Transverse Beam Dynamics III

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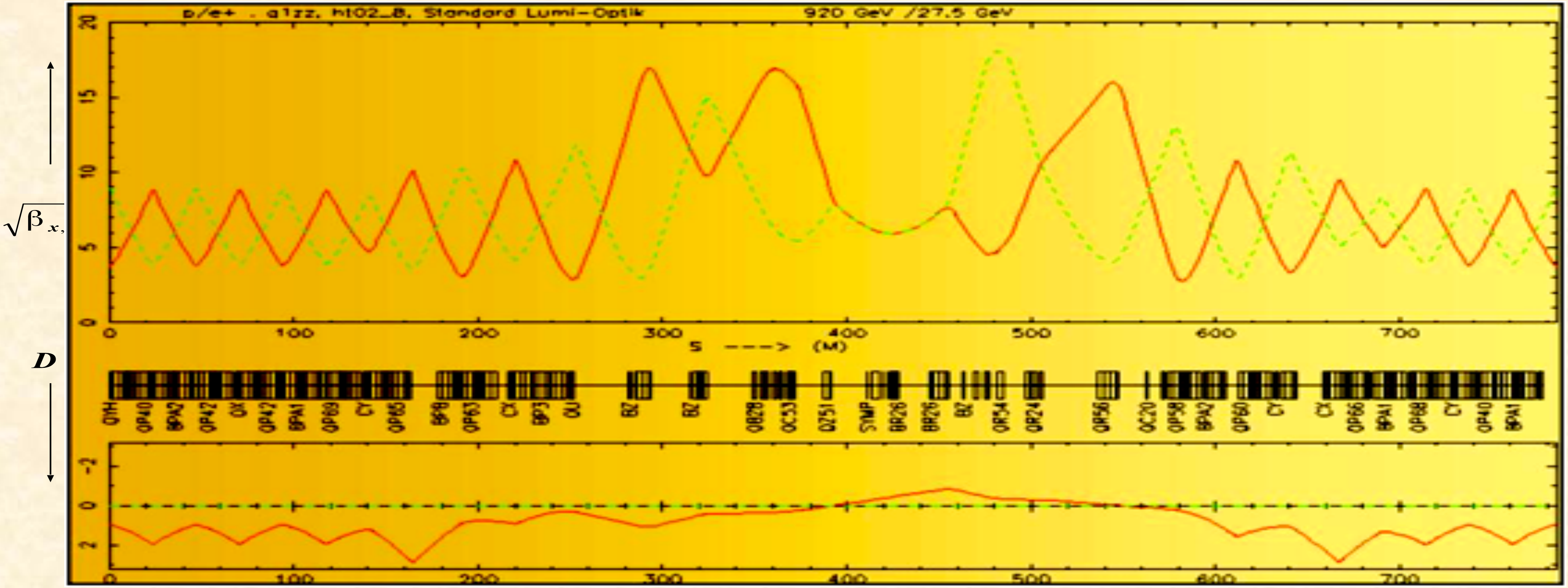
The „not so ideal“ World
Lattice Design in Particle Accelerators



1952: Courant, Livingston, Snyder:

Theory of strong focusing in particle beams

Recapitulation: ...the story with the matrices !!!

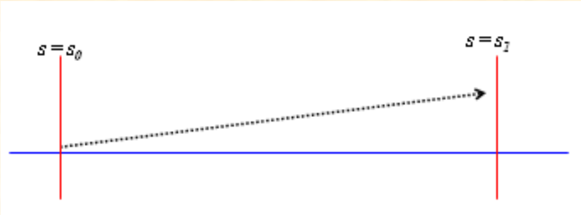
Equation of Motion:

$$x'' + K\,x = 0 \qquad K = \frac{1}{\rho^2} - k \quad \text{... hor. plane:}$$

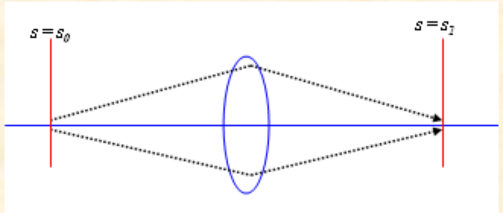
$$K = k \quad \text{... vert. Plane:}$$

Solution of Trajectory Equations

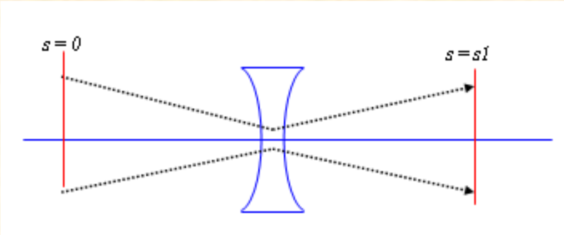
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}\,l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}\,l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}\,l) & \cos(\sqrt{|K|}\,l) \end{pmatrix}$$



$$M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}\,l) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}\,l) \\ \sqrt{|K|}\sinh(\sqrt{|K|}\,l) & \cosh(\sqrt{|K|}\,l) \end{pmatrix}$$

$$M_{total} = M_{QF} * M_D * M_B * M_D * M_{QD} * M_D * \quad \ldots$$

Recapitulation: ***...and for the complete particle ensemble the betas and epsilons !!!***

general solution of Hill equation

$$(1) \quad x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \phi)$$

$$(2) \quad x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} * \left\{ \alpha(s) * \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi) \right\}$$

from (1) we get

$$\cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\varepsilon} * \sqrt{\beta(s)}}$$

Insert into (2) and solve for ε

$$\alpha(s) = \frac{-1}{2} \beta'(s)$$

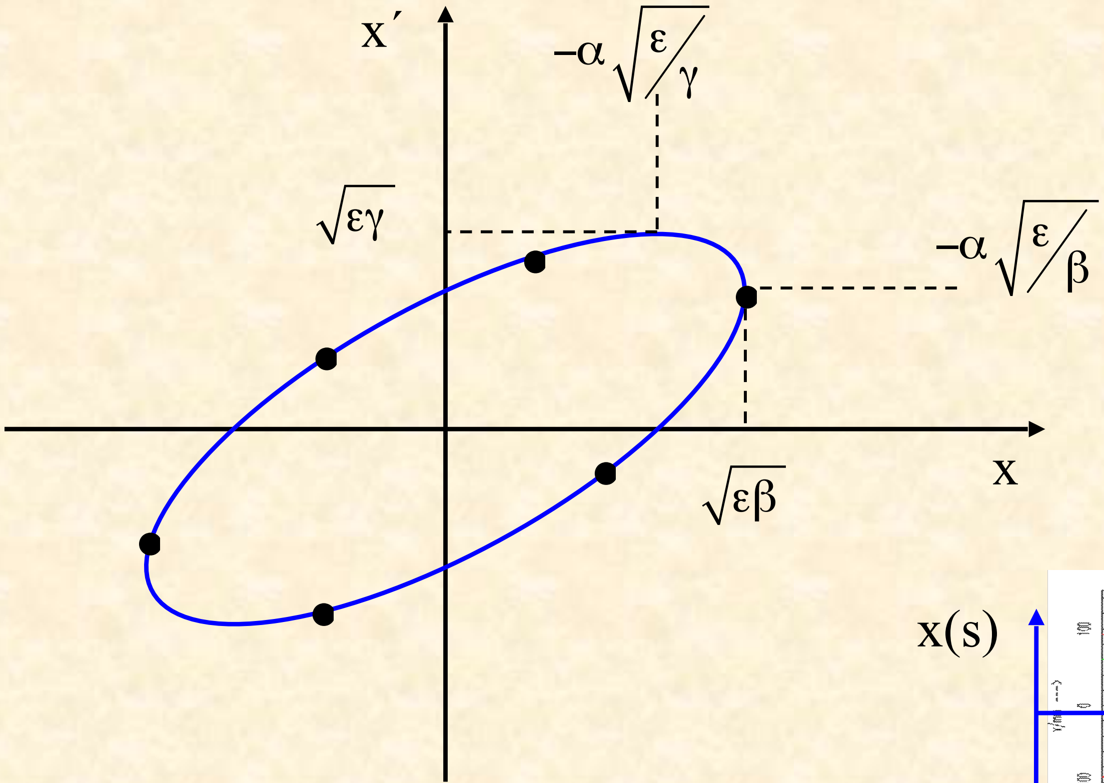
$$\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}$$

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s) x'(s)^2$$

**** ε is a constant of the motion … it is independent of „s"***
**** parametric representation of an ellipse in the x x' space***
**** shape and orientation of ellipse are given by α, β, γ***

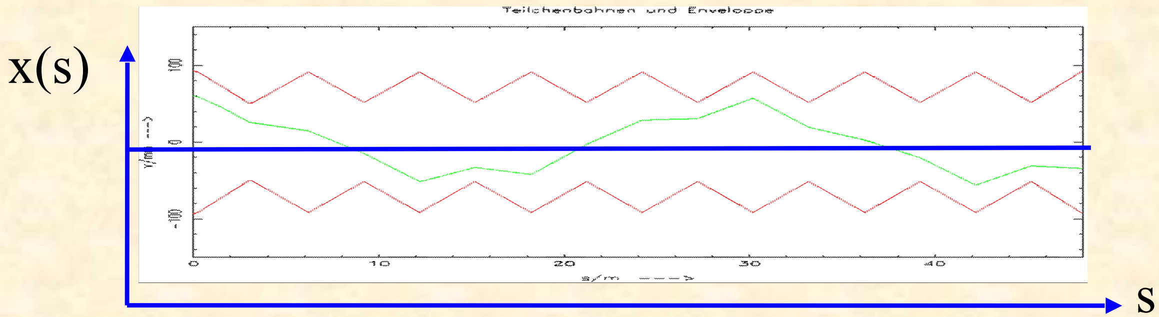
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



***Liouville:* in reasonable storage rings**
area in phase space is constant.

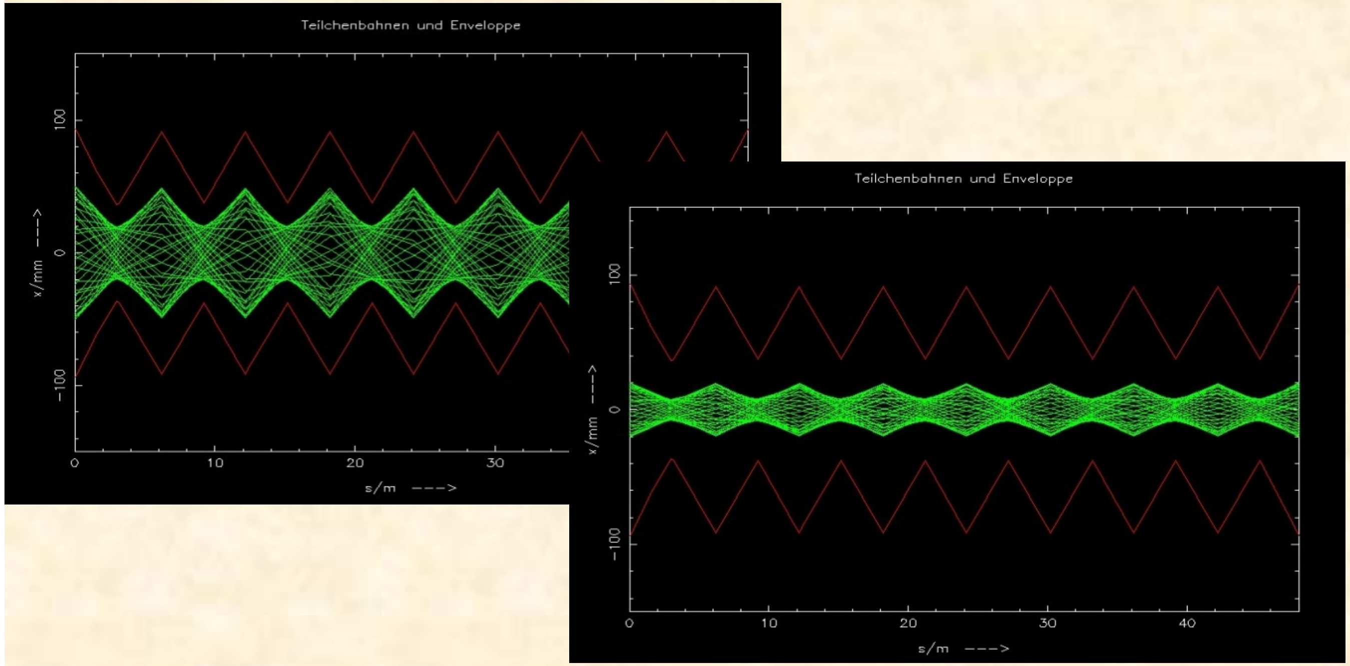
A = π*ε=const



ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.

***Scientifiquely speaking:* area covered *in transverse x, x´ phase space* … and it is constant !!!**

Emittance of the Particle Ensemble:



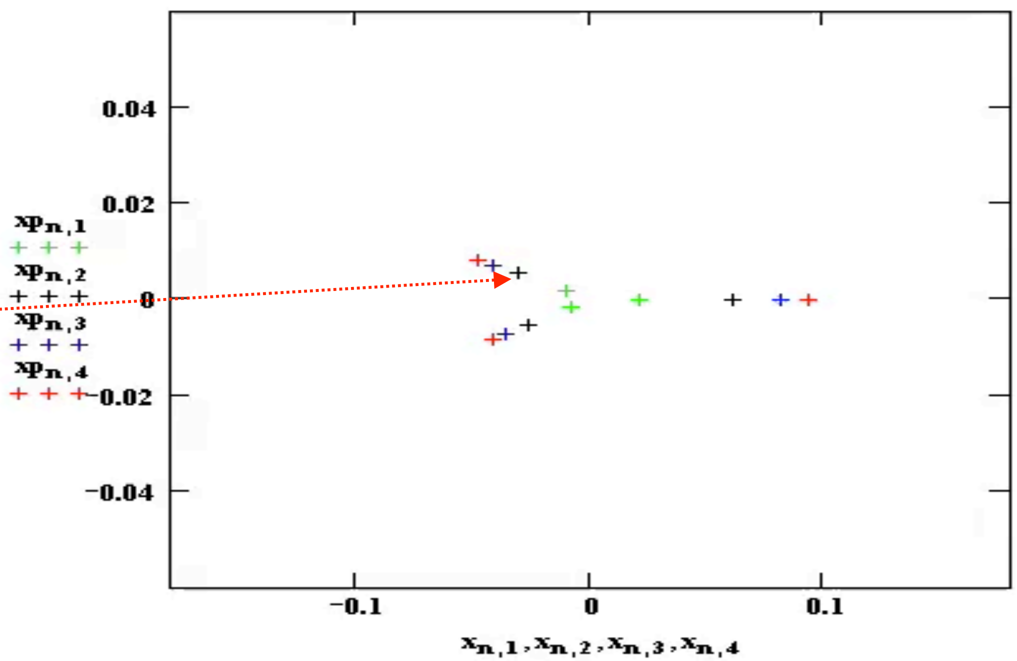
Example: LHC

beam parameters in the arc

$$\beta(x) \approx 180\ m$$

$$\varepsilon \approx 5*10^{-10}\ rad \cdot m \qquad (\leftrightarrow 1\sigma)$$

$$\sigma = \sqrt{\varepsilon\beta} \approx 0.3\ mm$$



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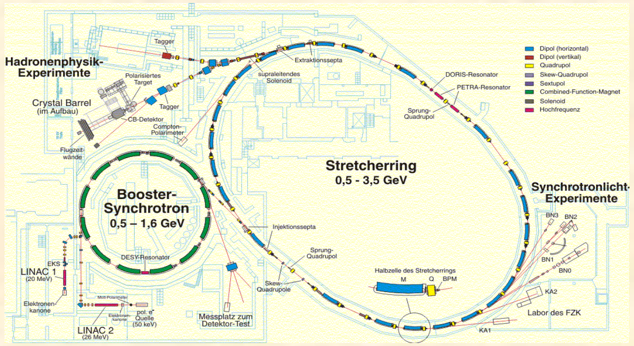
Periodic Lattices

transfer matrix for particle trajectories as a function of the lattice parameters



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M = \begin{pmatrix} \sqrt{\dfrac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s \beta_0}\,\sin\psi_s \\ \dfrac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\dfrac{\beta_0}{\beta_s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

„This rather formidable looking matrix simplifies considerably if we consider one complete turn …“

One Turn Matrix

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

$$\psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

ψ_{turn} = phase advance per period

Stability in a periodic structure requires $\leftrightarrow$ $|Trace(M)| < 2$

Transformation of α, β, γ

consider two positions in the storage ring: s_0 , s

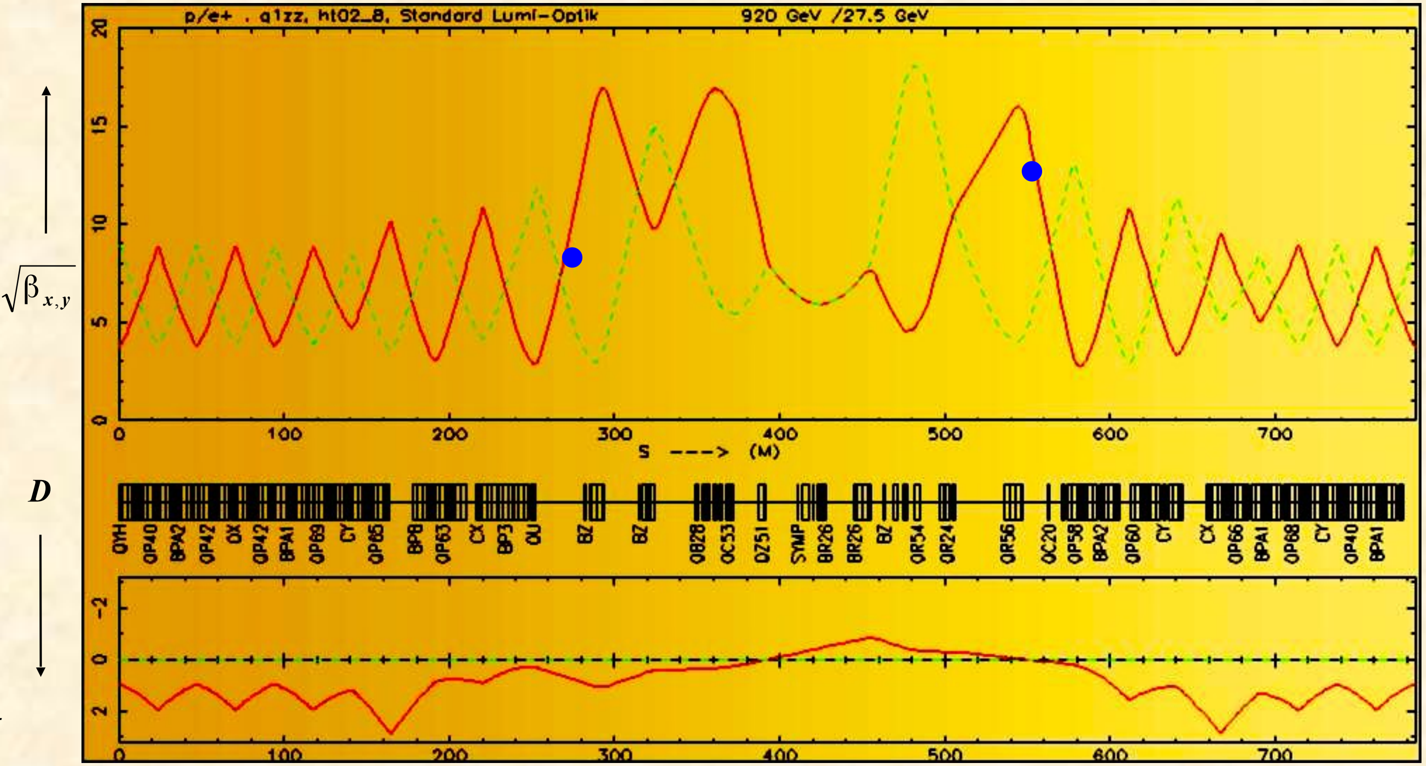
$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \qquad \text{where} \; \dots \quad M = M_{QF} \cdot M_{QD} \cdot M_B \cdot M_{Drift} \cdot M_{QF} \cdot \dots$$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \quad \longleftrightarrow \quad \boldsymbol{M}_{foc} = \begin{pmatrix} \cos(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}) & \frac{1}{\sqrt{|\boldsymbol{K}|}} \sin(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}) \\ -\sqrt{|\boldsymbol{K}|} \sin(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}) & \cos(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}) \end{pmatrix}$$

since ε = const (Liouville):

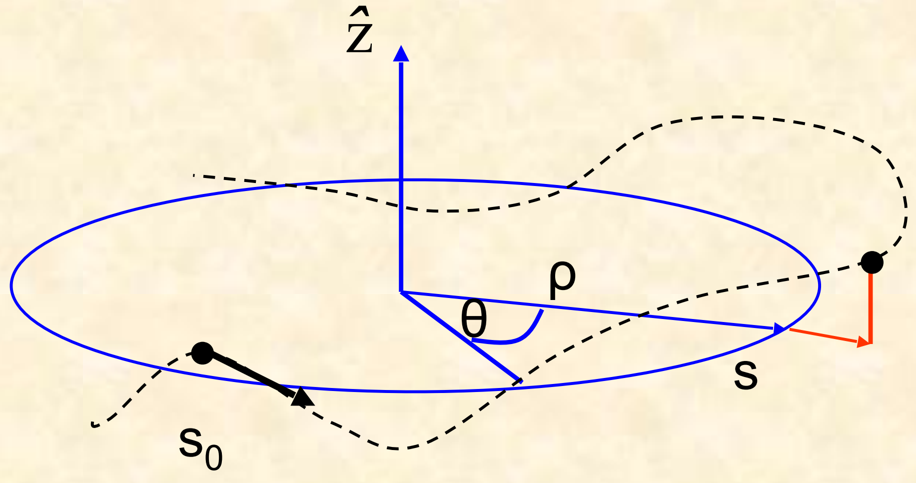
$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$



Beta function in a storage ring

express x_0, x'_0 as a function of x, x'.



$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

... remember W = CS′-SC′ = 1

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_s$$

$$M^{-1} = \frac{1}{det(M)} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

$= 1$

$$\Rightarrow \quad x_0 = S'x - Sx'$$
$$x'_0 = -C'x + Cx'$$

inserting into ε

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 (Cx' - C'x)^2 + 2\alpha_0 (S'x - Sx')(Cx' - C'x) + \gamma_0 (S'x - Sx')^2$$

sort via x, x′and compare the coefficients to get ....

$$\beta(s) = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0$$

in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

!

1.) this expression is important

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

4.) go back to point 1.)

Résumé:

equation of motion:

$$x''(s) + K(s)\,x(s) = 0 \;, \qquad K = \frac{1}{\rho^2} - k$$

general solution of Hill´s equation:

$$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

phase advance & tune:

$$\psi_{12}(s) = \int_{s1}^{s2} \frac{1}{\beta(s)}\,ds \;, \qquad Q(s) = \frac{1}{2\pi} \oint \frac{1}{\beta(s)}\,ds$$

emittance:

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

transfer matrix from s_1 —> s_2:

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s \beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

matrix for 1 turn:

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

stability criterion:

$$|Trace(M)| < 2$$

Transfer Matrix M

Transformation of particle coordinates:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

1.) using matrix notation in magnet parameters:

$$\boldsymbol{M}_{total} = \boldsymbol{M}_{QF} * \boldsymbol{M}_D * \boldsymbol{M}_B * \boldsymbol{M}_D * \boldsymbol{M}_{QD} * \boldsymbol{M}_D * \quad \ldots$$

$$\boldsymbol{M}_{foc} = \begin{pmatrix} \cos(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}) & \frac{1}{\sqrt{|\boldsymbol{K}|}}\sin(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}) \\ -\sqrt{|\boldsymbol{K}|}\sin(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}) & \cos(\sqrt{|\boldsymbol{K}|}\boldsymbol{l}) \end{pmatrix}$$

2.) using matrix notation in Twiss form:

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

3.) Transformation of Twiss parameters:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

III. Lattice Design in Particle Accelerators

„… how to build a storage ring“

High energy accelerators —> circular machines

somewhere in the lattice we need a number of dipole magnets, that are bending the design orbit to a closed ring

Geometry of the ring:

centrifugal force = Lorentz force

$$e * v * B = \frac{mv^2}{\rho}$$

$$\rightarrow e * B = \frac{mv}{\rho} = p / \rho$$

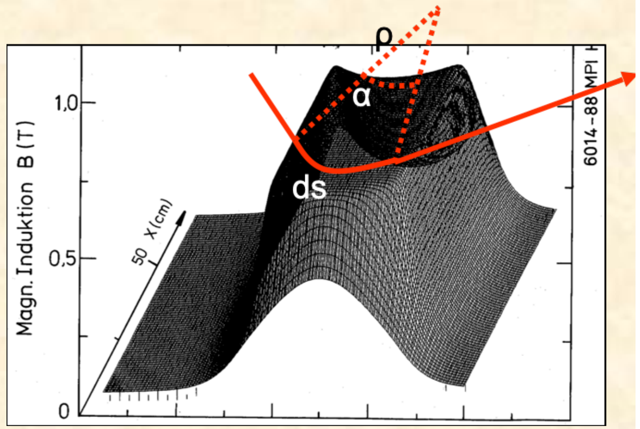
$$\rightarrow B * \rho = p / e$$

p = momentum of the particle,
ρ = curvature radius

Bρ= beam rigidity

Example: heavy ion storage ring TSR
8 dipole magnets of equal bending strength

19.) Lattice Design:



field map of a storage ring dipole magnet

$$B\rho = p/q$$

Circular Orbit: ***dipole magnets to define the geometry and the particle momentum (… energy)***

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{Bdl}{B\rho}$$

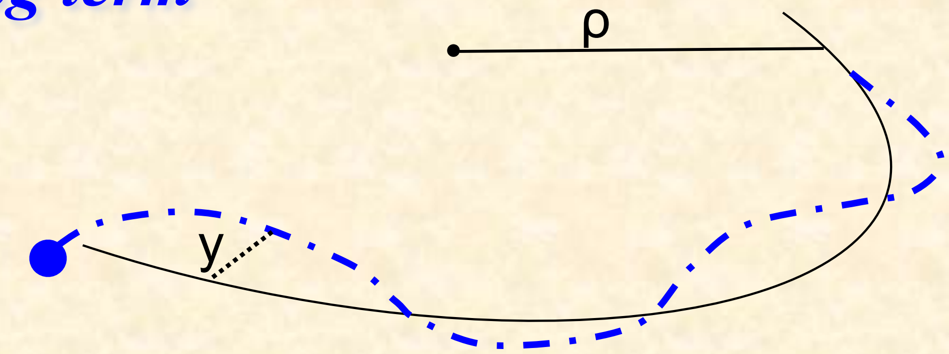
The integrated B-field of the dipoles determines the particle momentum

The number of dipoles is determined by the momentum of the beam … or vice versa.

—> tutorial exercise

$$\int Bdl = 2\pi\,\frac{p}{q}$$

20.) Focusing forces … weak focusing term



$$x'' + K * x = 0$$

$$K = \frac{1}{\rho^2} - k \qquad \textit{hor. plane}$$

$$K = k \qquad \textit{vert. plane}$$

dipole magnet $\frac{1}{\rho} = \frac{B}{p/q}$

quadrupole magnet $k = \frac{g}{p/q}$

Example: LHC Ring:

Bending radius: $\rho = 2.8\ km$

Quadrupol Gradient: $g = 220\ T/m$

$k = 9.4 * 10^{-3}\ /m^2$

$1/\rho^2 = 1.3 * 10^{-7}\ /m^2$

For estimates in large accelerators the weak focusing term $1/\rho^2$ can in general be neglected

Solution for a focusing magnet

$$x(s) = x_0 * \cos(\sqrt{K} * s) + \frac{x'_0}{\sqrt{K}} * \sin(\sqrt{K} * s)$$

$$x'(s) = -x_0 \sqrt{K} * \sin(\sqrt{K} * s) + x'_0 * \cos(\sqrt{K} * s)$$

The Twiss parameters α, β, γ can be transformed through the lattice via the matrix elements defined above.

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_S = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

Question: „ What does that mean ???? “

Most simple example: ***drift space***

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

particle coordinates

$$\begin{pmatrix} x \\ x' \end{pmatrix}_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0 \quad \rightarrow \quad \begin{aligned} x(l) &= x_0 + l * x_0' \\ x'(l) &= x_0' \end{aligned}$$

transformation of twiss parameters:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_l = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

$$\beta(s) = \beta_0 - 2l * \alpha_0 + l^2 * \gamma_0$$

Stability ...?

$$trace(M) = 1 + 1 = 2$$

→A periodic solution doesn't exist in a lattice built exclusively out of drift spaces.

Lattice Design:

Arc: ***regular (periodic) magnet structure:***

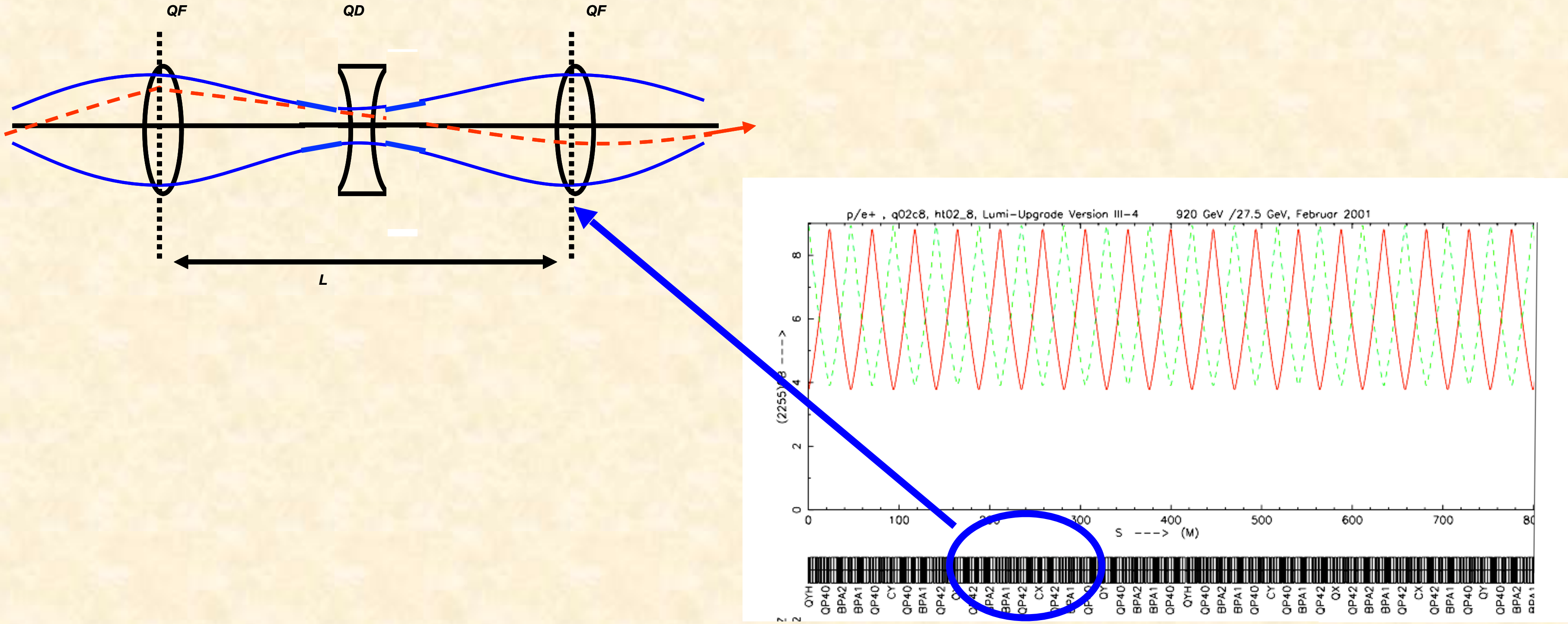
bending magnets to define the energy of the ring
main focusing & tune control, chromaticity correction,
multipoles for higher order corrections

Straight sections: ***drift spaces for injection, dispersion suppressors,***
low beta insertions, RF cavities, etc....
... and the high energy experiments if they cannot be avoided

21.) The FoDo-Lattice

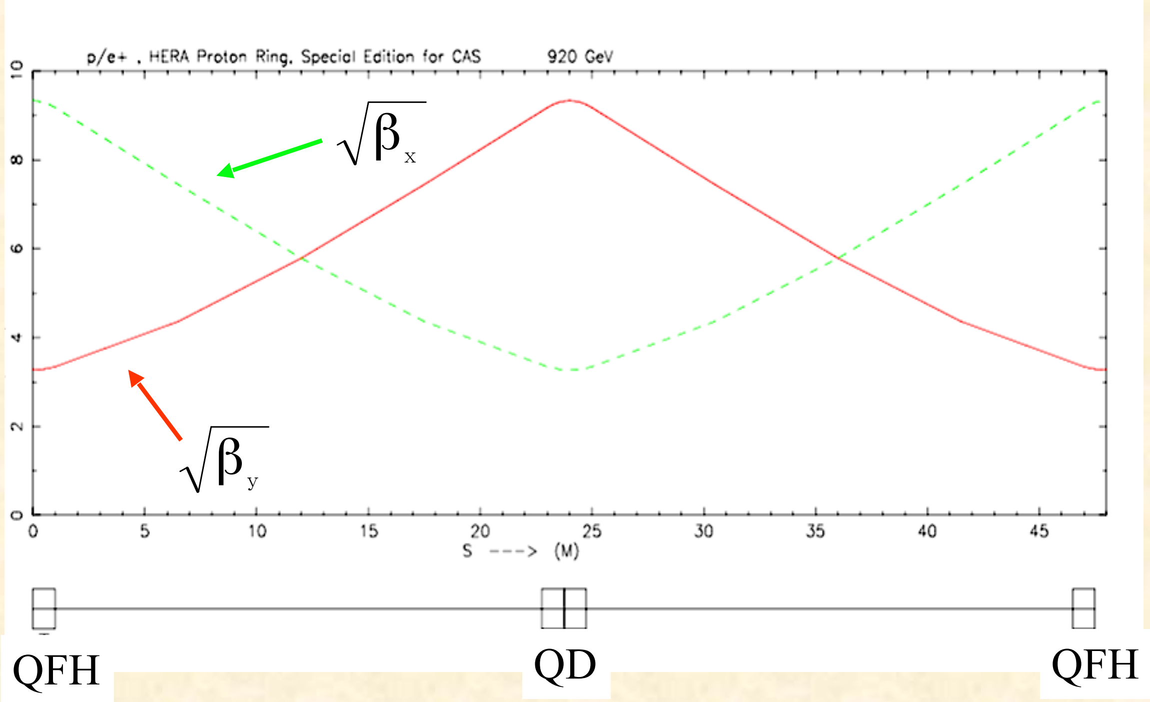
A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.

(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)



Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell $\mu = 45°$,
—> calculate the optics parameters for a periodic solution

Periodic Solution of a FoDo Cell



Output of the optics program:

Nr	Type	Length	Strength	β_x	α_x	φ_x	β_y	α_y	φ_y
		m	1/m2	m		1/2π	m		1/2π
0	Marker	0,000	0,000	11,611	0,000	0,000	5,295	0,000	0,000
1	QFH	0,250	-0,541	11,228	1,514	0,004	5,488	-0,781	0,007
2	QD	3,251	0,541	5,488	-0,781	0,070	11,228	1,514	0,066
3	QFH	6,002	-0,541	11,611	0,000	0,125	5,295	0,000	0,125
4	Marker	6,002	0,000	11,611	0,000	0,125	5,295	0,000	0,125

Qx= 0,125 Qy= 0,125

$$0.125 * 2\pi = 45^0$$

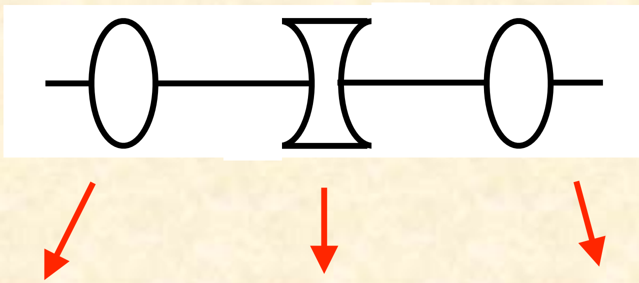
Can we understand what the optics code is doing ?

matrices

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l_q) & \frac{1}{\sqrt{K}}\sin(\sqrt{K} * l_q) \\ -\sqrt{K}\sin(\sqrt{K} * l_q) & \cos(\sqrt{K} * l_q) \end{pmatrix}, \qquad M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1_d \end{pmatrix}$$

strength and length of the FoDo elements K = +/- 0.54102 m^{-2}
lq = 0.5 m
ld = 2.5 m

The matrix for the complete cell is obtained by multiplication of the element matrices



$$M_{FoDo} = M_{QFH} \cdot M_{ld} \cdot M_{QD} \cdot M_{ld} \cdot M_{QFH}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for 1 period gives us all the information that we need !

1.) is the motion stable?
trace < 2

$$trace(M_{FoDo}) = 1.415 \rightarrow$$ 😊

2.) phase advance per cell

$$M(s) = \begin{pmatrix} \cos\psi_{cell} + \alpha_s \sin\psi_{cell} & \beta_s \sin\psi_{cell} \\ -\gamma_s \sin\psi_{cell} & \cos\psi_{cell} - \alpha_s \sin\psi_{cell} \end{pmatrix}$$

$$cos(\Psi_{cell}) = \frac{1}{2} Trace(M) = 0.707$$

$$\rightarrow \Psi_{cell} = cos^{-1}\left\{\frac{1}{2} Trace(M)\right\} = 45^o$$

3.) hor β-function

$$\beta = \frac{m_{12}}{sin\psi_{cell}} = 11.611\ m$$

4.) hor α-function

$$\alpha = \frac{m_{11} - cos\psi_{cell}}{sin\psi_{cell}} = 0$$

We can determine the Twiss parameters !!

… by calculating the matrix of a given periodic structure

… using the magnet parameters and the product matrix of the structure

… and compare with the (periodic) matrix in Twiss form.

The prize to pay … is hidden in these crazy functions like cos / sinh / cosh etc …

Can we do a bit easier ?
We can ... in thin lens approximation !

Matrix of a focusing quadrupole magnet:

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

If the focal length f is much larger than the length of the quadrupole magnet,

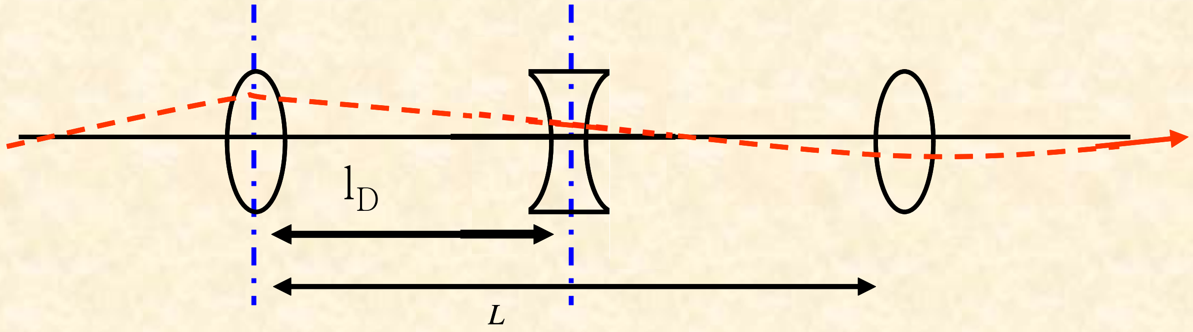
$$f = 1/kl_Q >> l_Q$$

the transfer matrix can be aproximated using $kl_q = const, \; l_q \rightarrow 0$

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix} \quad , \quad M_{QD} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

=> we keep the focusing strength $k \cdot l_q$ ***constant, but make the length zero.***

22.) FoDo in thin lens approximation



$$l_D = L/2$$
$$\tilde{f} = 2f$$

Calculate the matrix for a half cell, starting in the middle of a foc. quadrupole:

$$M_{halfCell} = M_{QD/2} * M_{lD} * M_{QF/2}$$

$$M_{halfCell} = \begin{pmatrix} 1 & 0 \\ 1/\tilde{f} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_D \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -1/\tilde{f} & 1 \end{pmatrix}$$

note: denotes the focusing strength of half a quadrupole, so

$$\tilde{f} = 2f$$

$$M_{halfCell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$

for the second half cell set f —> -f

FoDo in thin lens approximation

Matrix for the complete FoDo cell

$$M = \begin{pmatrix} 1+{l_D}/{\tilde{f}} & l_D \\ -{l_D}/{\tilde{f}^2} & 1-{l_D}/{\tilde{f}} \end{pmatrix} * \begin{pmatrix} 1-{l_D}/{\tilde{f}} & l_D \\ -{l_D}/{\tilde{f}^2} & 1+{l_D}/{\tilde{f}} \end{pmatrix}$$

$$M = \begin{pmatrix} 1-\frac{2l_D^2}{\tilde{f}^2} & 2l_D(1+\frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3}-\frac{l_D}{\tilde{f}^2}) & 1-2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Now we know, that the phase advance is related to the transfer matrix by

$$cos(\Psi_{cell}) = \frac{1}{2}\, trace(M) = \frac{1}{2}\cdot(2-\frac{4l_d^2}{\tilde{f}^2}) = 1-\frac{2l_d^2}{\tilde{f}^2} \qquad \textbf{(i)}$$

After some beer and with a little bit of trigonometric gymnastics

$$cos(x) = cos^2(x/2) - sin^2(x/2) = 1 - 2\cdot sin^2(x/2) \qquad \textbf{(ii)}$$

we can calculate the phase advance as a function of the FoDo parameter …
by comparing (i) and (ii)

$$cos(\Psi_{cell}) = 1 - 2 \cdot sin^2(\Psi_{cell}/2) = 1 - \frac{2l_d^2}{\tilde{f}^2} \rightarrow sin(\Psi_{cell}/2) = l_d/\tilde{f} = \frac{L_{cell}}{2\tilde{f}}$$

phase advance of a FoDo cell:
(in thin lens approx)

$$sin(\Psi_{cell}/2) = \frac{L_{cell}}{4f}$$

Example:
45-degree Cell

$$L_{Cell} = l_{QF} + l_D + l_{QD} + l_D$$
$$= 0.5m+2.5m+0.5m+2.5m = 6m$$

$$1/f = k*l_Q = 0.5m*0.541\ m^{-2} = 0.27\ m^{-1} \quad \text{—>}\ f = 3.7\ m$$

$$sin(\Psi_{cell}/2) = \frac{L_{cell}}{4f} = 0.405$$
$$\rightarrow \psi_{cell} = 47.8^\circ$$
$$\rightarrow \beta = 11.4\ m$$

Remember:
Exact calculation yields:

$$\rightarrow \psi_{cell} = 45^\circ$$
$$\rightarrow \beta = 11.6\ m$$

Stability in a FoDo structure

SPS Lattice

$$M_{FoDo} = \begin{pmatrix} 1-\frac{2l_D^2}{\tilde{f}^2} & 2l_D(1+\frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3}-\frac{l_D}{\tilde{f}^2}) & 1-2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Stability requires:

$$|Trace(M)| < 2 \qquad |Trace(M)| = |2-\frac{4l_d^2}{\tilde{f}^2}| < 2$$

$$\rightarrow f > \frac{L_{cell}}{4}$$

For stability the focal length has to be larger than a quarter of the FoDo cell length
... don't focus to strong !

Example:

45-degree Cell

$L_{Cell} = l_{QF} + l_D + l_{QD} + l_D$

$= 0.5m+2.5m+0.5m+2.5m = 6m$ —> $L_{cell}/4 = 1.5\ m$

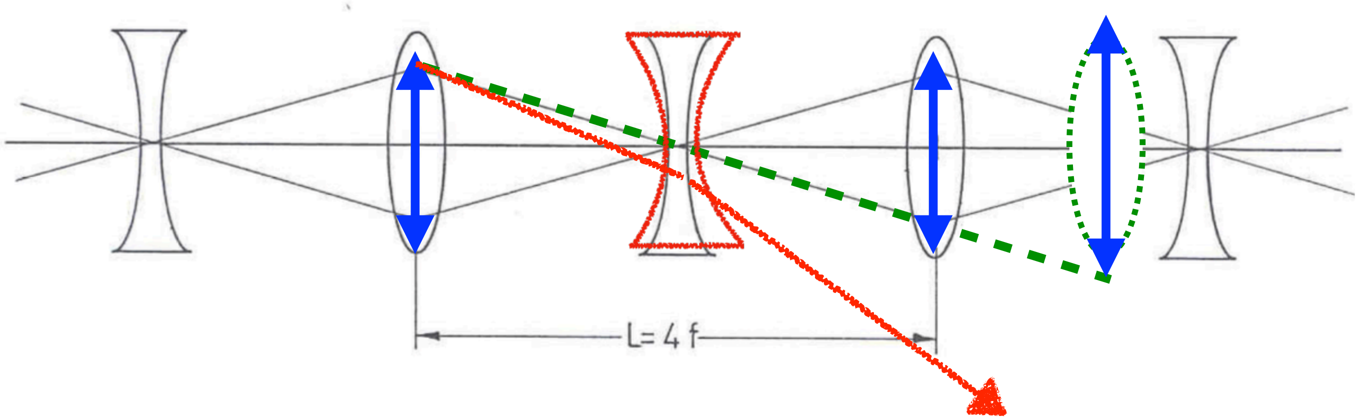
$1/f = k*l_Q = 0.5m*0.541\ m^{-2} = 0.27\ m^{-1}$ —> $f = 3.7\ m$

o.k.

The FODO cell

Stability condition $4f \geq L$, has a simple interpretation: ***—> do not focus too strong !!***

- It is well known from optics that an object at a distance $a = 2f$ from a focusing lens has its image at $b = 2f$



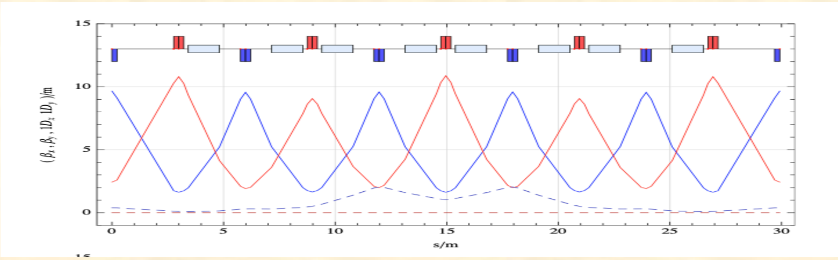
- The defocusing lenses have no effect if a point-like object is located exactly on the axis at distance $2f$ from a focusing lens, because they are traversed on the axis
- If however the lens system is moved further apart ($L > 4f$), this is no more true and the divergence of the light or particle beam is increased by every defocusing lens

23.) Lattice Types: Arc structure

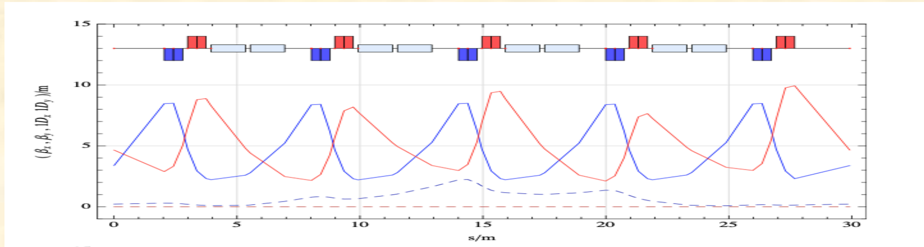
Magnet structures can be optimised to obtain certain properties of the beam optics

The most important ones:

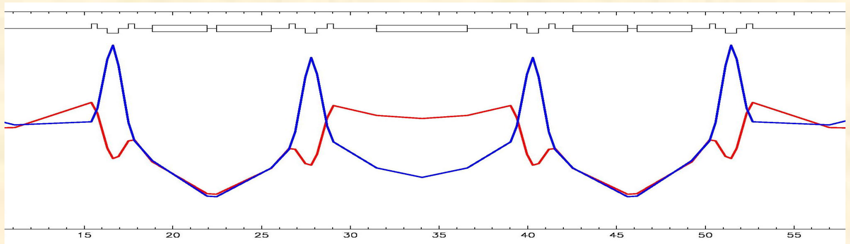
FoDo



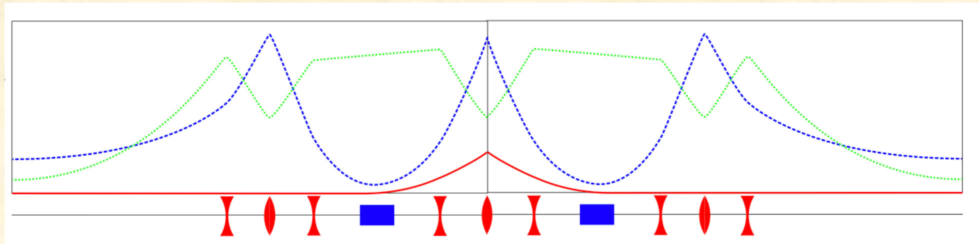
Doublets



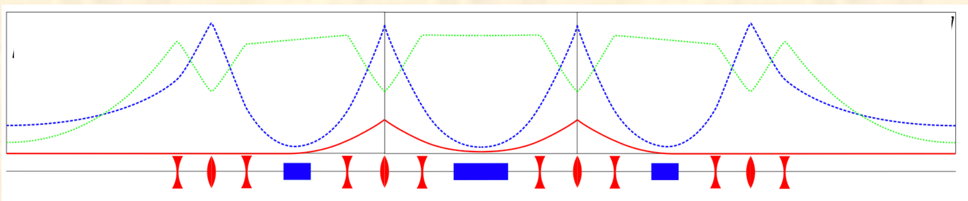
Triplett



Double Bend Achromat



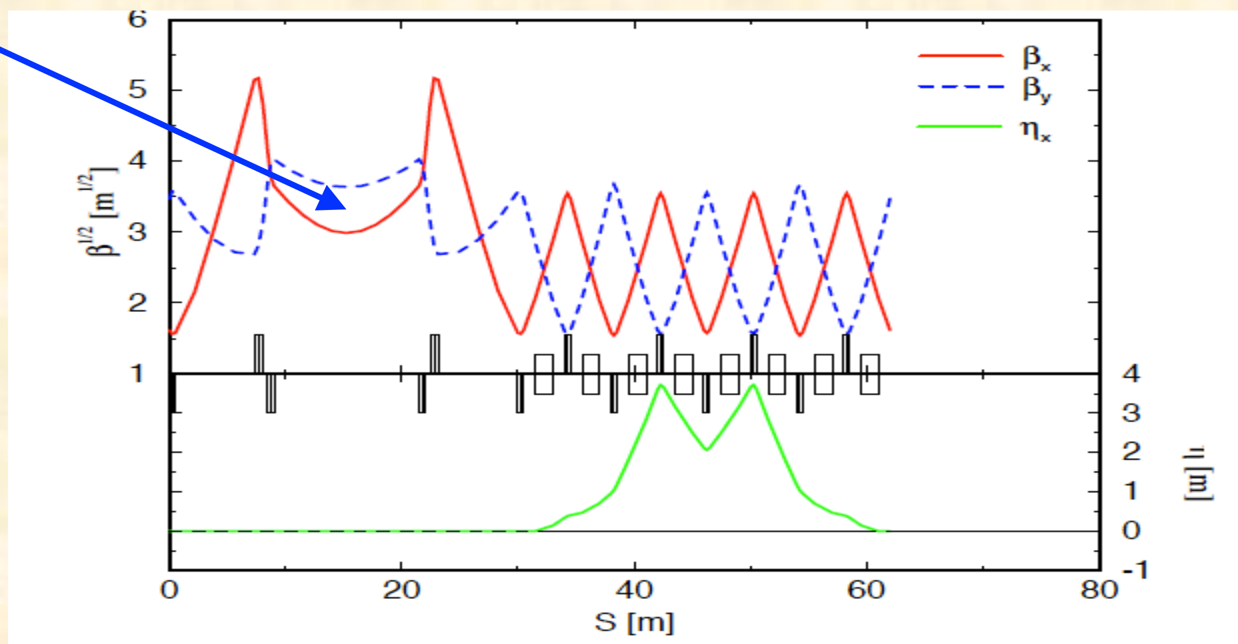
Triple Bend Achromat



Lattice Types: Insertions

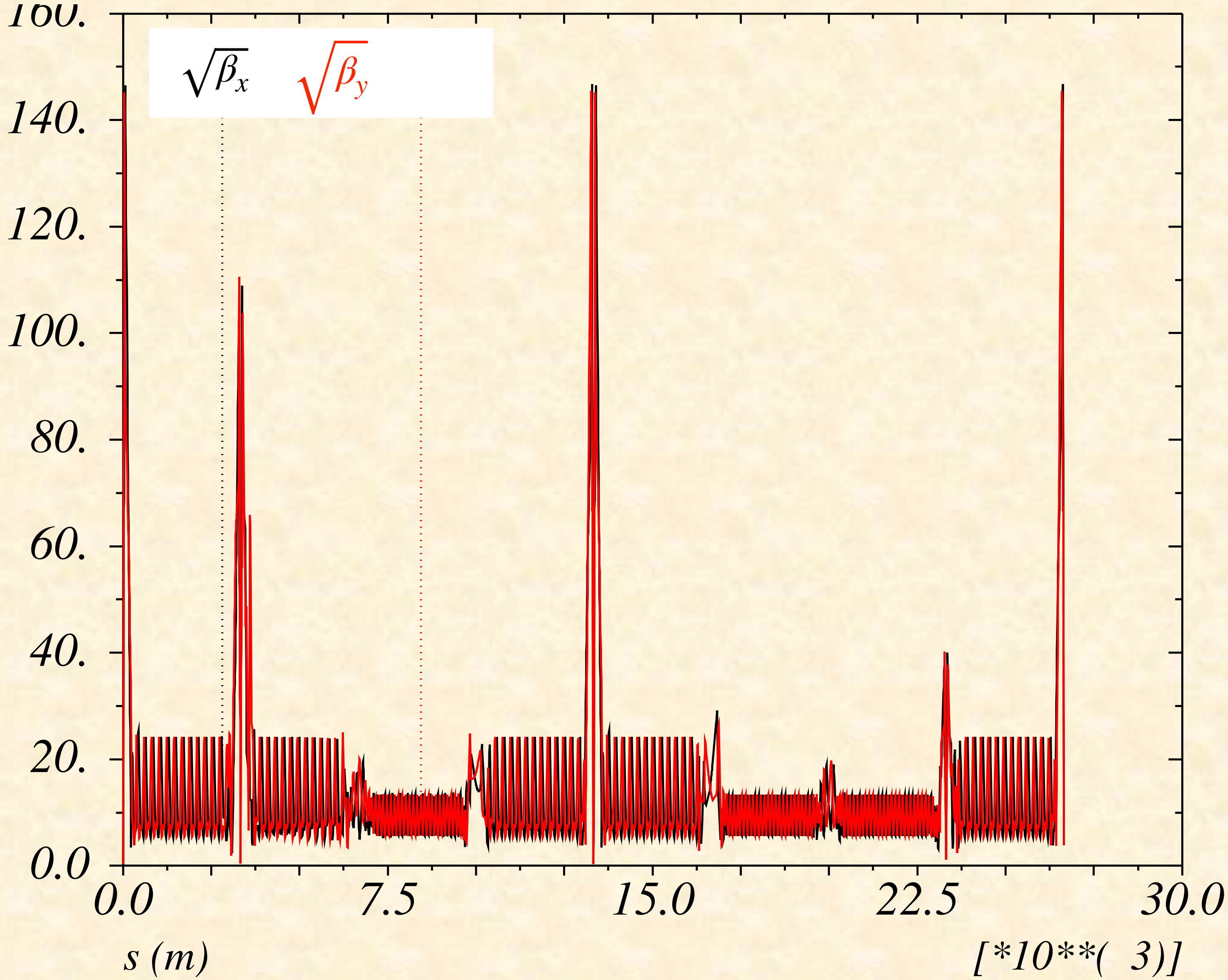
Magnet structures - lattices - can be optimised to obtain certain properties of the beam optics

Long symmetric Insertion



… a symmetric insertion that is optimised for small beta functions at the waist is called "Low-Beta-Insertion" or even "Mini-Beta-Insertion"

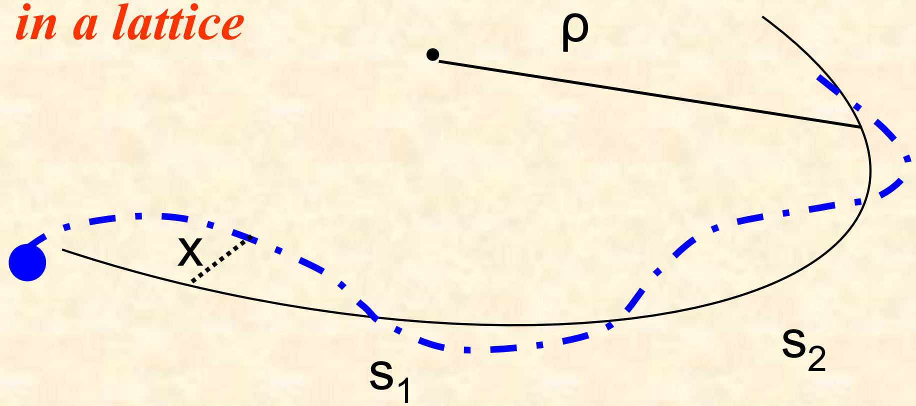
Back to the FoDo



Optics parameters of a FoDo:

Transformation of the coordinate vector (x, x′) in a lattice

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M_{s1,s2} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

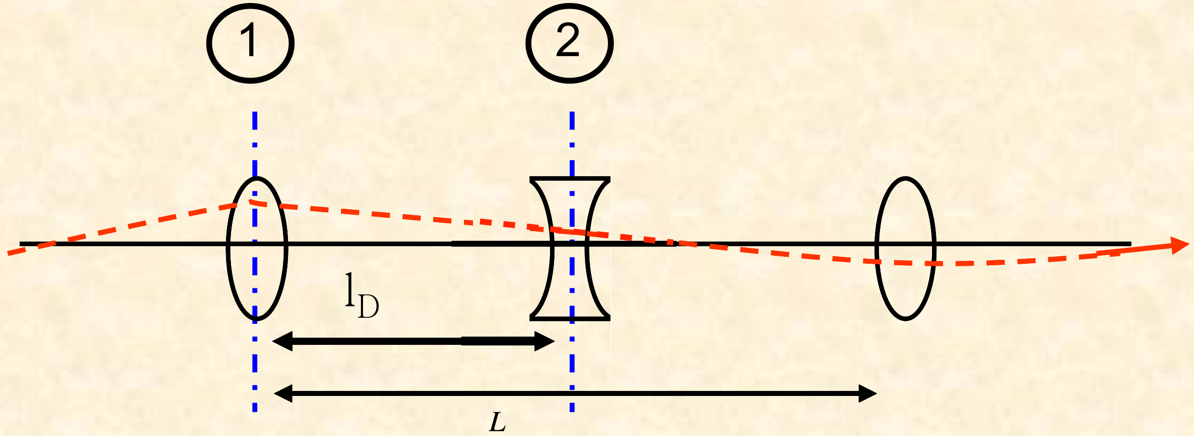


Transformation of the coordinate vector (x, x′) expressed as a function of the Optics parameters

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1 + \alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$

Transfer Matrix for half a FoDo cell:

$$M_{halfcell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$



Compare to the twiss parameter form of M

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1+\alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$

In the middle of a foc (defoc) quadrupole of the FoDo we always have $\alpha = 0$, and the half cell will lead us from β_{max} to β_{min}

$\beta_1 = \beta_{max}$ $\beta_2 = \beta_{min}$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_{min}}{\beta_{max}}}\cos\frac{\psi_{cell}}{2} & \sqrt{\beta_{min}\beta_{max}}\sin\frac{\psi_{cell}}{2} \\ \frac{-1}{\sqrt{\beta_{min}\beta_{max}}}\sin\frac{\psi_{cell}}{2} & \sqrt{\frac{\beta_{max}}{\beta_{min}}}\cos\frac{\psi_{cell}}{2} \end{pmatrix}$$

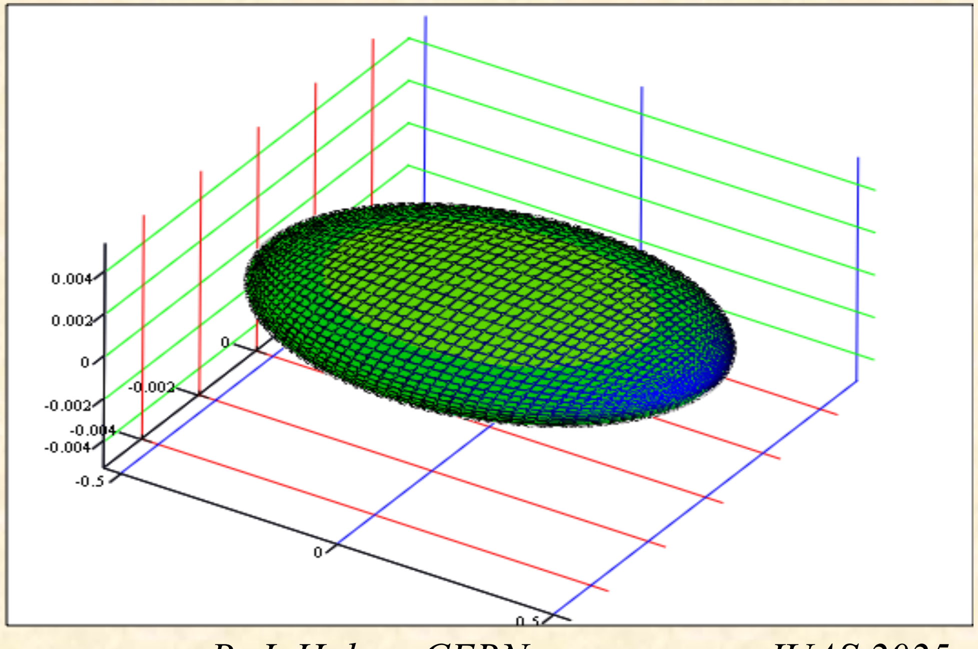
Solving for β_{max} and β_{min} and remembering that ….

$$\sin\frac{\psi_{cell}}{2} = \frac{l_d}{\tilde{f}} = \frac{L}{4f}$$

$$\left.\begin{aligned} \frac{m_{22}}{m_{11}} &= \frac{\beta_{max}}{\beta_{min}} = \frac{1 + l_d/\tilde{f}}{1 - l_d/\tilde{f}} = \frac{1 + \sin\frac{\psi_{cell}}{2}}{1 - \sin\frac{\psi_{cell}}{2}} \\ \frac{m_{12}}{m_{21}} &= \beta_{max}\cdot\beta_{min} = \tilde{f}^2 = \frac{l_d^2}{\sin^2\frac{\psi_{cell}}{2}} \end{aligned}\right\} \rightarrow$$

$$\beta_{max} = \frac{(1 + \sin\frac{\psi_{cell}}{2})\ L}{\sin\ (\psi_{cell})} \quad !$$

$$\beta_{min} = \frac{(1 - \sin\frac{\psi_{cell}}{2})\ L}{\sin\ (\psi_{cell})} \quad !$$



The maximum and minimum values of the β-function are solely determined by the phase advance and the length of the cell.

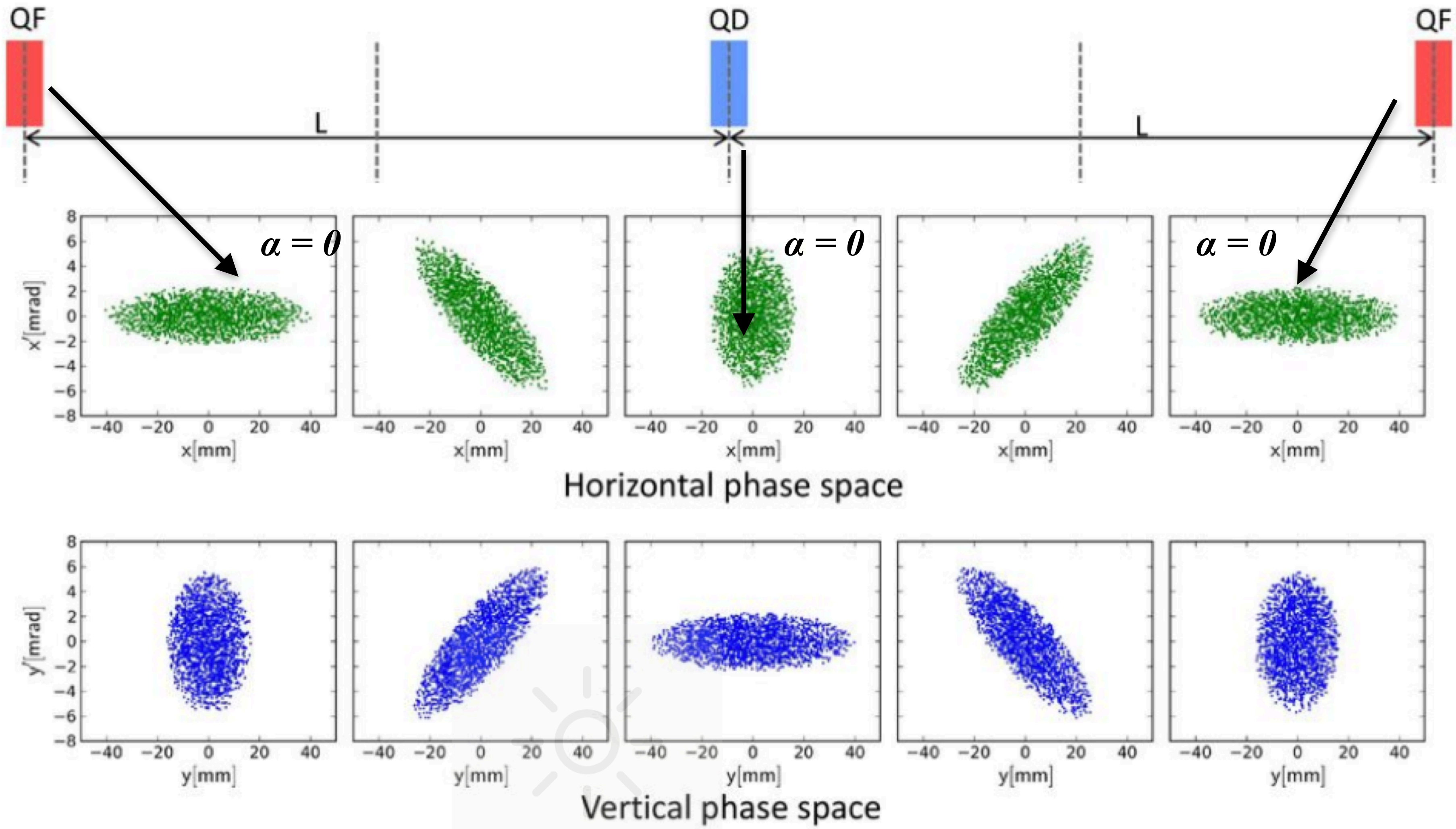
Longer cells lead to larger β

typical shape of a proton bunch in a FoDo Cell

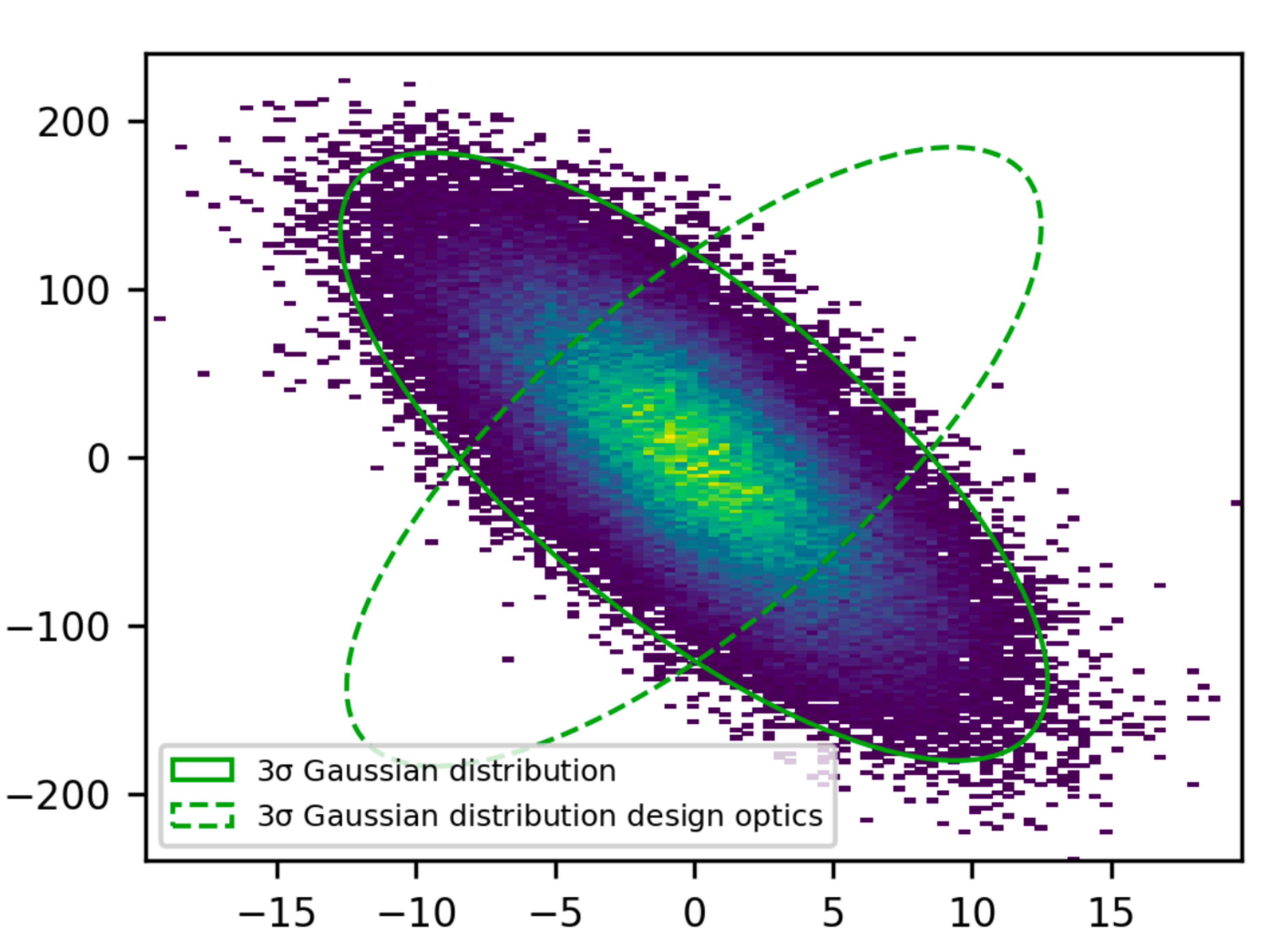
Phase Space Ellipse in a FoDo

The Twiss family α, β, γ determines shape & orientation of the ellipse

Phase space evolution in a FODO cell

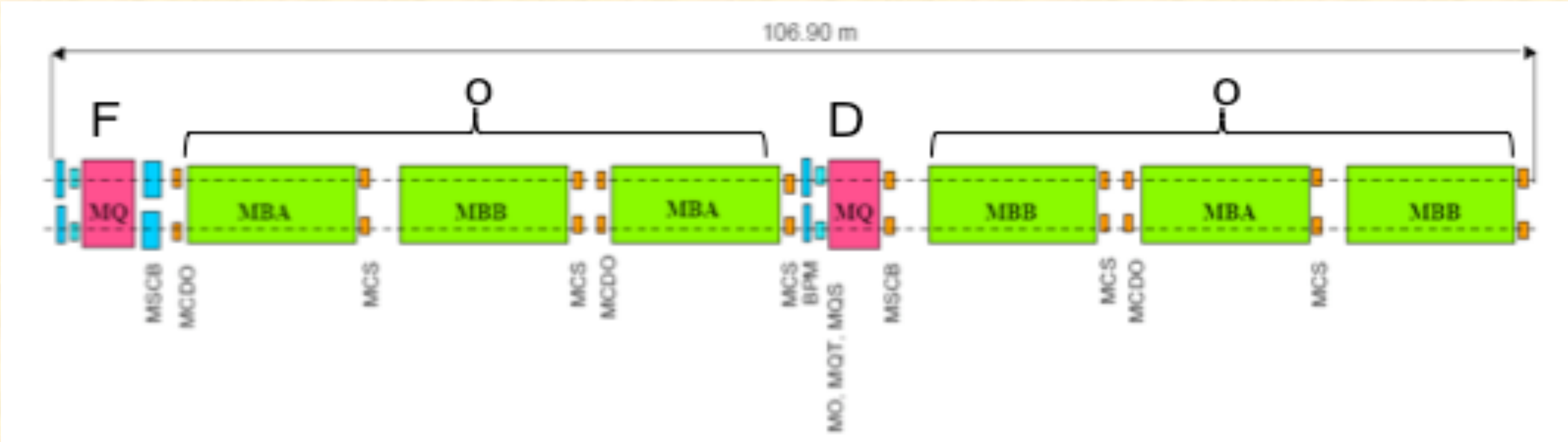


Phase Space Ellipse of a real beam in a FoDo



and now in reality …

LHC: Arc structure based on 90° FoDo in both planes



equipped with additional corrector coils

MB: main dipole
MQ: main quadrupole
MQT: Trim quadrupole
MQS: Skew trim quadrupole
MO: Lattice octupole (Landau damping)
MSCB: Skew sextupole
Orbit corrector dipoles
MCS: Spool piece sextupole
MCDO: Spool piece 8 / 10 pole
BPM: Beam position monitor + diagnostics

One Word about non-periodic Lattices:

- In the previous sections the Twiss parameters α, β, γ, and μ have been derived for a periodic, circular accelerator. The condition of periodicity was essential for the definition of the beta function (Hill's equation)
- Often, however, a particle beam moves only **once** along a **beam transfer line**, but one is nonetheless interested in quantities like beam envelopes and beam divergence
- In a circular accelerator α, β, and γ are completely determined by the magnet optics and the condition of periodicity (beam properties are not involved - only the beam emittance is chosen to match the beam size)
- In a transfer line, α, β, and γ are no longer uniquely determined by the transfer matrix, but they also depend on initial conditions which have to be specified in an adequate way

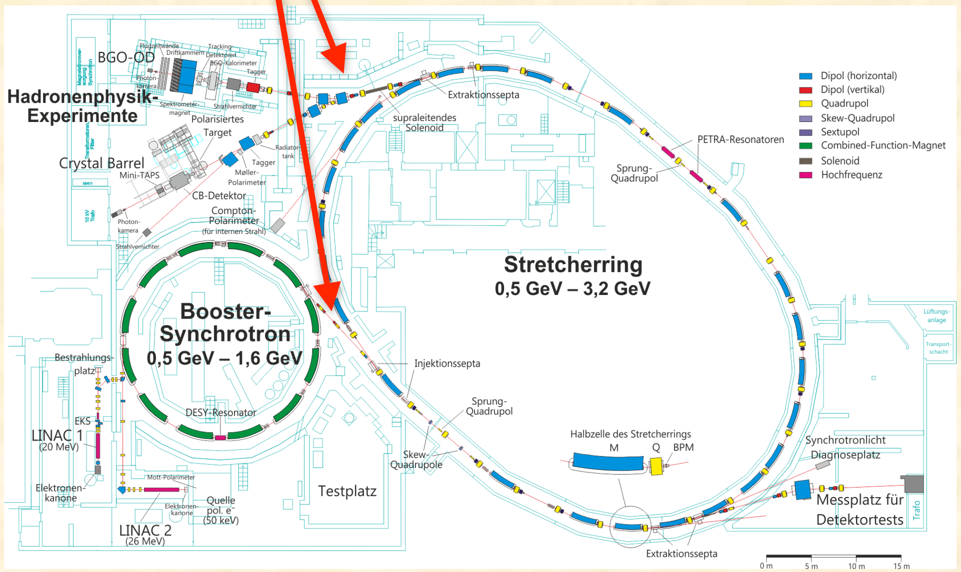
Transferring the Twiss family:

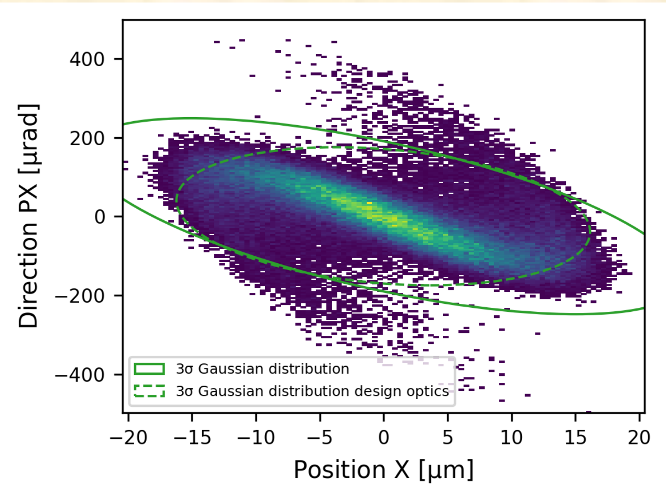
Rule for a non-periodic situation

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix}\beta_0\\ \alpha_0\\ \gamma_0\end{pmatrix}$$

However we have to determine (!) the initial Twisses

ELSA storage ring: start with the (uniquely) defined periodic Twiss.





LHeC: energy recovery linac
after the collision
—> fit … and guess: β, α, ε

Introduction to Transverse Beam Dynamics

III.) The „ not so ideal world “

Acceleration and Momentum Spread

Remember:

Beam Emittance and Phase Space Ellipse:

equation of motion: $x'' + K\,x = 0$

general solution of Hills equation: $x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\varphi)$

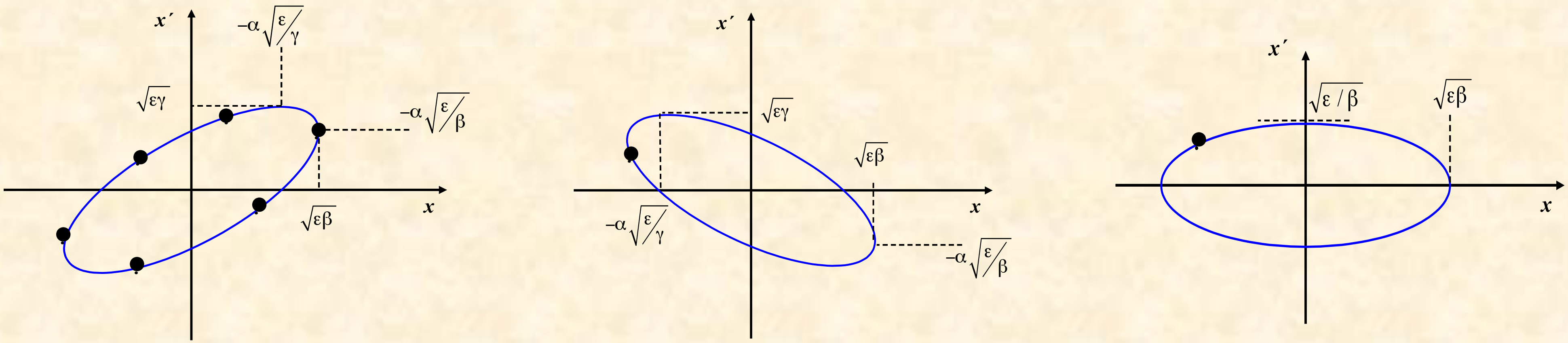
beam size: $\sigma = \sqrt{\varepsilon\beta} \approx \text{"}mm\text{"}$

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

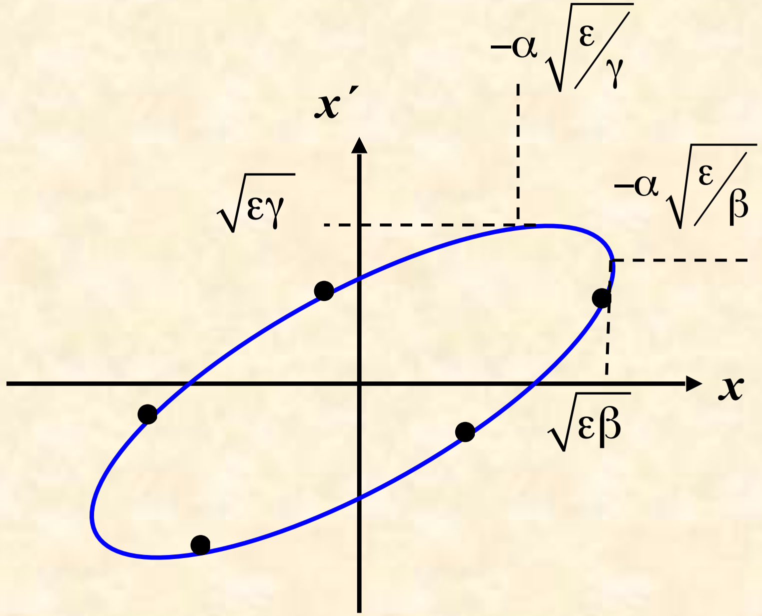
$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

* *ε is a constant of the motion … it is independent of „s“*
* *parametric representation of an ellipse in the x x‘ space*
* *shape and orientation of ellipse are given by α, β, γ*



24.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



***Beam Emittance** corresponds to the area covered in the x, x´ Phase Space Ellipse*

***Liouville:** Area in phase space is constant.*

But so sorry ... ε ≠ const !

Classical Mechanics:

phase space = diagram of the two canonical variables

position & momentum

x $\qquad$ p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = kin.\,Energy - pot.\,Energy$$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

Liouvilles Theorem:

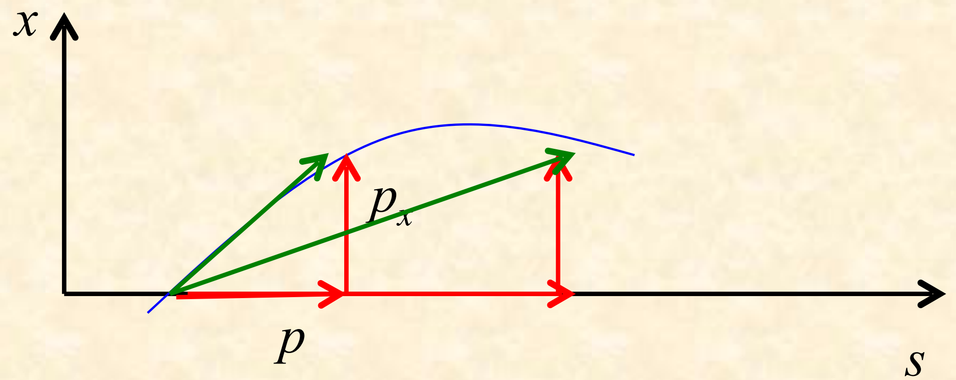
$$\int p\,dq = const$$

$$\int p_x dx = const$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt} \cdot \frac{dt}{ds} = \frac{\beta_x}{\beta} = \frac{p_x}{p}$$



$$\underbrace{\int x' dx}_{\varepsilon} = \frac{p_x dx}{p} \propto \frac{const}{m_0 c \;\gamma\beta}$$

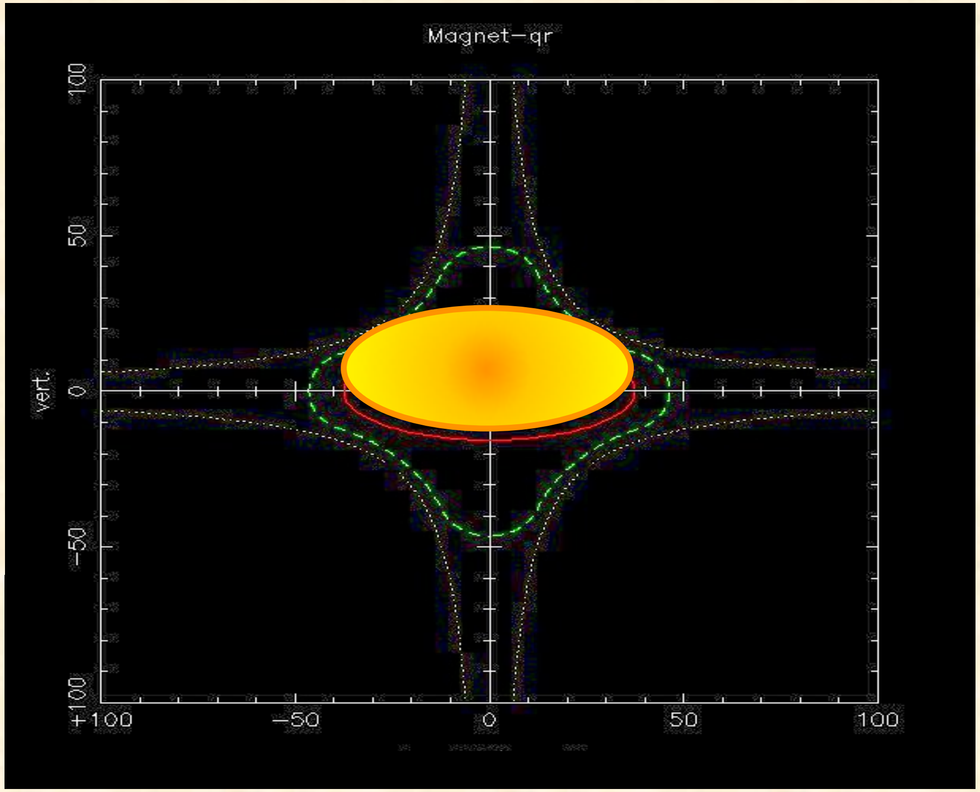
$$\Rightarrow \quad \varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

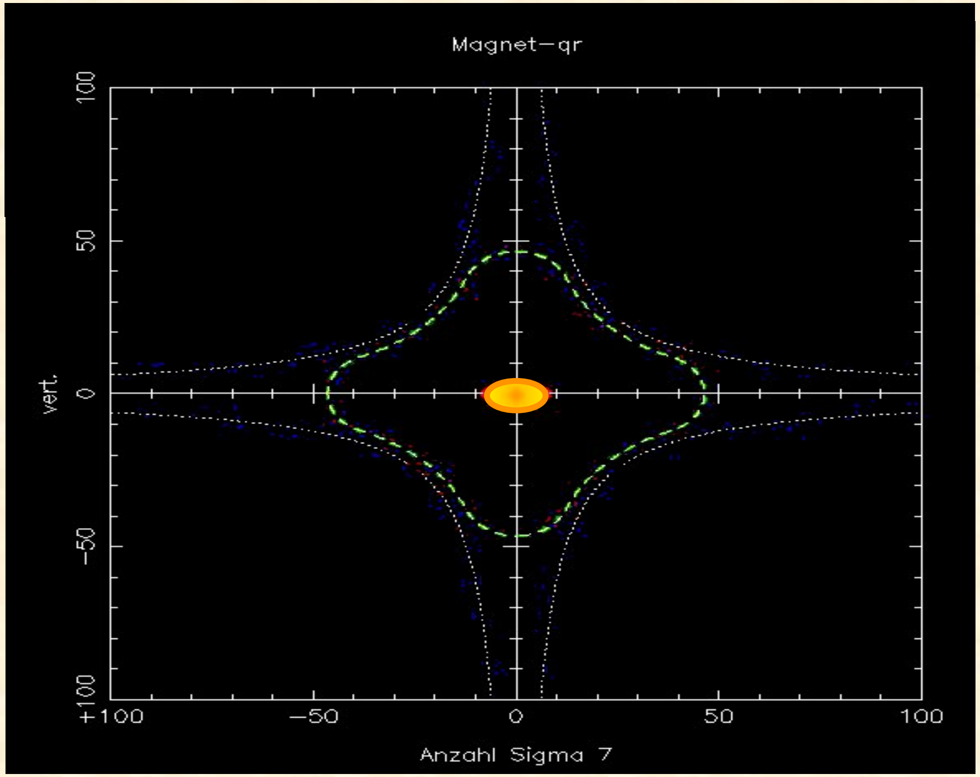
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance ε *(40GeV)* $= 1.2 * 10^{-7}$ *m rad*
ε *(920GeV)* $= 5.1 * 10^{-9}$ *m rad*



… at injection, E = 40 GeV



… and at E = 920

7 σ beam envelope at E = 40 GeV

Nota bene:

1.) A proton machine … or an electron linac … needs the highest aperture at injection energy !!!

as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

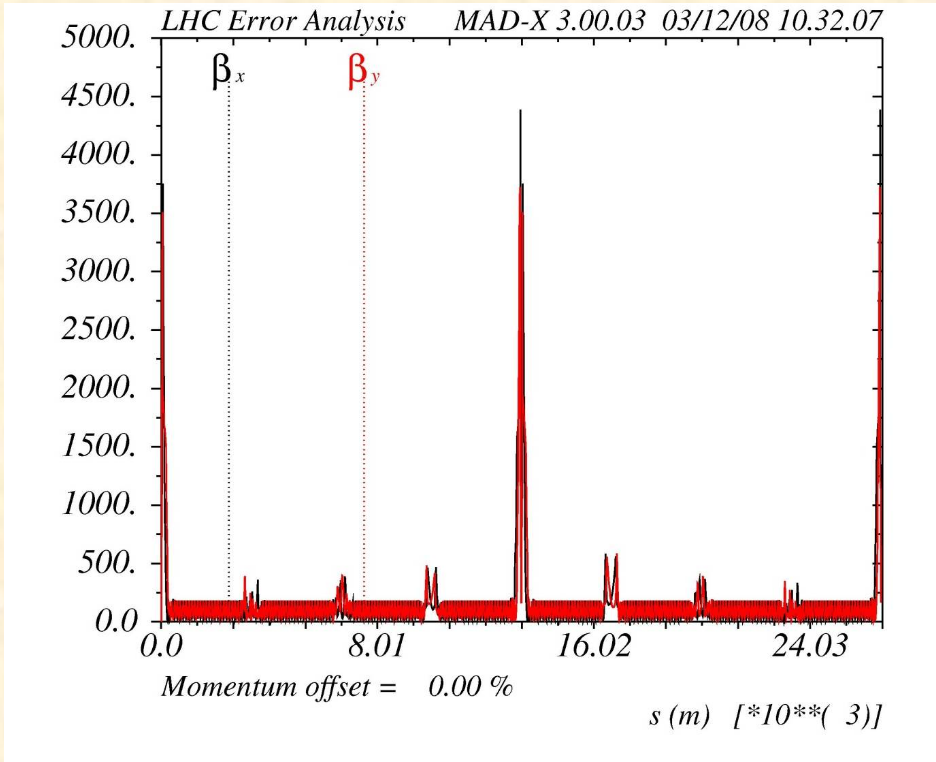
$$\sigma = \sqrt{\varepsilon\beta}$$

2.) At lowest energy the machine will have the major aperture problems,

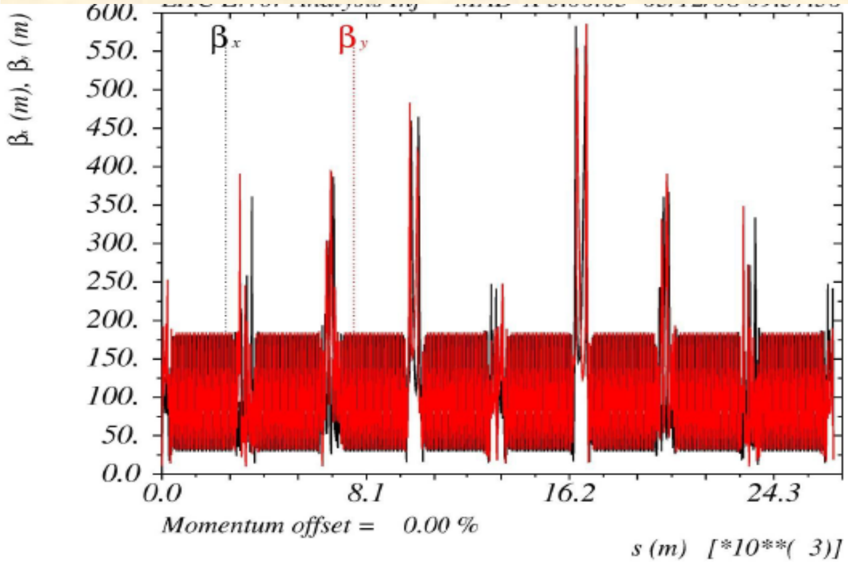
—> here we have to minimise $\hat{\beta}$

3.) we need different beam optics adopted to the energy:

A Mini Beta concept will only be adequate at flat top where the beam emittance is small !!



LHC mini beta optics at 7000 GeV



LHC injection optics at 450 GeV

Liouville during Acceleration

Protons

… shrink during acceleration

ATTENTION !!!

Electron beams in a storage ring are determined by light emission and behave completely different.

… they grow.

25.) The „ $\Delta p / p \neq 0$“ Problem

ideal accelerator: ***all particles will see the same accelerating voltage.***

—> $\Delta p / p = 0$

„nearly ideal“ accelerator: ***Cockroft Walton or van de Graaf***

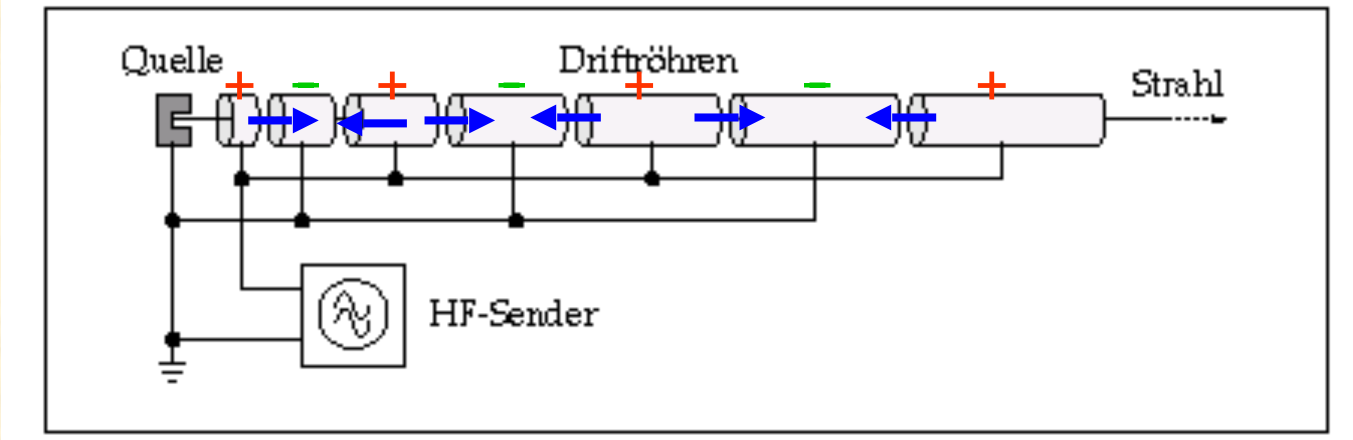
$\Delta p / p \approx 10^{-5}$

Vivitron, Straßbourg, inner structure of the acc. section

MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

Linear Accelerator

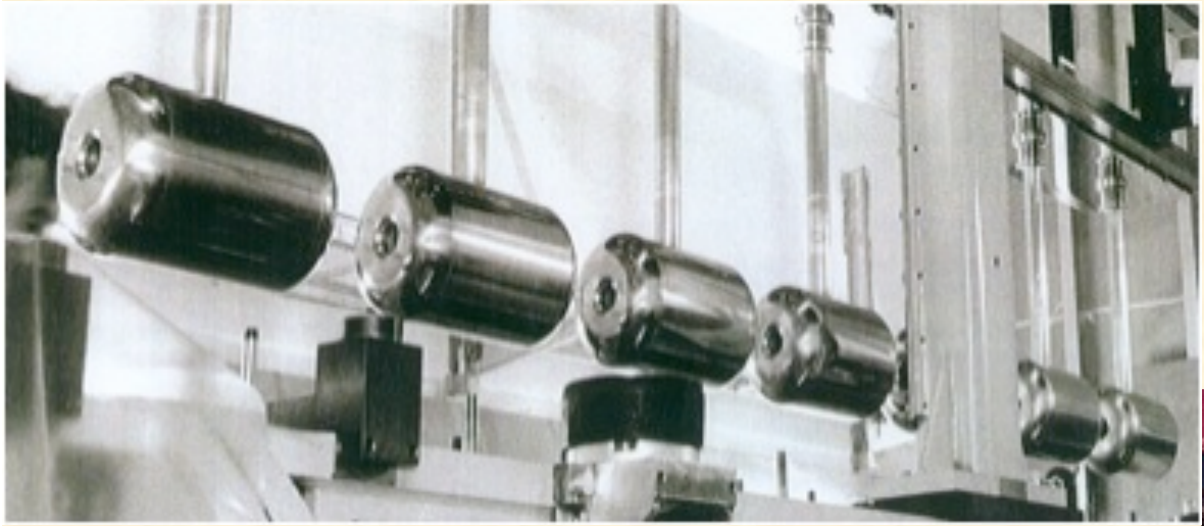
1928, Wideroe: schematic Layout:



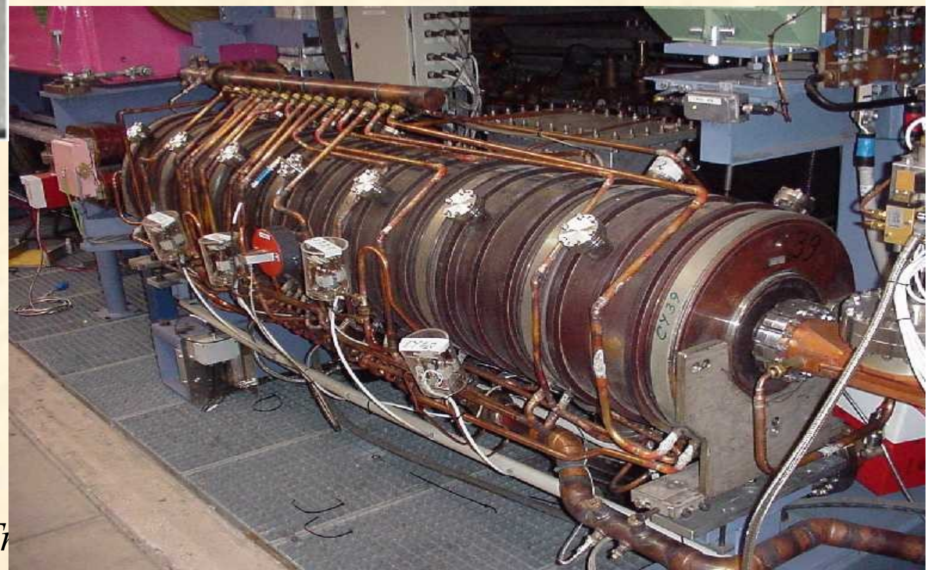
Energy Gain per „Gap":

$$W = q\,U_0 \sin \omega_{RF} t$$

drift tube structure at a proton linac



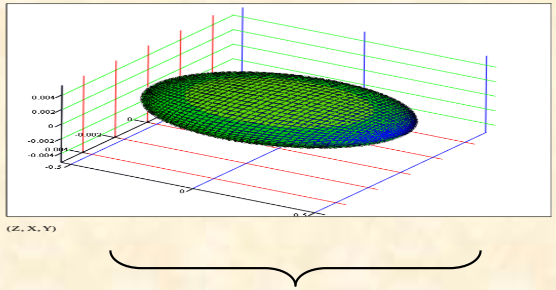
500 MHz cavities in an electron storage ring



* ***RF Acceleration:** multiple application of the same acceleration voltage; brillant idea to gain higher energies*

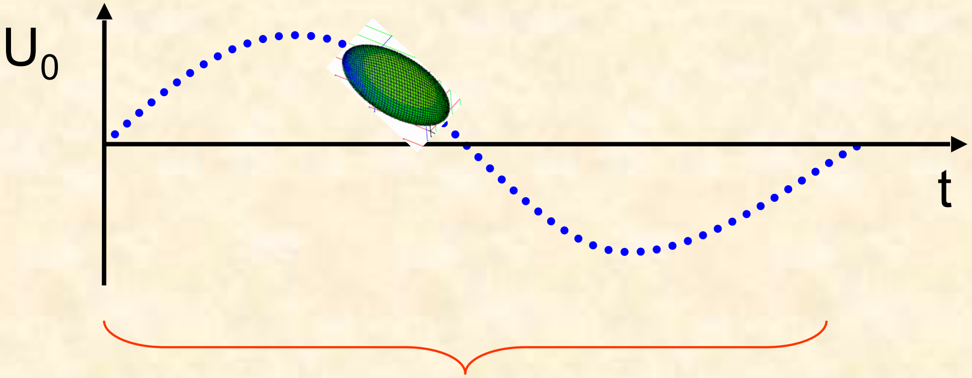
Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Bunch length of Electrons ≈ 1cm

Example: HERA RF:



$\lambda = 60\ cm$

$$\nu = 500\,MHz$$
$$c = \lambda\,\nu$$
$$\Big\}\quad \lambda = 60\ cm$$

$$\sin(90^o) = 1$$
$$\sin(84^o) = 0.994$$

$$\frac{\Delta U}{U} = 6.0\ \ 10^{-3}$$

typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1.0\ \ 10^{-3}$$

Dispersive and Chromatic Effects: $\Delta p/p \neq 0$

Are there any Problems ???

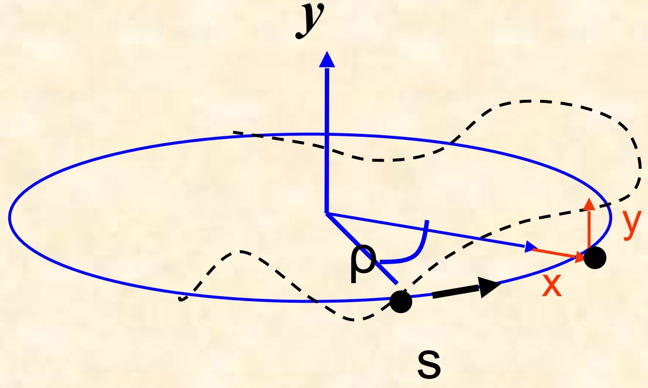
Sure there are !!!

font colours chosen for pedagogical reasons

26.) Dispersion: trajectories for $\Delta p / p \neq 0$

***Question:* do you remember last session, page 12 ? … sure you do**

***Force* acting on the particle**



$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = -\ eB_y v$$

remember:* $x \approx mm$, $\rho \approx m$ … —> *develop for small x

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = -\ eB_y v$$

***consider only linear fields, and change independent variable:* $t \rightarrow s$**

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho}(1-\frac{x}{\rho}) = -\frac{eB_0}{mv} + \frac{exg}{mv}$$

$$p = p_0 + \Delta p$$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error $\Delta p << p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx -\underbrace{\frac{eB_0}{p_0}}_{-\frac{1}{\rho}} + \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{exg}{p_0}}_{k*x} - \underbrace{xeg\frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0}\frac{1}{\rho} + x \cdot k$$

$$\boldsymbol{x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0}\frac{1}{\rho}}$$

***Momentum spread** of the beam adds a term on the r.h.s. of the equation of motion.*
*—> **inhomogeneous differential equation.***

Dispersion

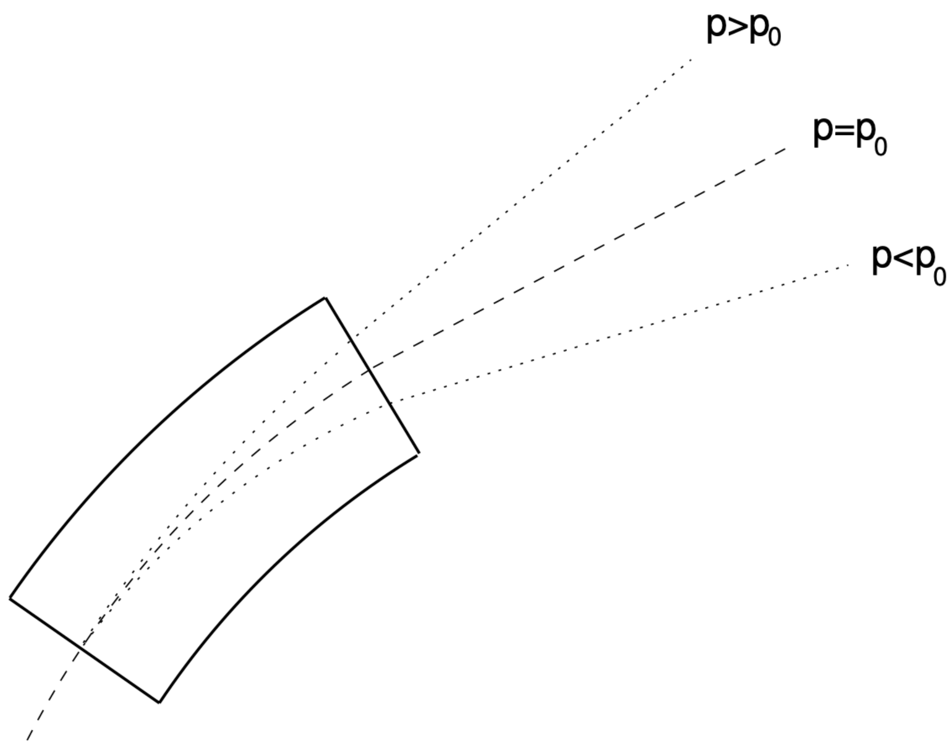
What if particles in a bunch have different momenta?
Remember beam rigidity:

$$B\rho = \frac{P}{q} \tag{100}$$

Orbit:

$$x(s) = D(s)\frac{\Delta P}{P_0} \tag{101}$$

where $D(s)$ is the dispersion function, an intrinsic property of dipole magnets.



Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

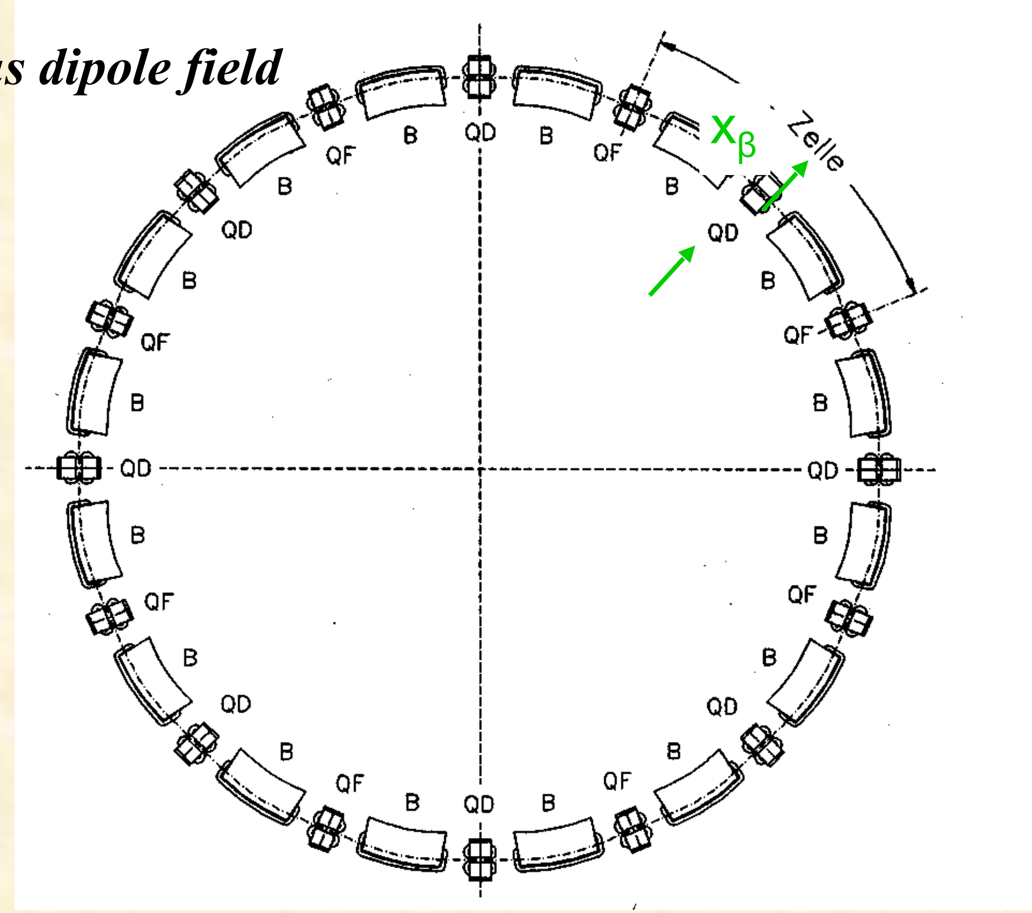
**** is that special orbit, an ideal particle would have for Δp/p = 1***

**** the orbit of any particle is the sum of the well known x_β and the dispersion***

**** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

Dispersion

Example: homogeneous dipole field



for Δp/p > 0

$$= D(s) \cdot \frac{\Delta p}{p}$$

Matrix formalism:

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x'_0 + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}_0$$

Resume´:

beam emittance $\varepsilon \propto \frac{1}{\beta\gamma}$

beta function in a drift $\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$

… and for α = 0 $\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$

particle trajectory for Δp/p ≠ 0
inhomogenious equation

$$\boldsymbol{x}'' + \boldsymbol{x}\left(\frac{1}{\rho^2} - \boldsymbol{k}\right) = \frac{\Delta \boldsymbol{p}}{\boldsymbol{p}_0}\frac{1}{\rho}$$

… and its solution

$$x(s) = x_\beta(s) + D(s)\cdot\frac{\Delta p}{p}$$