Transverse Beam Optics IV

Bernhard Holzer, CERN

Errors in Field and Gradient

The „ überhaupt nicht ideal world “

During magnet test campaign, the 7 MJ stored in one magnet were

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \dfrac{1}{\rho} \cdot \dfrac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

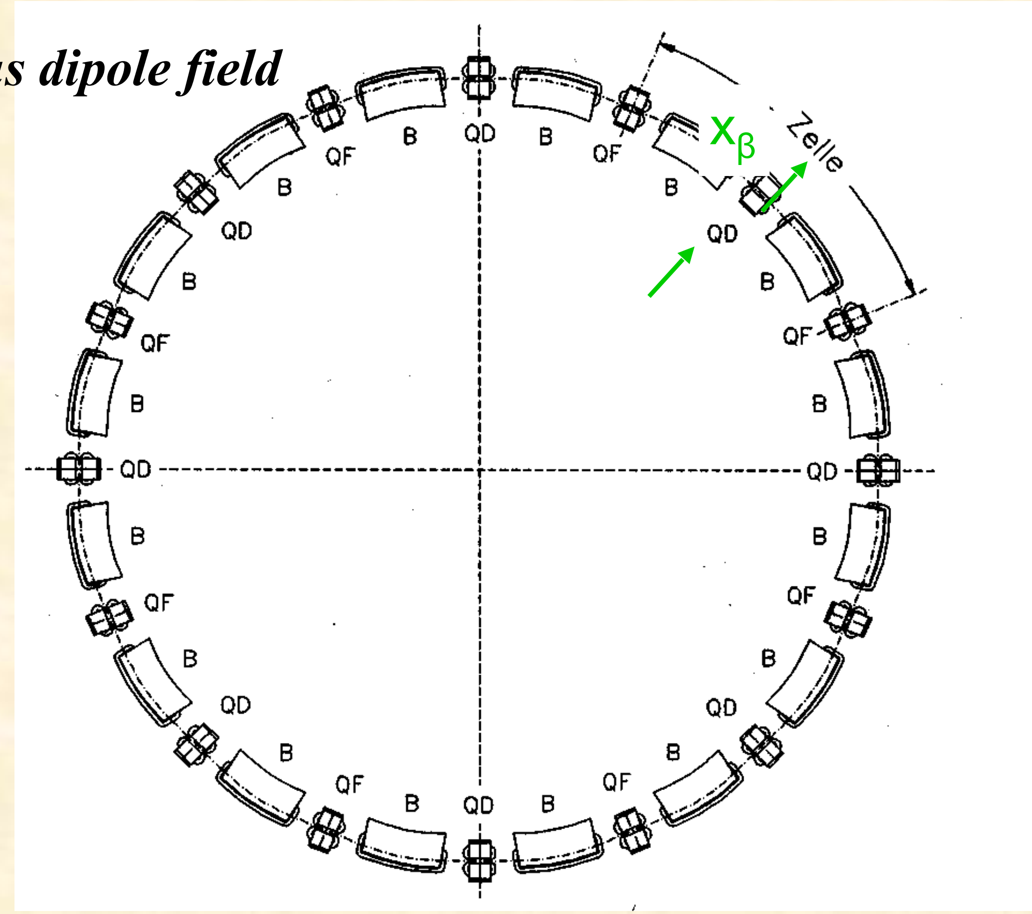
**** is that special orbit, an ideal particle would have for $\Delta p/p = 1$***

**** the orbit of any particle is the sum of the well known x_β and the dispersion***

**** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

Dispersion

Example: homogeneous dipole field



bit for Δp/p >

$$D(s) \cdot \frac{\Delta p}{p}$$

Matrix formalism:

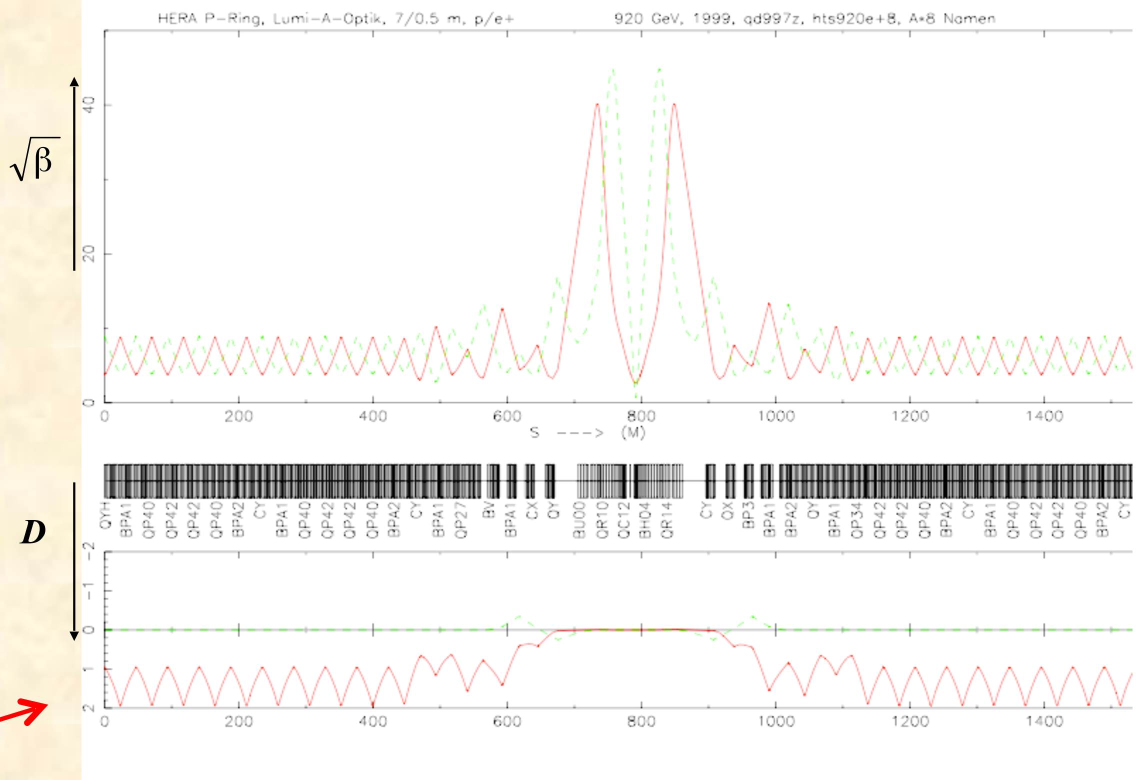
$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}_0$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p / p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p / p \end{pmatrix}_0$$



Example

$$x_\beta = 1 \dots 2\, mm$$

$$D(s) \approx 1 \dots 2\, m$$

$$\Delta p / p \approx 1 \cdot 10^{-3}$$

Amplitude of Orbit oscillation
contribution due to Dispersion ≈ beam size
—> Dispersion must vanish at the collision point

!

Calculate D, D´: ... takes a couple of sunny Sunday evenings !

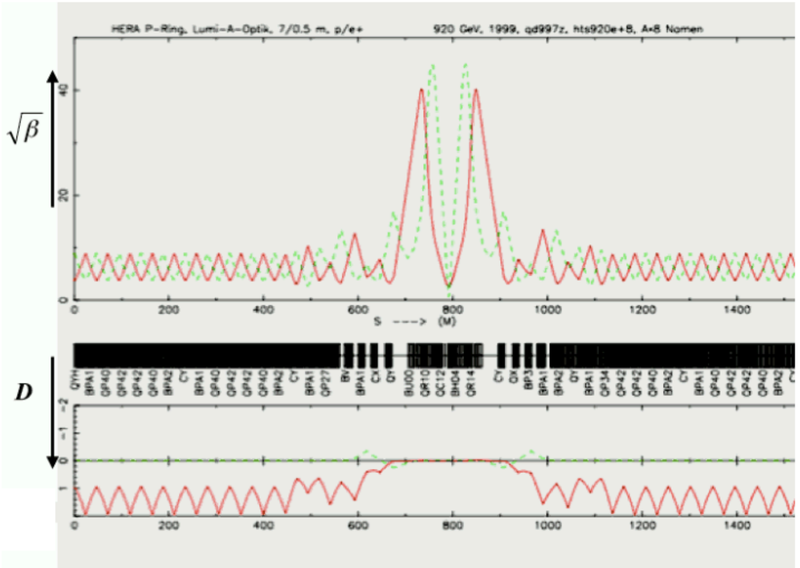
$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\,d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\,d\tilde{s}$$

Dispersion and Beam Size:

Super-position of two Gaussian distributions

In this example from the HERA storage ring (DESY) we see the Twiss parameters and the dispersion near the interaction point. In the periodic region,

$$x_\beta(s) = 1 \dots 2 \text{ mm}$$
$$D(s) = 1 \dots 2 \text{ m}$$
$$\Delta P / P_0 \approx 1 \cdot 10^{-3}$$



Remember:

$$x(s) = x_\beta(s) + D(s)\,\frac{\Delta P}{P_0}$$

Beware: the dispersion contributes to the beam size:

$$\sigma_x = \sqrt{\sigma_{x_\beta}^2 + \text{std}\left(D \cdot \frac{\Delta P}{P_0}\right)^2} = \sqrt{\epsilon_{\text{geometric}} \cdot \beta + D^2 \cdot \frac{\sigma_P^2}{P_0^2}}$$

- We need to suppress the dispersion at the IP !
- We need a special insertion section: a *dispersion suppressor*

Dispersion:

Example: Drift

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$M_{Drift} = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D(s) = S(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\,d\tilde{s}}_{=0} - C(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\,d\tilde{s}}_{=0}$$

Dispersion:

Example: Drift

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$M_{Drift} = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D(s) = \underbrace{S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})d\tilde{s}}_{=0} - \underbrace{C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})d\tilde{s}}_{=0}$$

$$D(s) = \underbrace{S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})d\tilde{s}}_{\neq 0} - \underbrace{C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})d\tilde{s}}_{\neq 0}$$

Example: Dispersion in a Sector Dipole Magnet

Remember: ***Matrix of a magnetic element***

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}$$

in general: $K = k - \frac{1}{\rho^2}$

… but in a dipole, as $k = 0$ … the focusing properties are

$$M_{foc} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix}$$

calculate the „D“ elements for the marix a Sector Dipole Magnet

$$D(s) = S(s)\int_{s0}^{s1}\frac{1}{\rho}C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1}\frac{1}{\rho}S(\tilde{s})d\tilde{s}$$

$$D(s) = (\rho\sin\frac{l}{\rho})*\frac{1}{\rho}*(\rho\sin\frac{l}{\rho}) - \cos\frac{l}{\rho}*\frac{1}{\rho}*\rho\cdot(-\cos\frac{l}{\rho}+1)*\rho$$

$$D(s) = \rho\sin^2\frac{l}{\rho} + \rho\cos\frac{l}{\rho}*(\cos\frac{l}{\rho}-1)$$

$$\boldsymbol{D(s) = \rho\cdot(1-\cos\frac{l}{\rho})} \quad , \quad \boldsymbol{D'(s) = \sin\frac{l}{\rho}}$$

Dispersion elements in a sector dipole magnet

$$M_{dipole} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} & D \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} & D' \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{s2} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} & \rho*(1-\cos\frac{l}{\rho}) \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} & \sin\frac{l}{\rho} \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{s1}$$

Dispersion in a Dipole:

Nota bene: ***even an ideal particle with $x = x' = 0$ will start to oscillate if it passes a dipole magnet and has a momentum error $\Delta p/p$.***

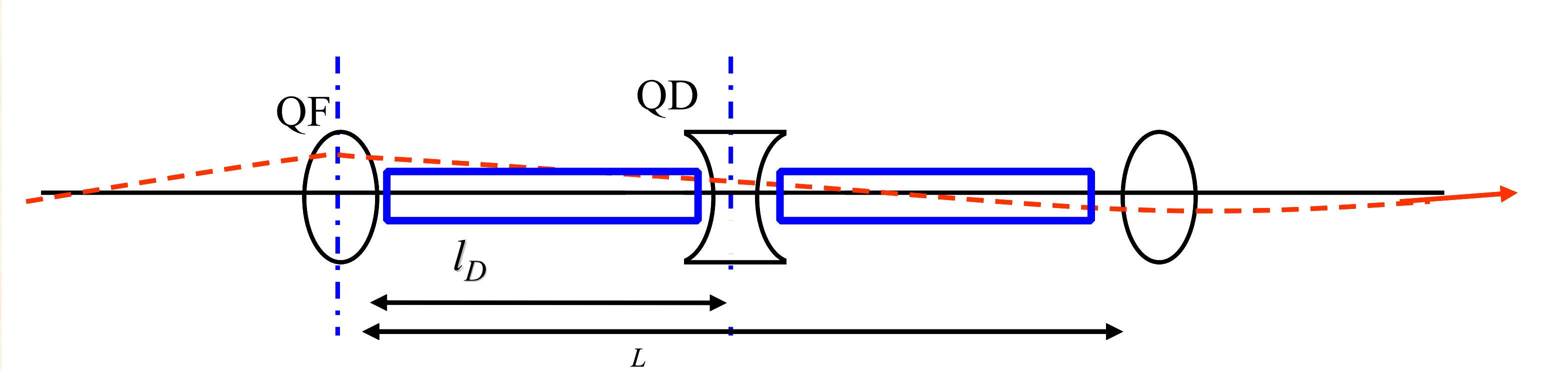
A dispersion trajectory will obey the same focusing forces (i.e. will be transferred by the same matrices) as a normal betatron oscillation.

Dipole magnets are described by a 3 x 3 matrix and carry as element m_{13} and m_{23} the dispersion driving expressions.

We get the full trajectory amplitude by multiplying the matrix with the vector

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_f = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} & D \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} & D' \\ 0 & 0 & 1 \end{pmatrix}_{i\to f} \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_i$$

27.) Dispersion in a FoDo Cell:



!! we have now introduced dipole magnets in the FoDo:

—> we still neglect the weak focusing contribution $1/\rho^2$

—> but take into account $1/\rho$ for the dispersion effect

assume: length of the dipole = l_D

Calculate the matrix of the FoDo half cell in thin lens approximation:

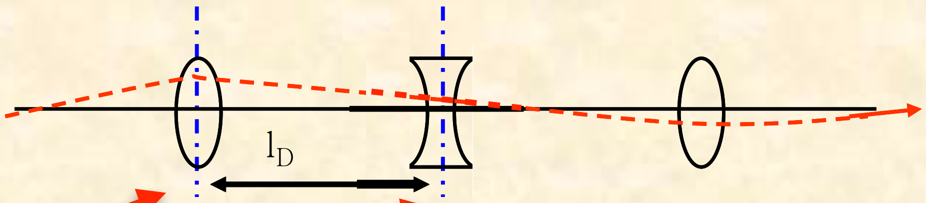
in analogy to the derivations of $\hat{\beta}, \check{\beta}$

** thin lens approximation:* $f = \frac{1}{k\ell_Q} >> \ell_Q$

** length of quad negligible* $\ell_Q \approx 0, \rightarrow \ell_D = \frac{1}{2}L$

** start at half quadrupole* $\frac{1}{\tilde{f}} = \frac{1}{2f}$

Matrix of the half cell



$$M_{HalfCell} = M_{\frac{QD}{2}} * M_B * M_{\frac{QF}{2}}$$

$$M_{Half\ Cell} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\tilde{f}} & 1 \end{pmatrix} * \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ \frac{-1}{\tilde{f}} & 1 \end{pmatrix}$$

$$M_{Half\ Cell} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 - \frac{\ell}{\tilde{f}} & \ell \\ \frac{-\ell}{\tilde{f}^2} & 1 + \frac{\ell}{\tilde{f}} \end{pmatrix}$$

calculate the dispersion terms D, D´ from the matrix elements

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

$$D(\ell) = \ell * \frac{1}{\rho} * \int_0^{\ell} \left(1 - \frac{s}{\tilde{f}}\right) ds - \left(1 - \frac{\ell}{\tilde{f}}\right) * \frac{1}{\rho} * \int_0^{\ell} s \; ds$$

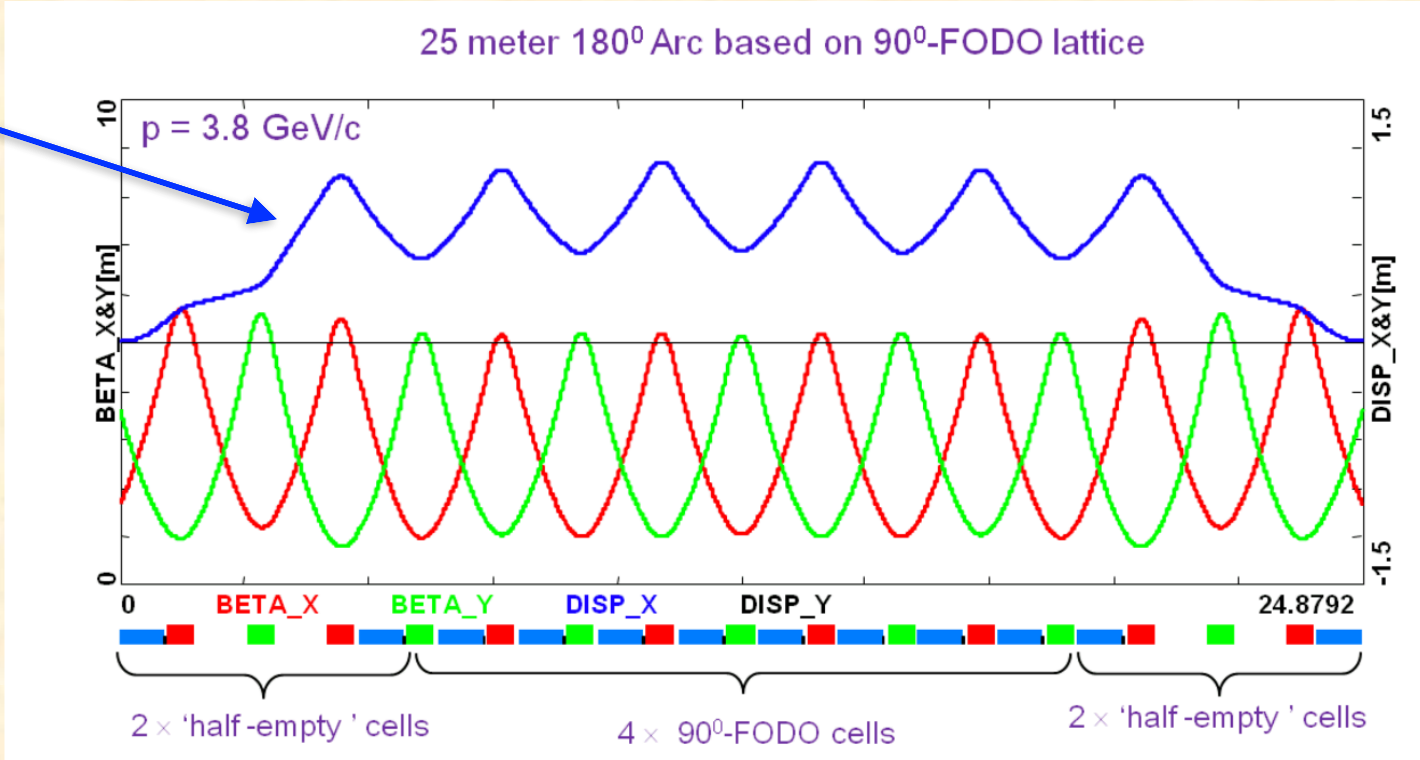
S(s) C(s) C(s) S(s)

$$D(\ell) = \frac{\ell}{\rho}\left(\ell - \frac{\ell^2}{2\tilde{f}}\right) - \left(1 - \frac{\ell}{\tilde{f}}\right) * \frac{1}{\rho} * \frac{\ell^2}{2} \qquad = \frac{\ell^2}{\rho} - \frac{\ell^3}{2\tilde{f}\rho} - \frac{\ell^2}{2\rho} + \frac{\ell^3}{2\tilde{f}\rho}$$

$$D(\ell) = \frac{\ell^2}{2\rho}$$

in full analogy on derives for D´:

$$D'(s) = \frac{\ell}{\rho}\left(1 + \frac{\ell}{2\tilde{f}}\right)$$



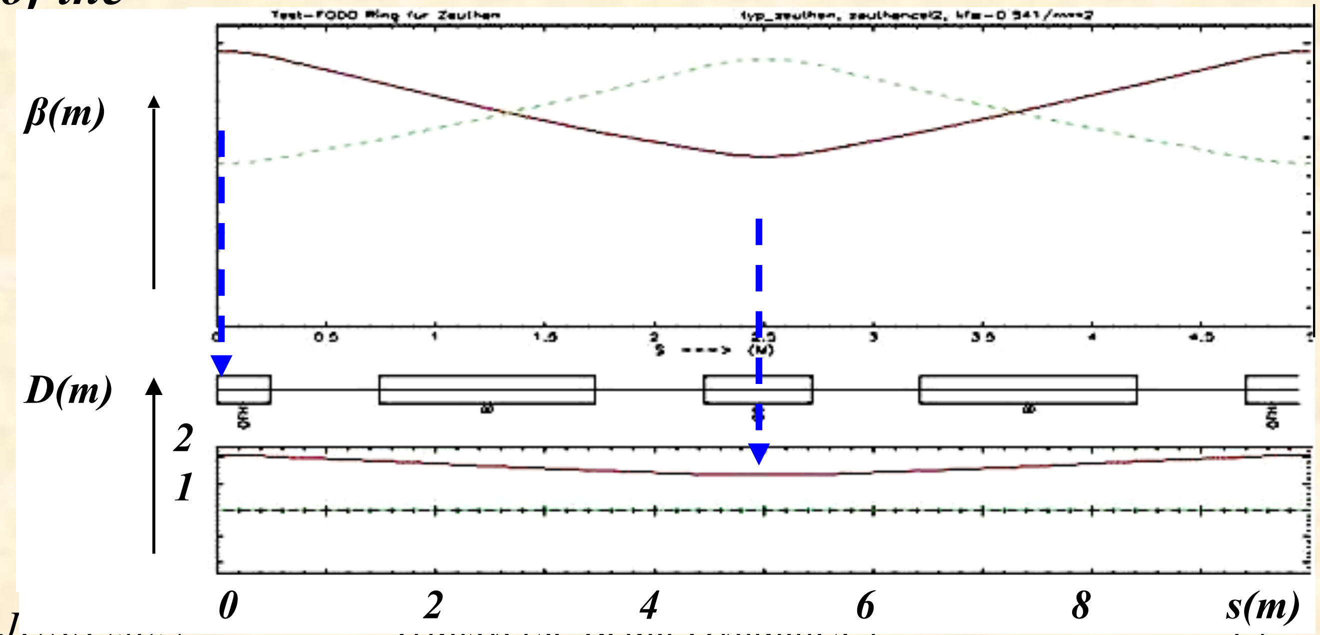
B. J. Holzer, CERN

and we get the complete matrix of the half cell including the dispersion terms D, D´

$$M_{halfCell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{\ell}{\tilde{f}} & \ell & \frac{\ell^2}{2\rho} \\ \frac{-\ell}{\tilde{f}^2} & 1+\frac{\ell}{\tilde{f}} & \frac{\ell}{\rho}(1+\frac{\ell}{2\tilde{f}}) \\ 0 & 0 & 1 \end{pmatrix}$$

***boundary conditions** for the transfer from the center of the foc. to the center of the defoc. quadrupole*

$$\begin{pmatrix} \check{D} \\ 0 \\ 1 \end{pmatrix} = M_{1/2} * \begin{pmatrix} \hat{D} \\ 0 \\ 1 \end{pmatrix}$$



Optimisation of the FoDo Phase advance
- - Dispersion - -

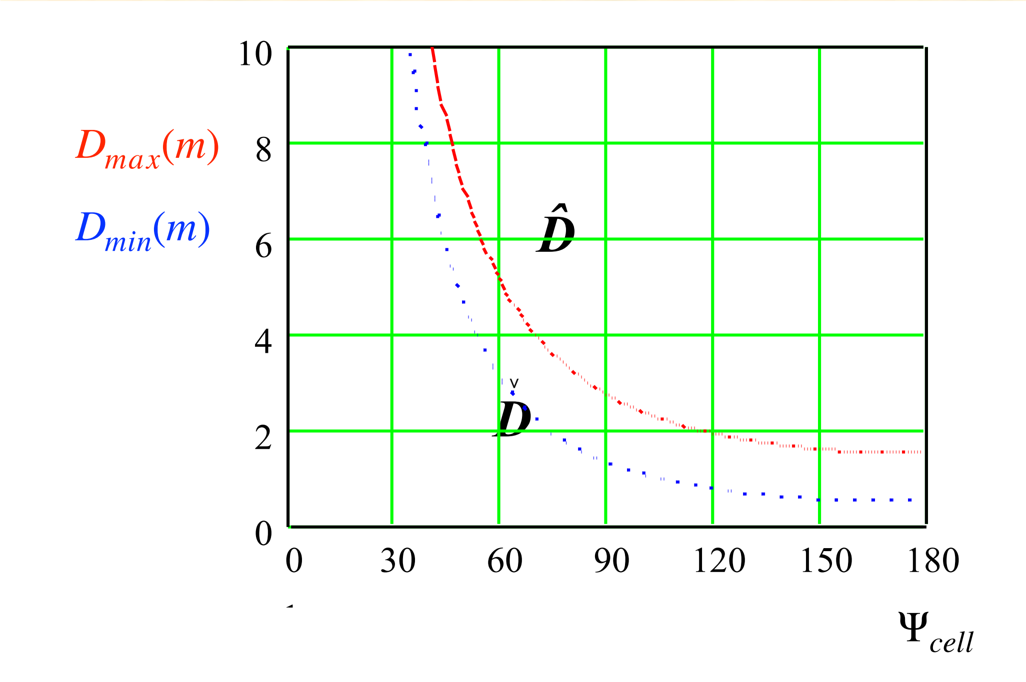
$$\rightarrow \check{D} = \hat{D}(1-\frac{\ell}{\tilde{f}}) + \frac{\ell^2}{2\rho}$$

$$\rightarrow 0 = -\frac{\ell}{\tilde{f}^2} * \hat{D} + \frac{\ell}{\rho}(1+\frac{\ell}{2\tilde{f}})$$

$$D_{max} = \frac{l^2}{r} \cdot \frac{1+\frac{1}{2}\sin\frac{\psi_{cell}}{2}}{\sin^2\frac{\psi_{cell}}{2}} \qquad D_{min} = \frac{l^2}{r} \cdot \frac{1-\frac{1}{2}\sin\frac{\psi_{cell}}{2}}{\sin^2\frac{\psi_{cell}}{2}}$$

where ψ_{cell} denotes the phase advance of the full cell and

$$1/f = \sin(\frac{\psi_{cell}}{2})$$



Nota bene:

! ***small dispersion needs strong focusing***
→ large phase advance

!! ↔ there is an optimum phase for small β

!!! ...do you remember the stability criterion?
½ trace = cos ψ ↔ ψ < 180°

!!!! … life is not easy

Optimisation of the FoDo Phase advance
- - Betafunctions - -

remember:

$$\beta_{max} = \frac{(1 + sin\frac{\psi_{cell}}{2})\ L}{sin\ (\psi_{cell})}$$

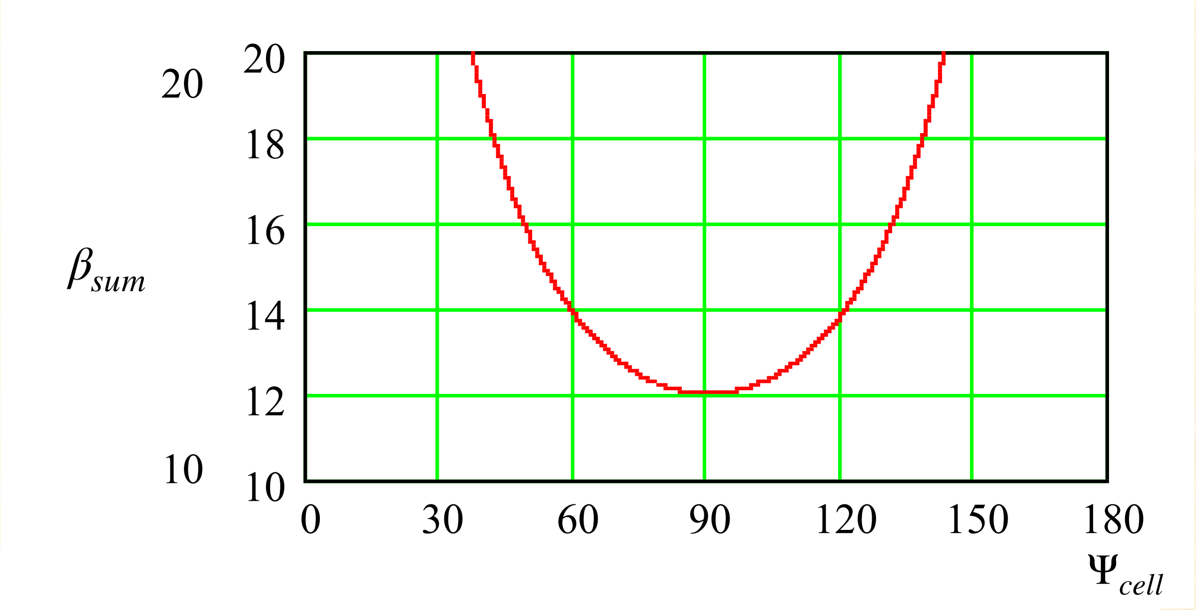
$$\beta_{min} = \frac{(1 - sin\frac{\psi_{cell}}{2})\ L}{sin\ (\psi_{cell})}$$

$$\beta_{max} + \beta_{min} = \frac{(1 + sin\frac{\psi_{cell}}{2})\ L}{sin\ (\psi_{cell})} + \frac{(1 - sin\frac{\psi_{cell}}{2})\ L}{sin\ (\psi_{cell})}$$

$$\beta_{max} + \beta_{min} = \frac{2L}{sin\ (\psi_{cell})}$$

search for the phase advance μ that results in a minimum of the sum of the beta's

—> require $\frac{d}{d\psi_{cell}}\left(\frac{2L}{sin\ (\psi_{cell})}\right) = 0$



$$\frac{L}{\sin^2\psi_{cell}} * \cos\psi_{cell} = 0$$

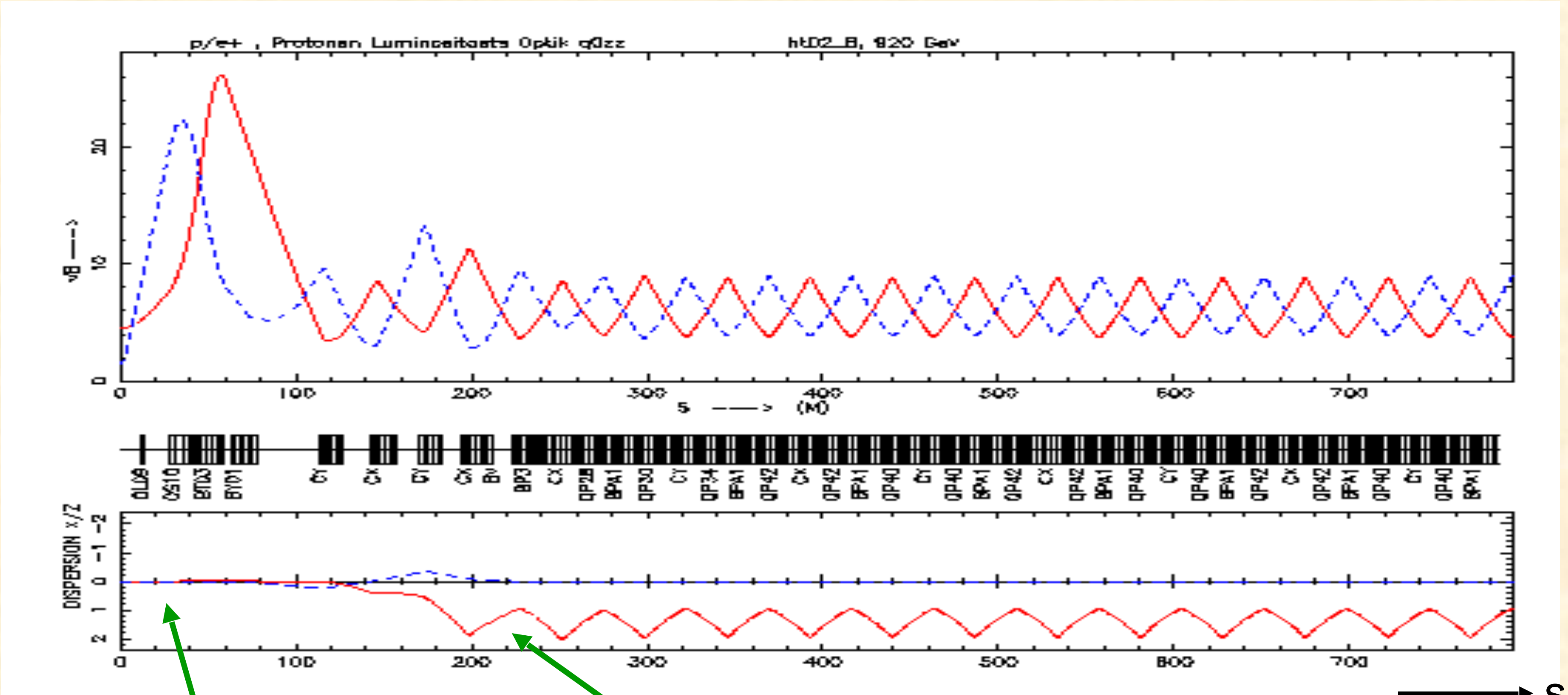
$$\rightarrow \quad \psi_{cell} = 90^\circ$$

Example: Dispersion, calculated by an optics code for a real machine

$$x_D = D(s) * \frac{\Delta p}{p}$$

** D(s) is created by the dipole magnets*

… and afterwards focused by the quadrupole fields



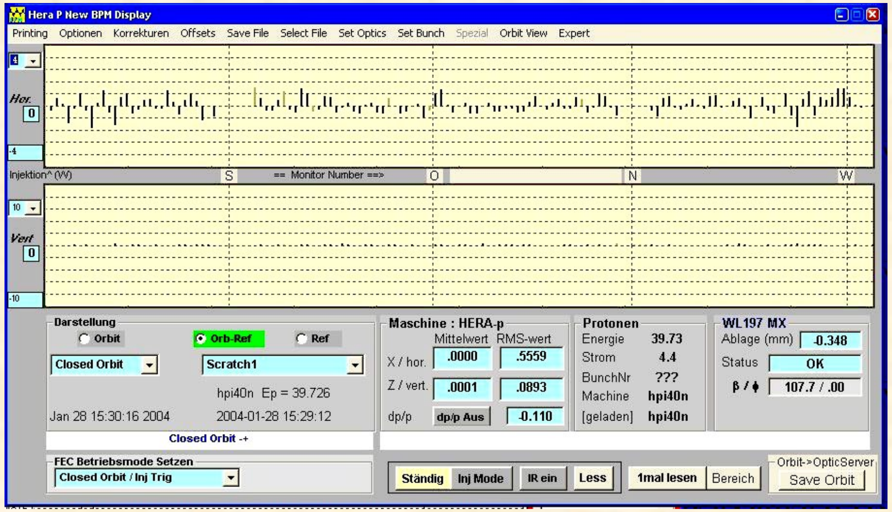
s

Mini Beta Section,

—> no dipoles !!!

D(s) ≈ 1 … 2 m

Dispersion is visible



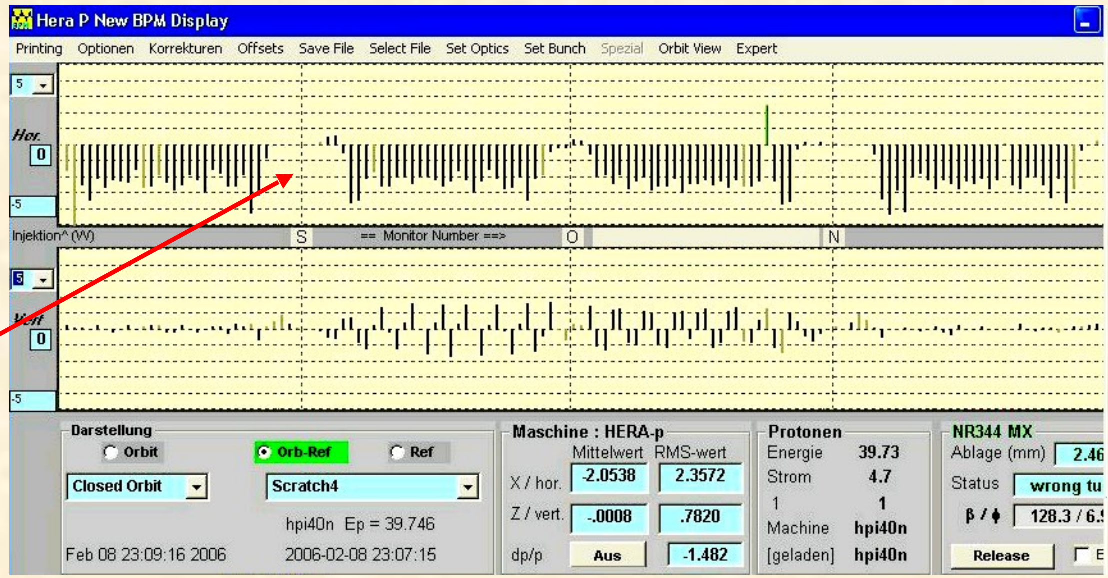
HERA Standard Orbit

HERA Dispersion Orbit

dedicated energy change of the stored beam

—> closed orbit is moved to a dispersions trajectory

$$x_D = D(s) * \frac{\Delta p}{p}$$



Attention: at the Interaction Points we require D=D´= 0

Dispersion

***For one magnet element or a sequence** the dispersion is part of the (3x3) matrix and the trajectory is calculated including the effect* $\Delta p/p \neq 0$

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_f = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} & D \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} & D' \\ 0 & 0 & 1 \end{pmatrix}_f \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_i$$

Example: Single Dipole magnet or Transfer Line.

Dispersion

For one magnet element or a sequence the dispersion is part of the (3x3) matrix and the trajectory is calculated including the effect $\Delta p/p \neq 0$

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_f = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} & D \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} & D' \\ 0 & 0 & 1 \end{pmatrix}_f \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_i$$

Example: Single Dipole magnet or Transfer Line.

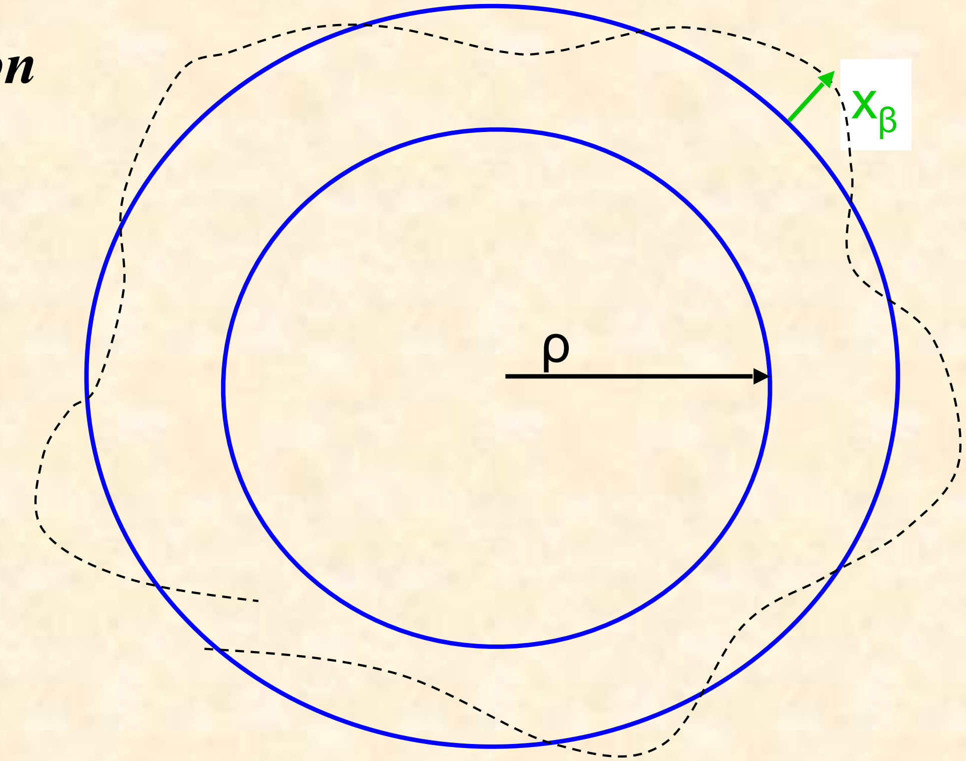
For one full turn - or a periodic part of it - we have to require periodicity conditions for the closed orbit as well as for the "periodic" dispersion η, η' **.**

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{periodic} \equiv \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} := \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix}_{turn} \cdot \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$$

The periodic dispersion $D_{periodic} \equiv \eta$ ***defines the closed orbit for particles with*** $\Delta p/p \neq 0$.
Around this new closed orbit the trajectories perform the good old betatron oscillations.

28.) Momentum Compaction Factor:

The dispersion function relates the momentum error of a particle to the horizontal orbit coordinate.



inhomogeneous differential equation

$$x'' + K(s) * x = \frac{1}{\rho}\frac{\Delta p}{p}$$

general solution

$$x(s) = x_\beta(s) + D(s)\frac{\Delta p}{p}$$

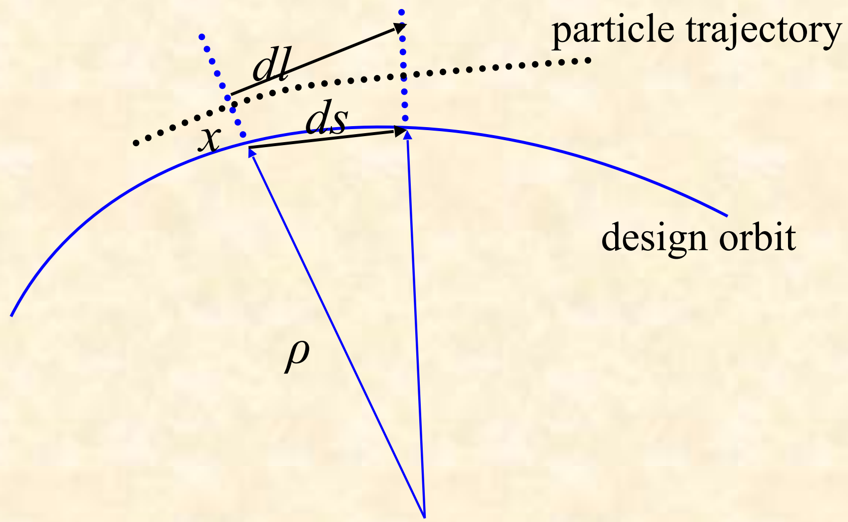
But it does much more:

it changes the length of the off - energy - orbit !!

Momentum Compaction Factor:

particle with a displacement x to the design orbit has different path length dl ...

$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$



$$\rightarrow \quad dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$

circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember: $\quad x_{\Delta E}(s) = D(s)\frac{\Delta p}{p_0}$

$$\delta l = l_{\Delta E} - l_0 = \frac{\Delta p}{p_0} \oint \left(\frac{D(s)}{\rho(s)}\right) ds$$

** The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.*

Momentum Compaction Factor:

Definition: $$\frac{\delta l}{L_0} = \alpha_p \frac{\Delta p}{p_0}$$

$$\rightarrow \quad \alpha_p = \frac{1}{L_0} \oint \left(\frac{D(s)}{\rho(s)} \right) ds$$

For first estimates assume: $\frac{1}{\rho} = const$

$$\int_{dipoles} D(s)\, ds = \Sigma\, (l_{dipoles}) * \langle D \rangle_{dipole}$$

$$\alpha_p = \frac{1}{L_0} \Sigma(l_{dipoles}) < D > \frac{1}{\rho} = \frac{1}{L_0} 2\pi\rho < D > \frac{1}{\rho} \quad \longrightarrow \quad \alpha_p \approx \frac{2\pi}{L_0} \langle D \rangle \approx \frac{\langle D \rangle}{R}$$

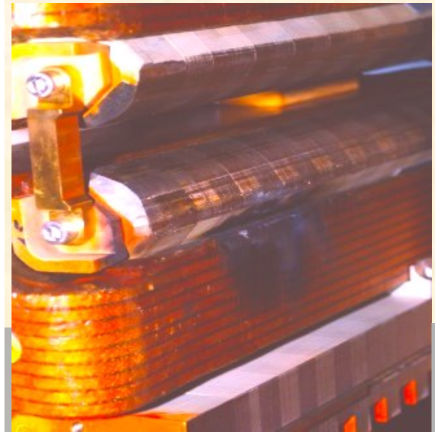
Assume: $v \approx c$

$$\rightarrow \quad \frac{\delta T}{T_0} = \frac{\delta l}{L_0} = \alpha_p \frac{\Delta p}{p_0}$$

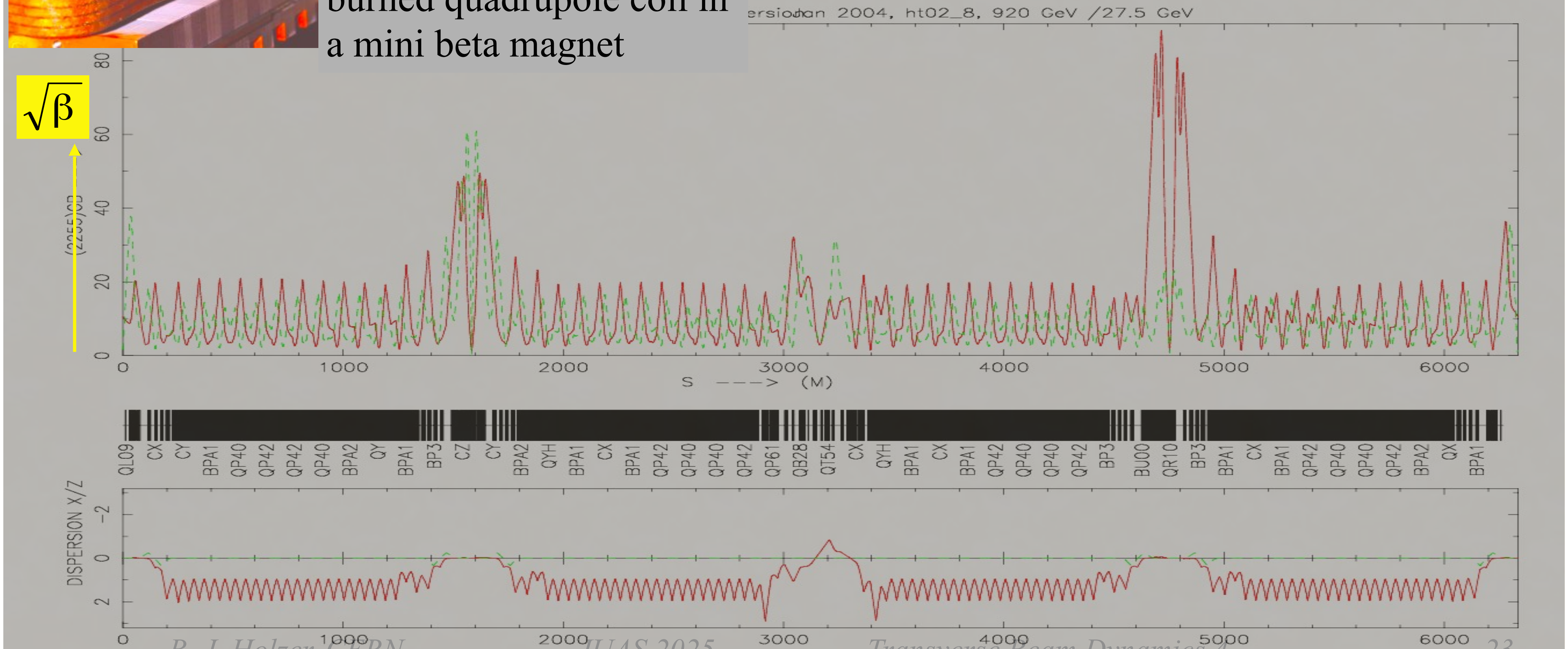
α_p combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

Introduction to Transverse Beam Optics

29.) Errors in Field and Gradient



burned quadrupole coil in a mini beta magnet



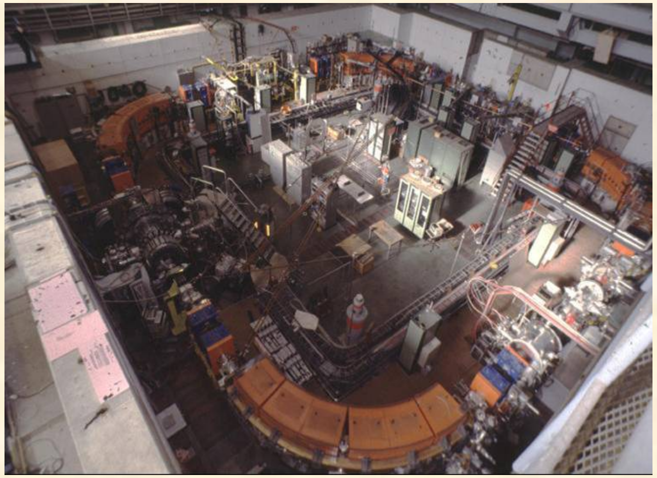
Errors in Field and Gradient
- life is not so easy -

The derivation of the equation of motion is based on the presumption that

… in our accelerator there are only linear magnetic fields ….

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k * x + \frac{1}{2!} m x^2 + \frac{1}{3!} n x^3 + \ldots$$

dipole ***quadrupole***



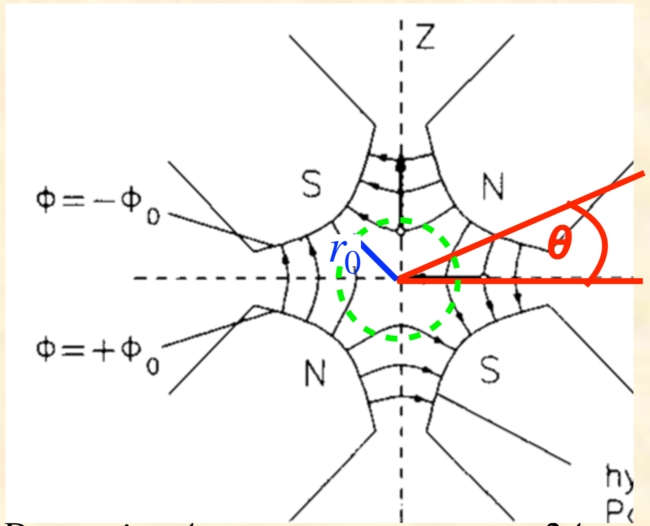
linear magnet structure of LEAR (CERN)

Multipole expansion of magnetic field:

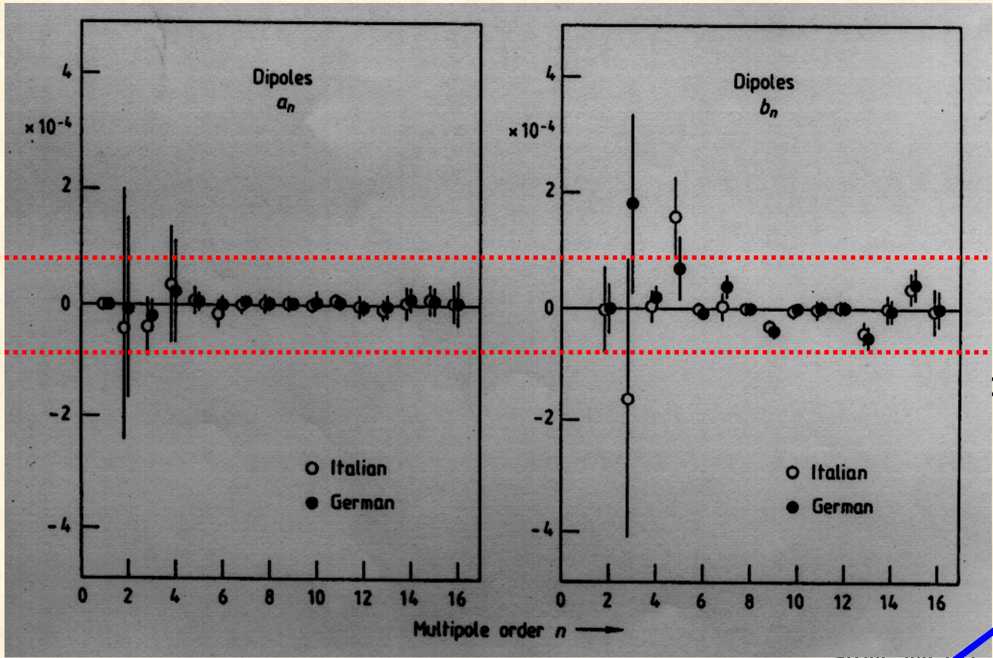
$$B_\theta(r,\theta) = B_{main} \cdot \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} [b_n \cos(n\theta) + a_n \sin(n\theta)]$$

example:
mid plane —> $\theta = 0$,
radius = ref radius r_0

$$b_n = \frac{B_{multipole}}{B_{main}}$$



Example: *HERA multipole coefficients of sc. dipole magnets*



$$B_\theta(r,\theta) = B_{main} \cdot \sum_{n=1}^{\infty} (\frac{r}{r_0})^{n-1}[b_n\ cos(n\theta) + a_n\ sin(n\theta)]$$

$b_n\ ,\ a_n \approx 1 \ldots 2*10^{-4}$

CD_col := 0.0000

D_col := 0.0000 ; a1U_MQXCD_col := 0.0000 ; a1R_MQXCD_col := 0.0000 ;
D_col := 0.0000 ; a2U_MQXCD_col := 0.0000 ; a2R_MQXCD_col := 0.0000 ;
D_col := 0.0000 ; a3U_MQXCD_col := 0.8900 ; a3R_MQXCD_col := 0.8900 ;
a4M_MQXCD_col := 0.0000 ; a4U_MQXCD_col := 0.6400 ; a4R_MQXCD_col := 0.6400 ;
a5M_MQXCD_col := 0.0000 ; a5U_MQXCD_col := 0.4600 ; a5R_MQXCD_col := 0.4600 ;
a6M_MQXCD_col := 0.0000 ; a6U_MQXCD_col := 1.2700 ; a6R_MQXCD_col := 0.3300 ;
a7M_MQXCD_col := 0.0000 ; a7U_MQXCD_col := 0.2100 ; a7R_MQXCD_col := 0.2100 ;
a8M_MQXCD_col := 0.0000 ; a8U_MQXCD_col := 0.1600 ; a8R_MQXCD_col := 0.1600 ;
D_col := 0.0800 ;
CD_col :=
0.0000 ;
a11M_MQXCD_col := 0.0000 ; a11U_MQXCD_col := 0.0300 ; a11R_MQXCD_col := 0.0300 ;
a12M_MQXCD_col := 0.0000 ; a12U_MQXCD_col := 0.0200 ; a12R_MQXCD_col :=
0.0100 ;
a14M_MQXCD_col := 0.0000 ; a14U_MQXCD_col := 0.0300 ; a14R_MQXCD_col :=
0.0100 ;
a15M_MQXCD_col := 0.0000 ; a15U_MQXCD_col := 0.0000 ; a15R_MQXCD_col :=
0.0000 ;

;
b5M_MQXCD_col := 0.0000 ; b5U_MQXCD_col := 0.4600
;
b6M_MQXCD_col := 0.0000 ; b6U_MQXCD_col := 1.7700
;
b7M_MQXCD_col := 0.0000 ; b7U_MQXCD_col := 0.2100
;
b8M_MQXCD_col
;
b9M_MQXCD_col := 0.0000 ; b9U_MQXCD_col := 0.0800
;
b10M_MQX
0.0600 ;
b11M_MQXCD_col := 0.0000 ; b11U_MQXCD_col := 0.0300
0.0300 ;
b12M_MQXCD_col := 0.0000 ; b12U_MQXCD_col := 0.0200
0.0200 ;
b13M_MQXCD_col := 0.0000 ; b13U_MQXCD_col := 0.020
0.0100 ;

Example: *LHC multipole coefficients of sc. triplet quadrupoles*

general rule: multipole errors should be in the range of „some 10 -4"

30.) Sources of field errors

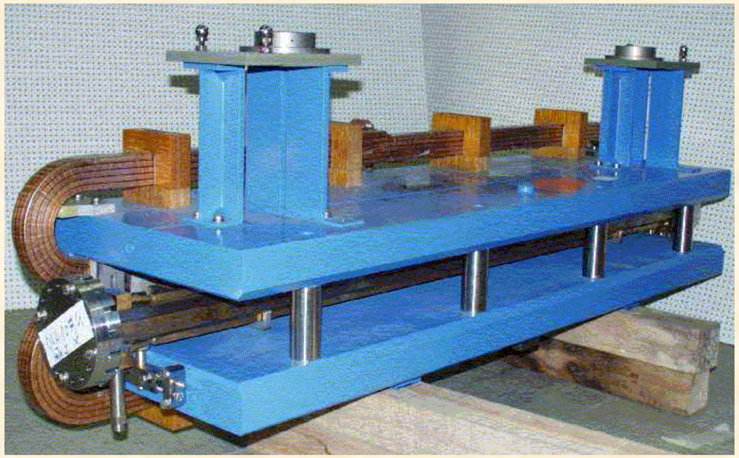
1.) power supply errors:

dipole error: ***remember from lecture N° 1:***

$$B = \frac{\mu_0 \, n \, I}{h}$$

2.) error in dipole strength: the gap

$$B = \frac{\mu_0 \, n \, I}{h}$$



Yoke production: laminations, made by stamping out of steel sheet.
variations of gap „h" by wear out of die or use of multiple dies

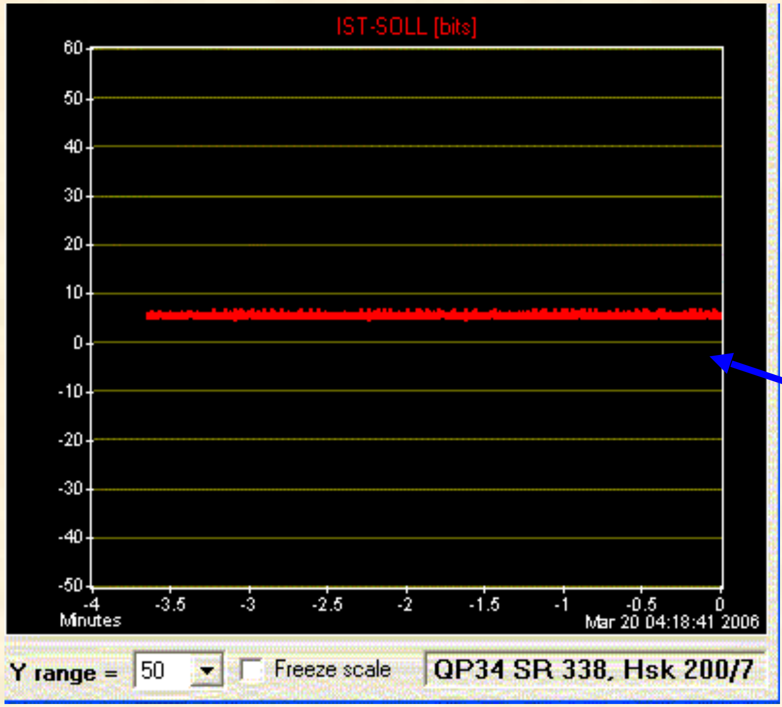
Tolerance:

$$\left.\begin{aligned} h &= 5\ cm \\ \Delta h &= 25\ \mu m \end{aligned}\right\} \quad \frac{\Delta B}{B} = \left|\frac{\Delta h}{h}\right| = \frac{25 \mu m}{5 cm} = 5*10^{-4}$$

Sources of field errors

power supply stability:

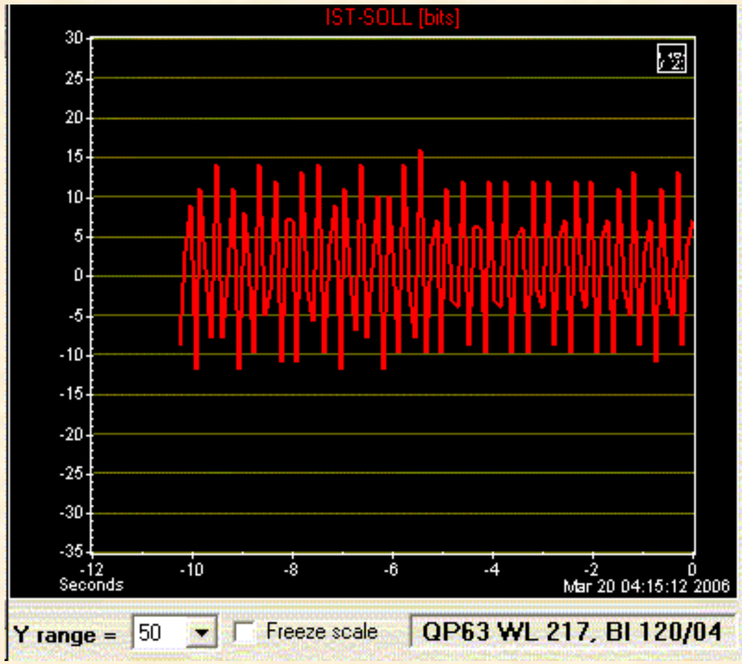
16 bit digital electronic for current control and stabilisation



survey of power supply electronics: bit stability

$$2^{16} = 65536 \qquad \frac{1bit}{2^{16}} \leftrightarrow 1.5*10^{-5}$$

require $\dfrac{\Delta I}{I} \leq 5*10^{-5}$



$$\Delta I \approx \pm 12 \quad bit$$

$$\frac{\Delta I}{I} \approx 1 \ldots 2*10^{-4}$$

Dipole Magnet errors: closed orbit distortion

The sum of all dipole magnets in a ring defines a curve that we call closed orbit.
perfect situation ↔ design orbit

normalised effect on the beam:

$$\int \frac{Bdl}{B\rho} = \frac{L_0}{\rho} = \alpha = 2\pi$$

effect of single dipole magnet error:

$$\int \frac{(Bds)}{B\rho} = \int \frac{1}{\rho} ds$$

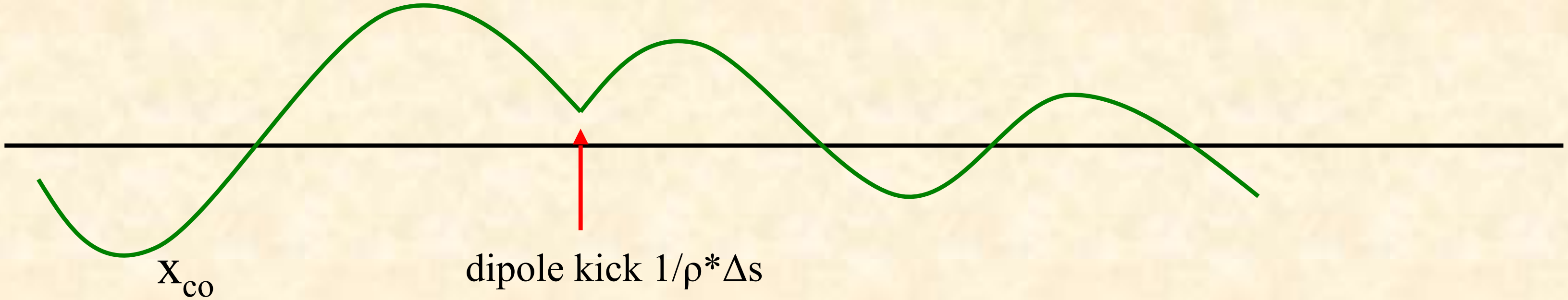
A dipole error will cause a distortion of the closed orbit, that will „run around" the storage ring, being observable everywhere … but – if small enough – still will lead to a closed orbit !!

Assume one single dipole error in a linac,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M_{lattice} * \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix}_{s0}$$

Overall amplitude of a single particle trajectory: $x = x_{co}(s) + x_\beta(s) + x_D(s)$

Calculation of Orbit Distortion in a circular machine:



periodicity condition still has to be fulfilled: ***we still get (!) a closed orbit***

in any case: ***distorted orbit will be a betatron oscillation.***

$$x_d(s) = a\sqrt{\beta(s)} * \cos(\psi(s) - \varphi)$$

a = orbit amplitude, φ = initial phase

put starting conditions: $s = 0 \;,\; \psi(s) = 0$

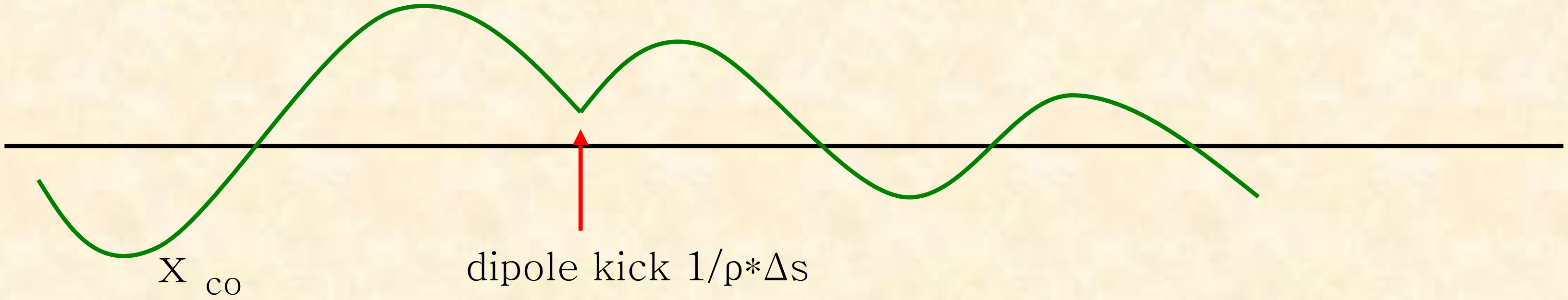
boundary condition (1): $x_d(s+L) = x_d(s)$

boundary condition (2): $x'_d(s+L) + \dfrac{\Delta s}{\rho} = x'_d(s)$

periodic closed orbit at the place of the distortion, (s = 0, ψ = 0)

Closed Orbit Distortion

Calculation of Orbit Distortion in a circular machine:



periodicity condition still has to be fulfilled: ***we still get (!) a closed orbit***

in any case: ***distorted orbit will be a betatron oscillation.***

$$x_d(s) = a\sqrt{\beta(s)} * \cos(\psi(s) - \varphi)$$

a = orbit amplitude, φ = initial phase

boundary condition (1): $x_d(s+L) = x_d(s)$ *periodic closed orbit*

$$\cancel{a\sqrt{\beta(s+L)}} * \cos(\psi(s) + 2\pi Q - \varphi) = \cancel{a\sqrt{\beta(s)}} * \cos(\psi(s) - \varphi)$$

$$\cos(2\pi Q - \varphi) = \cos(-\varphi) = \cos(\varphi)$$

$$\varphi = \pi Q$$

Calculation of Orbit Distortion:

angle x´:

$$x_d(s) = a\sqrt{\beta(s)} * \cos(\psi(s) - \varphi)$$

$$x'_d(s) = -a\sqrt{\beta} * \sin(\psi(s) - \varphi) * \psi'(s) + \frac{\beta'}{2\sqrt{\beta}} a * \cos(\psi(s) - \varphi)$$

remember: $\psi'(s) = \frac{1}{\beta}$

$$x'_d(s) = \frac{-a}{\sqrt{\beta}} \sin(\psi(s) - \varphi) + \frac{\beta'}{2\sqrt{\beta}} a * \cos(\psi(s) - \varphi)$$

boundary condition (2): $x'_d(s+L) + \frac{\Delta s}{\rho} = x'_d(s)$ ***at the place of the distortion, s = 0, ψ = 0***

$$\frac{-a}{\sqrt{\beta(s+L)}} \sin(2\pi Q - \varphi) + \frac{\beta'(s+L)}{2\sqrt{\beta(s+L)}} a * \cos(2\pi Q - \varphi) + \frac{\Delta s}{\rho} =$$

$$= \frac{-a}{\sqrt{\beta(s)}} \sin(-\varphi) + \frac{\beta'(s)}{2\sqrt{\beta(s)}} a * \cos(-\varphi)$$

periodicity: $\beta(s) = \beta(s+L), \quad \varphi = \pi Q$

$$\frac{-a}{\sqrt{\beta}}\sin(\pi Q)+\frac{\beta'}{2\sqrt{\beta}}a*\cos(\pi Q)+\frac{\Delta s}{\rho} \quad = \frac{-a}{\sqrt{\beta}}\sin(-\pi Q)+\frac{\beta'}{2\sqrt{\beta}}a*\cos(-\pi Q)$$

remember: $\sin(-x)=-\sin(x)\ ,\quad \cos(-x)=\cos(x)$

$$\frac{-a}{\sqrt{\beta}}\sin(\pi Q)+\cancel{\frac{\beta'}{2\sqrt{\beta}}a*\cos(\pi Q)}+\frac{\Delta s}{\rho} \quad = \frac{a}{\sqrt{\beta}}\sin(\pi Q)+\cancel{\frac{\beta'}{2\sqrt{\beta}}a*\cos(\pi Q)}$$

$$\frac{\Delta s}{\rho} \quad = \frac{2a}{\sqrt{\beta}}\sin(\pi Q) \qquad \longrightarrow \qquad a=\frac{\Delta s/\rho\,\sqrt{\beta}}{2\sin(\pi Q)}$$

put into orbit equation:

$$x_d(s)=a\sqrt{\beta(s)}*\cos(\psi(s)-\pi Q)=\frac{\delta_1*\sqrt{\beta(s)\beta_1}}{2\sin(\pi Q)}*\cos(\psi(s)-\pi Q)$$

where $\delta = \Delta s/\rho$
denotes the orbit kick

$$x_{co}(s)=\frac{\sqrt{\beta(s)}*\int\frac{1}{\rho_{s1}}\sqrt{\beta_{s1}}*\cos(|\psi_{s1}-\psi_s|-\pi Q)\,ds}{2\sin\pi Q}$$

Nota bene: * *orbit distortion is visible at any position „s" in the ring,*
… even if the dipole error is located at one single point „s1".

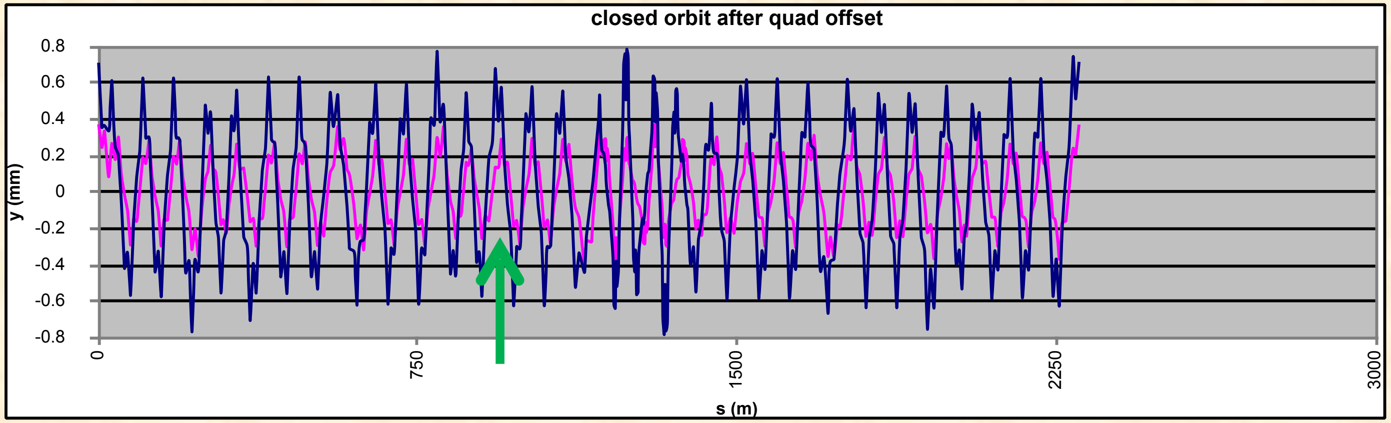
* *the β function describes the sensitivity of the beam to external fields*

* *the β function acts as amplification factor for the orbit amplitude at the given observation point*

* *in any case we (clearly …) will obtain a cosine-like orbit travelling around the ring … but being closed !!! after one turn.*

* *there is a resonance denominator*

$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{s1}} \sqrt{\beta_{s1}} * \cos(|\psi_{s1} - \psi_s| - \pi Q)\, ds}{2 \sin \pi Q}$$



PETRA III Light Source:

closed orbit error after offset of 0.3mm in 2 quadrupole magnets

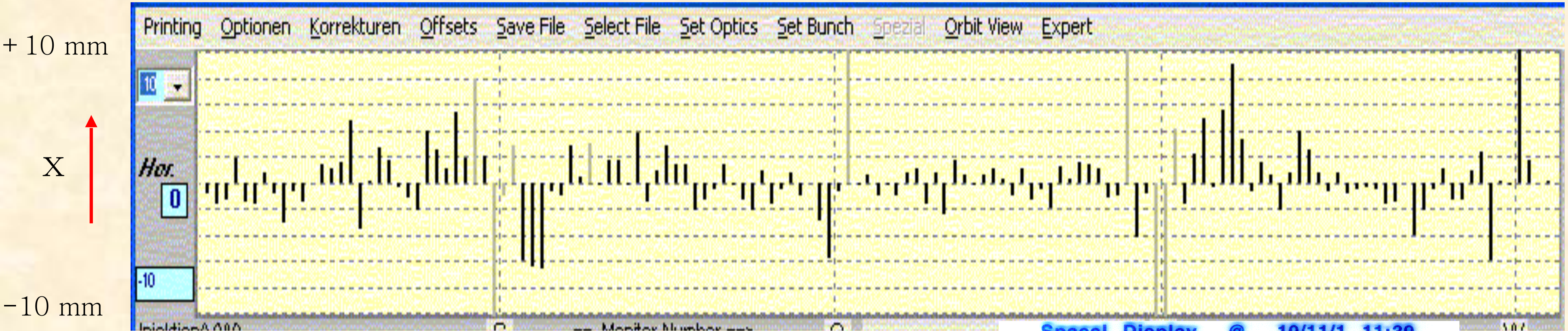
Example: *„bad orbit", i.e. closed orbit that contains large oscillation amplitudes*

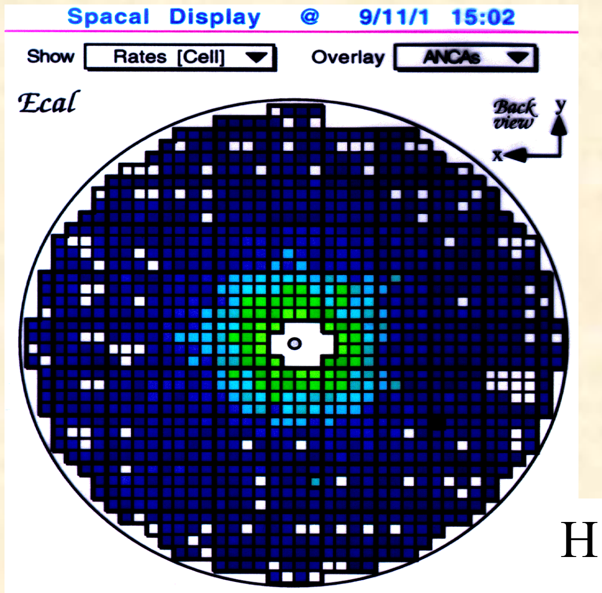
→eats up available magnet aperture

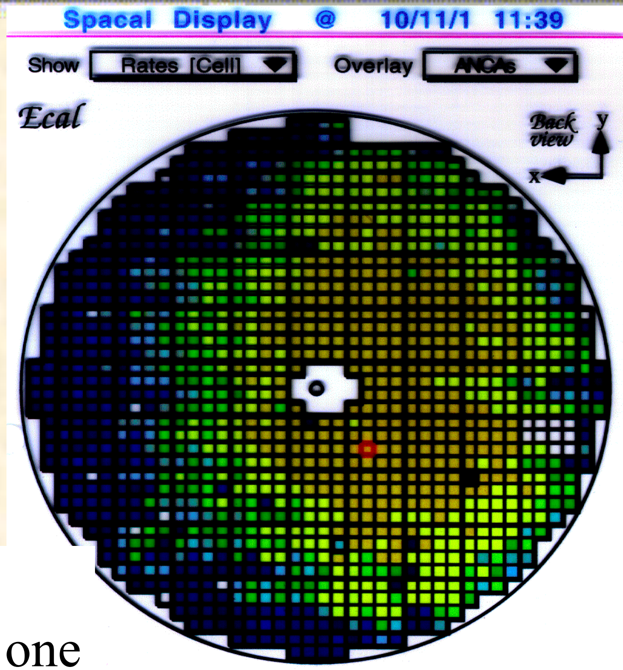
$$x(s) = x_\beta(s) + x_D(s) + x_{co}(s)$$

—> particle trajectories pass nonlinear field regions

—> detector components suffer from beam halo particles & light







H1 drift chamber during
a good run … or a „bad" one

30.) Finally: .... Resonances

closed orbit distortion:

$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{s1}} \sqrt{\beta_{s1}} * \cos(|\psi_{s1} - \psi_s| - \pi Q)\, ds}{2 \sin \pi Q}$$

remember from lecture 1: ***μ = phase advance per revolution in general measured and expressed in units of 2π … and called „Tune“ Q***

$$Q = \frac{\mu}{2\pi}$$

… and it depends on the focusing strength of the lattice cells.

Tune: number of oscillations per turn

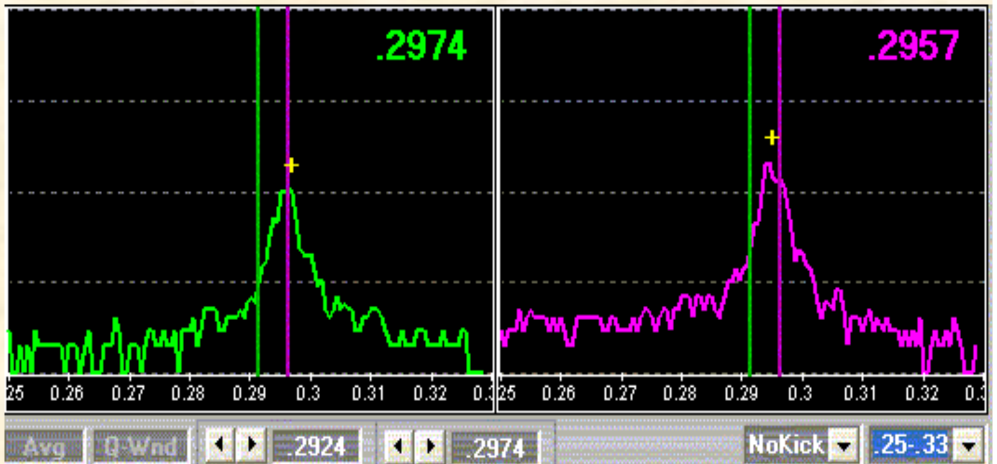
31.292
32.297

Relevant for beam stability:
non integer part

HERA revolution frequency: 47.3 kHz

$$0.292 * 47.3\ kHz = 13.81\ kHz$$

permanent tune measurement …and control in both planes



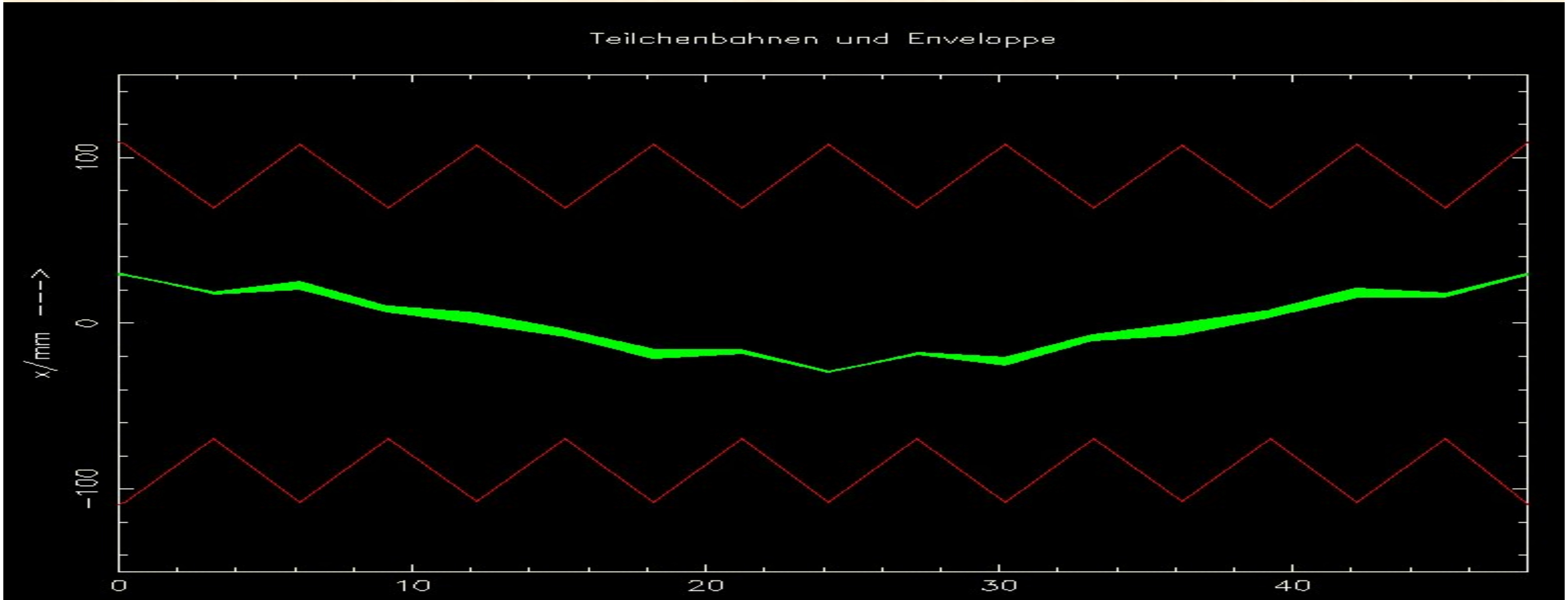
Resonances

$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{s1}} \sqrt{\beta_{s1}} * \cos(\psi_{s1} - \psi_s - \pi Q)\, ds}{2 \sin \pi Q}$$

Assume: Tune = integer $Q = 1 \rightarrow 2\sin\frac{\mu}{2} = 2\sin\pi = 0$

Integer tunes lead to a resonant increase of the closed orbit amplitude in presence of the smallest dipole field error.

Qualitatively spoken:



Tune & Resonances

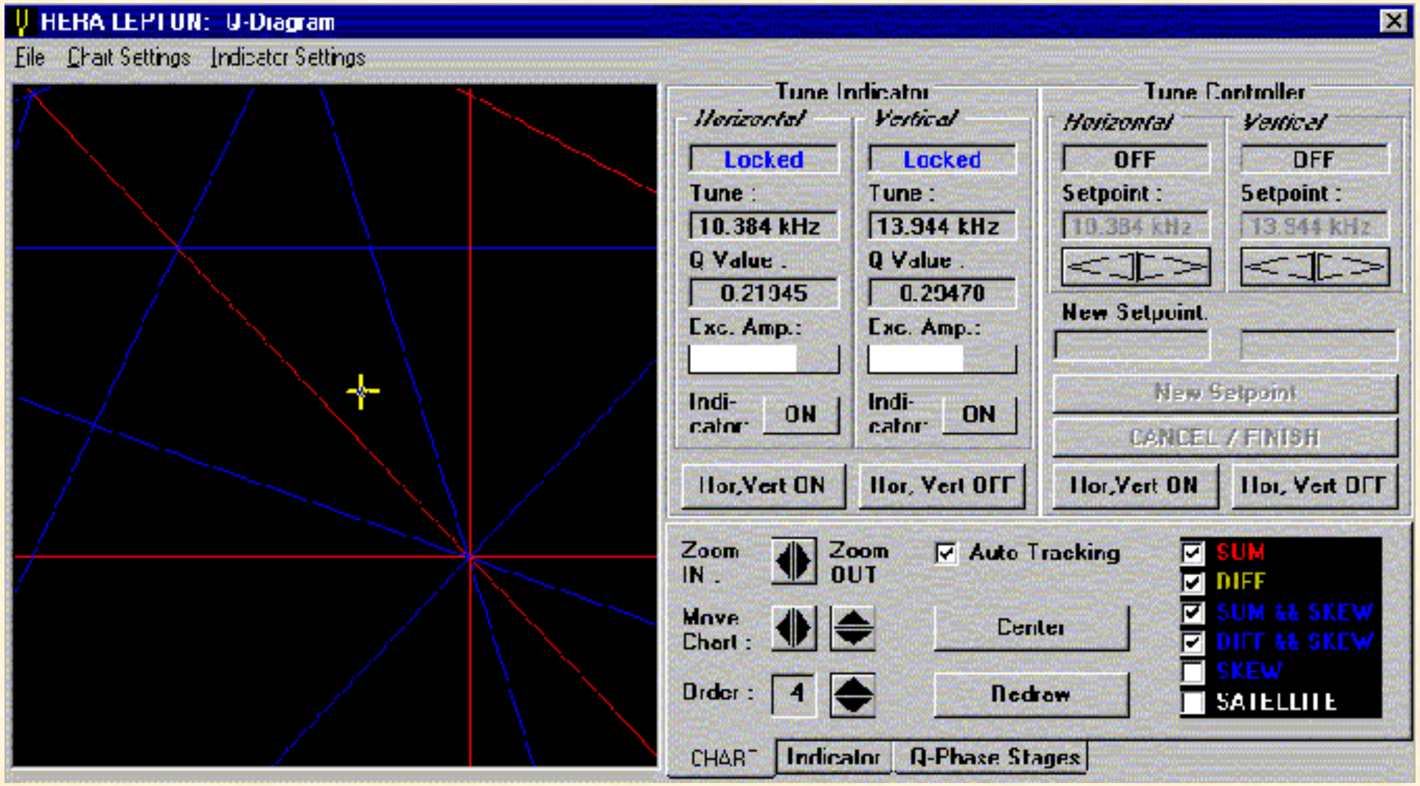
The particles – oscillating under the influence of the external magnetic fields – can be excited to infinite high amplitudes in case of resonant tunes—> particle loss within a short number of turns.

—> avoid large magnet errors

—> avoid forbidden tune values in both planes

$$m * Q_X + n * Q_Y = p$$

n,m,p = integer numbers



Examples:

n=1
m=0
p=1

$Q_x = 1$

integer resonance

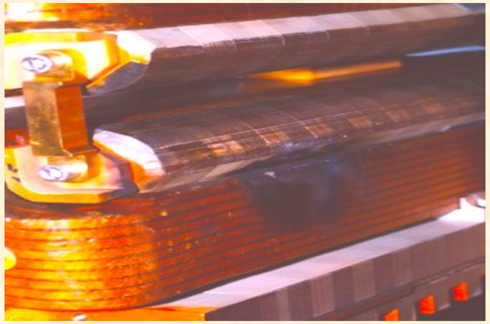
Examples:

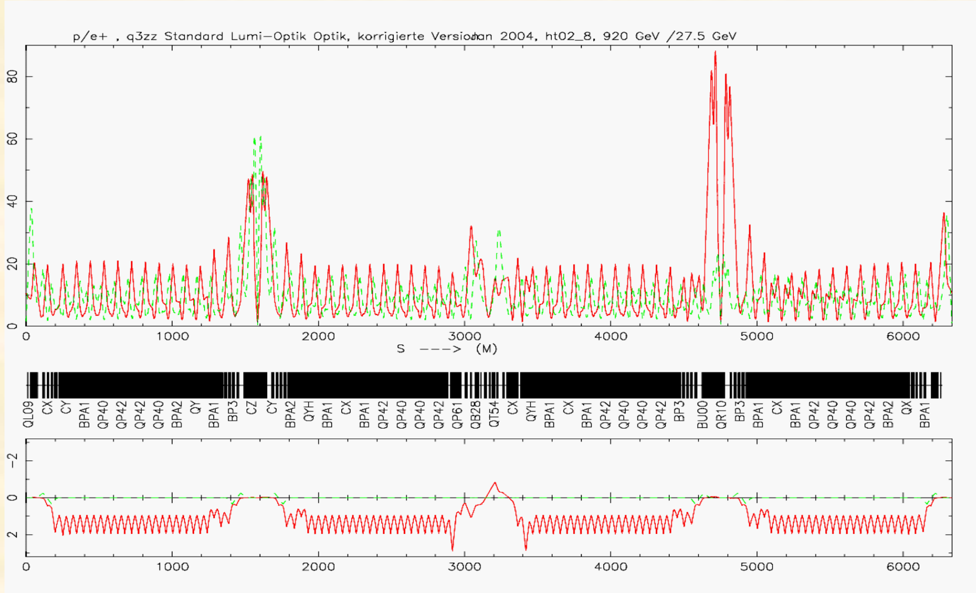
n=0
m=2
p=1

$Q_y = 1/2$

half-integer resonance

31.) Quadrupole Errors:





go back to Lecture I, page 1

single particle trajectory

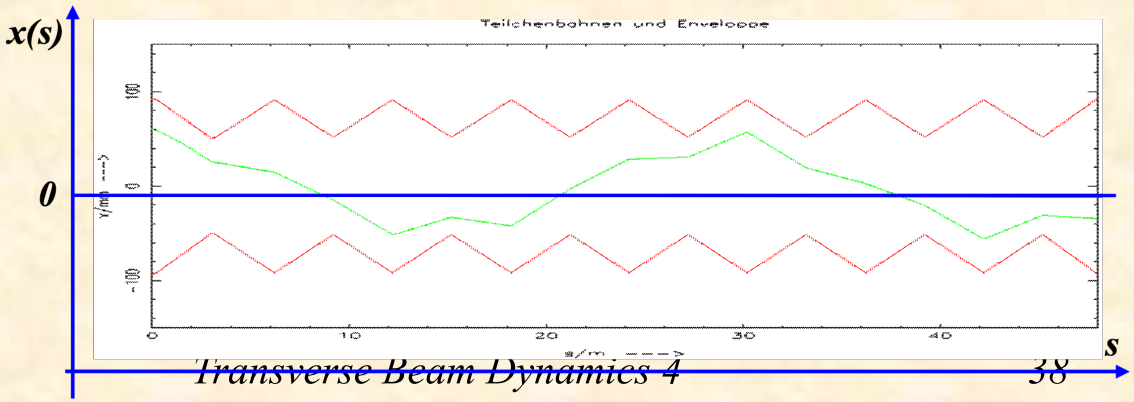
Solution of equation of motion

$$x = x_0 * \cos(\sqrt{k} * l) + x_0' * \frac{1}{\sqrt{k}} \sin(\sqrt{k} * l)$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M_{QF} * \begin{pmatrix} x \\ x' \end{pmatrix}_1 \qquad M_{QF} = \begin{pmatrix} \cos(\sqrt{k} * l) & \frac{1}{\sqrt{k}} \sin(\sqrt{k} * l) \\ -\sqrt{k} \sin(\sqrt{k} * l) & \cos(\sqrt{k} * l) \end{pmatrix}$$

Definition: phase advance of the particle oscillation per revolution in units of 2π is called *tune*

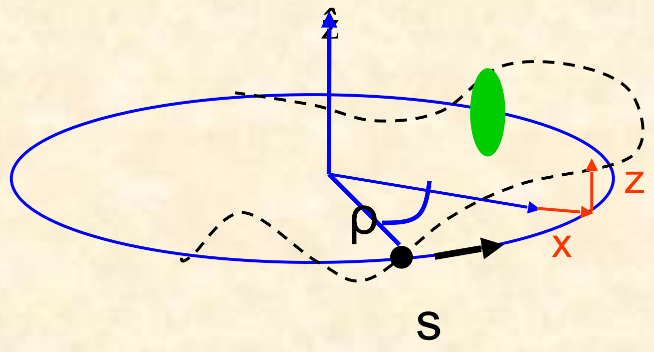
$$Q = \frac{\Delta\psi_{turn}}{2\pi} = \frac{\mu}{2\pi}$$



Quadrupole Error in the Lattice

optics perturbation described by thin lens quadrupole

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_s & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$



$$M_{dist} = M_{\Delta k} \cdot M_0 = \begin{pmatrix} 1 & 0 \\ -\Delta k ds & 1 \end{pmatrix} \cdot \begin{pmatrix} cos\psi_{turn} + \alpha sin\psi_{turn} & \beta sin\psi_{turn} \\ -\gamma sin\psi_{turn} & cos\psi_{turn} - \alpha sin\psi_{turn} \end{pmatrix}$$

quad ***ideal storage***

$$M_{dist} = \begin{pmatrix} cos\psi_0 + \alpha \cdot sin\psi_0 & \beta \cdot sin\psi_0 \\ -\Delta k ds \cdot (cos\psi_0 + \alpha sin\psi_0) - \gamma \cdot sin\psi_0 & -\Delta k ds \cdot \beta sin\psi_0 + cos\psi_0 - \alpha sin\psi_0 \end{pmatrix}$$

rule for getting the tune

$$Trace(M) = 2cos\psi = 2cos\psi_0 - \Delta k ds \beta \sin\psi_0$$

Quadrupole error —> Tune Shift

$\psi = \psi_0 + \Delta\psi$ ⟶ $cos\psi = cos(\psi_0 + \Delta\psi) = cos\psi_0 - \frac{\Delta k \; ds \; \beta \sin\psi_0}{2}$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$cos\psi_0 \cdot \underbrace{cos\Delta\psi)}_{\approx 1} - sin\psi_0 \cdot \underbrace{sin\Delta\psi)}_{\approx \Delta\psi} = cos\psi_0 - \frac{\Delta k ds \beta \sin\psi_0}{2}$$

$$\Delta\psi = \frac{\boldsymbol{kds}\,\beta}{2}$$

and referring to Q instead of ψ:

$$\psi = 2\pi \boldsymbol{Q}$$

$$\Delta \boldsymbol{Q} = \int_{s0}^{s0+l} \frac{\Delta \boldsymbol{k}(\boldsymbol{s})\beta(\boldsymbol{s})\boldsymbol{ds}}{4\pi}$$

! the tune shift is proportional to the β-function at the quadrupole

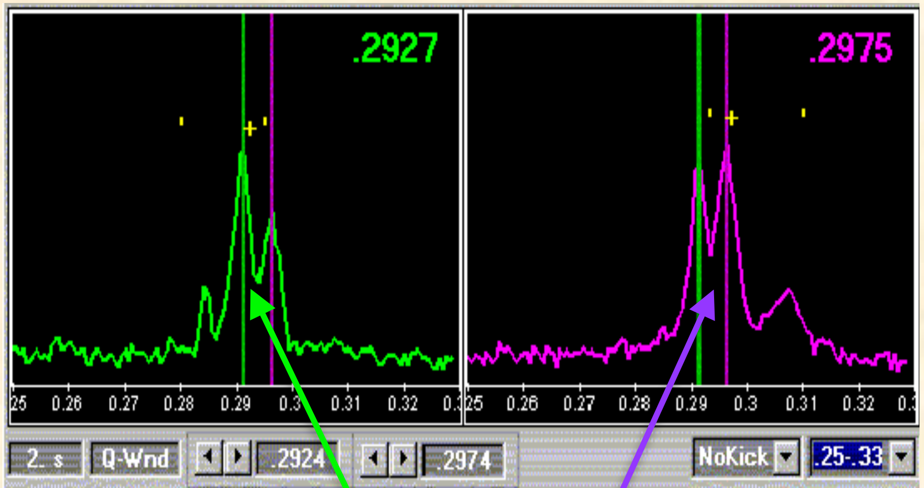
!! field quality, power supply tolerances etc are much tighter at places where β is large

!!! mini beta quads: β ≈ 1900 m
arc quads: β ≈ 80 m

!!!! β is a measure for the sensitivity of the beam

Example: deliberate change of quadrupole strength in a synchrotron:

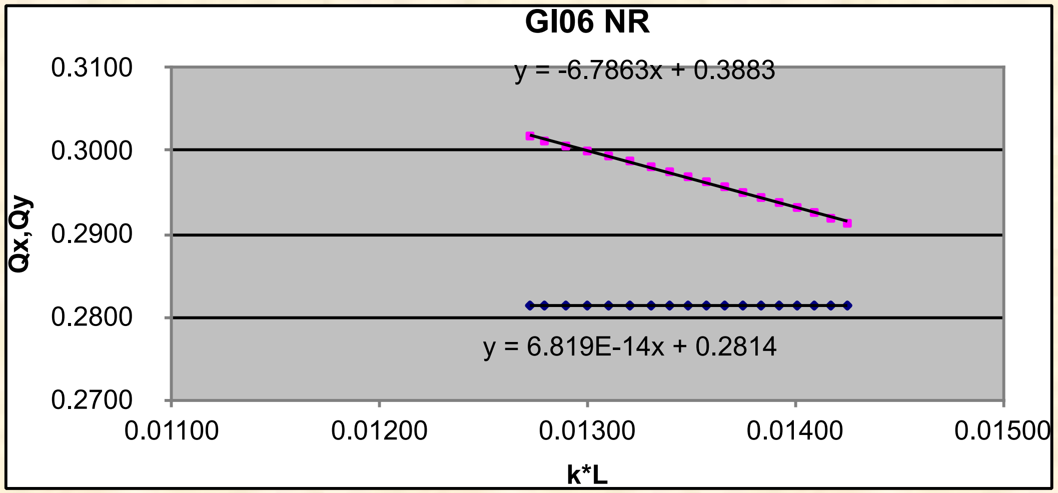
$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta K(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta K(s) * l_{quad} * \bar{\beta}}{4\pi}$$



tune spectrum ...

... for heaven's sake:

why do we get three peaks ????



tune shift as a function gradient change

Clearly there is another problem:

a focussing error at any location in the machine

... will shift the tune

... and distort the optics

... at any place in the ring

Example GA quadrupole:

burned quadrupole coil

Quadrupole error —> Beta Beat

$$\Delta\beta(s_0) = -\frac{\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta k\cos\left(2(\psi_{s1}-\psi_{s0})-2\pi Q\right)ds$$

orbit is not affected to first order !

Example LHC:

Many small quadrupole errors (gradient tolerances) add up

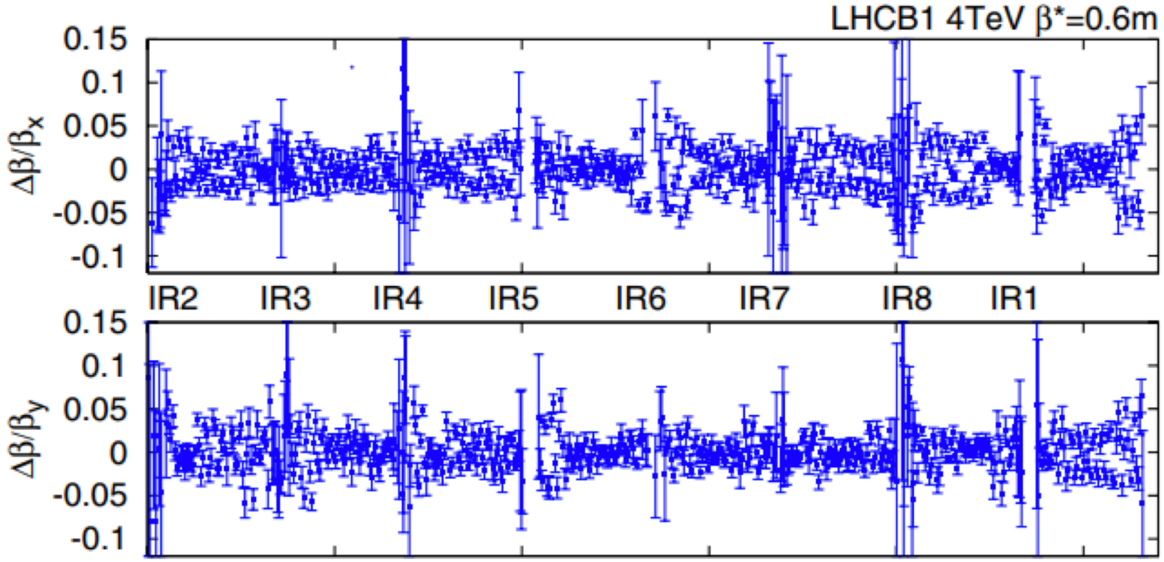
Tolerance limit: $\frac{\Delta\beta}{\beta} < 20\,\%$

Quadrupole error —> Beta Beat

A series of quadrupole errors Δk_i cause distortion of the β-function at s,

$$\frac{\Delta\beta}{\beta}(s) = \frac{1}{2\sin 2\pi Q}\sum_i \beta_i \Delta k_i \cos\left(2\pi Q - 2(\mu_i - \mu_s)\right) \qquad (132)$$

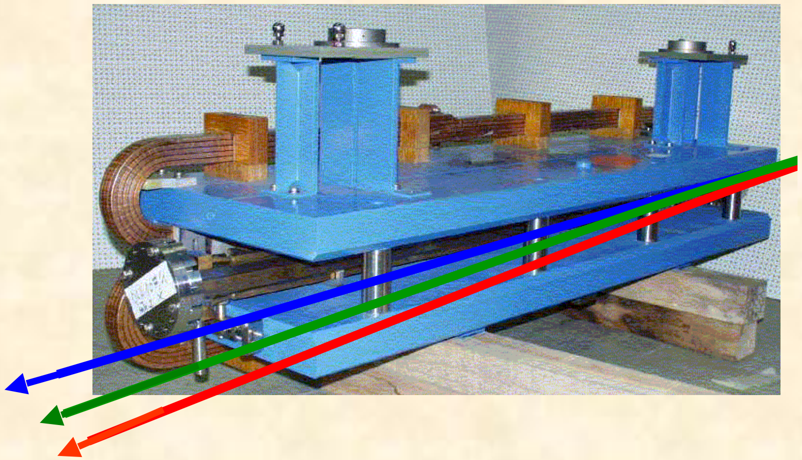
Unstable motion if Q is a half integer!



32.) Chromaticity:
A Quadrupole Error for $\Delta p/p \neq 0$

Influence of external fields on the beam: *prop. to magn. field & prop. zu 1/p*

dipole magnet $$\alpha = \frac{\int B\,dl}{p/e}$$



$$x_D(s) = D(s)\,\frac{\Delta p}{p}$$

focusing lens $$k = \frac{g}{p/e}$$

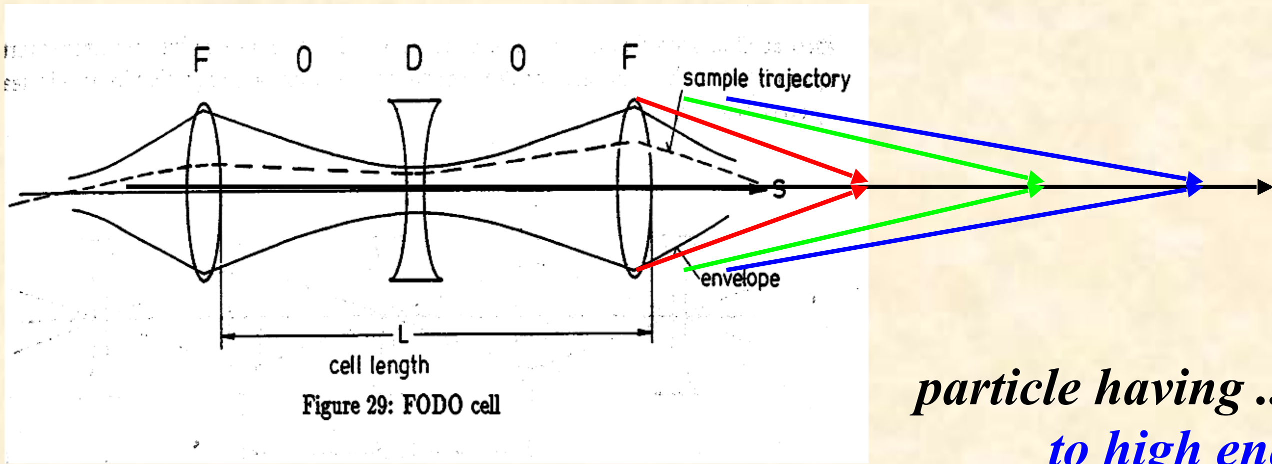


Figure 29: FODO cell

particle having ...
to high energy
to low energy
ideal energy

Chromaticity: Q' (… sometimes aka … "ξ")

$$k = \frac{g}{p/e} \qquad p = p_0 + \Delta p$$

in case of a momentum spread:

$$\boldsymbol{k} = \frac{\boldsymbol{eg}}{\boldsymbol{p}_0 + \Delta \boldsymbol{p}} \approx \frac{\boldsymbol{e}}{\boldsymbol{p}_0}\left(1 - \frac{\Delta \boldsymbol{p}}{\boldsymbol{p}_0}\right) \boldsymbol{g} = \boldsymbol{k}_0 + \Delta \boldsymbol{k}$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

… which acts like a quadrupole error in the machine and leads to a tune spread:

$$\Delta \boldsymbol{Q} = -\frac{1}{4\pi} \frac{\Delta \boldsymbol{p}}{\boldsymbol{p}_0} \boldsymbol{k}_0 \beta(\boldsymbol{s}) \boldsymbol{ds}$$

definition of chromaticity:

$$\Delta \boldsymbol{Q} = \boldsymbol{Q}' \frac{\Delta \boldsymbol{p}}{\boldsymbol{p}} \quad ; \qquad \boldsymbol{Q}' = -\frac{1}{4\pi} \oint \boldsymbol{k}(\boldsymbol{s}) \beta(\boldsymbol{s}) \boldsymbol{ds}$$

Where is the Problem ?

… what is wrong about Chromaticity:

Problem: chromaticity is generated by the lattice itself !!

Q' is a number indicating the size of the tune spot in the working diagram,
Q' is always created if the beam is focussed
—> it is determined by the focusing strength k of all quadrupoles

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

Example: LHC

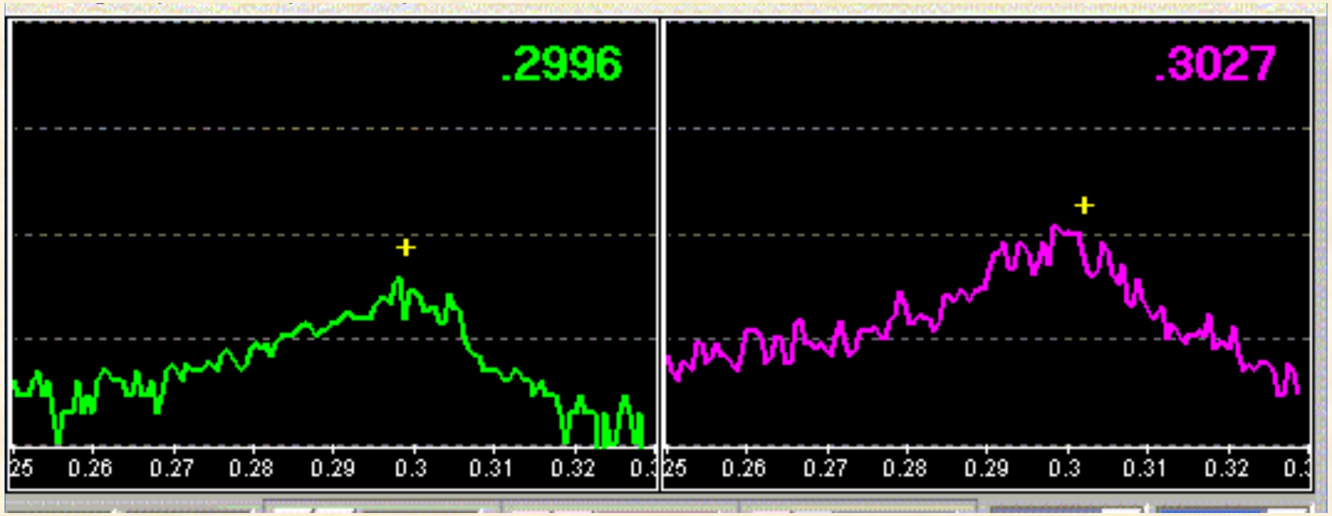
$Q' = 250$
$\Delta p/p = +/- 0.2 * 10^{-3}$
$\Delta Q = 0.256 \ldots 0.36$

→ *Some particles get very close to resonances and are lost*

in other words: the tune is not a point it is a pancake

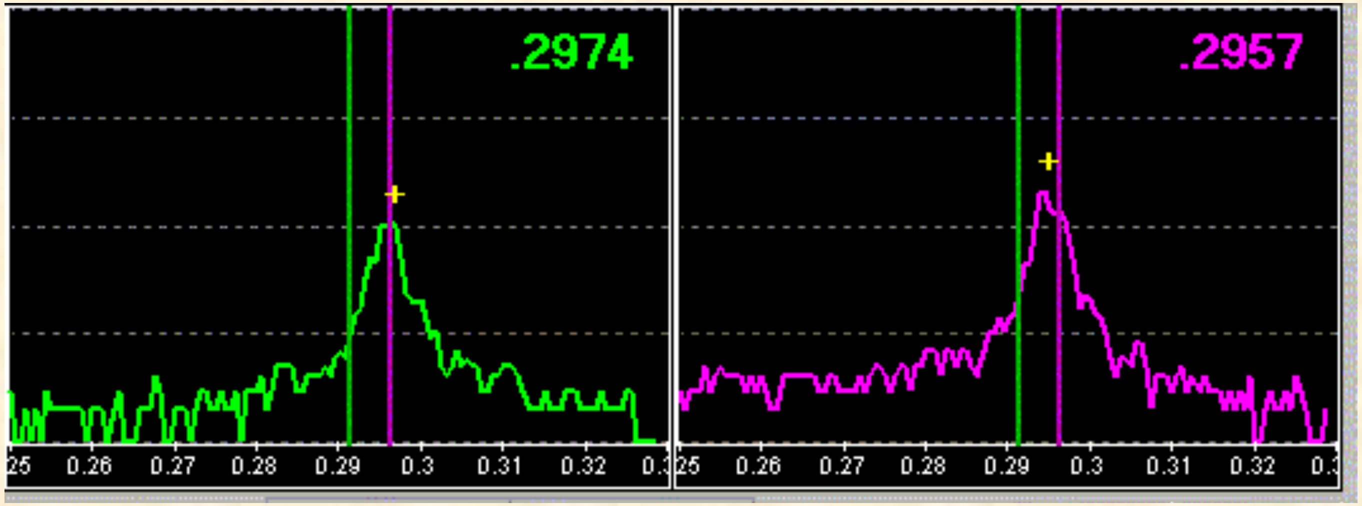
Betatron Tune: Qx, Qy

Effect of Chromaticity



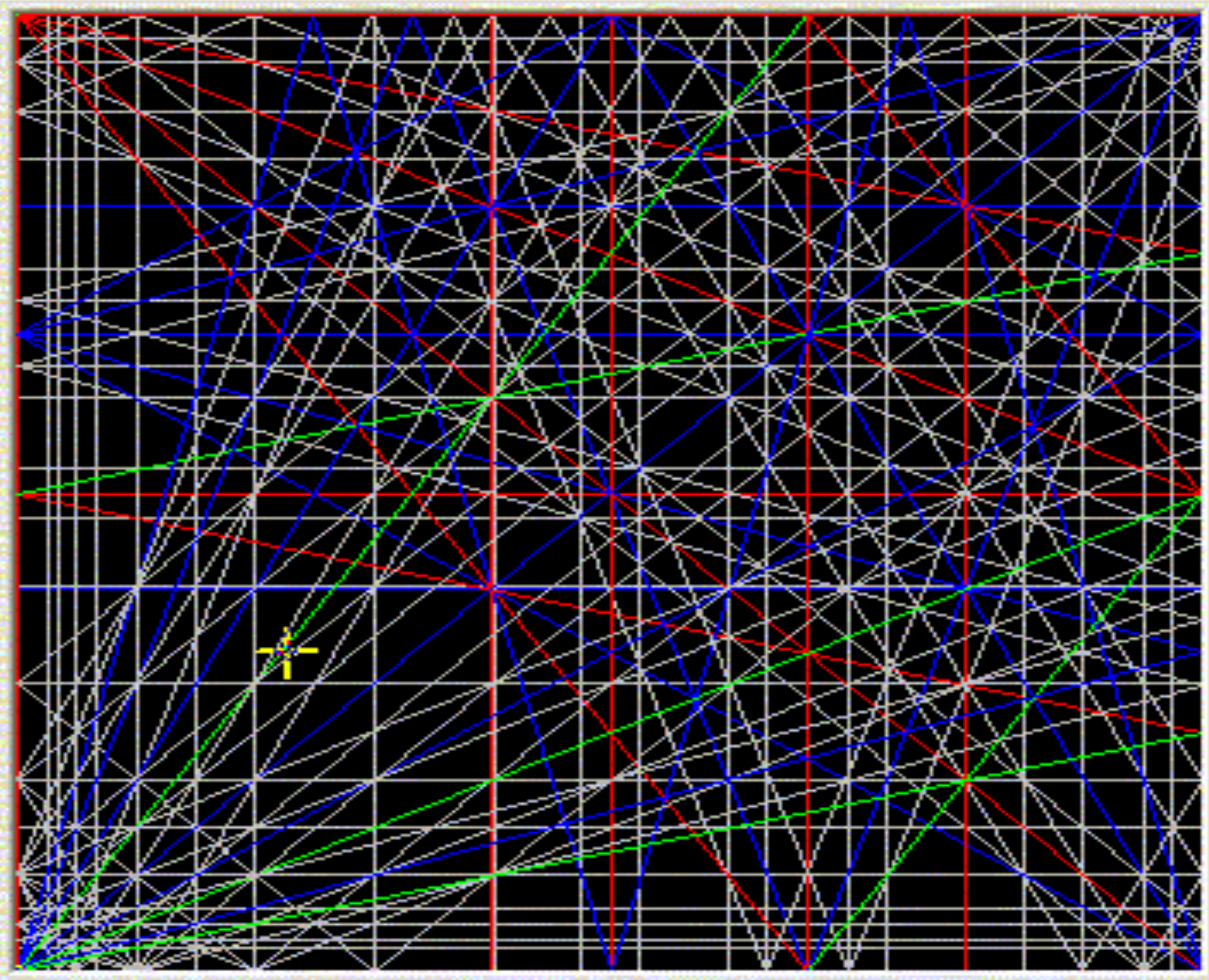
Tune signal for a nearly uncompensated cromaticity
(Q' ≈ 20)

Ideal situation: cromaticity well corrected,
(Q' ≈ 1)



Once more: Tune and Resonances

$m*Q_x+n*Q_y+l*Q_s = integer$



HERA e Tune diagram
up to 3rd order

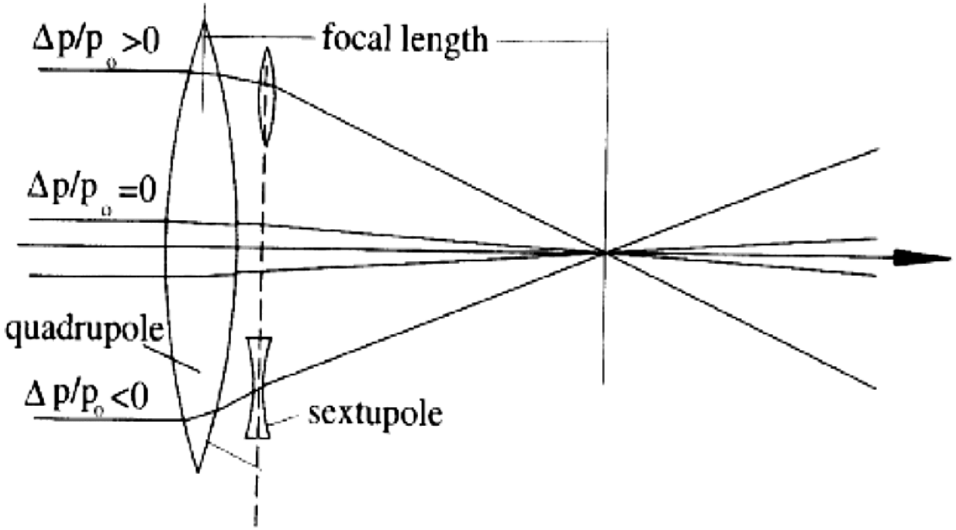
… and up to 7th order

Homework for the operateurs: find a nice place for the tune where against all probability the beam will survive

Correction of Q' :

Need: additional quadrupole strength for each momentum deviation Δp/p

Sextupoles, through a non-linear magnetic field, correct the effect of energy spread and focuses particles at a single location.



- Located in dispersive regions.
- Usually in arcs.
- Sextupole families.

Now is when the party starts

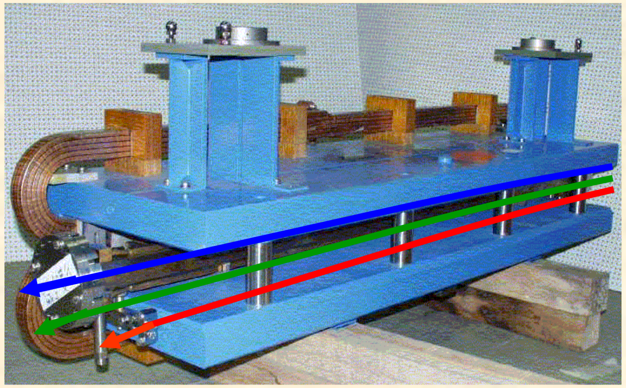
- Sextupoles introduce non-linear fields.
- ...i.e. they induce non-linear motion.
- resonances, tune shifts, chaotic motion.

Correction of Q' :

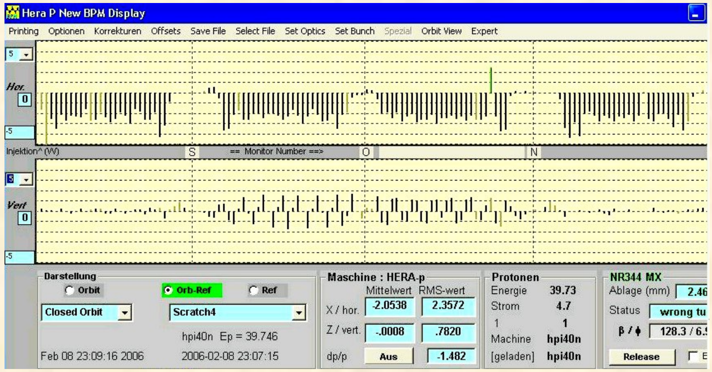
Need: additional quadrupole strength for each momentum deviation Δp/p

1.) sort the particles acording to their momentum

$$x_D(s) = D(s)\frac{\Delta p}{p}$$



… using the dispersion function



2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$\left.\begin{aligned} B_x &= \tilde{g}xz \\ B_z &= \frac{1}{2}\tilde{g}(x^2 - z^2) \end{aligned}\right\} \qquad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x$$

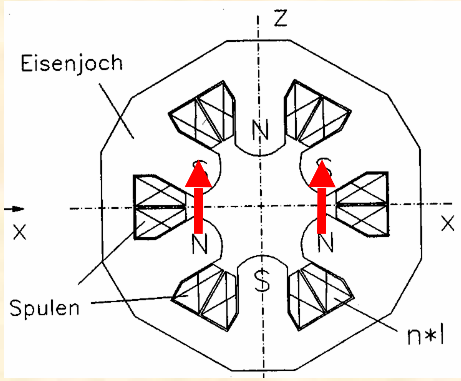
linear rising „gradient":

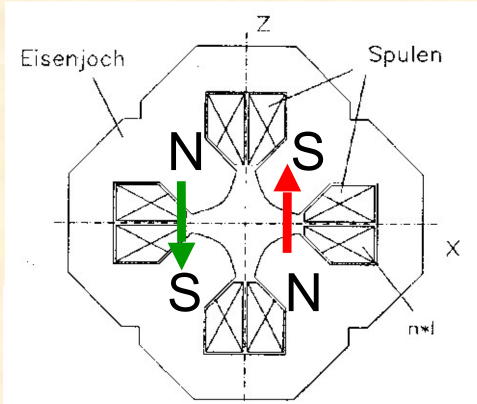
Correction of Q':

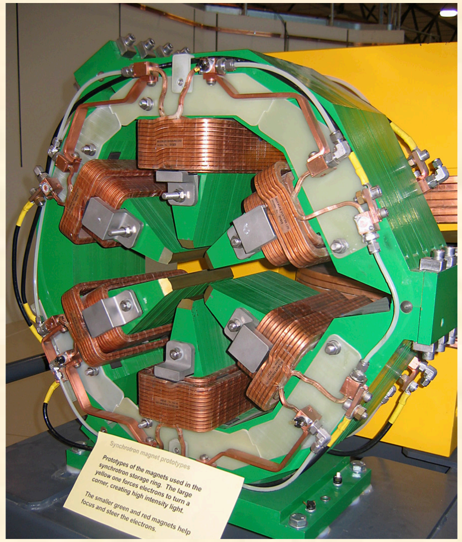
k_1 quadrupole strength being off-centre in a sextupole

$$k_1 = \frac{\tilde{g}}{p/e} \cdot x = k_2 \cdot x = k_2 \cdot D \cdot \frac{\Delta p}{p}$$

Sextupole Magnets:







corrected chromaticity

counter acting effect in the two planes

$$Q'_x = \frac{1}{4\pi} \cdot \left\{ -\oint \beta_x(s) \cdot |k_q(s)|\, ds + |k_2^F| \cdot l_{sext} \cdot \beta_x(s) \cdot D_{x_{sext}} - |k_2^D| \cdot l_{sext} \cdot \beta_x(s) \cdot D_{x_{sext}} \right\}$$

$$Q'_y = \frac{1}{4\pi} \cdot \left\{ \underbrace{-\oint \beta_y(s) \cdot |k_q(s)|\, ds}_{\text{"natural" chromaticity}} \underbrace{- |k_2^F| \cdot l_{sext} \cdot \beta_y(s) \cdot D_{x_{sext}} + |k_2^D| \cdot l_{sext} \cdot \beta_y(s) \cdot D_{x_{sext}}}_{\text{sextupole correction of chromaticity}} \right\}$$

"natural" chromaticity — *sextupole correction of chromaticity*

Resume´:

quadrupole error: **tune shift**

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k(s) * l_{quad} * \bar{\beta}}{4\pi}$$

beta beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q} \int_{s1}^{s1+l} \beta(s_1)\Delta k \cos\big(2(\psi_{s1} - \psi_{s0}) - 2\pi Q\big)ds$$

chromaticity

$$\Delta Q = Q' * \frac{\Delta p}{p}$$

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)\,ds$$

in a FoDo

$$Q'_{cell} = -\frac{1}{\pi}\tan\frac{\mu}{2}$$

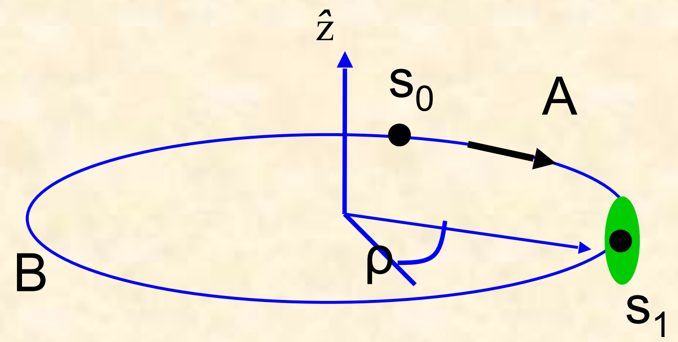
corrected chromaticty

$$Q'_x = \frac{-1}{4\pi} * \oint k_1(s)\beta(s)\,ds + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext}\, D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext}\, D_x^D \beta_x^D$$

Merci

Appendix 1: Quadrupole Errors and Beta Function

a quadrupole error will not only influence the oscillation frequency … „tune“
… but also the amplitude … „beta function“



$$M_{turn} = B * A \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

distorted matrix $M_{dist} = \begin{pmatrix} m^*_{11} & m^*_{12} \\ m^*_{21} & m^*_{22} \end{pmatrix} = B \begin{pmatrix} 1 & 0 \\ -\Delta k ds & 1 \end{pmatrix} A$

$$M_{dist} = B \begin{pmatrix} a_{11} & a_{12} \\ -\Delta k ds a_{11} + a_{12} & -\Delta k ds a_{12} + a_{22} \end{pmatrix}$$

$$M_{dist} = \begin{pmatrix} \sim & b_{11}a_{12} + b_{12}(-\Delta k ds a_{12} + a_{22}) \\ \sim & \sim \end{pmatrix}$$

the beta function is usually obtained via the matrix element „m12“, which is in Twiss form for the undistorted case

$$m_{12} = \beta_0 \sin 2\pi Q$$

and including the error:

$$m_{12}^* = \underbrace{b_{11}a_{12} + b_{12}a_{22}}_{m_{12} = \beta_0 \sin 2\pi Q} - b_{12}a_{12}\Delta k ds$$

$$(1) \quad m_{12}^* = \beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds$$

As M* is still a matrix for one complete turn we still can express the element m_{12} in twiss form:

$$(2) \quad m_{12}^* = (\beta_0 + d\beta)^* \sin 2\pi (Q + dQ)$$

Equalising (1) and (2) and assuming a small error

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = (\beta_0 + d\beta)^* \sin 2\pi (Q + dQ)$$

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = (\beta_0 + d\beta)^* \sin 2\pi Q \underbrace{\cos 2\pi dQ}_{\approx 1} + \cos 2\pi Q \underbrace{\sin 2\pi dQ}_{\approx 2\pi dQ}$$

$$\beta_0 \sin 2\pi Q - a_{12} b_{12} \Delta k ds = \beta_0 \sin 2\pi Q + \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q + d\beta_0 2\pi dQ \cos 2\pi Q$$

ignoring second order terms

$$- a_{12} b_{12} \Delta k ds = \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

remember: tune shift dQ due to quadrupole error: $dQ = \frac{\Delta k \beta_1 ds}{4\pi}$

(index „1" refers to location of the error)

$$- a_{12} b_{12} \Delta k ds = \frac{\beta_0 \Delta k \beta_1 ds}{2} \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

solve for dβ

$$d\beta_0 = \frac{-1}{2 \sin 2\pi Q} \left\{ 2 a_{12} b_{12} + \beta_0 \beta_1 \cos 2\pi Q \right\} \Delta k ds$$

express the matrix elements a_{12}, b_{12} in Twiss form

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} \left(\cos\psi_s + \alpha_0 \sin\psi_s \right) & \sqrt{\beta_s \beta_0} \sin\psi_s \\ \frac{(\alpha_0 - \alpha_s) \cos\psi_s - (1 + \alpha_0 \alpha_s) \sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta s}} \left(\cos\psi_s - \alpha_s \sin\psi_s \right) \end{pmatrix}$$

$$d\beta_0 = \frac{-1}{2\sin 2\pi Q}\left\{2a_{12}b_{12} + \beta_0\beta_1 \cos 2\pi Q\right\}\Delta k ds$$

$$a_{12} = \sqrt{\beta_0\beta_1}\sin\Delta\psi_{0\to 1}$$

$$b_{12} = \sqrt{\beta_0\beta_1}\sin(2\pi Q - \Delta\psi_{0\to 1})$$

$$\boldsymbol{d}\beta_0 = \frac{-\beta_0\beta_1}{2\sin 2\pi \boldsymbol{Q}}\left\{2\sin\Delta\psi_{01}\sin(2\pi \boldsymbol{Q} - \Delta\psi_{01}) + \cos 2\pi \boldsymbol{Q}\right\}\Delta \boldsymbol{k}\boldsymbol{d}\boldsymbol{s}$$

… after some TLC transformations … $= \cos(2\Delta\psi_{01} - 2\pi Q)$

$$\Delta\beta(s_0) = \frac{-\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta k \cos\left(2(\psi_{s1} - \psi_{s0}) - 2\pi Q\right)ds$$

Nota bene: ***! the beta beat is proportional to the strength of the error Δk***

!! and to the β function at the place of the error ,

!!! and to the β function at the observation point,
(… remember orbit distortion !!!)

!!!! there is a resonance denominator

Appendix 2: Dispersion

Solution of the inhomogenious equation of motion

Ansatz:

$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})d\tilde{s}$$

$$D'(s) = S' * \int \frac{1}{\rho} C\,dt + S\frac{1}{\rho}C - C' * \int \frac{1}{\rho} S\,dt - C\frac{1}{\rho}S$$

$$D'(s) = S' * \int \frac{C}{\rho}\,dt \;-\; C' * \int \frac{S}{\rho}\,dt$$

$$D''(s) = S'' * \int \frac{C}{\rho}\,d\tilde{s} + S'\frac{C}{\rho} \;-\; C'' * \int \frac{S}{\rho}\,d\tilde{s} \;-\; C'\frac{S}{\rho}$$

$$= S'' * \int \frac{C}{\rho}\,d\tilde{s} - C'' * \int \frac{S}{\rho}\,d\tilde{s} + \frac{1}{\rho}\underbrace{\left(CS' - S\,C'\right)}_{=\det M = 1}$$

---

remember: for Cs) and S(s) to be independent solutions the Wronski determinant has to meet the condition

$$W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} \neq 0$$

and as it is independent of the variable „s“

$$\frac{dW}{ds} = \frac{d}{ds}(CS' - SC') = CS'' - SC'' = -K(CS - SC) = 0$$

we get for the initital conditions that we had chosen …

$$\left.\begin{matrix} C_0 = 1, & C_0' = 0 \\ S_0 = 0, & S_0' = 1 \end{matrix}\right\} \qquad W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} = 1$$

$$D'' = S'' * \int \frac{C}{\rho} d\tilde{s} - C'' * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

remember: S & C are solutions of the homog. equation of motion:

$$S'' + K * S = 0$$
$$C'' + K * C = 0$$

$$D'' = -K * S * \int \frac{C}{\rho} d\tilde{s} + K * C * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

$$D'' = -K * \underbrace{\left\{ S \int \frac{C}{\rho} d\tilde{s} + C \int \frac{S}{\rho} d\tilde{s} \right\}}_{=D(s)} + \frac{1}{\rho}$$

$$D'' = -K * D + \frac{1}{\rho}$$

… or

$$D'' + K * D = \frac{1}{\rho}$$

qed