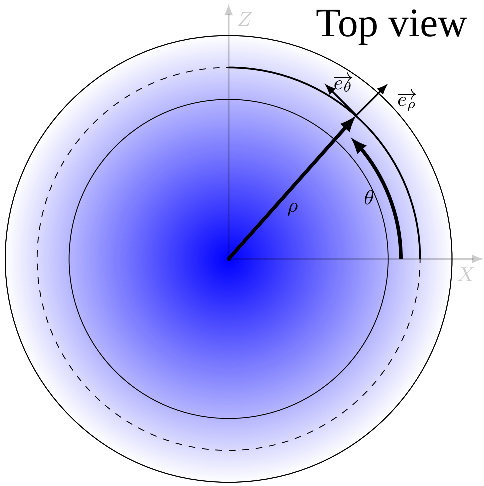
$$\mathcal{B}_y\rho = \frac{p_\theta}{q} = \frac{p}{q}$$

LIMITATIONS OF INDUCTION ACCELERATION

- The accelerator is covered by large magnets
  - Limited size of the accelerator
  - Saturation of the iron yoke

- The maximum energy reached is about 300 MeV with electrons (high energy lepton synchrotrons ~100s GeV!)





ELECTROMAGNETIC WAVE ACCELERATION

Combining Maxwell's equation in vacuum (no charge, no current)

$$\overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{E}} = 0 \qquad \text{Gauss' law}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{B}} = 0 \qquad \text{Flux/Thomson's law}$$

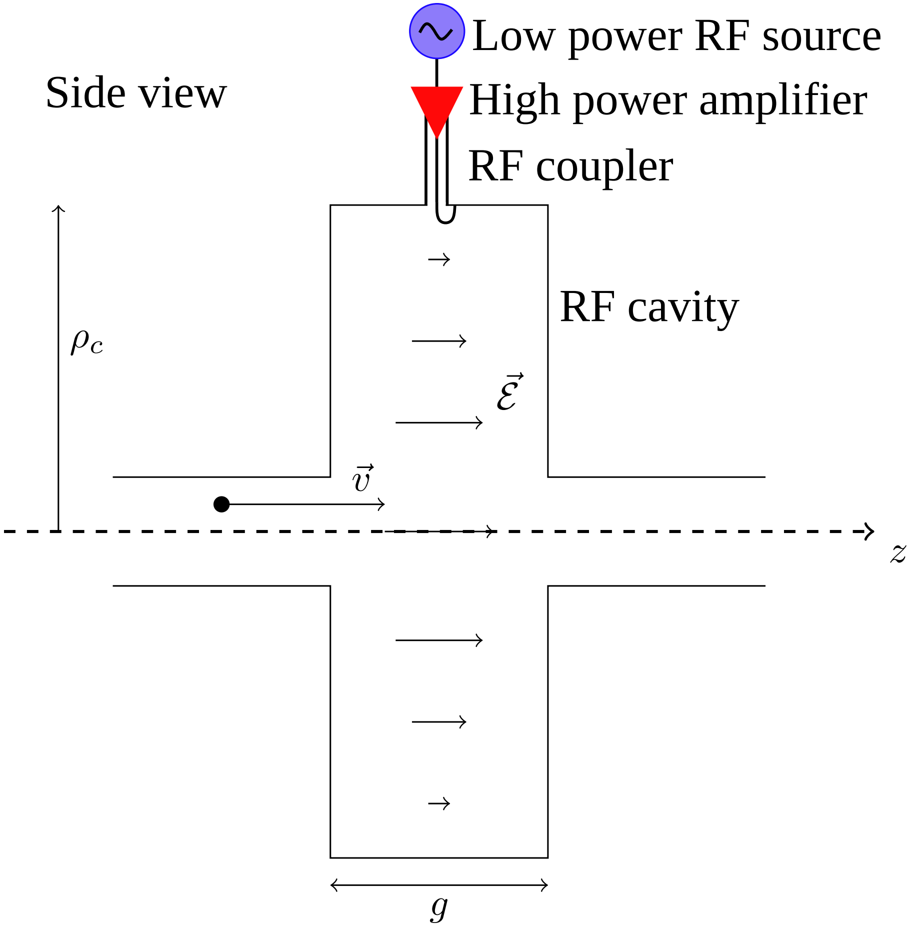
$$\overrightarrow{\nabla} \times \overrightarrow{\mathcal{E}} = -\frac{\partial \overrightarrow{\mathcal{B}}}{\partial t} \qquad \text{Faraday's law}$$

$$\overrightarrow{\nabla} \times \overrightarrow{\mathcal{B}} = \mu_0 \epsilon_0 \frac{\partial \overrightarrow{\mathcal{E}}}{\partial t} \qquad \text{Ampère's law}$$

an electric field can be obtained in the form of a wave

$$\Delta \overrightarrow{\mathcal{E}} - \frac{1}{c^2} \frac{\partial^2 \overrightarrow{\mathcal{E}}}{\partial t^2} = 0 \quad , \left(c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \right)$$

RF SYSTEMS



- An electromagnetic wave can be confined in a cavity, with an opening to let the beam pass through the oscillating electric field with

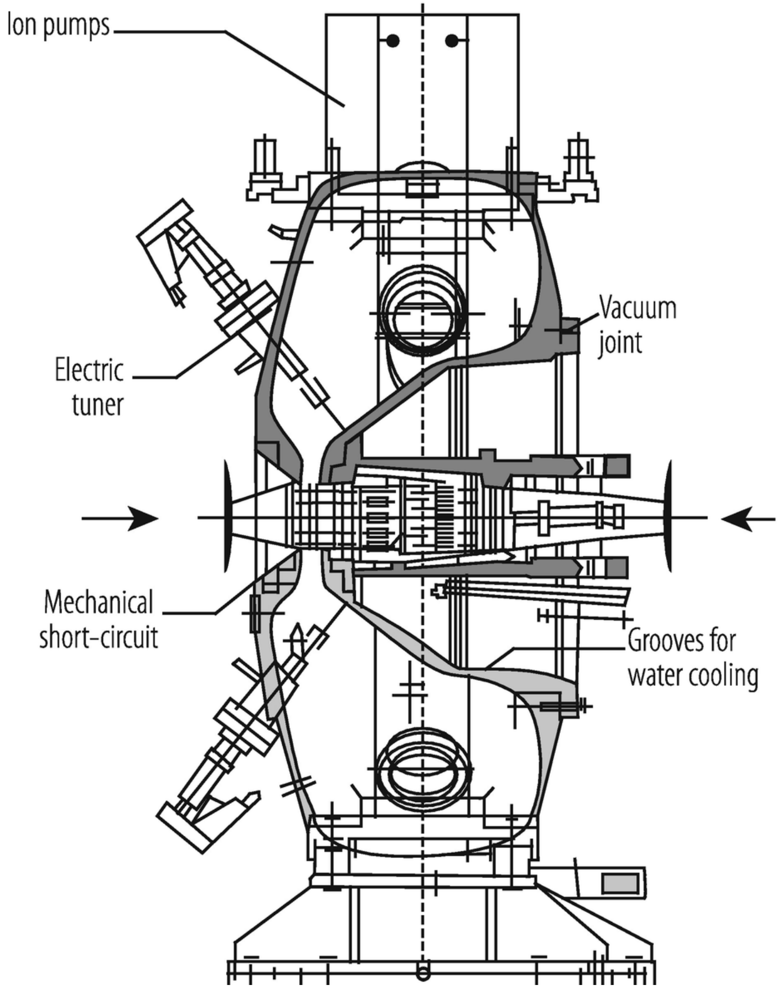
$$\vec{\mathcal{E}} = \mathcal{E}_z\left(\rho, z\right)\cos\left(\omega_{\mathrm{rf}} t\right)\vec{e_z}$$

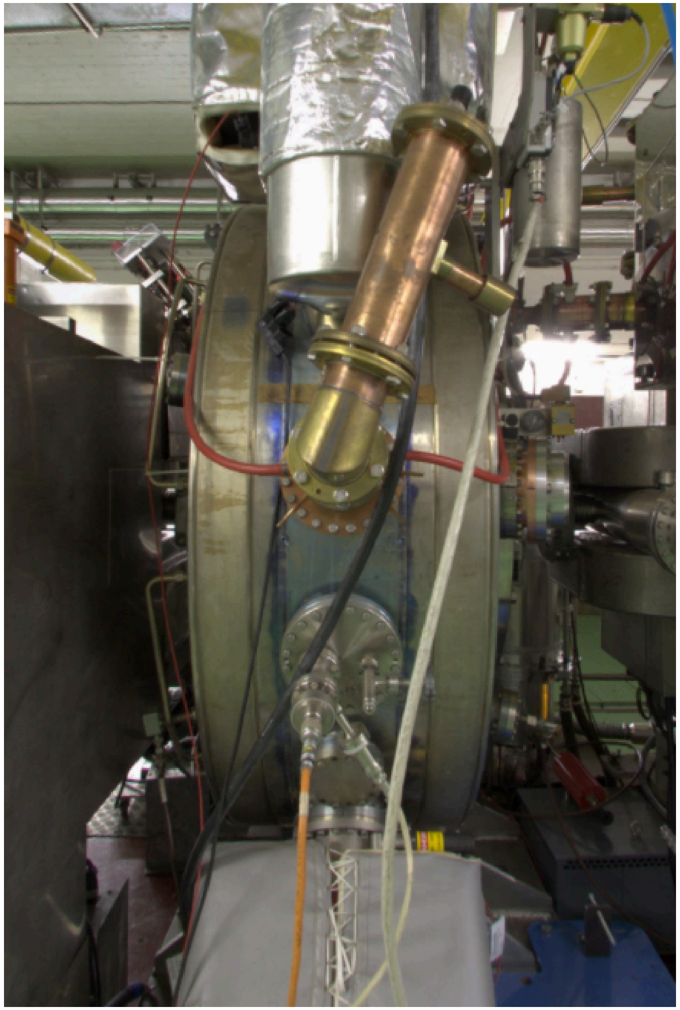
where $\omega_{\mathrm{rf}} = 2\pi f_{\mathrm{rf}}$ is the (angular) frequency of the field and depends on the geometry of the cavity.

- A low power RF signal is amplified and coupled to the cavity. The amplitude and phase of the wave with respect to the beam can be controlled.

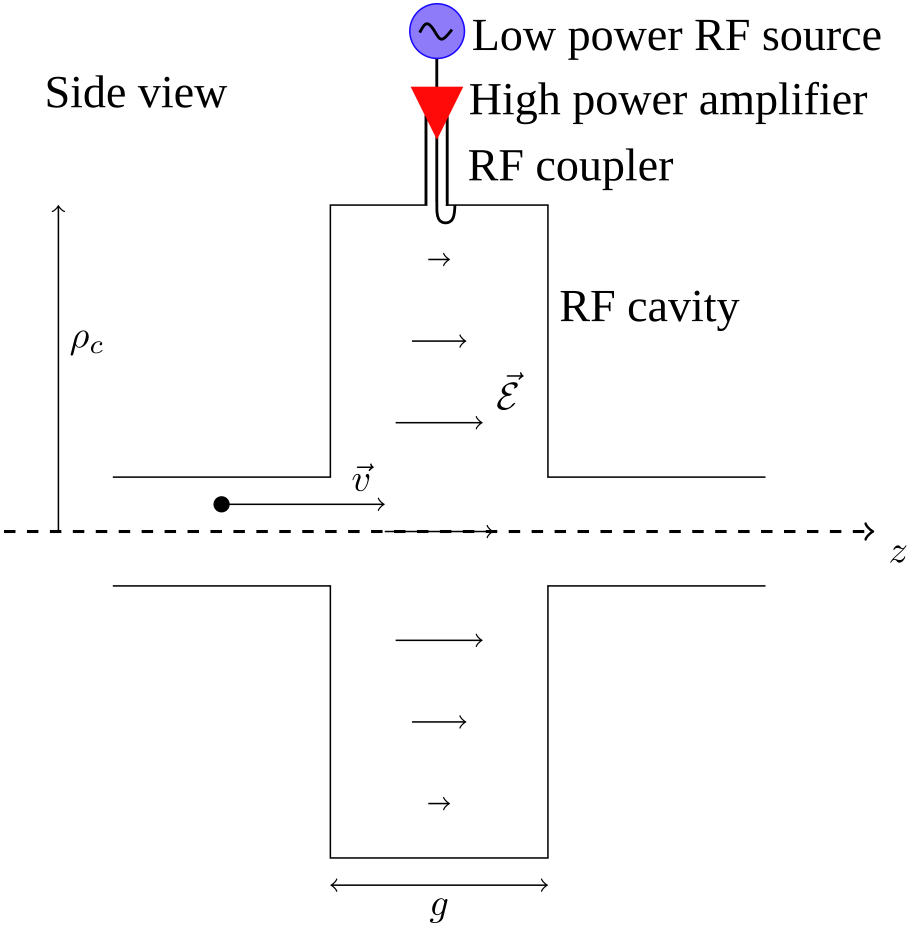
RF SYSTEMS

EXAMPLE OF REAL RF CAVITY IN THE PS (VIEW)





RF ACCELERATION



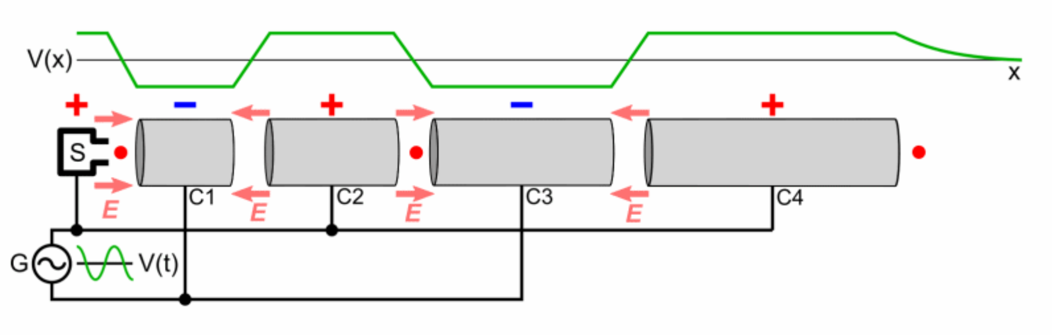
- The increment in energy of a particle passing through an RF cavity gap is

$$\begin{aligned}\delta E_{\text{rf}} &= \int q\mathcal{E}_z\left(\rho, z, t\right) dz \\ &= qV_{\text{rf}}\left(\rho, \tau\right)\end{aligned}$$

where V_{rf} is the total accelerating potential of a particle arriving at a time τ in the cavity (we will derive a relevant expression of V_{rf} during the next lesson!).

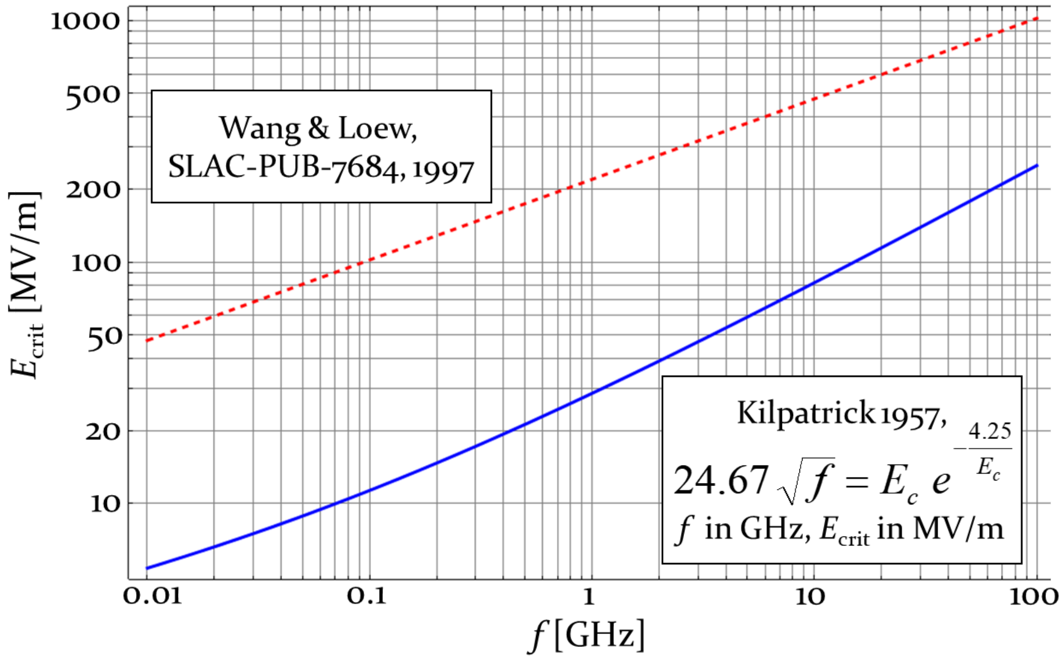
- Unlike electrostatic fields, cavities can be installed consecutively to accelerate the particles.

LINEAR ACCELERATORS (LINACS)



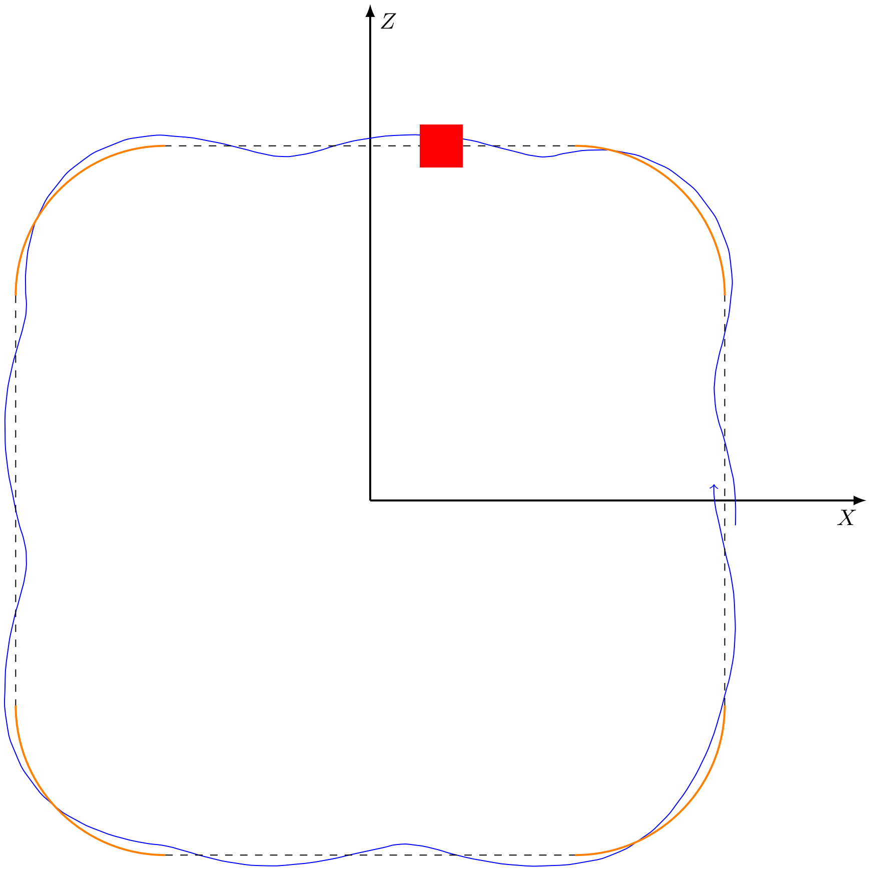
- The basic principle of linear accelerators is a single pass in many RF systems to accumulate energy.
- The distance between two accelerating gaps depends on the particle velocity (synchronism condition for Linacs).
- The maximum energy reach scales with the length of the linac and the RF accelerating gradient.
- Dedicated **JUAS Lecture on Linacs** and walk along LINAC4.

BREAKDOWN AND RF



- The maximum accelerating gradient in RF cavities is limited by breakdown.
- The observed frequency dependence was formulated empirically by Kilpatrick.
- Breakdown is dependent on the cavity surface quality and conditioning. Present cavities go beyond the Kilpatrick criterion (ratio expressed in "Kilpatrick" unit).
- Typical range for RF cavities $\sim \mathcal{O}\,(1\text{-}100\ \mathrm{MV/m})$

CIRCULAR ACCELERATORS



For circular accelerators the principle is to steer the beam back to the RF cavity and passing multiple time. We need to introduce the concept of **magnetic rigidity**.

MAGNETIC RIGIDITY

The applied force in bending magnets to shape a circular accelerator is

$$\vec{F}_{\mathcal{B}} = q\left(\vec{v} \times \vec{\mathcal{B}}\right)$$

which gives the vertical magnetic field required to keep particles with a given momentum on a given orbit

$$\mathcal{B}_y \rho = \frac{p}{q}$$

This relationship is called the magnetic rigidity or more trivially the "$\mathcal{B}\rho$".

MAGNETIC RIGIDITY

DERIVATION

The force from a magnetic field is always orthogonal to the particle direction, in cylindrical coordinates and with a constant $v = v_\theta$ (implying $p = p_\theta, \dot{m} = 0$), and a magnetic field $\overrightarrow{\mathcal{B}} = -\mathcal{B}_y \vec{e_y}$.

Demonstrate that

$$\mathcal{B}_y \rho = \frac{p_\theta}{q} = \frac{p}{q}$$

STRATEGY FOR CIRCULAR ACCELERATORS

Two possibilities to reach high energies with

$$\mathcal{B}_y \rho = \frac{p}{q}$$

- Increase $\mathcal{B}_y$ at fixed ρ $\rightarrow$ Synchrotron
- Increase ρ at fixed $\mathcal{B}_y$ $\rightarrow$ **Cyclotron (dedicated JUAS Lecture)**

- **In the next lessons, we will focus on the synchrotron design.**
- The maximum energy of a circular accelerator is in principle limited by the maximum $\mathcal{B}_y$ in the bending magnets or the radial size of the accelerator (e.g. FCC 100km!). The typical range for bending magnetic field is $\sim \mathcal{O}\left(1\text{-}10\ \mathrm{T}\right)$.
- In presence of **synchrotron radiation** (for lepton machines), the maximum energy is limited by RF power.

VERY HIGH GRADIENT ACCELERATION

- How do we go beyond the limits fixed by present accelerator technologies? Can we have more compact accelerators? Can we reach GV/m accelerating gradient using fields provided by lasers and plasmas?

→ **Follow the JUAS Seminar on AWAKE**

→ In the context of this lecture, we will concentrate on conventional **RF acceleration**.

MODULE 2: RELATIVISTIC KINEMATICS

→ **Recap on relativistic parameters**

→ **Useful differential relationships**

DEFINITION OF PARAMETERS

Reminder: we now assume that the momentum of the particle is $p \approx p_z$

Particle energy and momentum

$$E = E_{\mathrm{kin}} + E_0 = \sqrt{P^2 + E_0^2}$$

where E total energy, $E_0 = m_0 c^2$ rest energy (particle rest mass m_0), $p = P/c$ is the momentum

Relativistic parameters

$$\beta = \frac{v}{c} = \frac{P}{E}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{E_0}$$

where β relativistic velocity, γ Lorentz factor and $p = mv = \beta\gamma m_0 c$

UNITS

- The energies E, E_{kin}, E_0 and P can be expressed in eV
- The momentum p can be expressed in eV/c
- The mass m can be expressed in eV/c^2
- β and γ are unitless

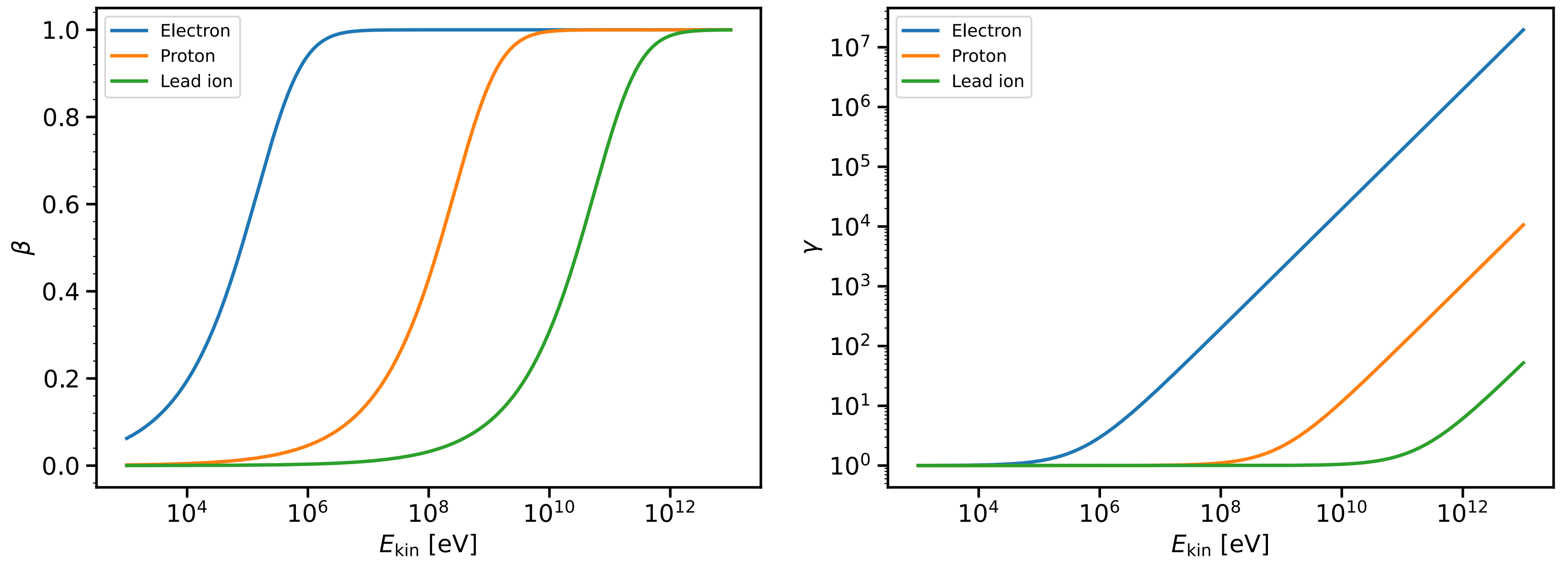
Practical magnetic rigidity formula

We will demonstrate that

$$p\,[\mathrm{GeV}/\mathrm{c}] \approx 0.3\; Z\; \mathcal{B}_y\,[\mathrm{T}]\,\rho\,[\mathrm{m}]$$

where Z is the number of elementary charges e ($Z = +1$ for protons and $Z = -1$ for electrons).

RELATIVISTIC PARAMETERS AND PARTICLE REST MASS



- Electrons can be considered with $v \approx c$ at moderate kinetic energy, but not heavier particles.
- The evolution of the particle velocity during acceleration has an important influence on the design of an accelerator.

USEFUL RELATIONSHIPS

Practical relationships that will be used in further derivations.

$$\beta^2 + \frac{1}{\gamma^2} = 1$$

Differential forms

$$\frac{dE}{dp} = \beta c = v$$

$$\frac{dp}{p} = \frac{1}{\beta^2}\frac{dE}{E} = \frac{1}{\beta^2}\frac{d\gamma}{\gamma}$$

$$\frac{dp}{p} = \gamma^2 \frac{d\beta}{\beta}$$

EXERCISES

- **Show that**

$$p\,[\mathrm{GeV/c}] \approx 0.3\ Z\ \mathcal{B}_y\,[\mathrm{T}]\,\rho\,[\mathrm{m}]$$

- **Compute the relativistic parameters for the following CERN machines**

Machine	E_0 [MeV]	E_{kin} [GeV]	E [GeV]	γ	β	p [GeV/c]	$\mathcal{B}_y\rho$ [Tm]
PSB inj (p+)		0.160					
PSB ext (p+)		2					
SPS ($^{208}\mathrm{Pb}^{82+}$)							86.4
LHC (p+)			7000				
LEP (e+/e-)			100				

$$m_p = 1.6726\times10^{-27}\ \mathrm{kg},\ m_e = 9.1094\times10^{-31}\ \mathrm{kg},\ u = 1.661\times10^{-27}\ \mathrm{kg}$$

- **Derive the differential relationships from the previous slide**

TAKE AWAY MESSAGE

- Acceleration in an RF gap:

$$\delta E = \int q\mathcal{E}_z\left(\rho, z, t\right) dz = qV_{\mathrm{rf}}\left(\rho, \tau\right)$$

- Magnetic rigidity:

$$\mathcal{B}_y \rho = \frac{p}{q} \quad \rightarrow \quad p\,[\mathrm{GeV/c}] \approx 0.3\ Z\ \mathcal{B}_y\,[\mathrm{T}]\,\rho\,[\mathrm{m}]$$

- Relativistic relationships ($P = p\ c$):

$$E = E_{\mathrm{kin}} + E_0 = \sqrt{P^2 + E_0^2}, \quad \beta = \frac{v}{c} = \frac{P}{E}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{E_0}$$

TAKE AWAY MESSAGE

- Relativistic relationships:

$$\beta^2 + \frac{1}{\gamma^2} = 1$$

- Relativistic differential relationships:

$$\frac{dE}{dp} = \beta c = v$$

$$\frac{dp}{p} = \frac{1}{\beta^2}\frac{dE}{E} = \frac{1}{\beta^2}\frac{d\gamma}{\gamma}$$

$$\frac{dp}{p} = \gamma^2 \frac{d\beta}{\beta}$$