Lévy α-stable generalization of the ReBB model

based on Universe **2023**, 9(8), 361 , Universe **2024**, 10(3), 127

& other resent results

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Outline

- **preliminaries, motivation for an improvement of the model**
- **the p=(q,d) Bialas-Bzdak model and its extended version**
- **motivation for the Lévy α-stable generalization**
- **Lévy α-stable generalization of the Bialas-Bzdak model**
- **an approximate simple Lévy α-stable model and fits to data**
- **relation between the parameters of the simple Lévy α-stable model and the full generalized model**

Preliminaries: ReBB model analysis of pp and p$\bar{p}$ data

- **the Real extended Bialas-Bzdak (ReBB) model describes elastic pp and p$\bar{p}$ $d\sigma/dt$ data in a statistically acceptable way (CL≥0.1%) in the kinematic region:**

$$546\ \text{GeV} \le \sqrt{s} \le 8\ \text{TeV}$$

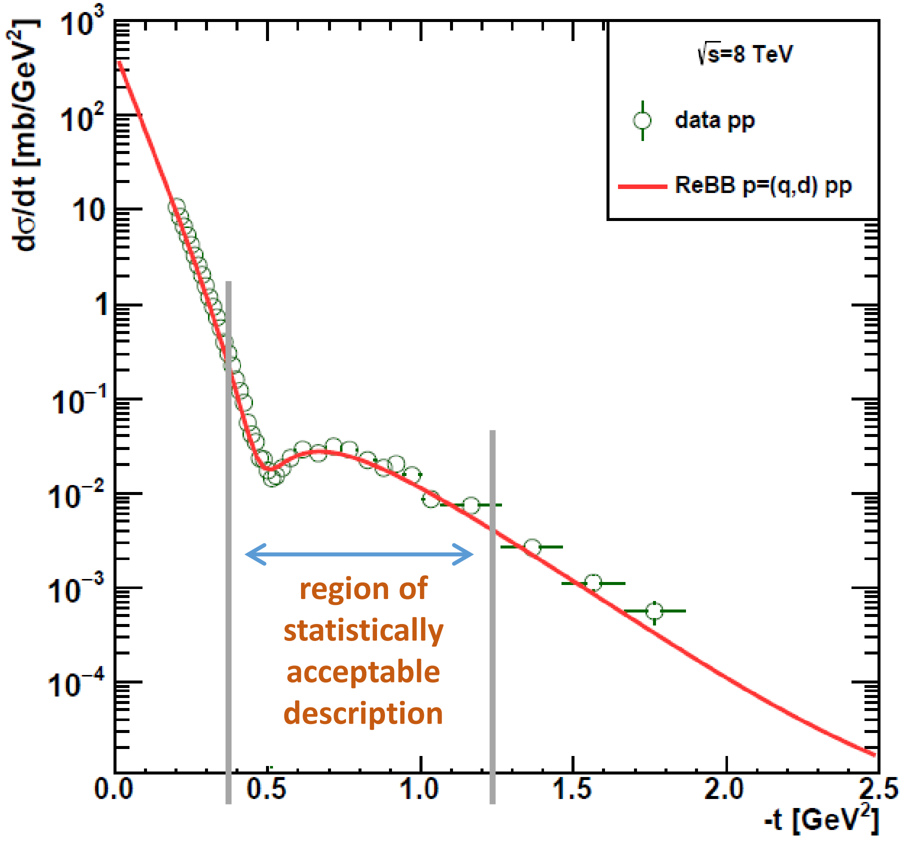
$$0.38\ \text{GeV}^2 \le -t \le 1.2\ \text{GeV}^2$$

- significant model dependent odderon signal is observed

- main goal: **to improve the ReBB model to have a statistically acceptable (CL≥0.1%) description to elastic pp and p$\bar{p}$ data in a wider kinematic range**

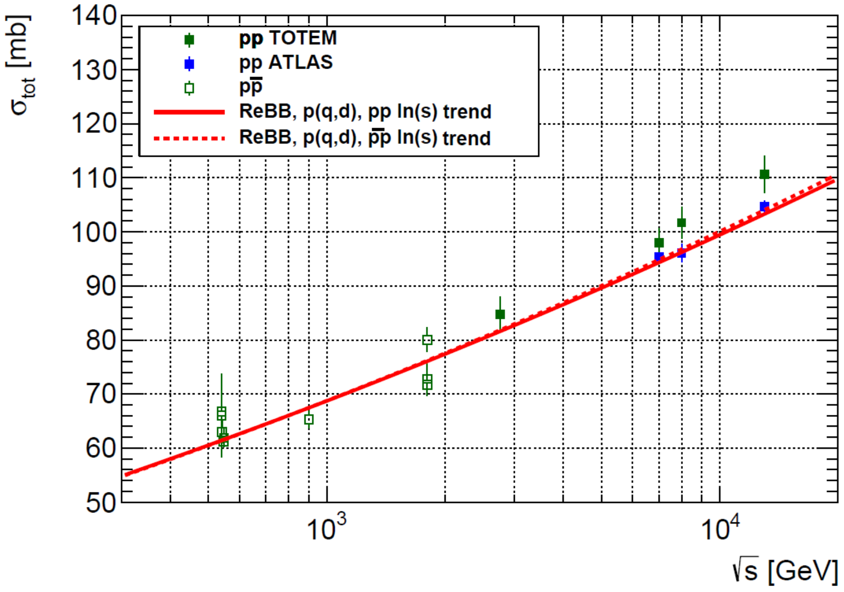
T. Csörgő, I. Szanyi, *Eur. Phys. J. C* ***81****, 611 (2021)*

I. Szanyi, T. Csörgő, *Eur. Phys. J. C* ***82****, 827 (2022)*

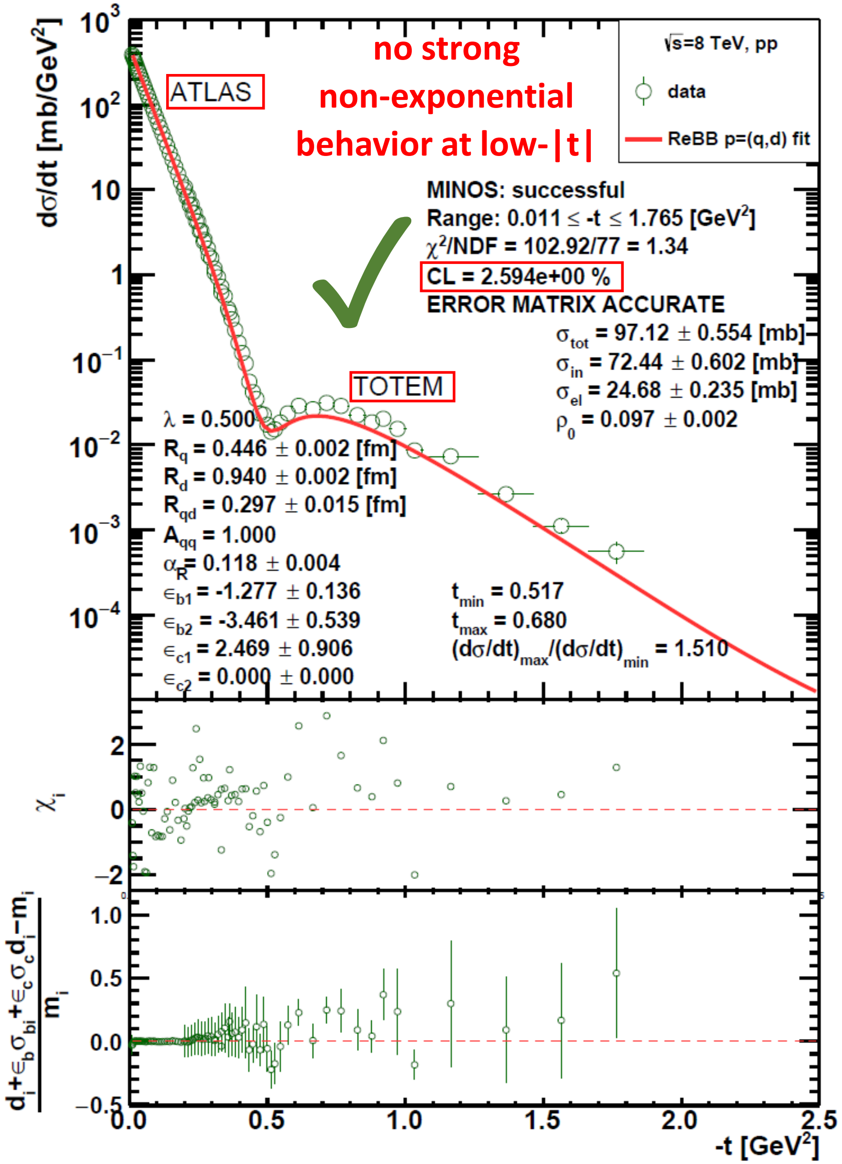


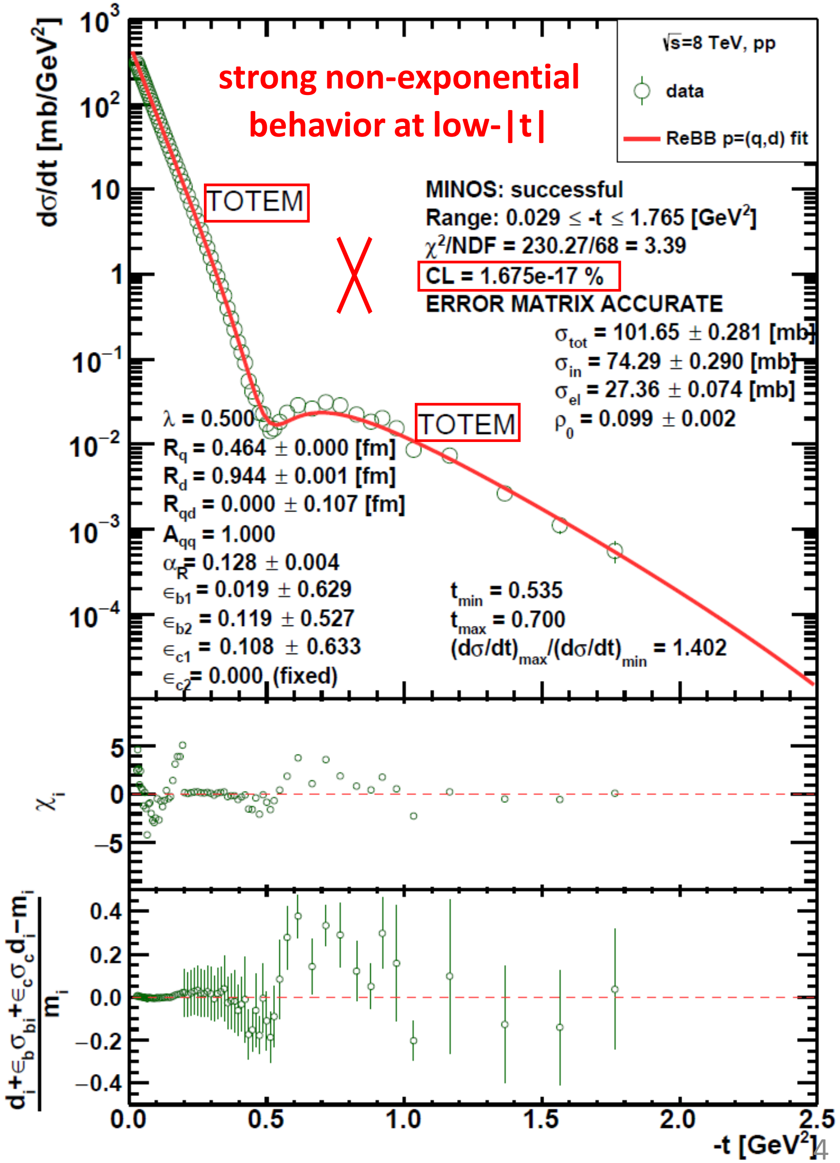
ReBB model description to the 8 TeV pp data

Motivation: study of low-|t| TOTEM-ATLAS discrepancy



the ReBB model fails to describe the TOTEM low-|t| and higher-|t| data simultaneously with CL≥ 0.1%





the ReBB model calibrated to $d\sigma_{el}/dt$ data in the higher-|t| domain perfectly describes the pp ATLAS σ_{tot} data being systematically below the TOTEM data

further studies needed within a model that describes the elastic data both at low-$|t|$ and high-$|t|$ with CL ≥ 0.1%

Unitarity and the elastic scattering amplitude

- the unitarity of the S-matrix expresses the conservation of probability

$$SS^{\dagger} = I$$

- **the unitaritiy relation in the impact parameter (b) representation at high energies is**

$$2\,\mathcal{Im}\,t_{el}(s,b) = |t_{el}(s,b)|^2 + \tilde{\sigma}_{in}(s,b)$$ ($\sqrt{s}$ is the CM energy)

$$0 \leq \tilde{\sigma}_{in}(s,b) \leq 1$$

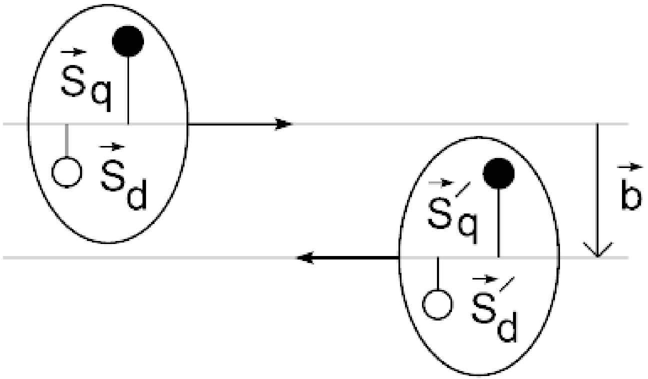
- **the elastic scattering amplitude $t_{el}(s,b)$ can be written as a solution of the unitarity equation in terms of the inelastic cross section $\tilde{\sigma}_{in}(s,b)$**

- **at a given energy $\tilde{\sigma}_{in}(s,b)$ is the probability of inelastic scattering as fuction of b and it can be calculated by using probability calculus and R. J. Glauber's multiple diffractive scattering theory**

The Bialas-Bzdak (BB) p=(q,d) model

A. Bialas, A. Bzdak, *Acta Phys.Polon. B 38, 159-168 (2007)*

- in the **Bialas-Bzdak (BB) p=(q,d) model** the proton is a **bound state of a constituent quark and constituent a diquark**
- the inelastic scattering probability of two protons at a fixed impact parameter vector ($\vec{b}$) and at fixed constituent transverse position vectors ($\vec{s}_q, \vec{s}_d, \vec{s}_q', \vec{s}_d'$) is given by a **Glauber expansion**:



Proton-proton collision in the quark-diquark model

$$\sigma(\vec{s}_q, \vec{s}_d; \vec{s}_q', \vec{s}_d'; \vec{b}) = 1 - \prod_{a \in \{q,d\}} \prod_{b \in \{q,d\}} \left[1 - \sigma_{ab}(\vec{b} + \vec{s}_b' - \vec{s}_a)\right]$$

- $\sigma_{ab}(\vec{x}) \equiv \frac{d^2\sigma_{ab}(\vec{x})}{dx^2}$ is the inelastic differential cross section (inelastic scattering probability) for the collision of two constituents at a fixed relative transverse position $\vec{x}$ of the constituents
- **the Glauber expansion sums the probabilities of all possible single and multiple binary inelastic collisions of the constituents (back scattering is prohibited)**
- the collision of two protons is inelastic if at least one constituent-constituent collision is inelastic

The Bialas-Bzdak (BB) p=(q,d) model

A. Bialas, A. Bzdak, *Acta Phys.Polon. B 38, 159-168 (2007)*

- the **probability of inelastic scattering** of protons at a fixed impact parameter vector ($\vec{b}$) is given by averaging over the constituent positions inside the protons:

$$\tilde{\sigma}_{in}(\vec{b}) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} d^2 s_q d^2 s'_q d^2 s_d d^2 s'_d \, D(\vec{s}_q, \vec{s}_d)\, D(\vec{s}'_q, \vec{s}'_d)\, \sigma(\vec{s}_q, \vec{s}_d; \vec{s}'_q, \vec{s}'_d; \vec{b})$$

- $D(\vec{s}_q, \vec{s}_d)$ is the (transverse) distribution of constituents inside a proton

- the scattering amplitude in the original BB model is considered to be completely imaginary by neglecting its relatively small real part

$$\tilde{t}_{el}(s,b) = i\left(1 - \sqrt{1 - \tilde{\sigma}_{in}(s,b)}\right)$$

$$b = |\vec{b}|$$

$$T(s,t) = 2\pi \int_0^{\infty} \tilde{t}_{el}(s,b)\, J_0(qb)\, b\, db$$

$$q = \sqrt{-t}$$

- the s-dependence of the amplitude happens through the s-dependencies of the model parameters

The Bialas-Bzdak (BB) p=(q,d) model

A. Bialas, A. Bzdak, *Acta Phys.Polon. B 38, 159-168 (2007)*

- **in the original BB model the distribution of constituents inside a proton is given in terms of products of Gaussians**

$$D(\vec{s}_q, \vec{s}_d) = \frac{1+\lambda^2}{R_{qd}^2 \pi} e^{-\frac{\vec{s}_q^{\,2}}{R_{qd}^2}} e^{-\frac{\vec{s}_d^{\,2}}{R_{qd}^2}} \delta^2(\vec{s}_d + \lambda \vec{s}_q)$$

$$\lambda = \frac{m_q}{m_d}$$

$$\vec{s}_d = -\lambda \vec{s}_q$$

$$\vec{s}_d' = -\lambda \vec{s}_q'$$

- **the constituent-constituent inelastic differential cross sections have also Gaussian shapes**

$$\sigma_{ab}(\vec{x}) = A_{ab} e^{-\frac{\vec{x}^2}{R_a^2 + R_b^2}}$$

$$a, b \in \{q, d\}$$

- **the constituent-constituent inelastic integrated cross sections are**

$$\sigma_{ab}^{int} = \iint \sigma_{ab}(\vec{x})\, d^2x = \pi A_{ab}(R_a^2 + R_b^2)$$

- **assuming that the diquark contains twice as many partons than the quark and the colliding constituents do not shadow each other,** $\sigma_{qq}^{int} : \sigma_{qd}^{int} : \sigma_{dd}^{int} = 1 : 2 : 4$ **and the number of free parameters reduces to five:** A_{qq}, λ, R_q, R_d, **and** R_{qd}

Real extended Bialas-Bzdak (ReBB) model

- **in the original BB model the differential cross section is zero around the position of the diffractive minimum**
- **as a solution, the elastic scattering amplitude was extended in a unitary manner leading to the Real extended Bialas-Bzdak (ReBB) model**

F. Nemes, T. Csörgő, M. Csanád, Int. J. Mod. Phys. A Vol. 30, 1550076 (2015)

$$\tilde{t}_{el}(s,b) = i\left[1 - e^{-\Omega(s,b)}\right]$$

$$\mathrm{Re}\Omega(s,b) = -1/2\ln[1 - \tilde{\sigma}_{in}(s,b)]$$

new free parameter

$$Im\Omega(s,b) = 0 \quad \rightarrow \quad Im\Omega(s,\vec{b}) = -\alpha_R \tilde{\sigma}_{in}(s,\vec{b})$$

$$\tilde{t}_{el}(s,b) = i\left(1 - \sqrt{1 - \tilde{\sigma}_{in}(s,b)}\right) \quad \rightarrow \quad \tilde{t}_{el}(s,b) = i\left(1 - e^{i\,\alpha_R\,\tilde{\sigma}_{in}(s,b)}\sqrt{1 - \tilde{\sigma}_{in}(s,b)}\right)$$

- **the ReBB model gives a statistically acceptable description (CL≥0.1%) to elastic pp and p$\bar{\mathrm{p}}$ scattering in the kinematic region:**

$$0.546\ \mathrm{TeV} \le \sqrt{s} \le 8\ \mathrm{TeV}\ \&\ 0.38\ \mathrm{GeV}^2 \le -t \le 1.2\ \mathrm{GeV}^2$$

T. Csörgő, I. Szanyi, *Eur. Phys. J. C* ***81****, 611 (2021)*

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Motivation for Lévy α-stable generalization

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- **the main goal is to have a statistically acceptable description in a wider kinematic range**

- **the TOTEM measurement at LHC at $\sqrt{s} = 8$ TeV excluded a purely exponential pp differential cross-section in the range of four-momentum transfer squared $0.027\ \mathrm{GeV}^2 \leq -t \leq 0.2\ \mathrm{GeV}^2$ with a significance greater than 7σ** TOTEM Collab., *Nucl. Phys. B, 899, 527 (2015)*

- **a simple model with Gaussian impact parameter amplitude yields a purely exponential t-distribution while a simple model with Levy α-stable impact parameter amplitude and $\alpha_L < 2$ yields a non-exponential t-distribution**

$$\widetilde{T}_{el}(s,b) = \frac{i+\rho_0(s)}{2}\sigma_{tot}(s)\frac{1}{2\pi B_0(s)}e^{-\frac{1}{2}\frac{b^2}{B_0(s)}} \rightarrow \widetilde{T}_{el}(s,b) = \frac{i+\rho_0(s)}{2}\sigma_{tot}(s)\frac{1}{4\pi^2}\int d^2q e^{-i\vec{q}\cdot\vec{b}} e^{-\frac{1}{2}\left|q^2 B_L(s)\right|^{\alpha_L(s)/2}}$$

$$T_{el}(s,t) = \int d^2b e^{i\vec{q}\cdot\vec{b}}\,\widetilde{T}_{el}(s,b) \qquad \frac{d\sigma_{el}}{dt}(s,t) = \frac{1}{4\pi}\left|T_{el}(s,t)\right|^2 \qquad a(s) = \frac{1+\rho_0^2(s)}{16\pi}\sigma_{tot}^2(s)$$

$$\frac{d\sigma_{el}}{dt}(s,-t) = a(s)e^{-tB_0(s)} \rightarrow \frac{d\sigma_{el}}{dt}(s,-t) = a(s)e^{-\left|tB_L(s)\right|^{\alpha_L(s)/2}}$$

Gaussian vs Lévy α-stable distribution

- **the bivariate Gaussian distribution centered at $\vec{0}$ is**

$$G(\vec{x}|R_G) = \frac{1}{2\pi R_G^2} e^{-\frac{\vec{x}^2}{2R_G^2}}$$

- **the bivariate symmetric Levy α-stable distribution centered at $\vec{0}$ is**

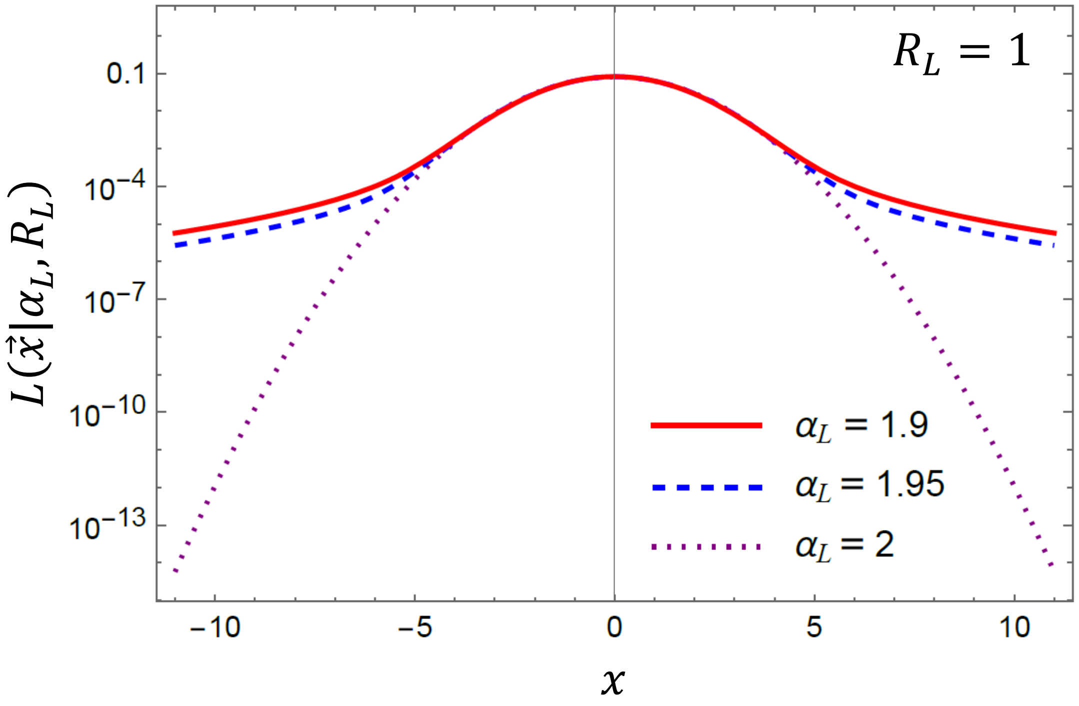
$$L(\vec{x}|\alpha_L, R_L) = \frac{1}{(2\pi)^2} \int d^2\, q\, e^{-i\vec{q}\cdot\vec{x}} e^{-|\vec{q}^2 R_L^2|^{\alpha_L/2}}$$

$$0 < \alpha_L \le 2$$

- **for $\alpha_L = 2$ the the Lévy α-stable distribution is the Gaussian distribution**

$$L(\vec{x}|\alpha_L = 2, R_L = R_G/\sqrt{2}) \equiv G(\vec{x}|R_G)$$

The emergence of Gaussian distribution can be explained by the central limit theorem. The emergence of Lévy α-stable distributions can be expected based on generalized central limit theorems.



The bivariate symmetric Levy α-stable distribution for $R_L = 1$ as a function of $x = |x|$

for large x and $\alpha_L < 2$,
$L(x|\alpha_L, R_L) \sim |x|^{-(1+\alpha_L)}$

Lévy α-stable distributions with $\alpha_L < 2$ have tails behaving asymptotically as a power law (infinite variance)

Lévy α-stable distributions in HEP

- **the application of Lévy α-stable distributions is not new in the field of high-energy physics**

- **the Cauchy-Lorentz or Breit-Wigner distribution ($\alpha_L = 1$ case) is used to model unstable particles**

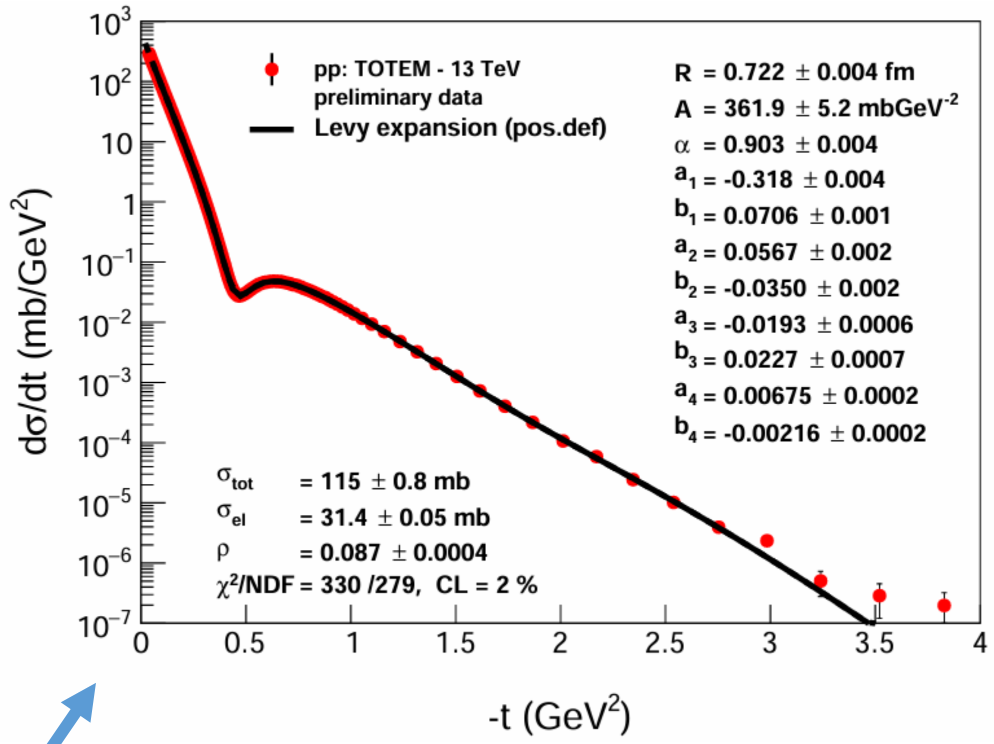
- **the application of stable distributions (with $0 < \alpha_L \leq 2$) is widespread in heavy ion physics**

T. Csorgo, S. Hegyi, W. A. Zajc, *Eur. Phys. J. C 36 (2004) 67-78*

M. Csanád, D. Kincses, *Universe 10 (2024) 2, 54*

- **the Lévy expansion technique was successfully applied to describe elastic pp scattering**

T. Csörgő, R. Pasechnik, A. Ster, *Eur. Phys. J. C 79, 62 (2019)*



Description to pp elastic differential cross section data at 13 TeV using the Lévy expansion technique

Inelastic constituent-constituent collisons

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- **the inelastic differential cross section for the collision of two constituents can be written in terms of a convolution of their parton distributions**

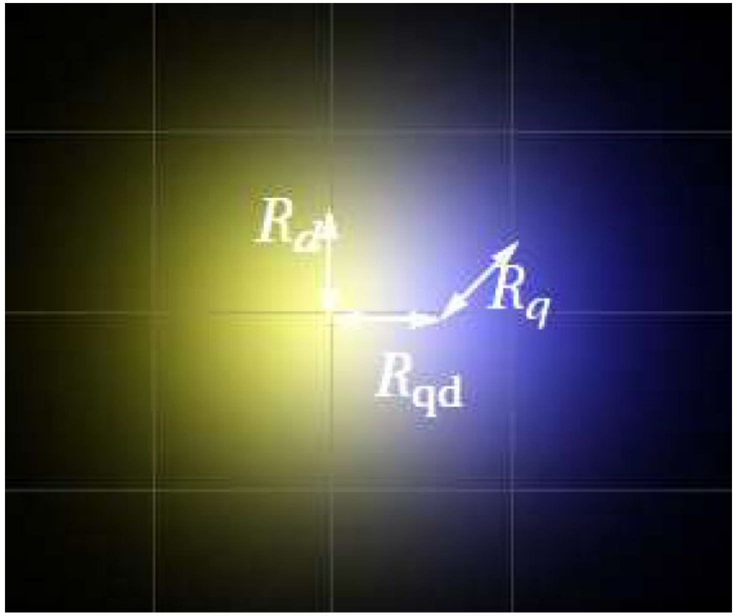
- **in the original BB model the parton distributions of the constituents are Gaussian distributions**

$$\sigma_{ab}(\vec{x}) = A_{ab}\pi S_{ab}^2 \int d^2 r_a G(\vec{r}_a | R_a/\sqrt{2}) G(\vec{x} - \vec{r}_a | R_b/\sqrt{2}) \equiv A_{ab}\pi S_{ab}^2 G(\vec{x} | S_{ab}/\sqrt{2})$$

$$\vec{x} = \vec{b} + \vec{s}_b' - \vec{s}_a$$

$$S_{ab}^2 = R_a^2 + R_b^2$$

$$a, b \in \{q, d\}$$



The picture of the proton in the quark-diquark model

The quark-diquark distribution

- **in the original BB model $D(\vec{s}_q, \vec{s}_d)$, the distribution of constituents inside a proton is given in terms of products of Gaussians**

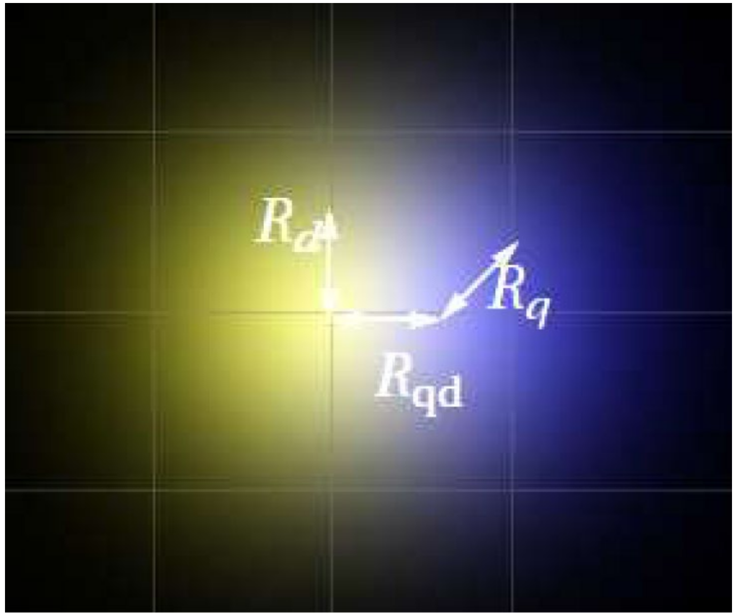
- **considering the relative distance** between the quark and diquark ($\vec{s}_q - \vec{s}_d$) one can write $D(\vec{s}_q, \vec{s}_d)$ in terms of a single Gaussian distribution:

$$D(\vec{s}_q, \vec{s}_d) = (1+\lambda)^2 G(\vec{s}_q - \vec{s}_d | R_{qd}/\sqrt{2})\delta^2(\vec{s}_q + \lambda\vec{s}_d)$$

$$\lambda = m_q/m_d$$

- the Dirac δ fixes the center of the mass of the proton making the calculations easier

- $D(\vec{s}_q, \vec{s}_d)$ is normalized as $\int d^2 s_q d^2 s_d D(\vec{s}_q, \vec{s}_d) = 1$



The picture of the proton in the quark-diquark model

Lévy α-stable generalized Bialas-Bzdak (LBB) model

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- **the parton distributions of the constituent quark and diquark are now Levy α-stable distributions and the inelastic differential cross section for the collision of two constituents is:**

$$\sigma_{ab}(\vec{x}) = A_{ab}\pi S_{ab}^2 \int d^2 r_a L(\vec{r}_a|\alpha_L, R_a/2) L(\vec{x} - \vec{r}_a|\alpha_L, R_b/2) \equiv A_{ab}\pi S_{ab}^2 L(\vec{x}|\alpha_L, S_{ab}/2)$$

$$S_{ab}^{\alpha_L} = R_a^{\alpha_L} + R_b^{\alpha_L}$$

- **the distribution of the constituents inside the proton is now given in terms of a Levy α-stabil distribution**:

$$D(\vec{s}_q, \vec{s}_d) = (1+\lambda)^2 L(\vec{s}_q - \vec{s}_d|\alpha_L, R_{qd}/2)\delta^2(\vec{s}_q + \lambda\vec{s}_d)$$

$$\int d^2 s_q d^2 s_d D(\vec{s}_q, \vec{s}_d) = 1$$

α_L is a new free parameter of the model and if $\alpha_L = 2$ the BB model with Gaussian distributions is recovered

Difficulties with LBB model

- $\tilde{\sigma}_{in}(\vec{b})$ **can be written as sum of 11 different terms that are integrals** of products of Lévy α-stable distributions

$$\tilde{\sigma}_{in}(\vec{b}) = \tilde{\sigma}_{in}^{qq}(\vec{b}) + 2\tilde{\sigma}_{in}^{qd}(\vec{b}) + \tilde{\sigma}_{in}^{dd}(\vec{b}) - [2\tilde{\sigma}_{in}^{qq,qd}(\vec{b}) + \tilde{\sigma}_{in}^{qd,dq}(\vec{b}) + \tilde{\sigma}_{in}^{qq,dd}(\vec{b}) + 2\tilde{\sigma}_{in}^{qd,dd}(\vec{b})] \\ +[\tilde{\sigma}_{in}^{qq,qd,dq}(\vec{b}) + 2\tilde{\sigma}_{in}^{qq,qd,dd}(\vec{b}) + \tilde{\sigma}_{in}^{dd,qd,dq}(\vec{b})] - \tilde{\sigma}_{in}^{qq,qd,dq,dd}(\vec{b})$$

- difficulties with the calculation of integrals of products of Lévy α-stable distributions
- the calculation is easy only if the integral can be written in a convolution form as in case of the leading order terms in $\tilde{\sigma}_{in}(s,\vec{b})$

Leading order terms in $\tilde{\sigma}_{in}$ in the LBB model

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$$
\begin{aligned}
\tilde{\sigma}_{in}^{qq}(\vec{b}) &= \pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} \times \\
&\times \int d^2 s_q d^2 s'_q L(\vec{s}_q|\alpha_L, R_{qd*}/2) L(\vec{s}_q{}'|R_{qd*}/2) L\left(\vec{b} + \vec{s}_q{}' - \vec{s}_q|\left(2R_q^{\alpha_L}\right)^{1/\alpha_L}/2\right) \\
&= \pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} L\left(\vec{b}|\alpha_L, \left(2R_{qd*}^{\alpha_L} + 2R_q^{\alpha_L}\right)^{1/\alpha_L}/2\right),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\sigma}_{in}^{qd}(\vec{b}) &= 2\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} \times \\
&\times \int d^2 s_q d^2 s'_q L(\vec{s}_q|R_{qd*}/2) L(\vec{s}_q{}'|R_{qd*}/2) L\left(\vec{b} - \lambda\vec{s}_q{}' - \vec{s}_q|\alpha_L, \left(R_q^{\alpha_L} + R_d^{\alpha_L}\right)^{1/\alpha_L}/2\right) \\
&= 2\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} L\left(\vec{b}|\alpha_L, \left((1+\lambda^{\alpha_L})R_{qd*}^{\alpha_L} + R_q^{\alpha_L} + R_d^{\alpha_L}\right)^{1/\alpha_L}/2\right),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\sigma}_{in}^{dd}(\vec{b}) &= 4\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} \times \\
&\times \int d^2 s_q d^2 s'_q L(\vec{s}_q|R_{qd*}/2) L(\vec{s}_q{}'|R_{qd*}/2) L\left(\vec{b} + \lambda(\vec{s}_q - \vec{s}_q{}')|\alpha_L, \left(2R_d^{\alpha_L}\right)^{1/\alpha_L}/2\right) \\
&= 4\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} L\left(\vec{b}|\alpha_L, \left(2\lambda^{\alpha_L}R_{qd*}^{\alpha_L} + 2R_d^{\alpha_L}\right)^{1/\alpha_L}/2\right).
\end{aligned}
$$

Difficulties with LBB model fits to the data

- since multivariate Lévy α-stable distributions can be given only in terms of special functions, it is hard to perform a numerical fitting procedure

- numerical calculations with the present form of the LBB model are time-consuming: a sequence of three integral calculations where the result of an integral is an integrand of the next integral → a relatively high computing capacity and improved analytic insight is needed to proceed with the full model

- **quick solution:** approximations that are valid at the low –t domain

- at low –t values, the original ReBB model had difficulties to describe the strongly non-exponential features of the experimental data on $d\sigma/dt$

- a simple model which is valid at the low –t domain easily illustrates the power of the Lévy α-stable generalization

Simple Lévy α-stable model for low-|t| pp $d\sigma/dt$

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- low-$|t|$ scattering corresponds to high-b scattering and at high b values $\tilde{\sigma}_{in}(s,b)$ is small
- leading order term in the Taylor expansion of the amplitude in $\tilde{\sigma}_{in}(s,b)$ dominates at low $-t$ values if α_R is small too

$$\tilde{t}_{el}(s,b) = i\left(1 - e^{i\,\alpha_R(s)\,\tilde{\sigma}_{in}(s,b)}\sqrt{1-\tilde{\sigma}_{in}(s,b)}\right) \rightarrow \tilde{t}_{el}(s,b) = \left(\alpha_R(s) + \frac{i}{2}\right)\tilde{\sigma}_{in}(s,b)$$

- motivated by the fact that the leading order terms in $\tilde{\sigma}_{in}(s,\vec{b})$ have Lévy α-stable shapes in the LBB model, $\tilde{\sigma}_{in}(s,\vec{b})$ is approximated with a single Lévy α-stable shape

$$\tilde{\sigma}_{in}(s,\vec{b}) = \tilde{c}(s)L\left(\vec{b}|\alpha_L(s), r(s)\right)$$

- **a simple Lévy α-stable model model for low-|t| pp $d\sigma/dt$ arises**

$$t_{el}(s,t) = \int d^2b\, e^{i\vec{q}\cdot\vec{b}}\tilde{t}_{el}(s,\vec{b}),\ |\vec{\Delta}| = \sqrt{-t} \rightarrow \frac{d\sigma}{dt}(s,t) = \frac{1}{4\pi}|t_{el}(s,t)|^2 = a(s)e^{-|tb(s)|^{\alpha_L(s)/2}}$$

- the model has three adjustable parameters, α_L, a, and b, to be determined at a given energy

Simple Lévy α-stable model and the data

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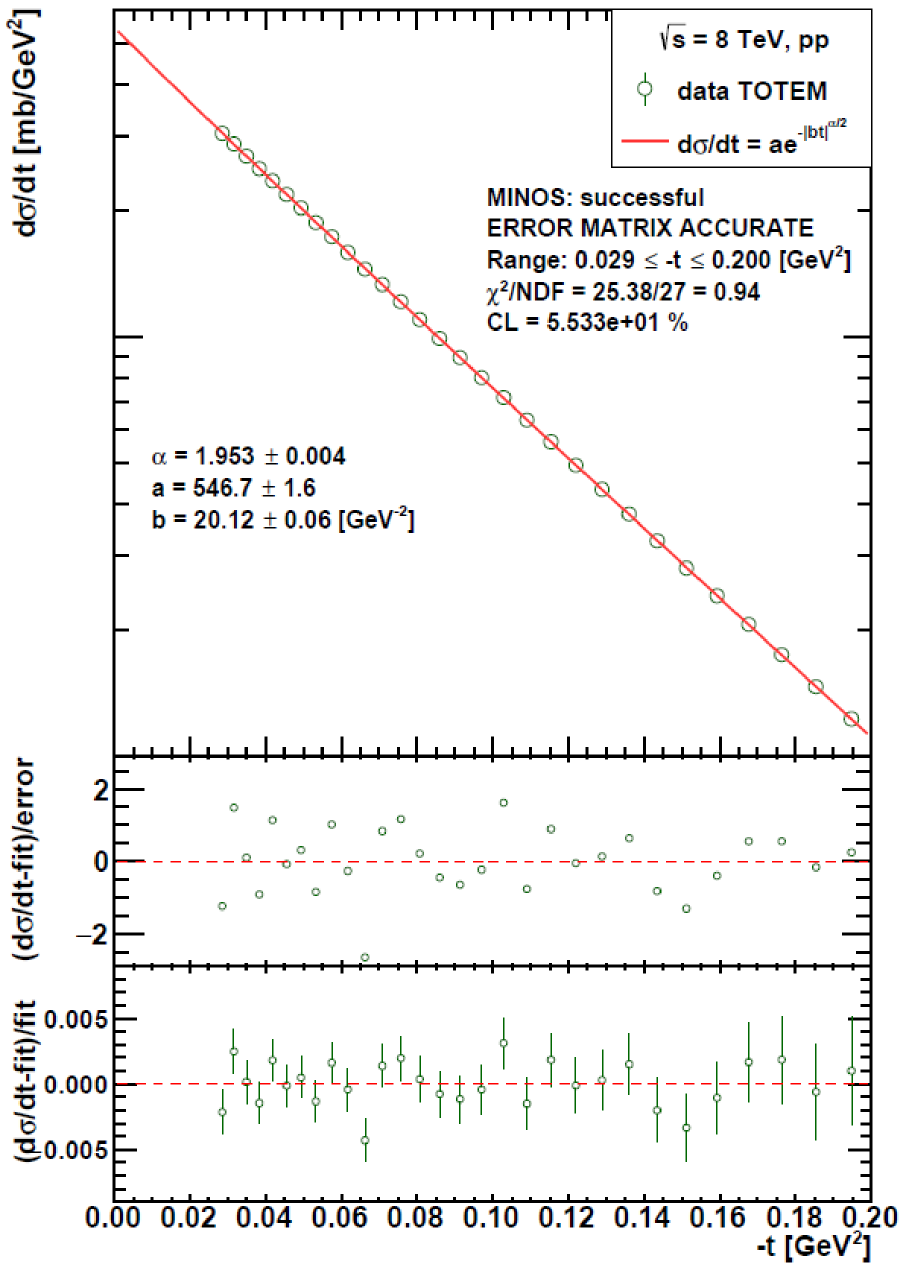

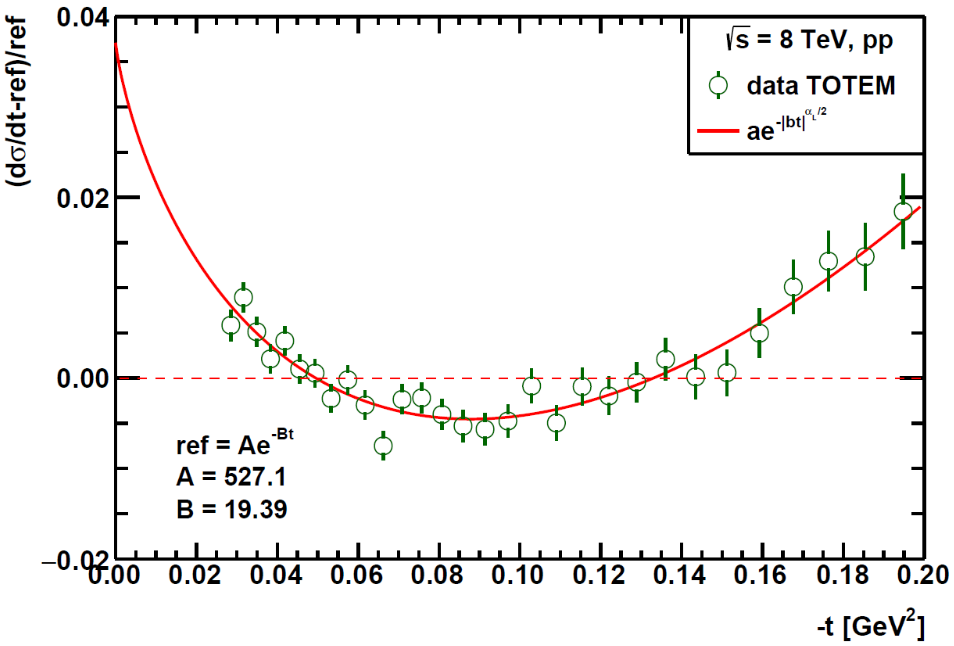

- the non-exponential Lévy α-stable model with α_L = 1.953 $\pm$ 0.004 represents the LHC TOTEM $\sqrt{s} = 8$ TeV low-$|t|$ differential cross section data with a confidence level of 55% (publieshed)
- similarly good description is obtained to all the LHC data on low-|t| pp (and p$\bar{\text{p}}$) $\boldsymbol{d\sigma/dt}$

Fits with simple Lévy α-stable model

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- fits to the existing pp and $p\bar{p}$ $d\sigma/dt$ data in the kinematic range:

$$546 \text{ GeV} \leq \sqrt{s} \leq 13 \text{ TeV}$$

$$0.02 \text{ GeV}^2 \leq -t \leq 0.15 \text{ GeV}^2$$

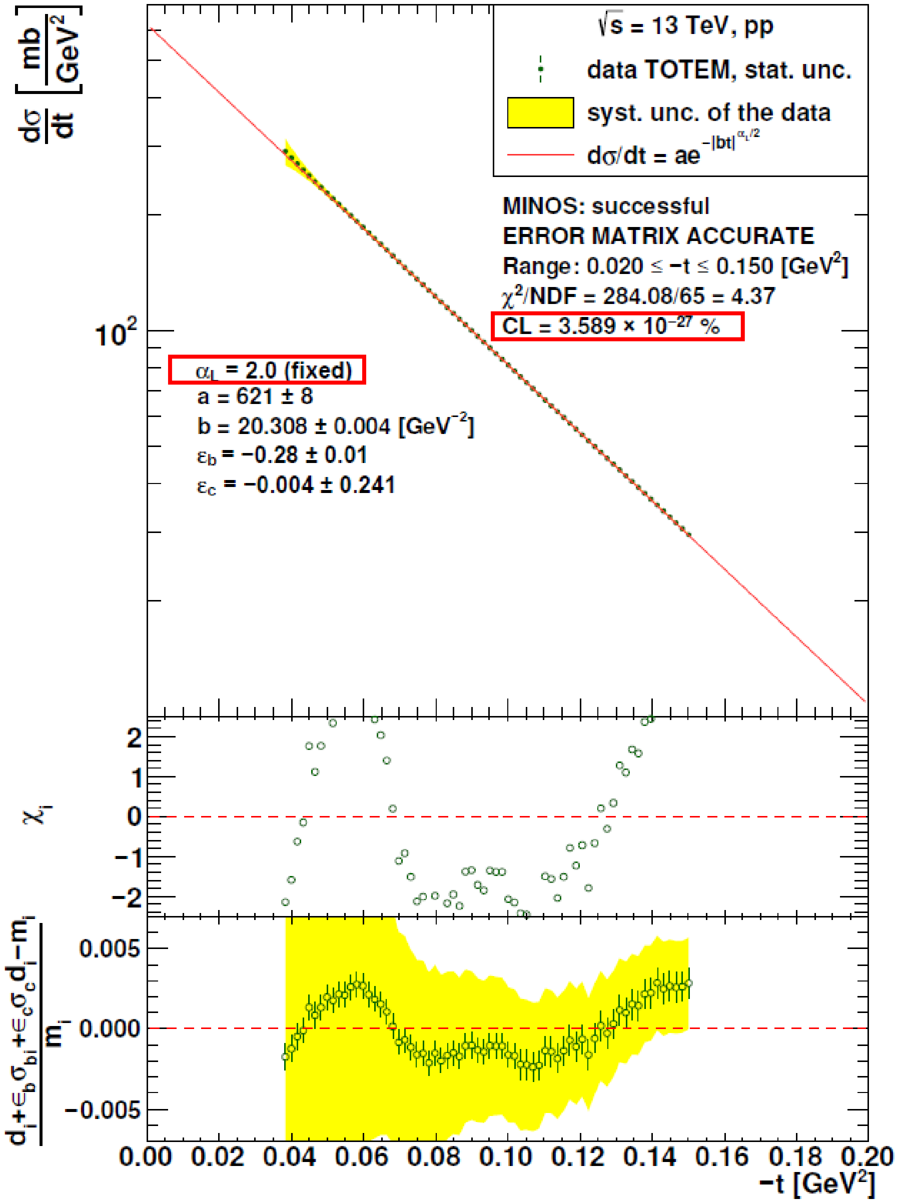
- the CL values of the fits range between 8.8% and 96%.
- statistical, systematic and normalization errors are taken into account using the χ^2 definition developed by PHENIX Collab.

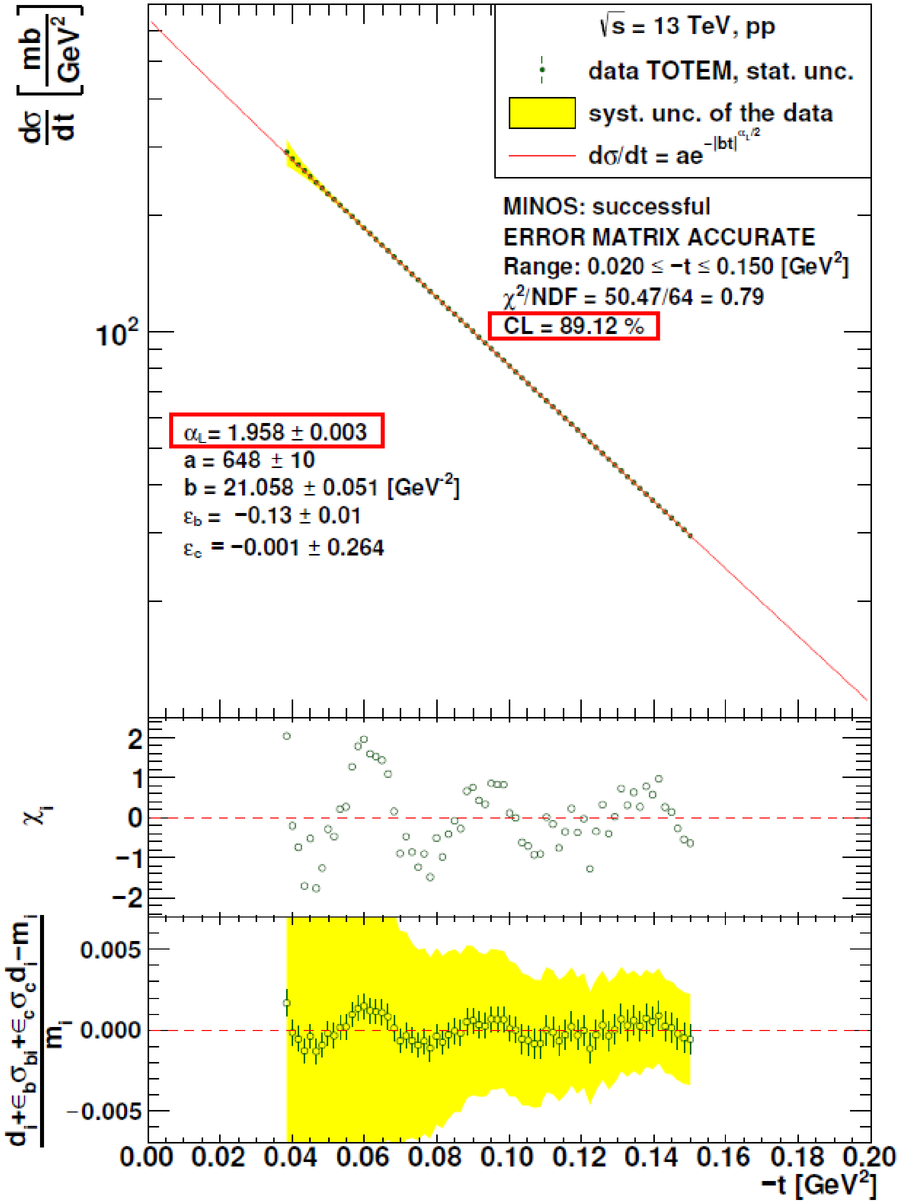
A. Adare *et al.* (PHENIX Collab.) *Phys. Rev. C 77, 064907*

$\sqrt{s}$, GeV	α_L	a, mb/GeV²	b, GeV⁻²	CL, %
546	1.93 ± 0.09	209 ± 15	15.8 ± 0.9	18.1
1800	2.0 ± 1.5	270 ± 24	16.2 ± 0.2	77.1
2760	1.600 ± 0.3	637 ± 252	28 ± 11	20.5
7000 (T)	1.95 ± 0.01	535 ± 30	20.5 ± 0.2	8.8
7000 (A)	1.97 ± 0.01	463 ± 13	19.8 ± 0.2	96.0
8000 (T1)	1.955 ± 0.005	566 ± 31	20.09 ± 0.08	43.86
8000 (T2)	1.90 ± 0.03	582 ± 33	20.9 ± 0.4	19.6
8000 (A)	1.97 ± 0.01	480 ± 11	19.9 ± 0.1	55.8
13000 (T1)	1.959 ± 0.006	677 ± 36	20.99 ± 0.08	76.5
13000 (T2)	1.958 ± 0.003	648 ± 10	21.06 ± 0.05	89.1
13000 (A)	1.968 ± 0.006	569 ± 17	20.84 ± 0.07	29.7

Values of the fitted parameters of the simple Lévy-α stable model at different energies

$\alpha_L = 2$ versus $\alpha_L < 2$ results @ 13 TeV



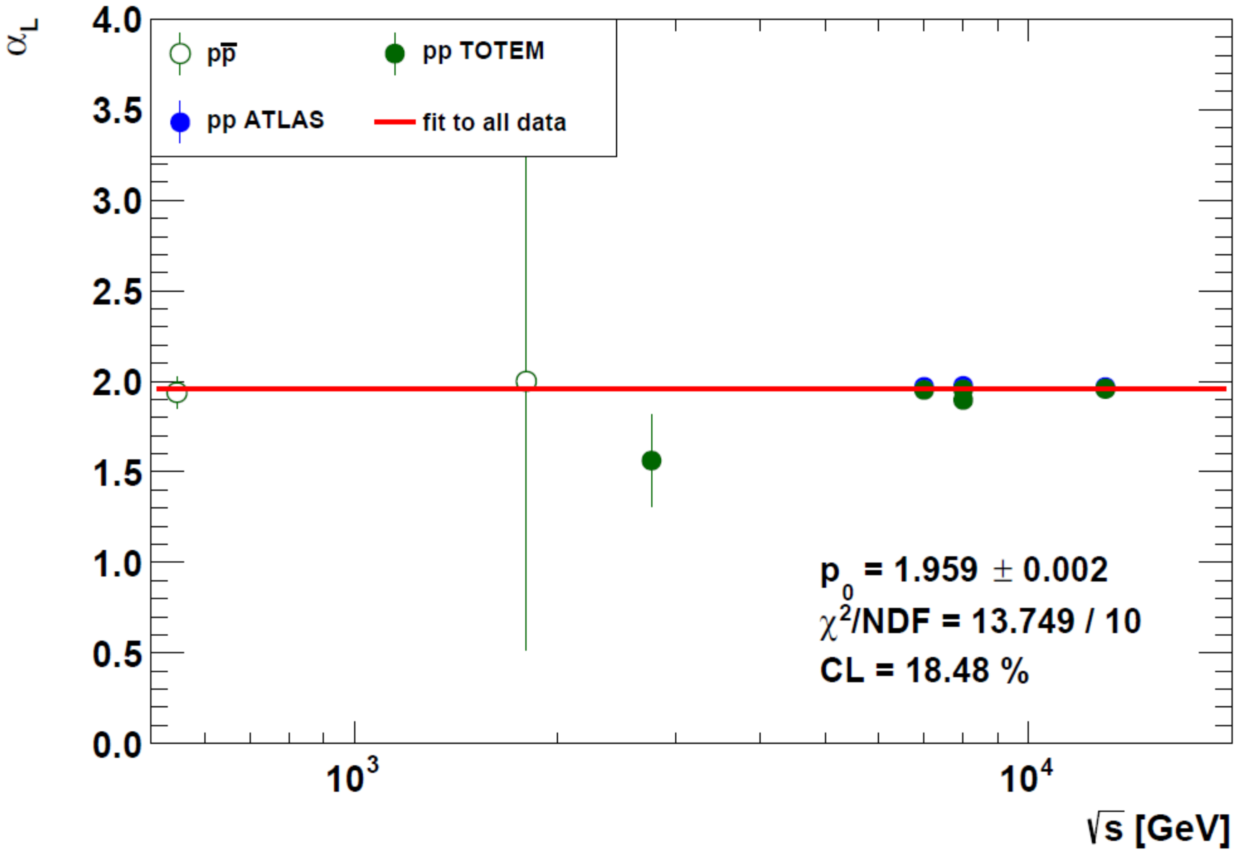


Energy dependence of the α_L parameter

Universe 2024, 10(3), 127

- the value of the α_L parameter does not depend on energy
- its value is 1.959 ± 0.002, i.e., slightly but in a statistical sense significantly different from 2

$\rightarrow$ strong non-exponential behavior at low $-t$ in the differential cross section, power law tail at high-$\vec{b}$ in $\tilde{\sigma}_{in}(s, \vec{b})$



Energy dependence of the α_L parameter of the simple Lévy-α stable model

Energy dependence of the optical point parameter

Universe 2024, 10(3), 127

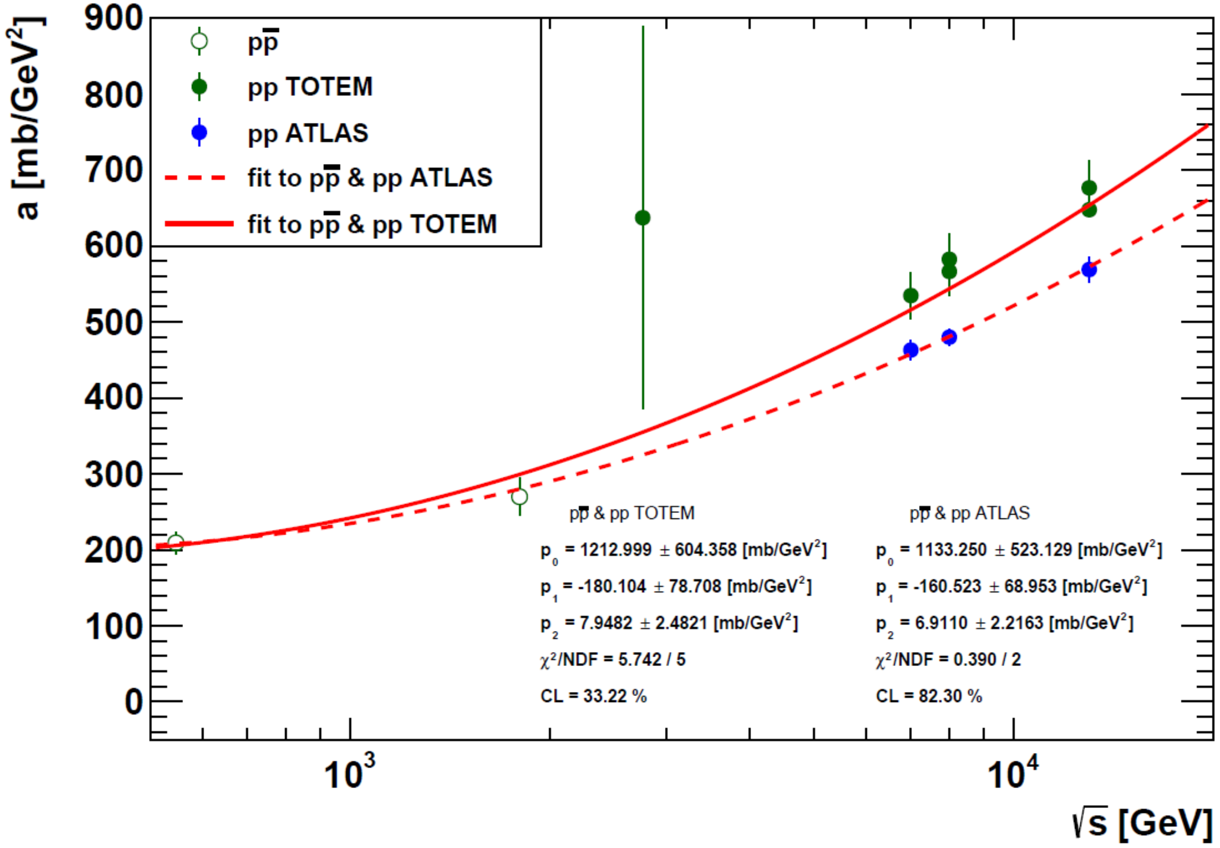
- the energy dependence of the a parameter is quadratically logarithmic:

$$a(s) = p_0 + p_1\, ln\frac{s}{1\ GeV^2} + p_2\, ln^2\frac{s}{1\ GeV^2}$$

- ATLAS and TOTEM data result slightly different energy dependencies

- reason: ATLAS and TOTEM use different methods to obtain the absolute normalization of the measurements

ATLAS Collab., ***Eur. Phys. J. C 83 (2023) 441***



Energy dependence of the a parameter of the simple Lévy-α stable model

Energy dependence of the slope parameter

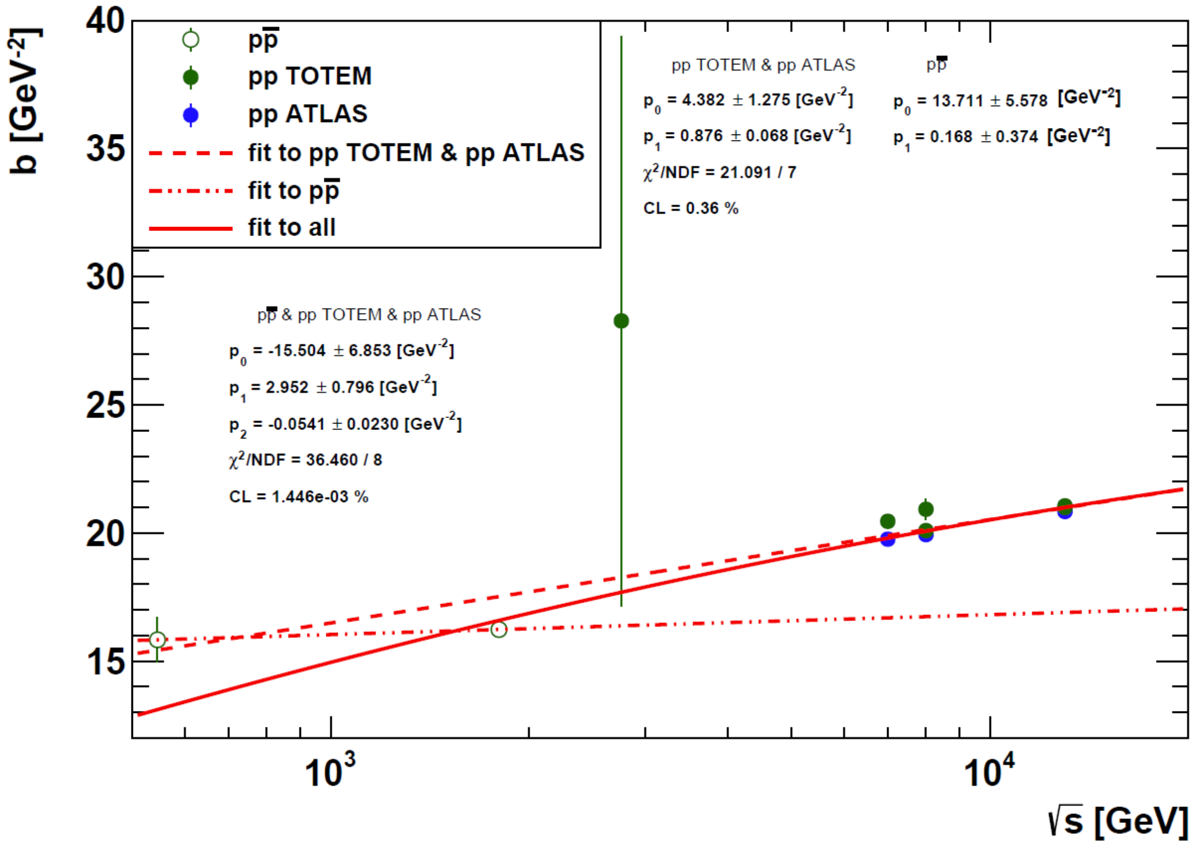
Universe 2024, 10(3), 127

- the energy dependence of the b parameter for TOTEM and ATLAS data together, and for $p\bar{p}$ data alone are linearly logarithmic:

$$b(s) = p_0 + p_1\, ln \frac{s}{1\ GeV^2}$$

- the LHC pp and the lower energy $p\bar{p}$ data do not lie on the same curve
- reason: the slope parameter data have a jump in the energy dependence around 3-4 TeV

TOTEM Collab., *Eur. Phys. J. C (2019) 79:103*



Energey dependence of the b parameter of the simple Lévy-α stable model

Simple Lévy α-stable & LBB model parameters

- parameters of the simple Levy α-stable model and the measurable quantities at $t \to 0$ can be approximately expressed in terms of the parameters of the LBB model *Universe 2023, 9(8), 361*
- only leading order terms in $\tilde{\sigma}_{in}(s,\vec{b})$ are considered; $A_{qq} = 1$ and $\lambda = 1/2$ are fixed

$$\frac{d\sigma}{dt}(s,t=0) = a(s) = \frac{81}{16}\pi\left(2R_q^{\alpha_L(s)}(s)\right)^{4/\alpha_L}\left(1+4\alpha_R^2(s)\right)$$

$$b(s) = \frac{1}{36}\left(\frac{4}{3}\right)^{2/\alpha_L(s)}\left(\left(2+2^{\alpha_L(s)}\right)R_{qd}^{\alpha_L(s)}(s) + 3^{\alpha_L(s)}\left(2R_d^{\alpha_L(s)}(s) + R_q^{\alpha_L(s)}(s)\right)\right)^{2/\alpha_L(s)}$$

(obtained after a Taylor expansion in $t^{\alpha_L/2}$)

$$\sigma_{tot}(s) = 9\pi\left(2R_q^{\alpha_L(s)}(s)\right)^{2/\alpha_L(s)}$$

$$\rho_0(s) = \frac{Re t_{el}(s,t=0)}{Im t_{el}(s,t=0)} = 2\alpha_R$$

$$\sigma_{el}(s) = \frac{a(s)}{b(s)}\Gamma\left(\frac{2+\alpha_L(s)}{\alpha_L(s)}\right)$$

- according to the analysis of elastic pp and p$\bar{\text{p}}$ data in the energy region $0.5\text{ TeV} \leq \sqrt{s} \leq 8\text{ TeV}$ only α_R is different for pp and p$\bar{\text{p}}$ scattering (T. Csörgő, I. Szanyi, *Eur. Phys. J. C* ***81***, *611 (2021)*)

- in the low-|t| approximation, difference between pp and p$\bar{\text{p}}$ scattering could be seen in the data on $\boldsymbol{d\sigma/dt}$, $\boldsymbol{\rho_0}$, $\boldsymbol{a}$ (optical point), and $\boldsymbol{\sigma_{el}}$, no difference in the data on $\boldsymbol{\sigma_{tot}}$ and $\boldsymbol{b}$

Summary

- the formal Lévy α-stable generalization of the Bialas-Bzdak model is done, the $\alpha_L = 2$ limit corresponds to the original model

- solution of difficult and complex technical (mathamatical and computational) problems is needed to fit the experimental data with the generalized model

- based on approximations a highly simplified Levy α-stable model of the pp (and $\mathrm{p\bar{p}}$) differential cross section is deduced and successfully fitted to the data in the low-|t| region

- the energy dependences of the parameters of the simple model are determined; the parameters of the simple model are related to the parameters of the Lévy α-stable generalized real extended Bialas-Bzdak (LBB) model

- final conclusion: the successful fit results indicate promising prospect for the future utility of the LBB model in describing experimental data

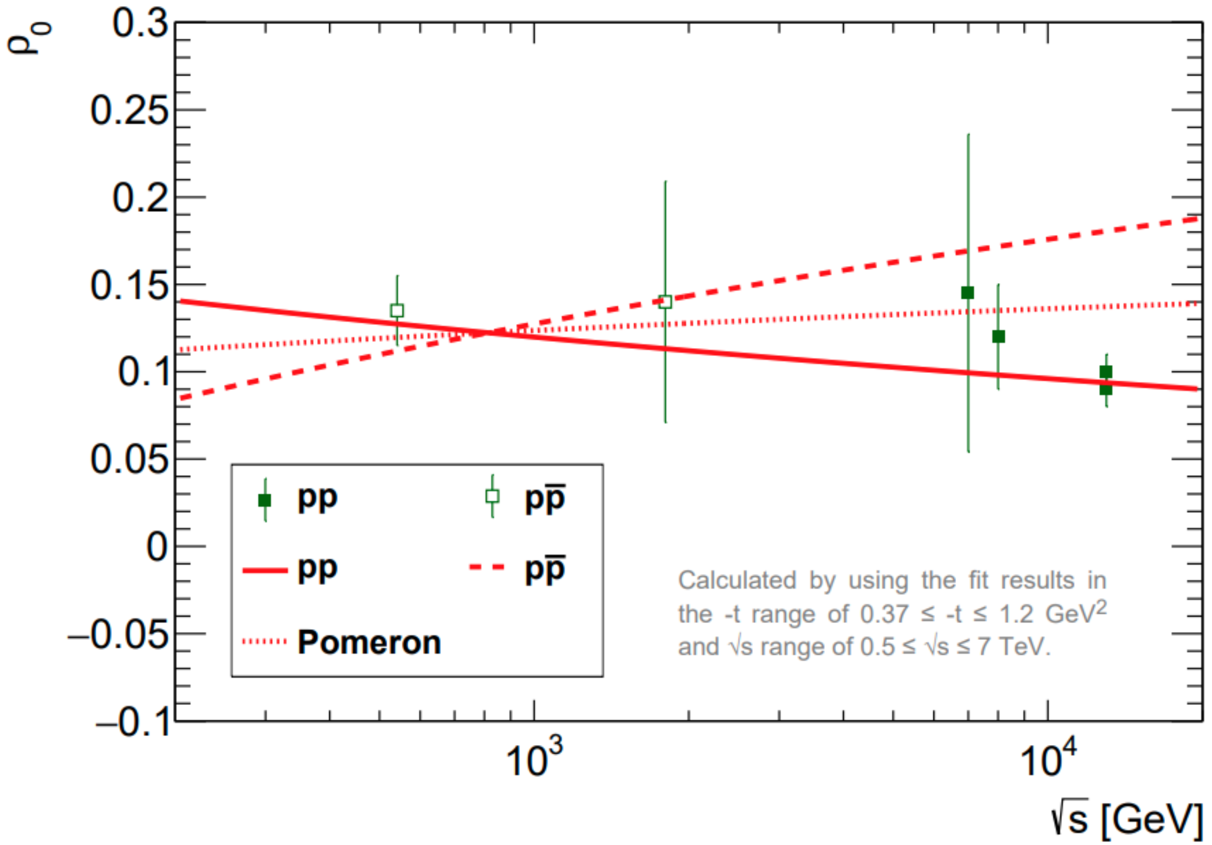
Thank you for your attention!

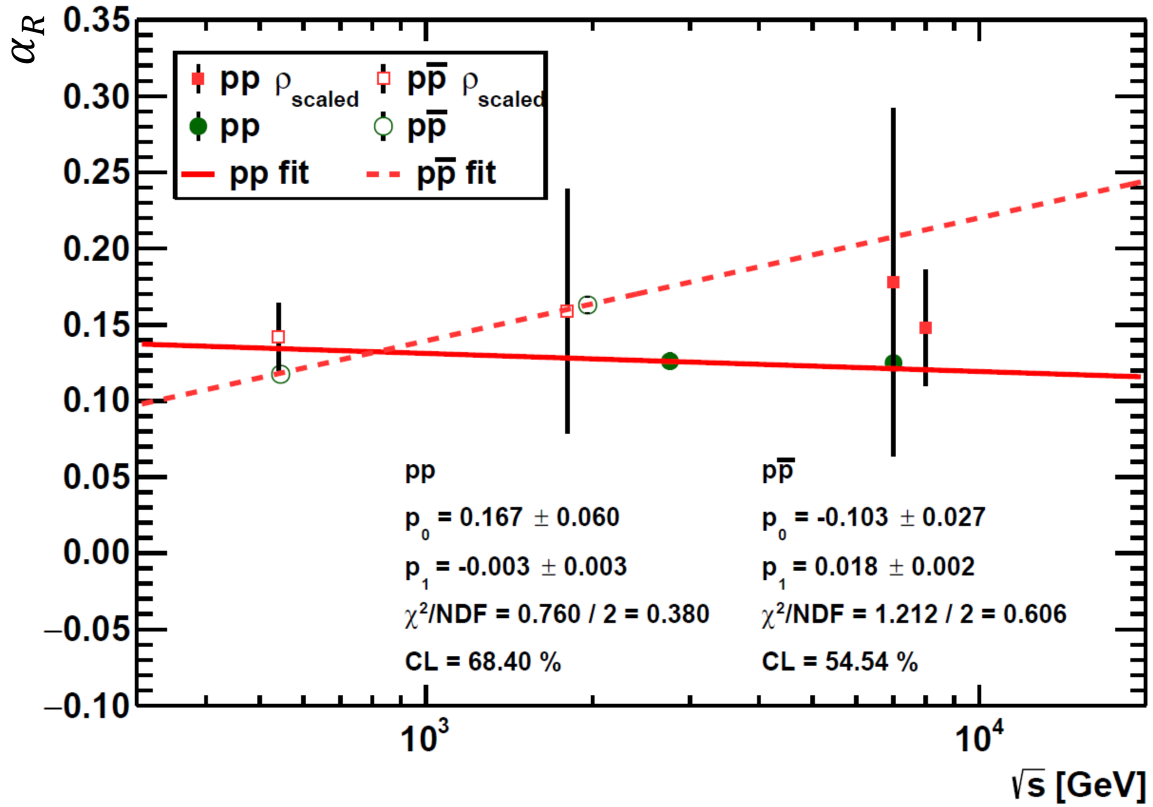
Backup slides

ρ_0 & α_R: connection between $t = 0$ and $t \neq 0$ data

- there is a connection between the ρ_0 parameter and the α_R parameter of the ReBB model regulating the real part of the scattering amplitude and the minimum-maximum structure of the $\mathrm{d}\sigma/dt$
- α_R is determined by the $\mathrm{d}\sigma/dt$ data at the minimum-maximum region but at the same time specifies the value of the ρ_0 in the ReBB model

T. Csörgő, I. Szanyi, *Eur. Phys. J. C* ***81***, *611 (2021)*





Most general term in $\tilde{\sigma}_{in}$

$$\tilde{\sigma}_{in}^{qq,qd,dq,dd}(\vec{b}) = \int d^2 s_q d^2 s'_q L\left(\vec{s}_q \left| R_{qd*}/2\right.\right) L\left(\vec{s}_q' \left| R_{qd*}/2\right.\right) \times \sigma_{qq}(\vec{s}_q, \vec{s}_q'; \vec{b}) \sigma_{qd}(\vec{s}_q, -\lambda\vec{s}_q'; \vec{b}) \sigma_{dq}(\vec{s}_q', -\lambda\vec{s}_q; \vec{b}) \sigma_{dd}(-\lambda\vec{s}_q, -\lambda\vec{s}_q'; \vec{b})$$

$$\sigma_{qq}(\vec{s}_q, \vec{s}_q'; \vec{b}) = \pi A_{qq}\left(2R_q^\alpha\right)^{2/\alpha} \times L\left(\vec{b} + \vec{s}_q' - \vec{s}_q | \alpha, \left(2R_q^\alpha\right)^{1/\alpha}/\sqrt{2}\right)$$

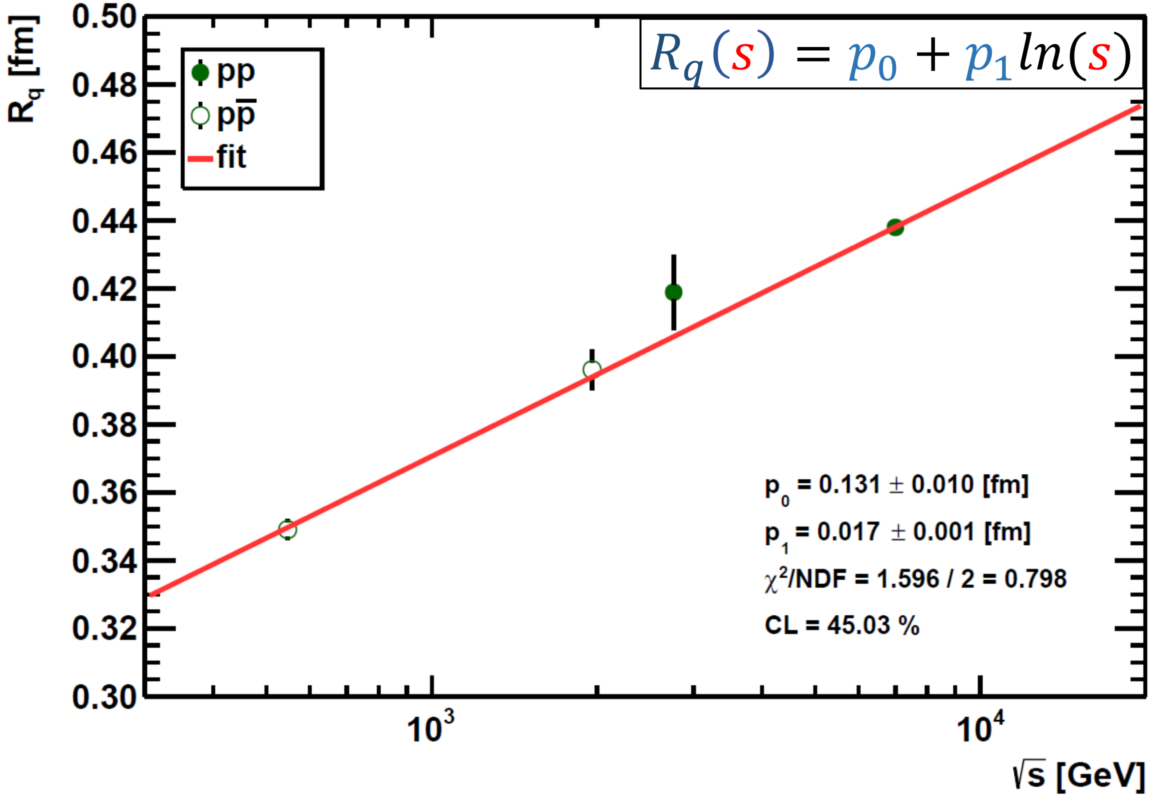
$$\sigma_{qd}(\vec{s}_q, \vec{s}_d'; \vec{b}) = 2\pi A_{qq}\left(2R_q^\alpha\right)^{2/\alpha} \times L\left(\vec{b} + \vec{s}_d' - \vec{s}_q \left| \alpha, \left(R_q^\alpha + R_d^\alpha\right)^{1/\alpha}/\sqrt{2}\right.\right)$$

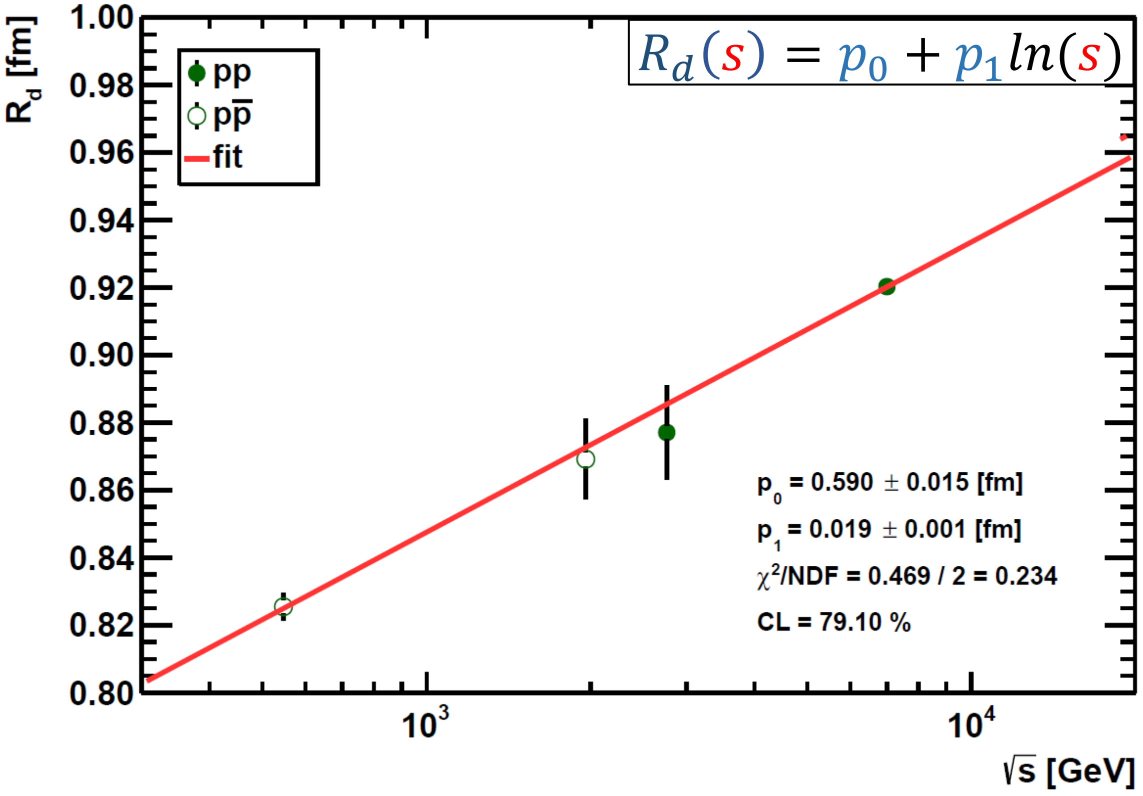
$$\sigma_{dd}(\vec{s}_d, \vec{s}_d'; \vec{b}) = 4\pi A_{qq}\left(2R_q^\alpha\right)^{2/\alpha} \times L(\vec{b} + \vec{s}_d' - \vec{s}_d | \alpha, (2R_d^\alpha)^{1/\alpha}/2)$$

$$\sigma_{dq}(\vec{s}_d, \vec{s}_q'; \vec{b}) = 2\pi A_{qq}\left(2R_q^\alpha\right)^{2/\alpha} \times L\left(\vec{b} + \vec{s}_q' - \vec{s}_d \left| \alpha, \left(R_q^\alpha + R_d^\alpha\right)^{1/\alpha}/2\right.\right)$$

Energy dependences of the ReBB model parameters

T. Csörgő, I. Szanyi, *Eur. Phys. J. C* 81, 611 (2021)



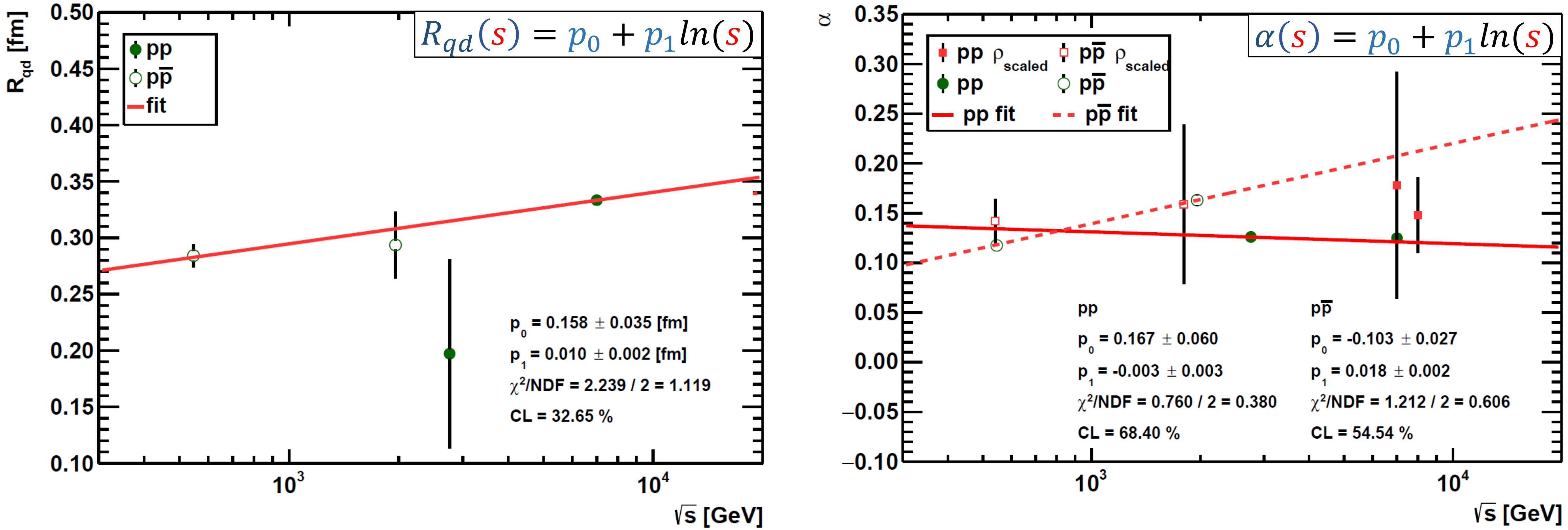


The energy dependences of the scale parameters, $R_q(s)$, $R_d(s)$, and $R_{qd}(s)$ are linear logarithmic and the same for pp and p$\bar{p}$ processes!

The energy dependence of the α parameter, $\alpha(s)$ is linear logarithmic too, but not the same for pp and p$\bar{p}$ processes!

Energy dependences of the ReBB model parameters

T. Csörgő, I. Szanyi, *Eur. Phys. J. C* 81, 611 (2021)



The energy dependences of the scale parameters, $R_q(s)$, $R_d(s)$, and $R_{qd}(s)$ are linear logarithmic and the same for pp and pp̄ processes!

The energy dependence of the α parameter, $\alpha(s)$ is linear logarithmic too, but not the same for pp and pp̄ processes!

Fit method

- least squares fitting with the method developed by the PHENIX collaboration
- this method is **equivalent to the diagonalization of the covariance matrix** if the experimental errors are separated into three different types:
  - **type A: point-to-point varying uncorrelated statistical and systematic errors**
  - **type B: point-to-point varying 100% correlated systematic errors**
  - **type C: point-independent, overall systematic uncertainties**
- **i.e least squares fitting with:**

A. Adare *et al.* (PHENIX Collab.) Phys. Rev. C 77, 064907

$$\chi^2 = \left(\sum_{j=1}^{M} \left(\sum_{i=1}^{n_j} \frac{\left(d_{ij} + \epsilon_{bj}\tilde{\sigma}_{bij} + \epsilon_{cj} d_{ij}\sigma_{cj} - th_{ij}\right)^2}{\tilde{\sigma}_{ij}^2} \right) + \epsilon_{bj}^2 + \epsilon_{cj}^2 \right) + \left(\frac{d_{\sigma_{tot}} - th_{\sigma_{tot}}}{\delta\sigma_{tot}} \right)^2 + \left(\frac{d_{\rho_0} - th_{\rho_0}}{\delta\rho_0} \right)^2$$

$$\tilde{\sigma}_{ij}^2 = \tilde{\sigma}_{aij} \left(\frac{d_{ij} + \epsilon_{bj}\tilde{\sigma}_{bij} + \epsilon_{cj} d_{ij}\sigma_{cj}}{d_{ij}} \right)$$

$$\tilde{\sigma}_{kij} = \sqrt{\sigma_{kij}^2 + (d'_{ij}\delta_k t_{ij})^2}, \qquad k \in \{a, b\}$$

- **minimization with CERN Root MINUIT, parameter error estimation by MINOS.**

Fit method

- the method takes into account (in M separately measured t ranges):
  - the t-dependent statistical (type A) and systematic (type B) errors (both vertical σ_k and horizontal $\delta_k t$) → ϵ_b parameters;
  - the t-independent σ_c normalization uncertainties (type C) → ϵ_c parameters;
  - the measured total cross-section $d_{\sigma_{tot}}$ and ratio d_{ρ_0} and their total uncertainties $\delta\sigma_{tot}$ and $\delta\rho_0$.

A. Adare *et al.* (PHENIX Collab.) Phys. Rev. C 77, 064907

- **i.e least squares fitting with:**

$$\chi^2 = \left(\sum_{j=1}^{M} \left(\sum_{i=1}^{n_j} \frac{\left(d_{ij} + \epsilon_{bj}\tilde{\sigma}_{bij} + \epsilon_{cj} d_{ij}\sigma_{cj} - th_{ij}\right)^2}{\tilde{\sigma}_{ij}^2} \right) + \epsilon_{bj}^2 + \epsilon_{cj}^2 \right) + \left(\frac{d_{\sigma_{tot}} - th_{\sigma_{tot}}}{\delta\sigma_{tot}} \right)^2 + \left(\frac{d_{\rho_0} - th_{\rho_0}}{\delta\rho_0} \right)^2$$

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Fit method

- the method takes into account (in M separately measured t ranges):
  - th … rtical
    σ_k …
  - th …
  - the measured total cross-section $d_{\sigma_{tot}}$ and ratio d_{ρ_0} and their total uncertainties $\delta\sigma_{tot}$ and $\delta\rho_0$.

ϵ_i-s must be considered as both measurements and fit parameters not effecting the NDF (since they have known central value of zero and known standard deviation of one)

A. Adare *et al.* (PHENIX Collab.) Phys. Rev. C 77, 064907

- **i.e least squares fitting with:**

$$\chi^2 = \left(\sum_{j=1}^{M}\left(\sum_{i=1}^{n_j}\frac{\left(d_{ij}+\epsilon_{bj}\tilde{\sigma}_{bij}+\epsilon_{cj}d_{ij}\sigma_{cj}-th_{ij}\right)^2}{\tilde{\sigma}_{ij}^2}\right)+\epsilon_{bj}^2+\epsilon_{cj}^2\right)+\left(\frac{d_{\sigma_{tot}}-th_{\sigma_{tot}}}{\delta\sigma_{tot}}\right)^2+\left(\frac{d_{\rho_0}-th_{\rho_0}}{\delta\rho_0}\right)^2$$

$$\tilde{\sigma}_{ij}^2 = \tilde{\sigma}_{aij}\left(\frac{d_{ij}+\epsilon_{bj}\tilde{\sigma}_{bij}+\epsilon_{cj}d_{ij}\sigma_{cj}}{d_{ij}}\right)$$

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Fit method

The PHENIX method is validated by evaluating the χ2 from a full covariance matrix fit of the √s = 13 TeV TOTEM differential cross-section data using the Lévy expansion method from T. Csörgő, R. Pasechnik, & A. Ster, Eur. Phys. J. C 79, 62 (2019).

- the t-independent σ_c normalization uncertainties → ϵ_c parameters;
- the measured total cross-section $d_{\sigma_{tot}}$ and ratio d_{ρ_0} and their total uncertainties $\delta\sigma_{tot}$ and $\delta\rho_0$.

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$$\tilde{\sigma}_{ij}^2 = \tilde{\sigma}_{aij}\left(\frac{d_{ij}+\epsilon_{bj}\tilde{\sigma}_{bij}+\epsilon_{cj}d_{ij}\sigma_{cj}}{d_{ij}}\right)$$

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- the t independent σ normalization uncertainties → ϵ_c parameters;

The PHENIX method and the fit with the full covariance matrix result in the same minimum within one standard deviation of the fit parameters.

$\delta\sigma_{tot}$ and $\delta\rho_0$.

A. Adare *et al.* (PHENIX Collab.) Phys. Rev. C 77, 064907

- **i.e least squares fitting with:**

$$\chi^2 = \left(\sum_{j=1}^{M}\left(\sum_{i=1}^{n_j}\frac{\left(d_{ij}+\epsilon_{bj}\tilde{\sigma}_{bij}+\epsilon_{cj}d_{ij}\sigma_{cj}-th_{ij}\right)^2}{\tilde{\sigma}_{ij}^2}\right)+\epsilon_{bj}^2+\epsilon_{cj}^2\right)+\left(\frac{d_{\sigma_{tot}}-th_{\sigma_{tot}}}{\delta\sigma_{tot}}\right)^2+\left(\frac{d_{\rho_0}-th_{\rho_0}}{\delta\rho_0}\right)^2$$

$$\tilde{\sigma}_{ij}^2 = \tilde{\sigma}_{aij}\left(\frac{d_{ij}+\epsilon_{bj}\tilde{\sigma}_{bij}+\epsilon_{cj}d_{ij}\sigma_{cj}}{d_{ij}}\right)$$

$$\tilde{\sigma}_{kij} = \sqrt{\sigma_{kij}^2+\left(d'_{ij}\delta_k t_{ij}\right)^2}, \qquad k \in \{a,b\}$$

- **minimization with CERN Root MINUIT, parameter error estimation by MINOS.**

Proportionality between $\rho_0(s)$ and $\alpha(s)$

$$t_{el}(s,b) = i\left(1 - e^{i\,\alpha\,\tilde{\sigma}_{in}(s,b)}\sqrt{1-\tilde{\sigma}_{in}(s,b)}\right)$$

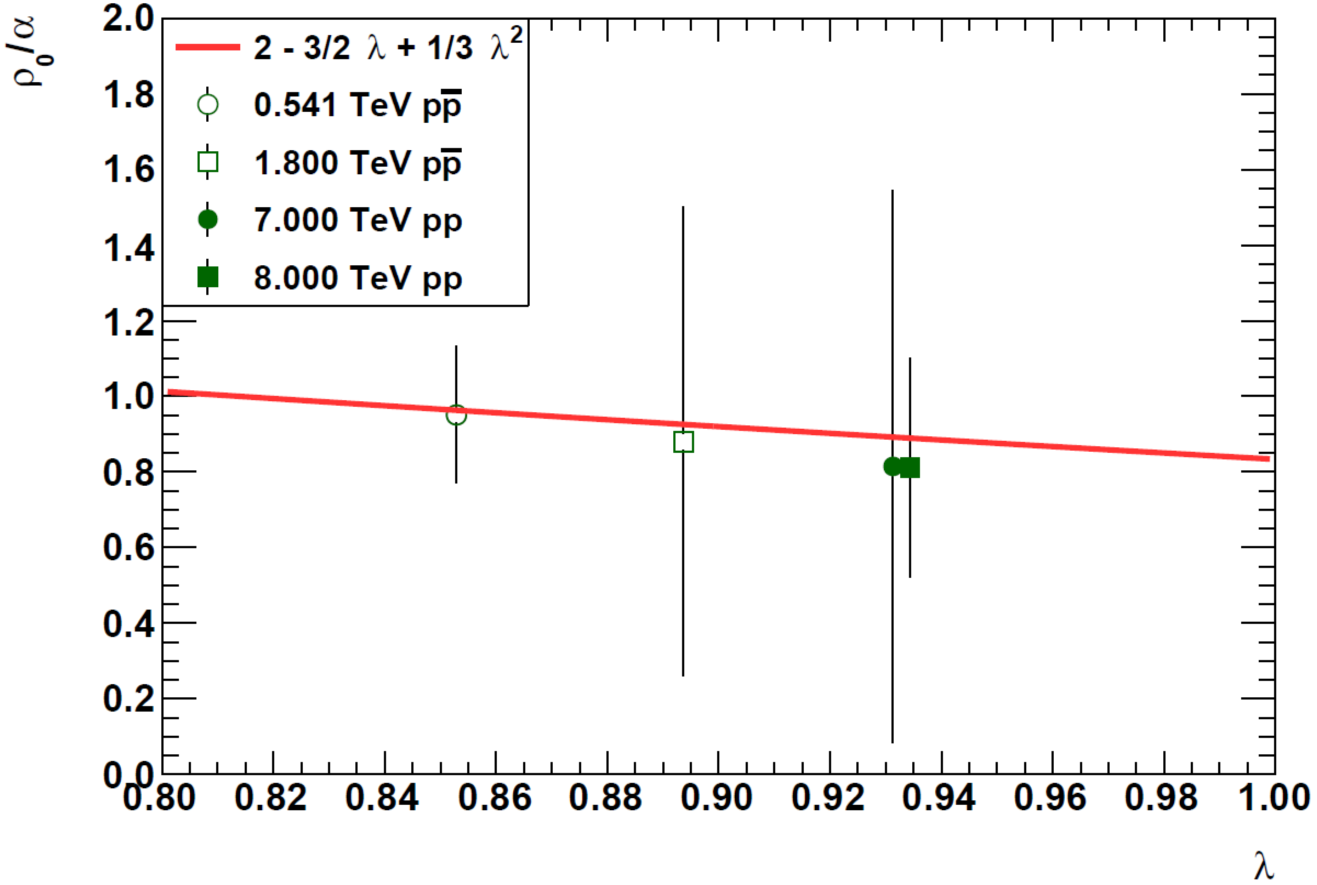
$$\alpha\tilde{\sigma}_{in} \ll 1$$

$$\mathrm{Im}\, t_{el}(s,b) \simeq \lambda(s)\exp\left(-\frac{b^2}{2R^2(s)}\right)$$

$$\rho_0(s) = \alpha(s)\left(2 - \frac{3}{2}\lambda(s) + \frac{1}{3}\lambda^2(s)\right)$$

$$\lambda(s) = \mathrm{Im}\, t_{el}(s, b=0)$$

→ by rescaling one can get additional α parameter values at energies where ρ_0 is measured (and vice versa)



The dependence of ρ_0/α on $\lambda = \mathrm{Im}\, t_{el}(s, b=0)$ in the TeV energy range. The data points are generated numerically by using the trends of the ReBB model scale parameters and the experimentally measured ρ-parameter values.

Measurable quantities

- **differential cross section:**

$$\frac{d\sigma}{dt}(s,t) = \frac{1}{4\pi}|T(s,t)|^2$$

- **total, elastic and inelastic cross sections:**

$$\sigma_{tot}(s) = 2ImT(s,t=0)$$

$$\sigma_{el}(s) = \int_{-\infty}^{0} \frac{d\sigma(s,t)}{dt} dt$$

$$\sigma_{in}(s) = \sigma_{tot}(s) - \sigma_{el}(s)$$

- **ratio ρ_0:**

$$\rho_0(s) = \lim_{t\to 0} \rho(s,t) \equiv \frac{ReT(s,t\to 0)}{ImT(s,t\to 0)}$$

- **slope of dσ/dt:**

$$\mathrm{B}(\mathrm{s},\mathrm{t}) = \frac{\mathrm{d}}{\mathrm{dt}}\left(\ln \frac{d\sigma}{\mathrm{dt}}(\mathrm{s},\mathrm{t})\right)$$

$$B_0(s) = \lim_{t\to 0} \mathrm{B}(\mathrm{s},\mathrm{t})$$