Coulomb & strong interactions in Fourier space
— work in progress —

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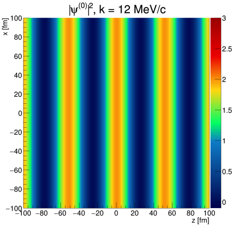
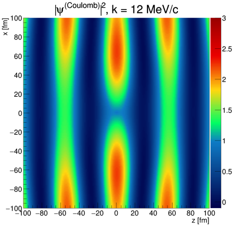
Eötvös University, Budapest

Day of Femtoscopy
October 30, 2024, Gyöngyös, Hungary





Outline

- Skipping most of the ,,introduction": HBT correlations, Coulomb effect...
  From $D(\mathbf{r})$ pair source, $C(\mathbf{k}) = \int d\mathbf{r}\, D(\mathbf{r}) |\psi_{\mathbf{k}}(\mathbf{r})|^2$.
- Coulomb (& strong interaction) effect: essential for precision measurements
- Possible treatments:
  - Gamow correction (simple, old, very off by today's standards)
  - Coulomb correction pre-calculated for some source (e.g. 5 fm spherical Gaussian): *still widely used, even for 3D measurements*. **Need better!**
    NB: if source is spherical in LCMS (Longitudinally Co-Moving System), then it is not in PCMS (Pair Co-Moving System; where Coulomb is calculated)
  - Calculation of interacting $C(\mathbf{k})$ as above on the fly: time-consuming if not impossible (even nowadays)
  - Memory lookup-tables, interpolating formulas...
- Need for new method: precision measurements, non-Gaussian (e.g. Lévy) sources
  Our new method: works directly with Fourier transform of source
  - Spherically symmetric case: done & programmed, using routinely
  - Generalizations: in the works, some expounded here

Some basic formulas

- Momentum variables: $\mathbf{K} := \frac{1}{2}(\mathbf{p}_1+\mathbf{p}_2)$, $\mathbf{k}\equiv\mathbf{k}^* = \frac{m_2\mathbf{k}_1 - m_1\mathbf{k}_2}{m_1+m_2}$;
- Coordinate variables: $\mathbf{r} := \mathbf{r}_1 - \mathbf{r}_2$, $\boldsymbol{\rho} = \frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{m_1+m_2}$. (Working in 3D setting now.)
- source function: $S(\mathbf{r}_1,\mathbf{p}_1)$; $\quad N_1(\mathbf{p}_1) = \int d\mathbf{r}_1\, S(\mathbf{r}_1,\mathbf{p}_1)$
- pair wave function: $\psi^{(2)}(\mathbf{r}_1,\mathbf{r}_2)$; assume $\psi^{(2)}(\mathbf{r}_1,\mathbf{r}_2) = e^{2i\mathbf{K}\boldsymbol{\rho}}\psi_{\mathbf{k}}(\mathbf{r})$
- pair mom. distr.: $N_2(\mathbf{p}_1,\mathbf{p}_2) = \int d\mathbf{r}_1 d\mathbf{r}_2\, S(\mathbf{r}_1,\mathbf{p}_1)S(\mathbf{r}_2,\mathbf{p}_2)\left|\psi^{(2)}(\mathbf{r}_1,\mathbf{r}_2)\right|^2$
- corr. function: $C(\mathbf{p}_1,\mathbf{p}_2) = \dfrac{N_2(\mathbf{p}_1,\mathbf{p}_2)}{N_1(\mathbf{p}_1)N_1(\mathbf{p}_2)}$
- pair source: $D(\mathbf{r},\mathbf{K}) = \int d\boldsymbol{\rho}\, S\left(\boldsymbol{\rho}+\frac{\mathbf{r}}{2},\mathbf{K}\right)S\left(\boldsymbol{\rho}-\frac{\mathbf{r}}{2},\mathbf{K}\right)$, $\quad \underline{D(\mathbf{r},\mathbf{K}) = D(-\mathbf{r},\mathbf{K})}$
- normalization: $\int d\mathbf{r}_1\, S(\mathbf{r}_1,\mathbf{p}_1) = 1 \quad\Rightarrow\quad \int d\mathbf{r}\, D(\mathbf{r}) = 1$ (from now on, $\mathbf{K}$ suppressed)
- Thus $\Rightarrow$ $\underline{\underline{C(\mathbf{k}) = \int d\mathbf{r}\, D(\mathbf{r})|\psi_{\mathbf{k}}(\mathbf{r})|^2}}$; $\overset{Bowler-Sinyukov}{\Rightarrow}$ $\underline{C(\mathbf{k}) = 1-\lambda+\lambda\int d\mathbf{r}\, D(\mathbf{r})|\psi_{\mathbf{k}}(\mathbf{r})|^2}$.

Wave functions — Coulomb interaction

- Basic quantities: η Sommerfeld parameter, $|\mathcal{N}_r|^2$ & $|\mathcal{N}_a|^2$ repulsive & attractive Gamow factors (NB: use $\mathcal{N}_r$ for identical particles), $\delta_l^c \equiv \arg\Gamma(l+1+i\eta)$ Coulomb phase shift

$$\eta \equiv \alpha_{\rm em}\frac{mc}{\hbar k}, \qquad \mathcal{N}_r := e^{-\pi\eta/2}\Gamma(1+i\eta), \quad \mathcal{N}_a := e^{\pi\eta/2}\Gamma(1-i\eta) \qquad \Rightarrow \qquad |\mathcal{N}_r|^2 = \frac{2\pi\eta}{e^{2\pi\eta}-1}, \quad |\mathcal{N}_a|^2 = \frac{2\pi\eta}{1-e^{-2\pi\eta}}.$$

- Coulomb wave function $\psi_{\mathbf{k}}(\mathbf{r})$:

Attractive ($\Rightarrow$non-identical): $\mathcal{N}_a^* e^{-ikr}\mathbf{M}\big(1+i\eta, 1, i(kr+\mathbf{kr})\big)$,

Repulsive, non-identical: $\mathcal{N}_r^* e^{-ikr}\mathbf{M}\big(1-i\eta, 1, i(kr+\mathbf{kr})\big)$,

Repulsive, symmetrized: $\frac{\mathcal{N}_r^*}{\sqrt{2}} e^{-ikr}\Big[\mathbf{M}\big(1-i\eta, 1, i(kr+\mathbf{kr})\big)+\mathbf{M}\big(1-i\eta, 1, i(kr-\mathbf{kr})\big)\Big]$,

Repulsive, antisymm'd: $\frac{\mathcal{N}_r^*}{\sqrt{2}} e^{-ikr}\Big[\mathbf{M}\big(1-i\eta, 1, i(kr+\mathbf{kr})\big)-\mathbf{M}\big(1-i\eta, 1, i(kr-\mathbf{kr})\big)\Big]$,

- Here, $\mathbf{M}(a,b,z)$ is the (reduced) confluent hypergeometric function

$$\mathbf{M}(a,b,z) = \frac{1}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(b+n)}\frac{z^n}{n!}$$

Wave functions — strong interaction (s-wave)

- For strong interaction: only s-wave for now (questionable around origin?); only $\Delta_0^s(k)$ phase shift $\Leftrightarrow$ $f_s(k)$ scattering amplitude; $\sin\Delta_0^s e^{i\Delta_0^s} = kf_s(k)$:
- $\psi_{\mathbf{k}}(\mathbf{r})$ in case of strong interaction but no Coulomb:
  Non-identical: $e^{i\mathbf{kr}} + \frac{e^{-ikr}}{r} f_s^*(k)$,
  Symmetrized: $\frac{1}{\sqrt{2}}\left[e^{i\mathbf{kr}} + e^{-i\mathbf{kr}} + 2\frac{e^{-ikr}}{r} f_s^*(k)\right]$,
  Antisymm'd: $\frac{1}{\sqrt{2}}\left[e^{i\mathbf{kr}} - e^{-i\mathbf{kr}}\right]$.
- $\psi_{\mathbf{k}}(\mathbf{r})$ in case of strong + Coulomb (with notation $\mathcal{S}_r \equiv 2i\sin\Delta_0^s e^{-i\Delta_0^s} e^{-2i\delta_0^c} e^{\pi\eta/2}$ for repulsive case; for attractive case, $\mathcal{S}_a$ is obtained by replacing η with $-\eta$ in $\mathcal{S}_r$):
  Attractive (non-identical): $\mathcal{N}_a^* e^{-ikr}\Big[\mathbf{M}\big(1+i\eta, 1, i(kr+\mathbf{kr})\big) + \mathcal{S}_a \cdot U(1-i\eta, 2, 2ikr)\Big]$,
  Repulsive, non-identical: $\mathcal{N}_r^* e^{-ikr}\Big[\mathbf{M}\big(1-i\eta, 1, i(kr+\mathbf{kr})\big) + \mathcal{S}_r \cdot U(1-i\eta, 2, 2ikr)\Big]$,
  Repulsive, symmetrized or antisymm'd: add or subtract $\mathbf{r} \leftrightarrow -\mathbf{r}$ and divide by $\sqrt{2}$.
- Here $U(a, b, z)$ is Tricomi's function (with l'Hospital's rule needed for $b \in \mathbb{Z}$):

$$U(a, b, z) = \frac{\pi}{\sin(\pi b)}\left\{\frac{\mathbf{M}(a, b, z)}{\Gamma(a+1-b)} - \frac{\mathbf{M}(a+1-b, 2-b, z)}{z^{b-1}\Gamma(a)}\right\}.$$

„Roadmap"

- Possible cases (11; cases of no Coulomb & no strong interaction omitted):

	No Coulomb	Repulsive Coulomb	Attractive Coulomb
Non-identical	– Strong Yes	– Strong Yes – Strong No	Strong Yes Strong No
Symmetrized (Bose-Einstein)	– Strong Yes	– Strong Yes – Strong No	X
Antisymmetrized (Fermi-Dirac)	– Strong Yes	– Strong Yes – Strong No	X

- Another distinction: $S(\mathbf{r})$ & $D(\mathbf{r})$ spherically symmetric vs. non-spherical (besides functional form of $S(\mathbf{r})$ & $D(\mathbf{r})$, of course)
- Calculating $C(\mathbf{k})=\int d\mathbf{r}\, D(\mathbf{r})|\psi_{\mathbf{k}}(\mathbf{r})|^2$ possible by
  - brute-force integration: suffers from oscillations in $\psi_{\mathbf{k}}(\mathbf{r})$,
  - Monte Carlo integration: same problem
  - **new methodology based on Fourier transformation** worked out & programmed for only Coulomb repulsion, spherically symmetric (Lévy & Gaussian) source, symmetrized wave function in Eur. Phys. J. C **83**, 1015 (2023) `arXiv:2308.10745`, as well as `github.com/csanadm/CoulCorrLevyIntegral`.
  - Generalization to non-spherical case: math done (see below), programming in progress. . .
  - Generalization to strong interaction: in the works.

Fourier transform method

- Assume that source is easily expressed as Fourier transform

$$D(\mathbf{r}) := \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{qr}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int d^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{qr}}$$

  E.g. Gaussian: $f(\mathbf{q}) = \exp\left(-\frac{1}{2}\mathbf{q}\mathbf{R}^2\mathbf{q}\right)$; if spherically symmetric: $f(\mathbf{q}) \equiv f_s(q) = e^{-q^2R^2/2}$,
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- Mathematical idea: „interchange integrals". Not trivial since

$$C_2(\mathbf{k}) = \int d^3\mathbf{r}\, |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}}$$

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- Tasks at hand:

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  - Step 1: Calculate this last **r**-integral, independently of $f(\mathbf{q})$, i.e. of $D(\mathbf{r})$,

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$$\underline{\underline{C_2(\mathbf{k})}} = \int d^3\mathbf{r}\, |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3}\, f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \underline{\underline{\lim_{\lambda \to 0} \int \frac{d^3\mathbf{q}}{(2\pi)^3}\, f(\mathbf{q}) \int d^3\mathbf{r}\, e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2}} \quad ✓$$

- Tasks at hand:
  - Step 1: Calculate this last **r**-integral, independently of $f(\mathbf{q})$, i.e. of $D(\mathbf{r})$,
  - Step 2: *Then* simplify the result of the $\lambda \to 0$ limit of the **q**-integral.

Fourier transform method

- Assume that source is easily expressed as Fourier transform

$$D(\mathbf{r}) := \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int d^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

E.g. Gaussian: $f(\mathbf{q}) = \exp\left(-\frac{1}{2}\mathbf{q}\mathbf{R}^2\mathbf{q}\right)$; if spherically symmetric: $f(\mathbf{q}) \equiv f_s(q) = e^{-q^2R^2/2}$,
Lévy source: $f(\mathbf{q}) = \exp\left(-\frac{1}{2}|\mathbf{q}\mathbf{R}^2\mathbf{q}|^{\alpha/2}\right)$; spherically symmetric: $f(\mathbf{q}) \equiv f_s(q) = e^{-(qR)^\alpha/2}$,

- Mathematical idea: „interchange integrals". Not trivial since

$$C_2(\mathbf{k}) = \int d^3\mathbf{r}\, |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \neq \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) \int d^3\mathbf{r}\, e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \quad ⚡⚡⚡$$

Instead, by means of regularization ($\lambda \in \mathbb{R}^+$, $\lambda \to 0$):

$$C_2(\mathbf{k}) = \int d^3\mathbf{r}\, |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \lim_{\lambda\to 0} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) \int d^3\mathbf{r}\, e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \quad ✓$$

- Tasks at hand:
  - Step 1: Calculate this last **r**-integral, independently of $f(\mathbf{q})$, i.e. of $D(\mathbf{r})$,
  - Step 2: *Then* simplify the result of the $\lambda \to 0$ limit of the **q**-integral.
  - Result: *functional* of $f(\mathbf{q})$; not simple integral transform.

Spherically symmetric, only Coulomb, symmetrized (already done)

- If $f(\mathbf{q}) \equiv f_s(q)$, solid angle integrals come first, resulting in

$$C_2(Q) = \frac{|\mathcal{N}|^2}{2\pi^2} \lim_{\lambda\to 0} \int_0^\infty \mathrm{d}q\, q^2 f_s(q) \Big[\mathcal{D}_{1\lambda s}(q) + \mathcal{D}_{2\lambda s}(q)\Big], \qquad \text{where}$$

$$\mathcal{D}_{1\lambda s}(q) = \int d^3\mathbf{r}\, \frac{\sin(qr)}{qr} e^{-\lambda r} M\big(1+i\eta, 1, -i(kr+\mathbf{kr})\big) M\big(1-i\eta, 1, i(kr+\mathbf{kr})\big),$$

$$\mathcal{D}_{2\lambda s}(q) = \int \mathrm{d}^3\mathbf{r}\, \frac{\sin(qr)}{qr} e^{-\lambda r} M\big(1+i\eta, 1, -i(kr-\mathbf{kr})\big) M\big(1-i\eta, 1, i(kr+\mathbf{kr})\big).$$

- Using method of A. Nordsieck, *Phys. Rev.* 93, 785 (1954), with $\mathcal{F}_+(x) \equiv {}_2F_1(i\eta, 1+i\eta, 1, x)$

$$\mathcal{D}_{1\lambda s}(q) = \tfrac{4\pi}{q} \mathrm{Im}\Big[\tfrac{1}{(\lambda-iq)^2}\big(1+\tfrac{2k}{q+i\lambda}\big)^{2i\eta} \mathcal{F}_+\big(\tfrac{4k^2}{(q+i\lambda)^2}\big)\Big],$$

$$\mathcal{D}_{2\lambda s}(q) = \tfrac{4\pi}{q} \mathrm{Im}\Big[\tfrac{(\lambda-iq-2ik)^{i\eta}(\lambda-iq+2ik)^{-i\eta}}{(\lambda-iq)^2+4k^2}\Big];$$

$\Rightarrow$ **Step 1 completed**.

- **Step 2:** for $\lambda\to 0$, integral transforms with $\mathcal{D}_{1\lambda s, 2\lambda s}$ become $\mathcal{A}_{1s,2s}$ functionals of f_s:

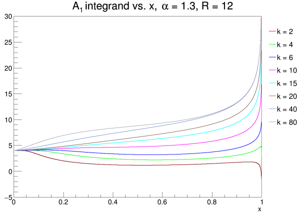
$$C_2(Q) = |\mathcal{N}|^2 \left(1 + f_s(2k) + \frac{\eta}{\pi}\left[\mathcal{A}_{1s} + \mathcal{A}_{2s}\right]\right), \quad \text{where}$$

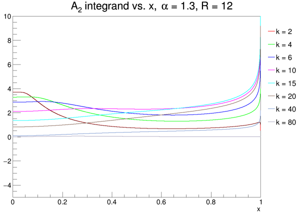
$$\mathcal{A}_{1s} = -\frac{2}{\eta} \int_0^\infty \mathrm{d}q\, \frac{f_s(q)-f_s(0)}{q} \mathrm{Im}\left[\left(1+\frac{2k}{q}\right)^{2i\eta} \mathcal{F}_+\left(\frac{4k^2}{q^2} - i0\right)\right],$$

$$\mathcal{A}_{2s} = -\frac{2}{\eta} \int_0^\infty \mathrm{d}q\, \frac{f_s(q)-f_s(2k)}{q-2k} \frac{q}{q+2k} \mathrm{Im}\frac{(q+2k)^{i\eta}}{(q-2k+i0)^{i\eta}}.$$

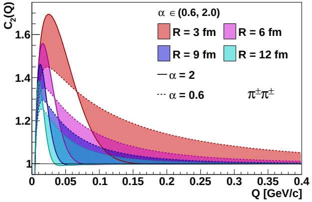
Spherically symmetric, only Coulomb, symmetrized (already done)

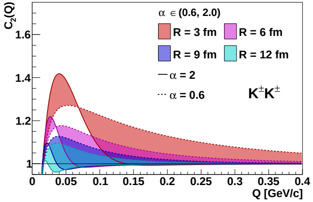
- Numerical implementation: one final integral needed for $\mathcal{A}_{1s}$ and for $\mathcal{A}_{2s}$; transformed from $q \in \mathbb{R}_0^+$ to $x \in [0, 1]$; used Gauss-Krohnrod algorithm from `C++ boost` library, since integrands are „well-behaved":





- Example result (for Lévy sources, various α and R values, $\pi^{\pm}\pi^{\pm}$ & $K^{\pm}K^{\pm}$):





Generalization: non-spherically symmetric, only Coulomb, symmetrized

- Notation: $\mathbf{q}_\pm \equiv \mathbf{q} \pm 2\mathbf{k}$. (NB: free $C^{(0)}(\mathbf{k}) = 1 + f(2\mathbf{k})$; thus $\mathbf{q}=2\mathbf{k}$ has important role.)
- Three-dimensional integrals remain at all stages:

$$C_2(\mathbf{k}) = \frac{|\mathcal{N}|^2}{(2\pi)^3} \lim_{\lambda\to 0} \int d^3\mathbf{q}\, f(\mathbf{q}) \Big[\mathcal{D}_{1\lambda}(\mathbf{q}) + \mathcal{D}_{2\lambda}(\mathbf{q})\Big], \qquad \text{where}$$

$$\mathcal{D}_{1\lambda s}(\mathbf{q}) = \int d^3\mathbf{r}\, e^{i\mathbf{qr}} e^{-\lambda r} M\big(1+i\eta, 1, -i(kr+\mathbf{kr})\big) M\big(1-i\eta, 1, i(kr+\mathbf{kr})\big),$$

$$\mathcal{D}_{2\lambda s}(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, e^{i\mathbf{qr}} e^{-\lambda r} M\big(1+i\eta, 1, -i(kr-\mathbf{kr})\big) M\big(1-i\eta, 1, i(kr+\mathbf{kr})\big).$$

- Again with same method, result is

$$\mathcal{D}_{1\lambda}(\mathbf{q}) = -\frac{d}{d\lambda} \frac{4\pi}{\lambda^2+q^2} \mathcal{U}_{1-} \mathcal{F}_+(x_1), \qquad \mathcal{D}_{2\lambda}(\mathbf{q}) = -\frac{d}{d\lambda} \frac{4\pi}{\lambda^2+q_+^2} \mathcal{U}_{2-} \mathcal{F}_+(x_2),$$

with

$$\mathcal{U}_{1\pm} := \frac{(\lambda^2+q^2)^{\pm 2i\eta}}{(\lambda^2+\mathbf{q}\mathbf{q}_\mp \pm 2ik\lambda)^{\pm 2i\eta}}, \qquad \mathcal{U}_{2\pm} := \frac{((\lambda^2+q^2)(\lambda^2+q_+^2))^{\pm i\eta}}{(\lambda^2+\mathbf{q}\mathbf{q}_+ \pm 2ik\lambda)^{\pm 2i\eta}},$$

and

$$x_1 := \frac{4(\mathbf{qk}-ik\lambda)^2}{(\lambda^2+q^2)^2} \in \mathbb{C}\setminus\mathbb{R}_0^+, \qquad \lim_{\lambda\to 0} x_1 := X_1 = 1 - \frac{\mathbf{q}\mathbf{q}_+}{q^2}\frac{\mathbf{q}\mathbf{q}_-}{q^2},$$

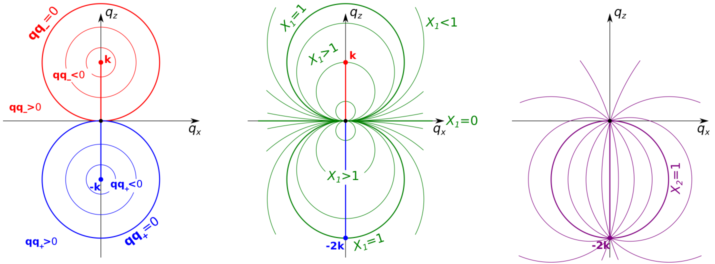
with

$$x_2 := \frac{4[k^2q^2-(\mathbf{qk})^2]}{(\lambda^2+q^2)(\lambda^2+q_+^2)} \in [0,1[\,, \qquad \lim_{\lambda\to 0} x_2 := X_2 = 1 - \Big(\frac{\mathbf{q}\mathbf{q}_+}{qq_+}\Big)^2.$$

- Step 1 is thus completed. For **step 2** (i.e. for the simplification of the $\lambda\to 0$ limit of the $\mathbf{q}$-integral), fix coordinate system to $\mathbf{k}$ vector, the final variable of $C(\mathbf{k})$ & introduce new variables that fit to level surfaces of X_1 and X_2.

Generalization: non-spherically symmetric, only Coulomb, symmetrized

- On the following plot of level surfaces, z axis $\parallel$ $\mathbf{k}$ (& φ azimuth in x–y plane):



$\mathbf{qq}_-$ & $\mathbf{qq}_+$ $\qquad$ X_1 (for $\mathcal{D}_{1\lambda}$) $\qquad$ X_2 (for $\mathcal{D}_{2\lambda}$)

- Parametrize $\mathbf{q}$ in this frame as

$$\underbrace{\mathbf{q}=\frac{2ka\beta}{a^2+\beta^2}\begin{pmatrix}a\cos\varphi\\ a\sin\varphi\\ \beta\end{pmatrix} \quad\Rightarrow\quad X_1=\frac{1-a\cdot i0}{a^2}}_{\text{for } \mathcal{D}_{1\lambda};\ a\in\mathbb{R},\ \beta\in\mathbb{R}^+}$$

$$\underbrace{\mathbf{q}=\frac{k(1-y)}{1+by^2}\begin{pmatrix}\sqrt{b}(1+y)\cos\varphi\\ \sqrt{b}(1+y)\sin\varphi\\ by-1\end{pmatrix} \quad\Rightarrow\quad X_2=\frac{4b}{(1+b)^2}}_{\text{for } \mathcal{D}_{2\lambda};\ b\in\mathbb{R}^+,\ y\in[-1,1]}$$

Non-spherically symmetric case: result

- $D(\mathbf{r})=D(-\mathbf{r}) \quad\Leftrightarrow\quad f(\mathbf{q})=f(-\mathbf{q})$. Result of Step 2 (simplifying the $\lambda\to 0$ limit)

$$C_2(Q)=|\mathcal{N}|^2\left(1+f_s(2\mathbf{k})+\frac{\eta}{\pi}\left[\mathcal{A}_1+\mathcal{A}_{2\mathcal{P}}+\mathcal{A}_{2\delta}\right]\right),\quad\text{where}$$

$$\pi\mathcal{A}_1=\int_{-\infty}^{\infty}\mathrm{d}a\int_0^{\infty}\mathrm{d}\beta\int_{-\pi}^{\pi}\mathrm{d}\varphi\,\frac{aY(-a)}{a+1}\mathrm{Re}\left\{\frac{|a|^{2i\eta}}{|a+1|^{2i\eta}}\left[2g_+^*\frac{2i\,f_1(-1,\beta,\varphi)}{\beta(\beta^2+1)(a-i)^2}-\right.\right.$$
$$\left.\left.-\mathcal{F}_-\left(\tfrac{1+a\cdot i0}{a^2}\right)\frac{f_1(a,\beta,\varphi)}{\beta(a^2+\beta^2)}\right]\right\},$$

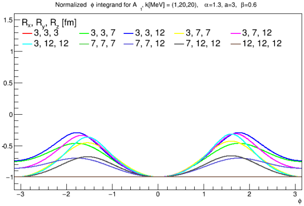
$$\mathcal{A}_{2\mathcal{P}}=-\frac{1}{\pi}\int_0^{\infty}\mathrm{d}b\int_{-1}^{1}\mathrm{d}y\int_{-\pi}^{\pi}\mathrm{d}\varphi\,\frac{1-y}{1+y}\,\frac{Y(b)}{b-1}\mathrm{Re}\left\{\frac{(b+1)^{2i\eta}}{|b-1|^{2i\eta}}\left[\mathcal{F}_-\left(\tfrac{4b}{(1+b)^2}\right)\frac{f_2(b,y,\varphi)}{1+by^2}\right.\right.$$
$$\left.\left.-2g_+^*\frac{2f_2(1,y,\varphi)}{(1+b)(1+y^2)}\right]\right\},,$$

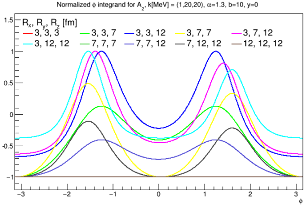
$$\mathcal{A}_{2\delta}=\Im(g_+)\frac{e^{2\pi\eta}-1}{2\pi\eta}\int_{\Omega}\mathrm{d}\mathbf{n}\,\frac{f(\mathbf{k}+k\mathbf{n})-f(2\mathbf{k})}{1-\hat{\mathbf{k}}\mathbf{n}}.$$

- Here besides $\mathcal{F}_+(x)$ we have another hypergeometric function, $\mathcal{F}_-(x):=(1+i\eta)\cdot{}_2F_1(-i\eta,1-i\eta,2,x)$.
  Also, $g_+:=\frac{\Gamma(-2i\eta)}{\Gamma(-i\eta)\Gamma(1-i\eta)}$.
- The point is: no $\lambda\to 0$ limit anymore!
- Result rightfully *seems* complicated; however, integrands are again ,,nice'' functions!

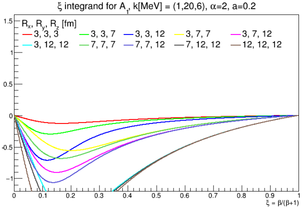
Present status: integrating...

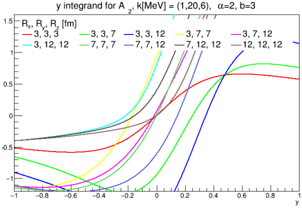
- No jumps, no wild oscillations... Again: Gauss-Krohnrod algorithm
- Example φ-integrands:





- Example β– and y-integrands (shrunk to a compact interval) after φ-integration:





Example calculations: illustrations

. . . This is where we are at the moment. . .

Summary and outlook

- Where are we?
- Efficient new method for Coulomb interacting HBT correlation function calculation
  - Calculations directly in momentum (Fourier) space
  - Cross-checked with previous direct calculations
  - Numerical implementation done, ready for use in data analysis
- Spherically symmetric sources:
  - Calculation done, programming done, checks done
  - Published: `arXiv:2308.10745`, Eur. Phys. J. C **83**, 1015 (2023) (talked about at WPCF 2023)
  - Prospective generalization (in fact, simplification) for non-identical particle correlations: only $\mathcal{D}_{1\lambda_s}$ (ie. $\mathcal{A}_1$) term needed
- Non-spherically symmetric sources:
  - Calculation done, programming is being done (as we speak; i.e. after)
  - Result (shortly summarized): efficient means of calculating valid 3D Coulomb-interacting correlation functions

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New exact analytic formulas for QM Coulomb problem! ☺

Thank you for your attention!