RF Engineering
Transmission Lines II &
Pillbox Resonator I

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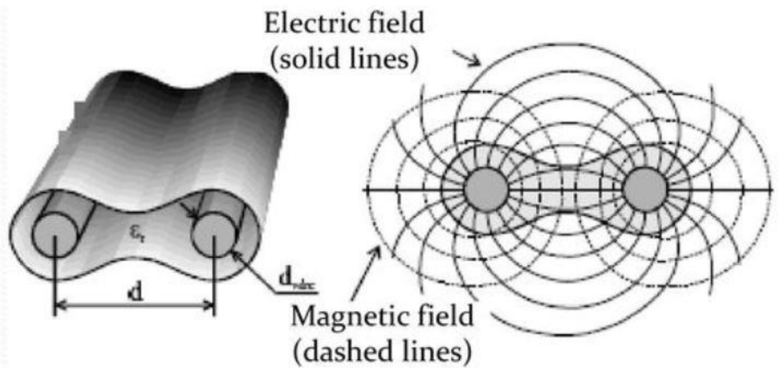
Transmission Lines and Waveguides

- Wave patterns in guided wave systems depend on the number of conductors used.

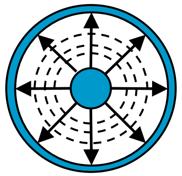
"Open" two-wire system (TEM)



Source: Ma, *Electromagnetic Waves and Applications Part III, univ. lecture*

TEM-propagation = 2 conductor system

Coaxial line (TEM)

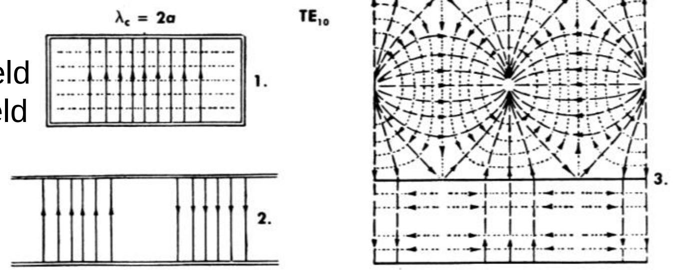


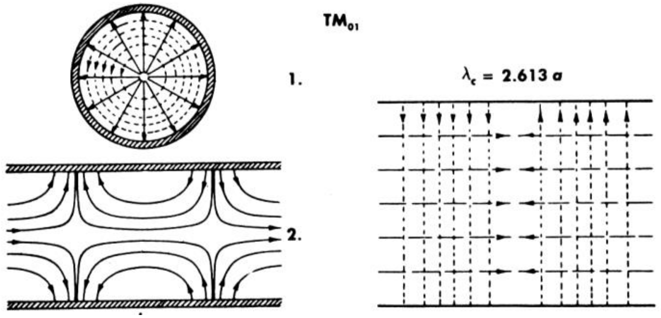
—— E-field ---H-field

Picture: Coaxial loads for the SPS 200 MHz cavity © CERN

Uniform waveguides rectangular, round, random cross-section (all non-TEM)

— E-field
---H-field





Source: Saad, *Microwave Engineers Handbook, vol. 1,* Artech House

Source: Zhang, *Electromagnetic Theory for Microwaves and Optoelectronics*, 2nd ed., Springer

Hollow Waveguides

Rectangular waveguide

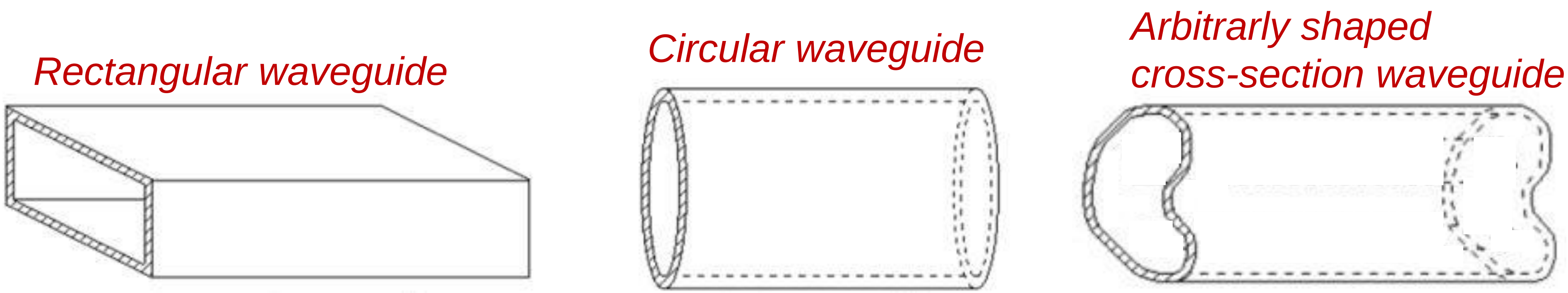
Circular waveguide

Arbitrarly shaped cross-section waveguide

Source: Zhang, *Electromagnetic Theory for Microwaves and Optoelectronics*, 2nd ed., Springer

- Waveguides exist in different shapes. They are just hollow metallic tubes with uniform cross-sections.
- The metallic waveguide is a completely enclosed system without any radiation losses.
- Further, losses due to a dielectric layer, as we have seen for the coaxial line do not exist, equally the inner conductor of the coaxial line is eliminated.
- Waveguides present a one-conductor system, so *no TEM-mode propagation* is possible. Propagation happens in TE-mode *(**T**ransverse-**E**lectric)* and TM-mode *(**T**ransverse-**M**agnetic).* These modes have a lower frequency limit, the so-called cut-off frequency below which propagation is not possible. The cut-off frequency depends on the dimensions of the waveguide cross-section.

Waveguides & Boundary conditions



Source: Zhang, *Electromagnetic Theory for Microwaves and Optoelectronics*, 2nd ed., Springer

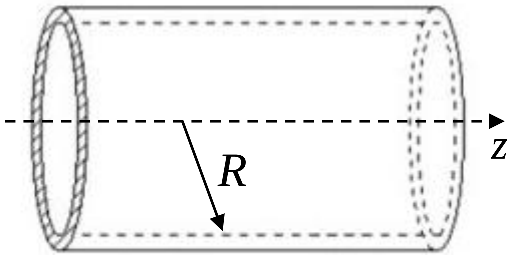
- Waves are following the waveguide shape, even if it is bend.
- For this, the propagating waves (formerly unbound) now need to fulfil boundary conditions on the walls of the waveguide.

- Remember? From EM-field description, we know that:

→ on *perfect conducting surfaces, the tangential electric field has to vanish* to fulfill the boundary condition. We speak of an “electric wall”: $E_{tangential} = 0.$

→ the walls of a waveguide can approximated as a perfect conductor (exception: loss calculation).

Circular (round) Waveguides (1/6)



- Like all transmission lines, also the circular waveguide is characterized by a propagation constant, an attenuation constant and a characteristic impedance.
- These quantities are derived by field theory analysis (see lecture of A. Mostacci). Propagation is usually assumed in z-direction.

- For a perfect conducting tube (no resistive attenuation) with radius R, we obtain propagation constants for the TE-mode and the TM-mode:

$$\beta_{nm} = \sqrt{\omega^2 \epsilon \mu - \left(\frac{p'_{nm} \text{ or } p_{nm}}{R} \right)^2}$$

p_{nm} → Roots of the Bessel-function for TM-mode $J_n(p_{nm}) = 0$

p'_{nm} → Roots of the derivative of the B.F. for TE-mode $J'_n(p'_{nm}) = 0$

- This leads to the so-called cut-off frequencies for the different modes:

$$f_{c,nm} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \left(\frac{p'_{nm} \text{ or } p_{nm}}{R} \right)$$

Circular (round) Waveguides (2/6)

Recall the Bessel function and the derivative of the Bessel function with their roots:

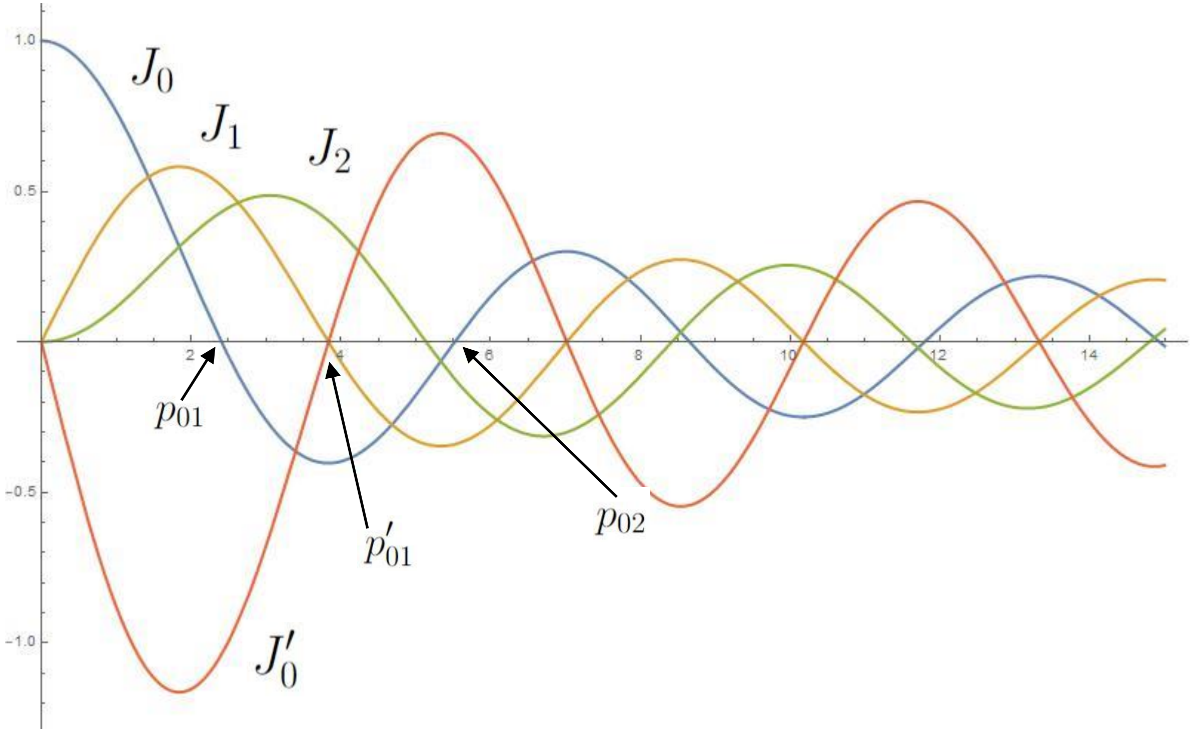


TABLE 3.3 Values of p'_{nm} for TE Modes of a Circular Waveguide

n	p'_{n1}	p'_{n2}	p'_{n3}
0	3.832	7.016	10.174
1	1.841	5.331	8.536
2	3.054	6.706	9.970

TABLE 3.4 Values of p_{nm} for TM Modes of a Circular Waveguide

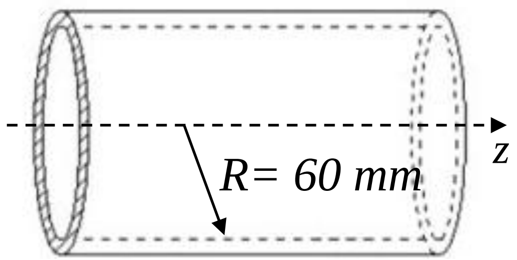
n	p_{n1}	p_{n2}	p_{n3}
0	2.405	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620

Source: Pozar, *Microwave Engineering*, 4th ed., Wiley

p_{01} = 1st root of Bessel function of first type J_0
p_{02} = 2nd root of Bessel function of first type J_0
p'_{01} = 1st root of derivative of Bessel function of first type J'_0

Circular (round) Waveguides (3/6)

Let's see an example.



→ Calculating cut-off frequencies for a circular waveguide with $R = 60\ mm$.

$$f_c(\mathrm{TE}_{11}) = \frac{1}{2\pi\sqrt{\mu_0\epsilon_0}}\frac{p'_{11}}{R} = \frac{2.9\cdot 10^8}{2\pi}\frac{1.841}{0.06} = 1.416\ \mathrm{GHz}$$

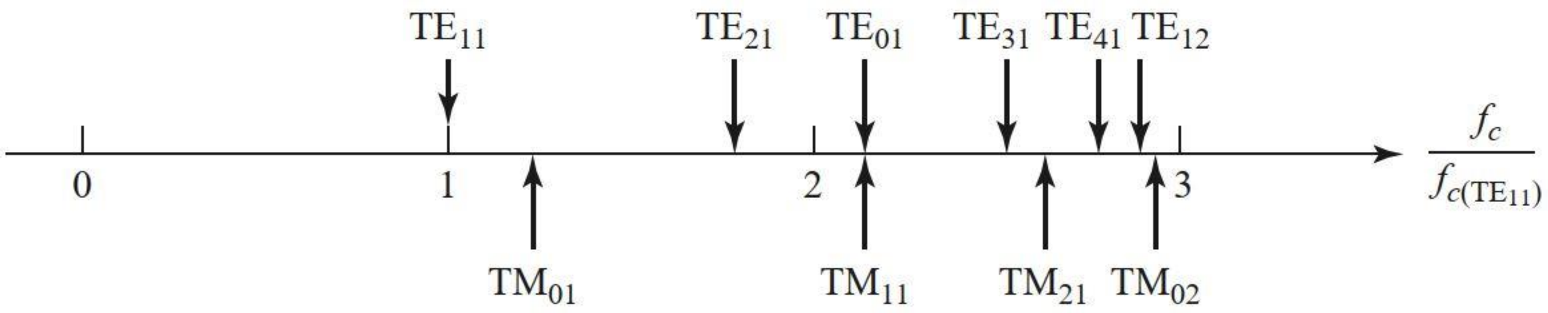
$$f_c(\mathrm{TM}_{01}) = \frac{1}{2\pi\sqrt{\mu_0\epsilon_0}}\frac{p_{01}}{R} = \frac{2.9\cdot 10^8}{2\pi}\frac{2.405}{0.06} = 1.850\ \mathrm{GHz}$$

Frequencies where these modes start propagating.

TE-mode TM-mode

$$f_{c,nm} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\left(\frac{p'_{nm} \text{ or } p_{nm}}{R}\right)$$

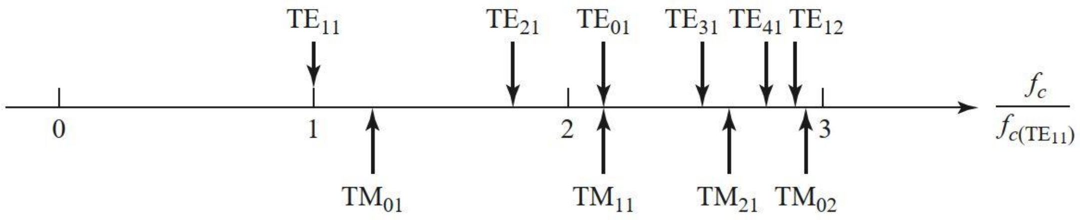
Chart of cut-off frequencies relative to f_c of TE_{11}-mode:



Source: Pozar, *Microwave engineering*, 4th ed., Wiley

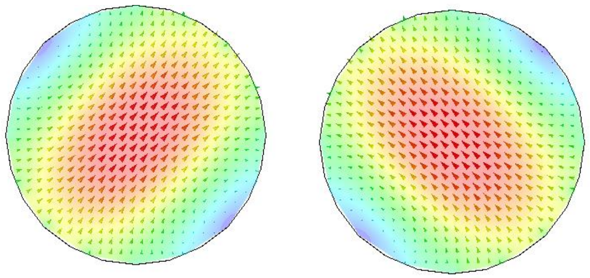
Circular (round) Waveguides (4/6)

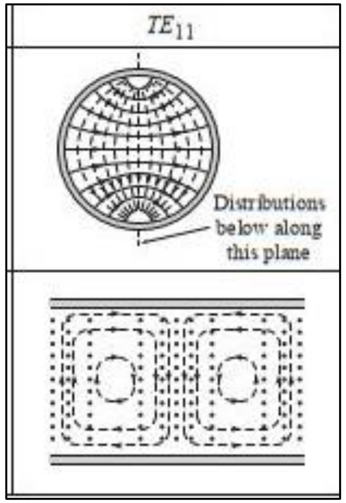
Waveguide modes and field patterns

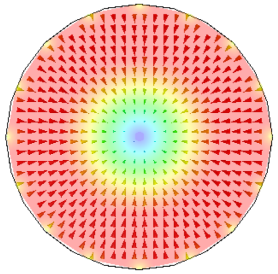


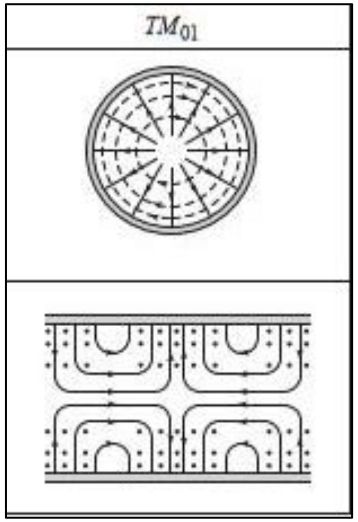
1st mode is TE_{11}. Electric field is transverse.
Mode has 2 polarisations (orientations of the electrical field).

- 2nd mode is TM_{01}. Magnetic field is transverse.





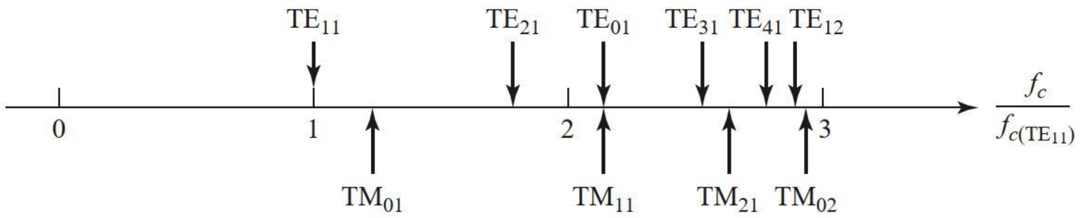




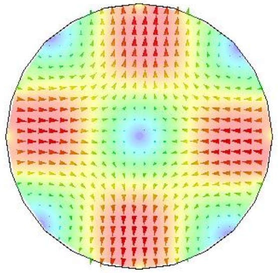
Simulaton pictures: courtesy E. Jensen, source pictures and chart: Pozar, *Microwave engineering*, 4th ed., Wiley

Circular (round) Waveguides (5/6)

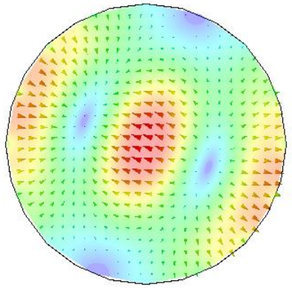
Waveguide modes and field patterns

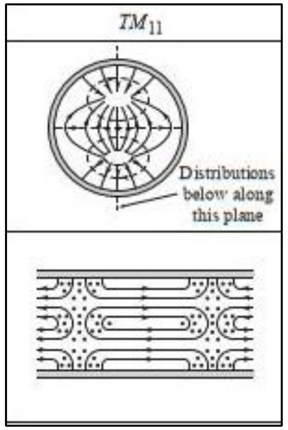


- 3^{rd} mode is TE_{21}.
  Electric field is transverse

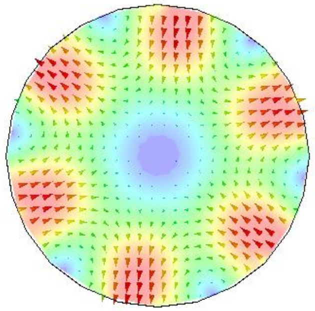


- 4^{th} mode is TM_{11}. Magnetic field is transverse.





- 6^{th} mode is TE_{31}.
  Electric field is transverse.



Further field plots can *e.g.* be found here: Lee, Lee and Chuang, *Plot of Modal Field Distribution in Rectangular and Circular Waveguides*. IEEE, Trans. Microwave Theory and Techniques, Vol. MTT-33, no. 3, March 1985

Simulaton pictures: courtesy E. Jensen, source pictures and chart: Pozar, *Microwave engineering*, 4^{th} ed., Wiley

Circular (round) Waveguides (6/6)

What is actually happening if the wave frequency in a waveguide is below cut-off?

Remember? Propagation constant is defined: $\gamma = \alpha + j\beta$

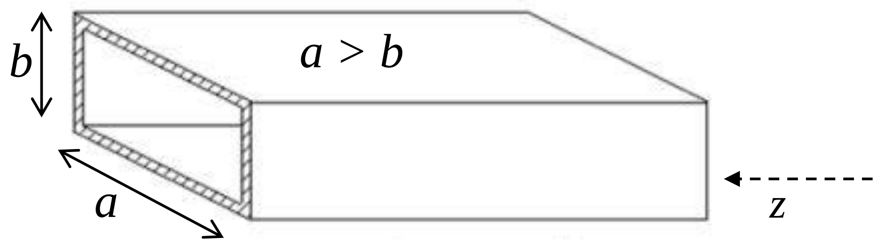
$$\beta_{nm} = \sqrt{\omega^2 \epsilon\mu - \underbrace{\left(\frac{p'_{nm} \text{ or } p_{nm}}{R}\right)^2}_{\epsilon\mu\ \omega_c^2}} \qquad \Rightarrow \beta^2 = \mu\epsilon(\omega^2 - \omega_c^2)$$

Now for frequencies below cut off, β would become complex: $\beta = j \cdot \sqrt{\mu\epsilon}\sqrt{\omega_c^2 - \omega^2}$

Then, the propagation constant γ becomes fully real, and the wave is damped even in the loss-less case ($\alpha=0$), since the propagation is following: $e^{-\beta z}$

This is called an evanescent wave = a wave that cannot propagate.

Rectangular Waveguides (1/6)



Source: Zhang, *Electromagnetic Theory for Microwaves and Optoelectronics*, 2nd ed., Springer

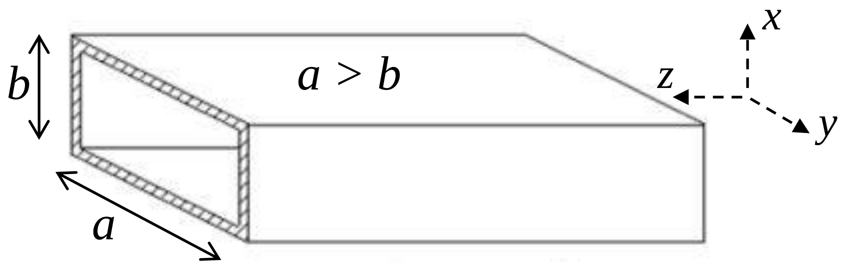
- All field quantities are derived by field theory analysis (see lecture of A. Mostacci).
- Wave propagation is usually assumed in z-direction.
- The rectangular waveguide is characterized by a propagation constant, attenuation constant and a characteristic impedance.

- We distinguish again between TE-modes and TM-modes. TEM-modes do not exist in a waveguide.
- We will see that the hollow rectangular waveguide can propagate TE- and TM-modes, and that these modes have cut-off frequencies below which propagation is not possible, similar to the TE- and TM-modes of the circular waveguide.

The *derivation* of the cut-off frequency is straightforward, please refer to textbooks, if you are interested. The field components are obtained by the separation method.

Rectangular Waveguides (2/6)

TE-modes



Source: Zhang, *Electromagnetic Theory for Microwaves and Optoelectronics*, 2nd ed., Springer

- TE-modes are characterized by a zero *electric* field in propagation direction. The electric field is in the transverse plane, only.
- Each mode has a cut-off frequency:

$$f_{c,mn} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \omega_{c,mn} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\,\pi}{a}\right)^2 + \left(\frac{n\,\pi}{b}\right)^2}$$

- TE_{10}-mode is the mode with the lowest cut-off frequency: $\omega_{c,10} = \frac{1}{2a\sqrt{\mu\epsilon}}$

The mode with the *lowest cut-off frequency* is called *the dominant mode*.

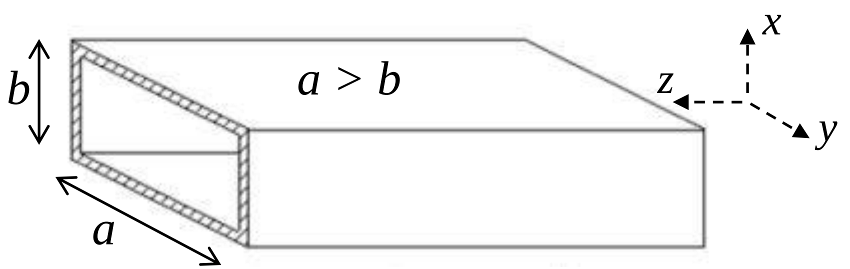
Note that if $n=m=0$, all field components become zero, there is no TE_{00} mode.

A waveguide is called *overmoded*, if more than one mode is propagating in the waveguide at the same time.

Rectangular Waveguides (3/6)

TM-modes

- TM-modes are characterized by a zero *magnetic* field in propagation direction. The magnetic field is in the transverse plane, only.



Source: Zhang, *Electromagnetic Theory for Microwaves and Optoelectronics*, 2nd ed., Springer

- Also in this case, each mode has a cut-off frequency (identical to TE):

$$f_{c,mn} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \omega_{c,mn} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\,\pi}{a}\right)^2 + \left(\frac{n\,\pi}{b}\right)^2}$$

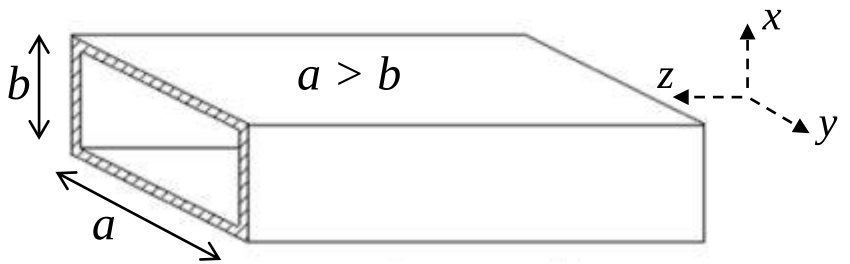
- TM_{11} has as cut-off frequency: $\omega_{c,11} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$

There is no TM_{00}-mode, neither do TM_{01} or TM_{10} exist. For $n=m=0$, all field components become zero.

Note that TE_{10}-mode remains the dominant mode, it is the mode with the lowest cut-off frequency.

Rectangular Waveguides (3/6)

TM-modes



Source: Zhang, *Electromagnetic Theory for Microwaves and Optoelectronics*, 2nd ed., Springer

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There is no TM_{00}-mode, neither do TM_{01} or TM_{10} exist. For $n=m=0$, all field components become zero.

Note that TE_{10}-mode remains the dominant mode, it is the mode with the lowest cut-off frequency.

Rectangular Waveguides (4/6)

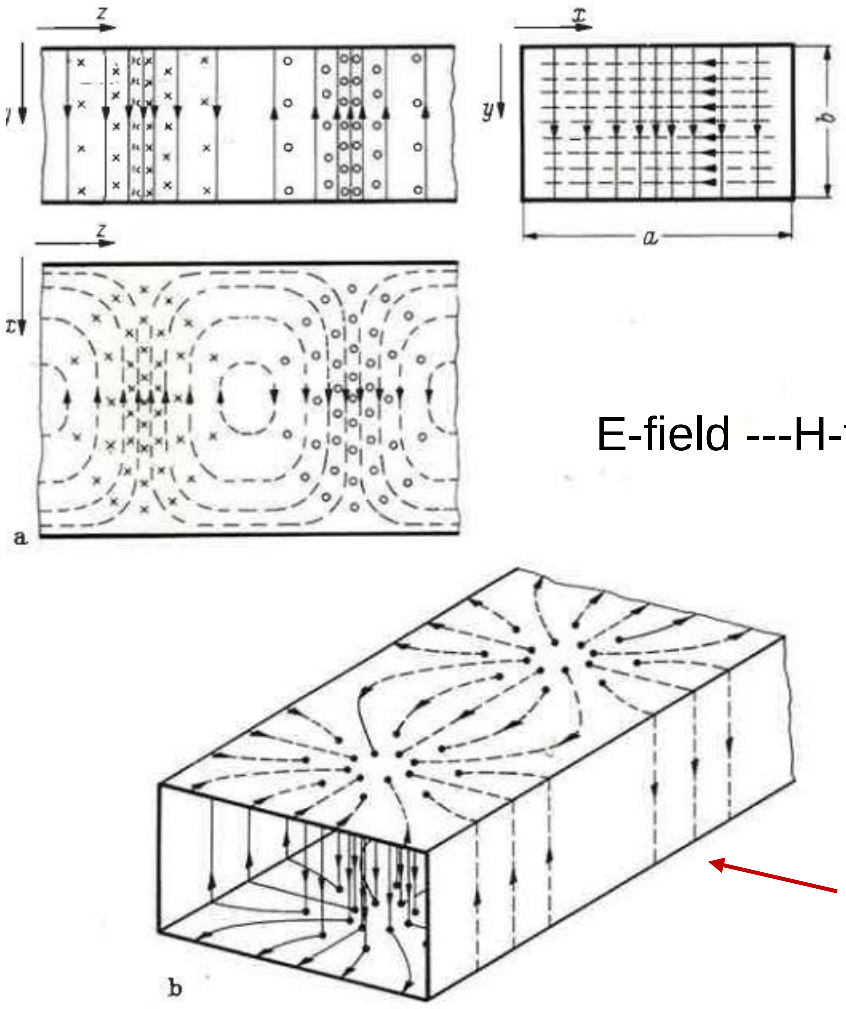
Field pattern for TE_{01}-mode

What do we actually see?

Side view narrow *(y/z-plane)*

Side view broad *(x/z-plane)*

View from front (x/y-plane)



E-field ---H-field

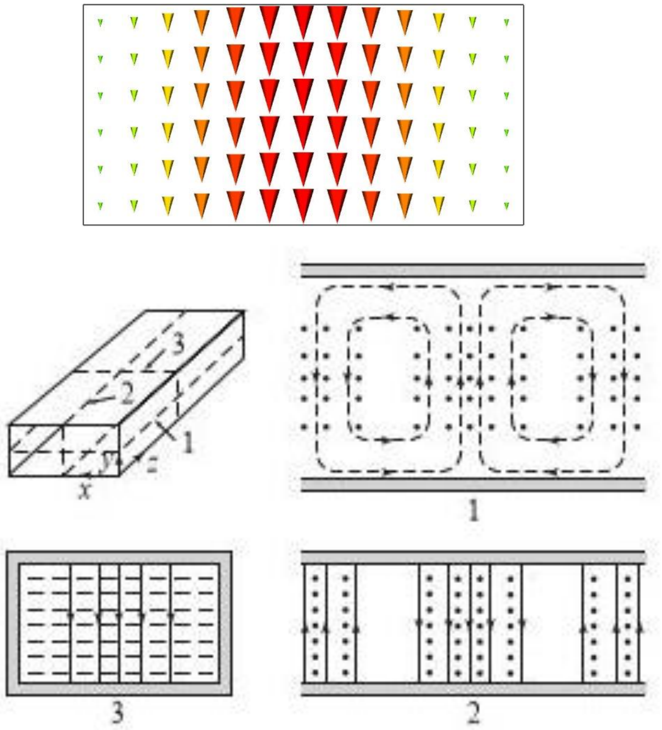
Wall currents and displacement currents inside the waveguide.

source: Zinke/Brunswig, *Hochfrequenztechnik,* 5th ed., Springer

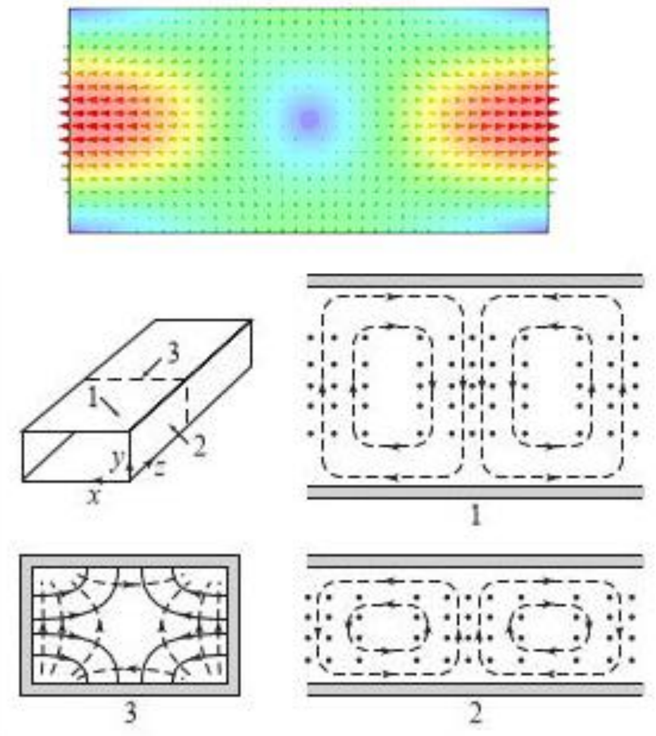
Rectangular Waveguides (5/6)

Fundamental and higher waveguide modes
and field patterns for TE-modes

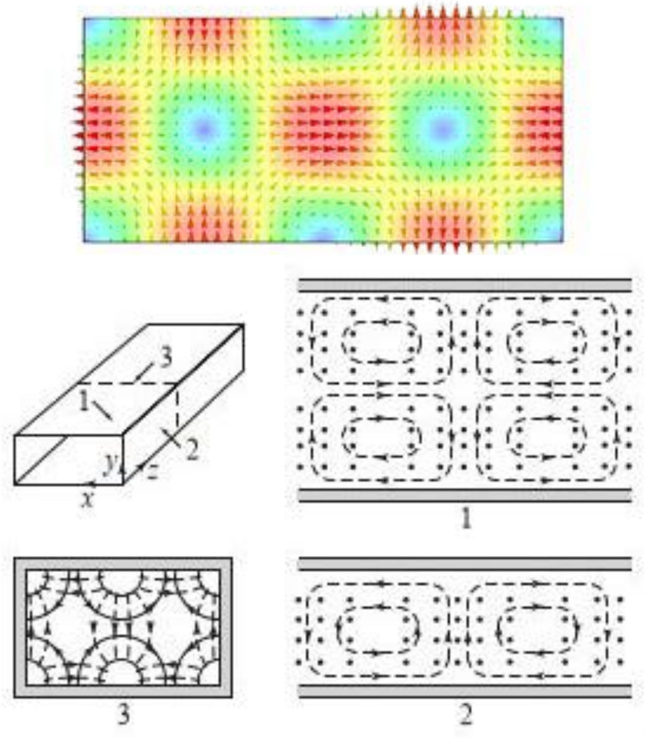
TE_{10}-mode



TE_{11}-mode



TE_{21}-mode

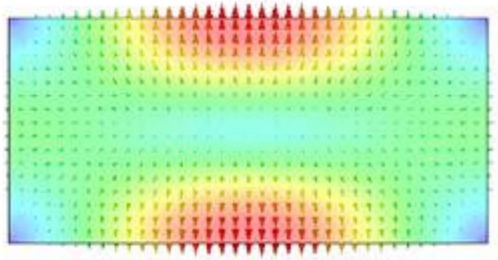


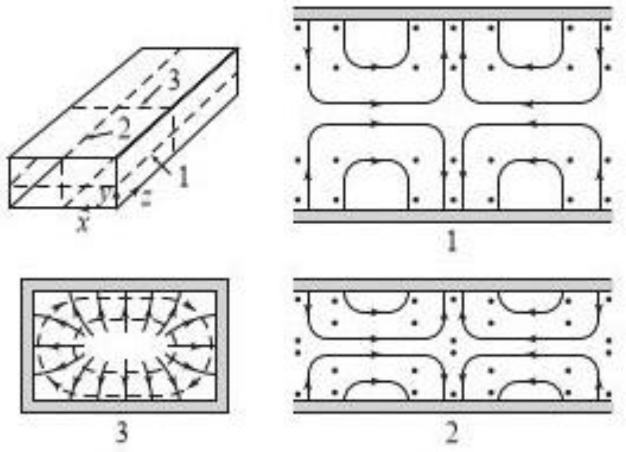
Simulaton pictures: courtesy E. Jensen, field pattern source: Pozar, *Microwave engineering*, 4th ed., Wiley

Rectangular Waveguides (6/6)

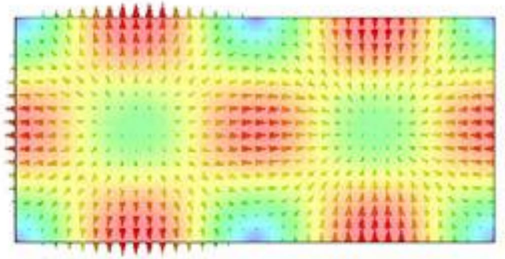
Fundamental and higher waveguide modes
and field patterns for TM-modes

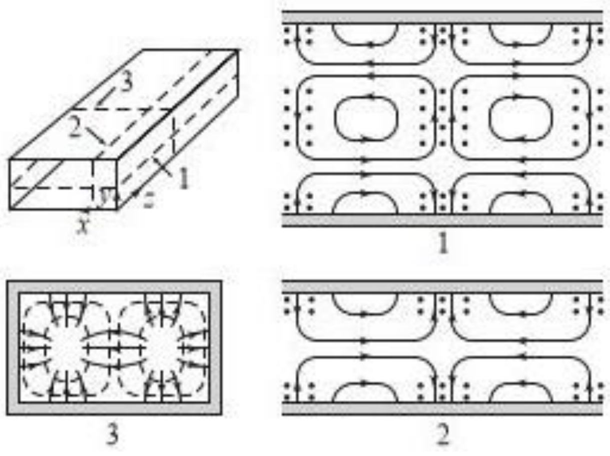
TM$_{11}$-mode



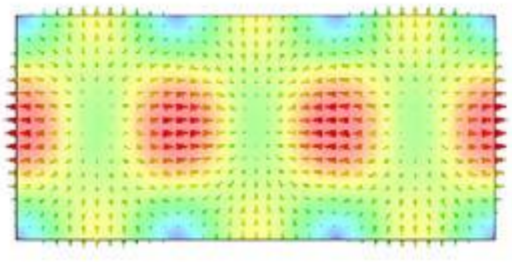


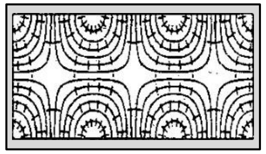
TM$_{21}$-mode





TM$_{31}$-mode





Simulaton pictures: courtesy E. Jensen, field pattern source: Pozar, *Microwave engineering*, 4th ed., Wiley, TM31-mode from Lee et al., MTT-33, 1985

Coaxial line installation

Amplifiers of the 200 MHz travelling wave system in the CERN SPS and their coaxial power lines leading to the underground installation (feeder lines)

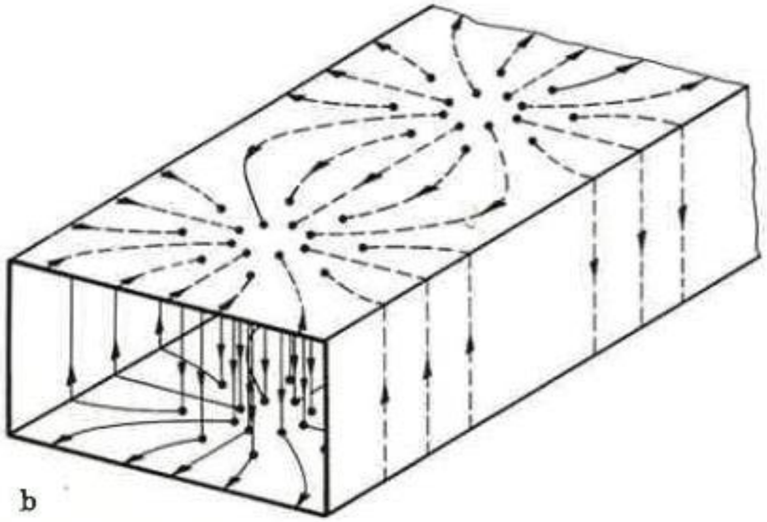
Quiz!

Test your knowledge.

Quiz no. 4

- **Waveguides are used for TEM signal transport**

  - true
  - false



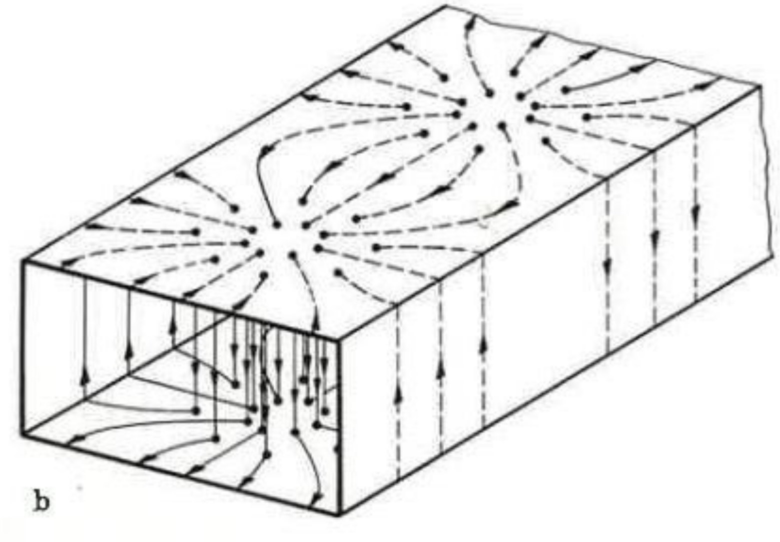
- **In a waveguide, lossy wall-currents exist in…?**

  - circular waveguides due to the shape of their outer conductor
  - rectangular waveguides due to the shape of their outer conductor
  - rectangular waveguides only, but not in circular waveguides and not in elliptical waveguides
  - all waveguides

Quiz no. 4

- **Waveguides are used for TEM signal transport**

  - true
  - false (correct answer)



- **In a waveguide, lossy wall-currents exist in…?**

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  - rectangular waveguides only, but not in circular waveguides and not in elliptical waveguides
  - all waveguides (correct answer)

Cavity resonators

Microwave resonators can be built from waveguides by closing the open ends.

We call this an *RF-resonator* or a *cavity*.

Source: Zhang, *Electromagnetic Theory for Microwaves and Optoelectronics*, 2^{nd} ed., Springer

- A cavity stores electric and magnetic energy inside the hollow body.
- The frequency of the electromagnetic field resonance depends on the cavity dimensions.

- Same as with the rectangular and the circular waveguide, the electromagnetic field has to fulfil the boundary conditions on the resonator walls. The field builds up in resonant modes.

- Cavities that are used for the acceleration of particles in our accelerators are mostly of a cylindrical and flat shape. This is why we call them pillbox cavities.

- We will later see how resonances can be excited in cavities by feeding RF-signal in, and how such a cavity can be used in accelerators for particle acceleration.

Cavity resonators and Q-value (1/2)

- Resonators are classified by their quality factor *Q*.
- The quality factor (or *Q-value*) can be used as a measure of "how well the cavity is resonating" (more details will come).

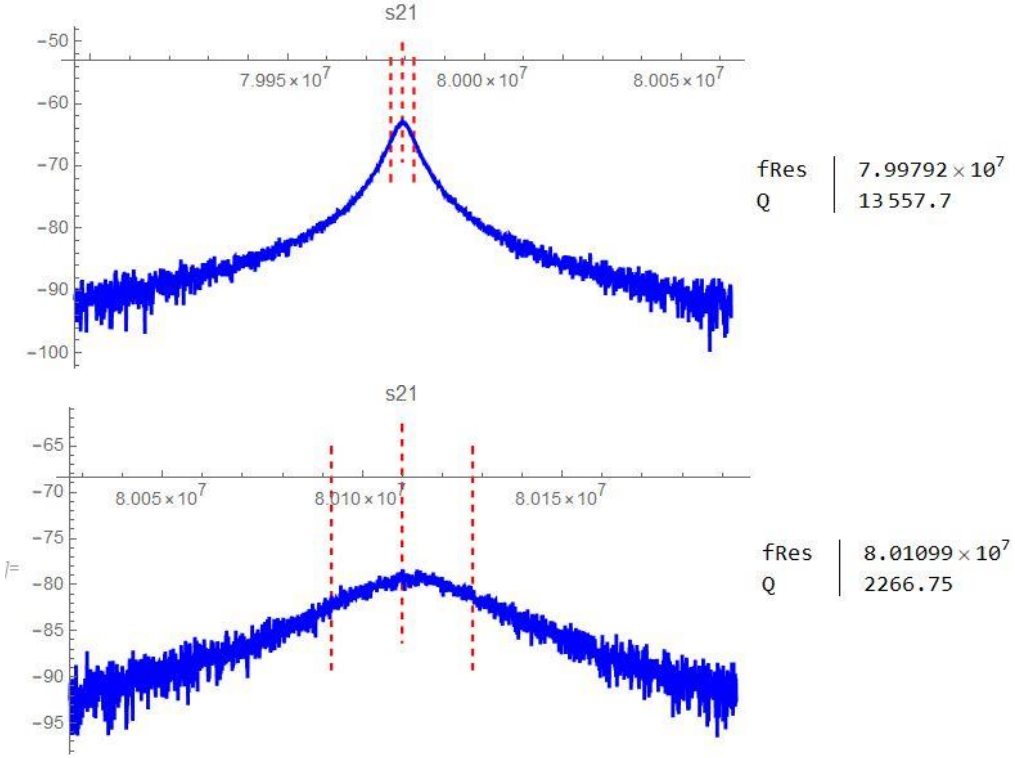
define $$Q_0 = 2\pi \frac{\text{energy stored in the resonator}}{\text{energy dissipated in the resonator}}$$

Q_0 = unloaded *Q*

- High *Q*-value is desired in accelerating cavities; *Q*-value is one of the accelerator efficiency figures-of-merit.
- *Q*-value reduces e.g. due to the power dissipated in the metallic walls or other loss mechanisms.
- The connection of the cavity resonator to the outer world will reduce its *Q*-value as well *(we say "it loads the cavity with an additional loss mechanism)*.
- If the cavity is filled with a dielectric, this will produce additional losses → we don't have this in accelerators.

Cavity resonators and Q-value (2/2)

Example: measurements taken on the PS 80MHz pillbox cavity

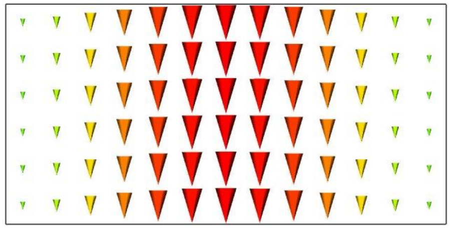


- We tested this cavity for *Q*-deterioration and shifting of its fundamental mode (~80 MHz).
- *Q*-values obtained from 3-dB-measurement (see dashed red lines, method will be introduced later).
  → Plot on the top has a higher *Q*-value than the plot below.

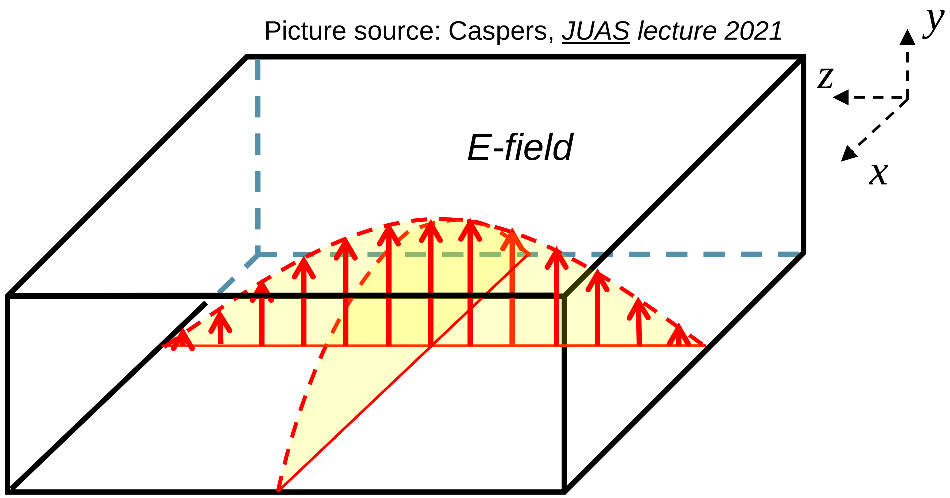
Rectangular Cavity Resonators (1/3)

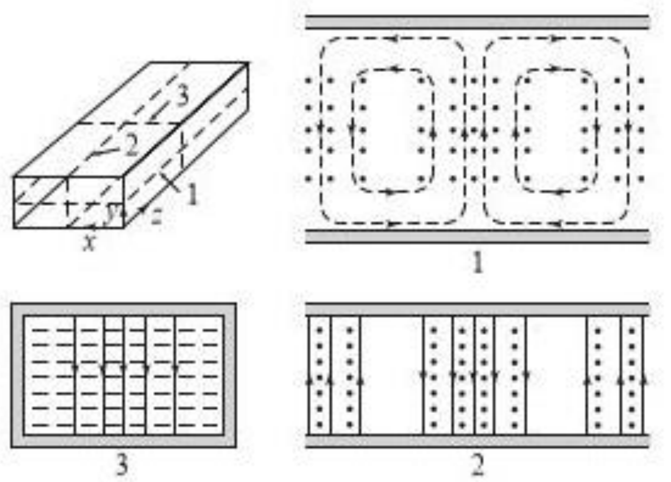
From the field pattern of the TE10-mode in the rectangular waveguide → imagine that the resonant field in a rectangular resonator will build up in a similar way.

Waveguide TE_{10}-mode



Picture source: Caspers, *JUAS lecture 2021*





Source: Pozar, *Microwave engineering*, 4th ed., Wiley

- The field is building up in a standing wave pattern, using sine and cosine functions.
- From the boundary condition on the resonator wall, the electric field has to be zero, thus we can see multiples of electric field maxima.
- The *mode indices* are counting the maxima of the field along one axis (we see either a TE_{xyz} or a TM_{xyz}- mode.
- The mode shown is a TE_{101}-mode.

Rectangular Cavity Resonators (2/3)

Source: Pozar, *Microwave Engineering*, 4th ed., Wiley

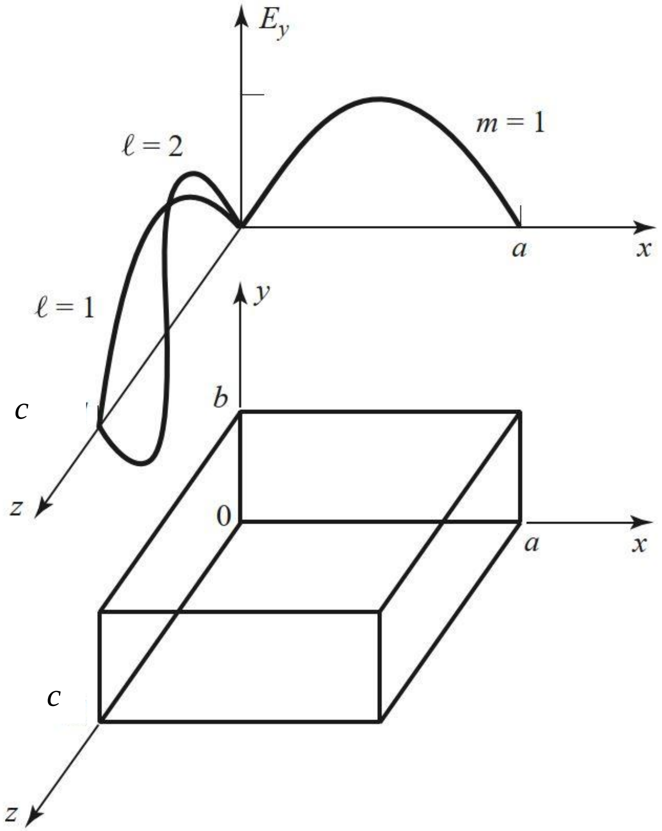
Formulae for a rectangular resonator
(for derivation see lecture of A. Mostacci or textbooks).

- resonant wavelength: $\lambda_0 = \dfrac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{l}{c}\right)^2}}$

- Resonant frequencies for TE_{mnl}-mode and TM_{nml}-modes:

$$f_{mnl} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2}$$

- Q-value for TE_{101}: $Q_{TE101} = \dfrac{\lambda_0}{\delta}\dfrac{b}{2}\dfrac{(a^2+b^2)^{3/2}}{c^3(a+2b)+a^3(c+2b)}$



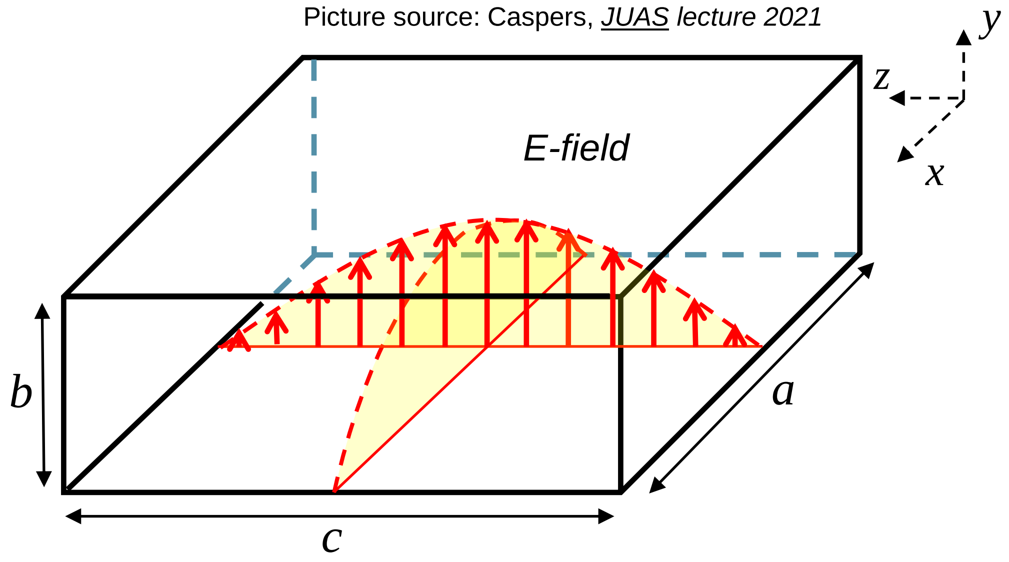
TE_{101}-mode is the lowest resonant mode, hence also the dominant mode in this resonator.

Rectangular Cavity Resonators (3/3)

For the case a=c, the formulae simplifies strongly.

- resonant wavelength: $\lambda_0 = \sqrt{2}a$
- Q-value for TE_{101}: $Q_0 = \frac{1}{\delta}\frac{ab}{a+2b}$

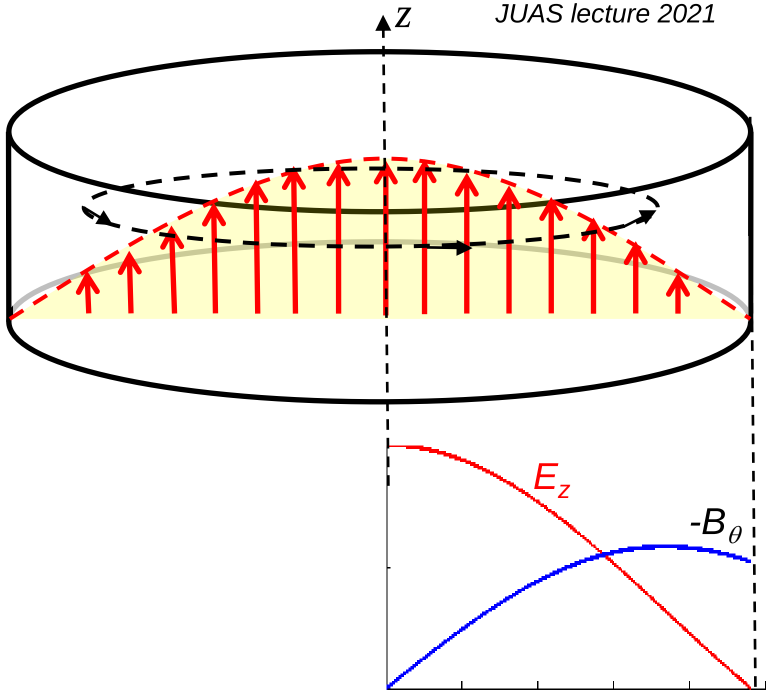
Picture source: Caspers, *JUAS lecture 2021*



Pillbox Cavity Resonators (1/6)

A cylindrical cavity resonator can be seen as a section of circular waveguide shorted at both ends.

Picture source: Caspers, *JUAS lecture 2021*



From the TE_{11}-mode (dominant mode for circular waveguide)

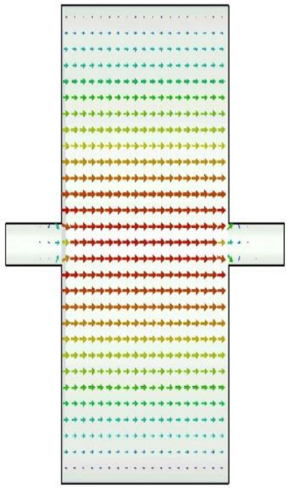
→ TE_{111}-mode is the dominant TE-mode for a cylindrical cavity.

→ TM_{010}-mode is the dominant TM-mode for a cylindrical cavity.

→ *TM_{010}-mode mode is used for acceleration as it has a large electric field along the z-axis.*

→ *Electric field is described by Besselfunction J_0.*

→ *TM_{010}-mode and pillbox mode indices?*

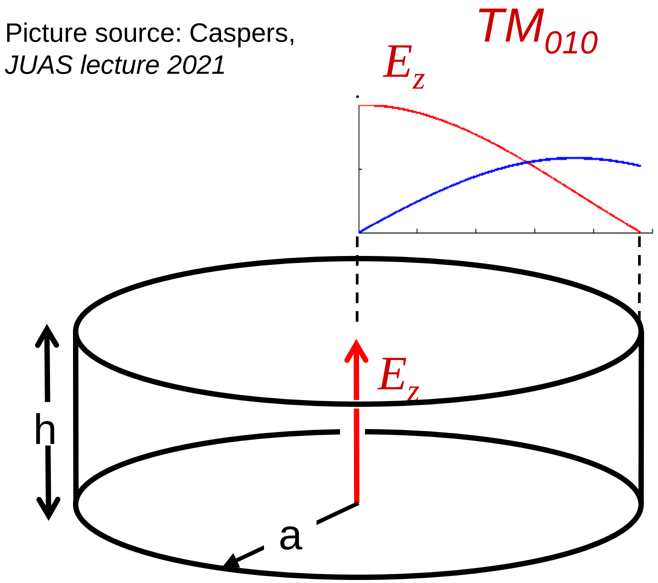


Pillbox cavity is flat compared to a "long" cylindrical cavity.
Also, beam-pipe connection is needed.

Simulation: P. Kramer

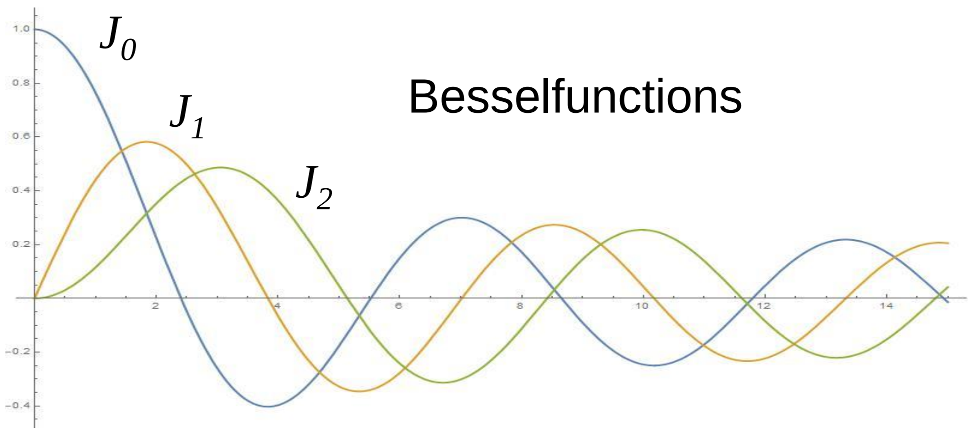
Pillbox Cavity Resonators (2/6)

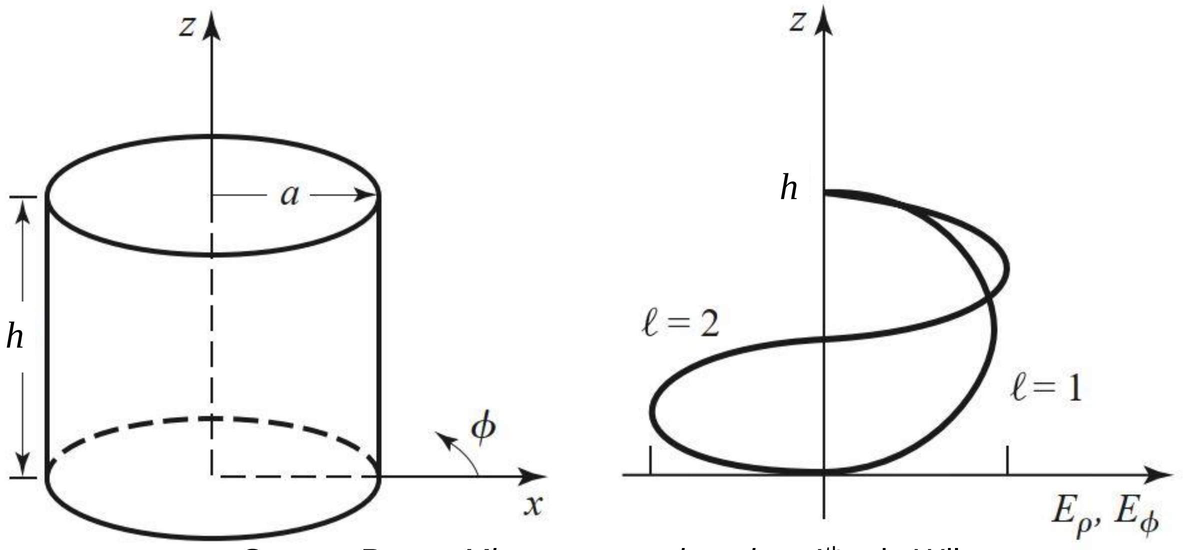
Picture source: Caspers, *JUAS lecture 2021*



Mode indices of a pillbox cavity are as follows:

- The electric field on the cavity's axis following the Besselfunction J_0
- The *first mode-index gives the order of the Bessel function*.
- The *second mode-index indicates the root of the Bessel function* which is the number of its zero-crossings.
- The *third mode-index is the number of half waves (maxima) along the z-axis*.

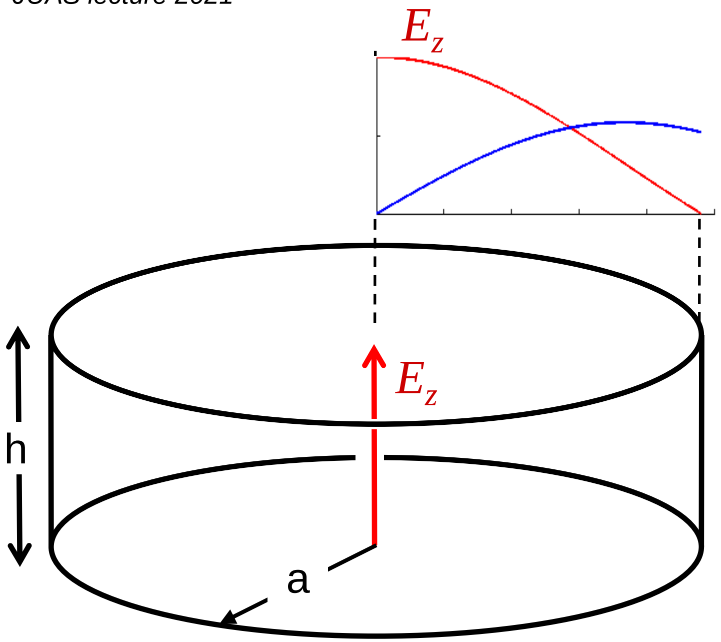




Source: Pozar, *Microwave engineering*, 4th ed., Wiley

Pillbox Cavity Resonators (3/6)

Picture source: Caspers, *JUAS lecture 2021*



Formulae for a circular cavity resonator (for derivation see lecture of A. Mostacci or textbooks).

- Resonant frequencies for TE_{nml}-mode:

$$f_{\mathrm{TE},nml} = \frac{c_0}{2\pi}\sqrt{\left(\frac{p'_{nm}}{a}\right)^2 + \left(\frac{l\pi}{h}\right)^2}$$

- Resonant frequency for TM_{nml}-modes:

$$f_{\mathrm{TM},nml} = \frac{c_0}{2\pi}\sqrt{\left(\frac{p_{nm}}{a}\right)^2 + \left(\frac{l\pi}{h}\right)^2}$$

Note that the TM_{010}- mode is *independent of the cavity height h* (until the TE_{111} mode shows up, roughly at *2a/h=1*).

Pillbox Cavity Resonators (4/6)

Recall the Bessel function and the derivative of the Bessel function with their roots:

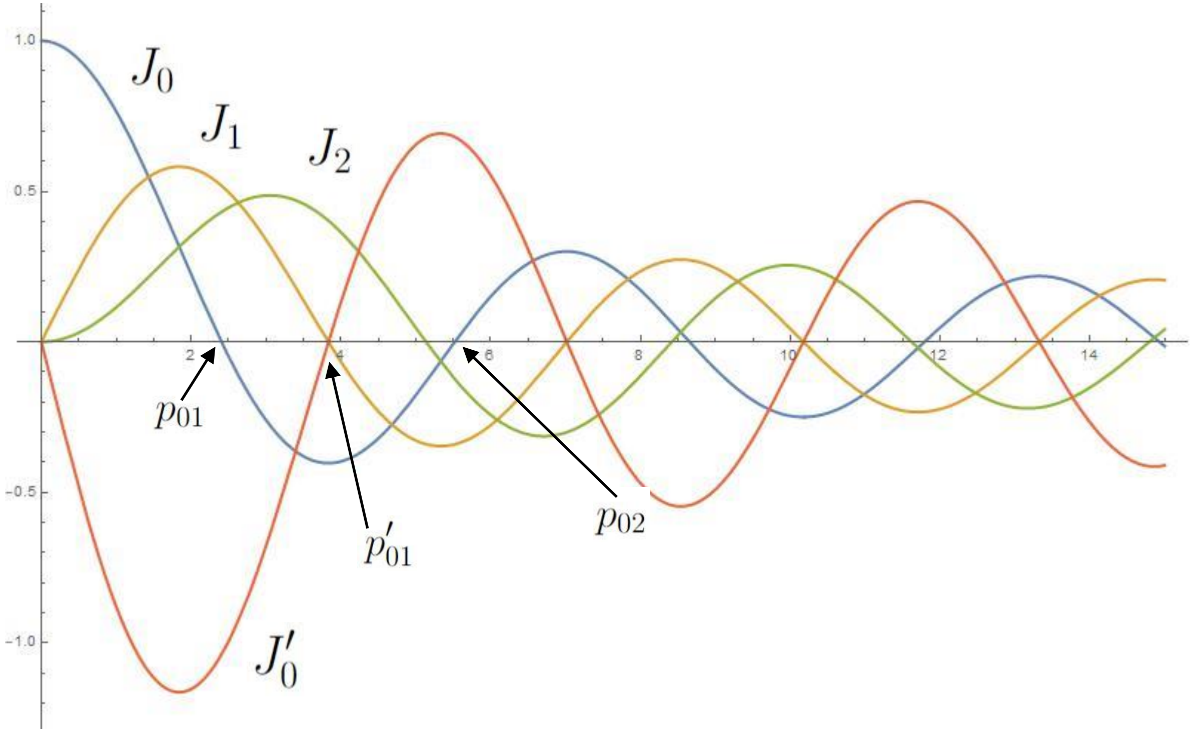


TABLE 3.3 Values of p'_{nm} for TE Modes of a Circular Waveguide

n	p'_{n1}	p'_{n2}	p'_{n3}
0	3.832	7.016	10.174
1	1.841	5.331	8.536
2	3.054	6.706	9.970

TABLE 3.4 Values of p_{nm} for TM Modes of a Circular Waveguide

n	p_{n1}	p_{n2}	p_{n3}
0	2.405	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620

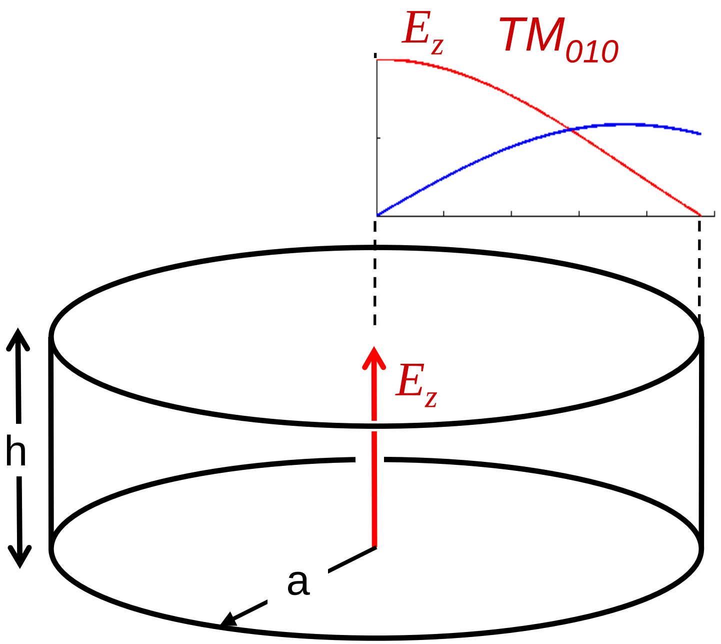
Source: Pozar, *Microwave Engineering*, 4th ed., Wiley

p_{01} = 1st root of Bessel function of first type J_0

p_{02} = 2nd root of Bessel function of first type J_0

p'_{01} = 1st root of derivative of Bessel function of first type J'_0

Pillbox Cavity Resonators (5/6)



For the case of the TM_{010}-mode, we have no dependence on cavity height, so we get a much simpler formulae:

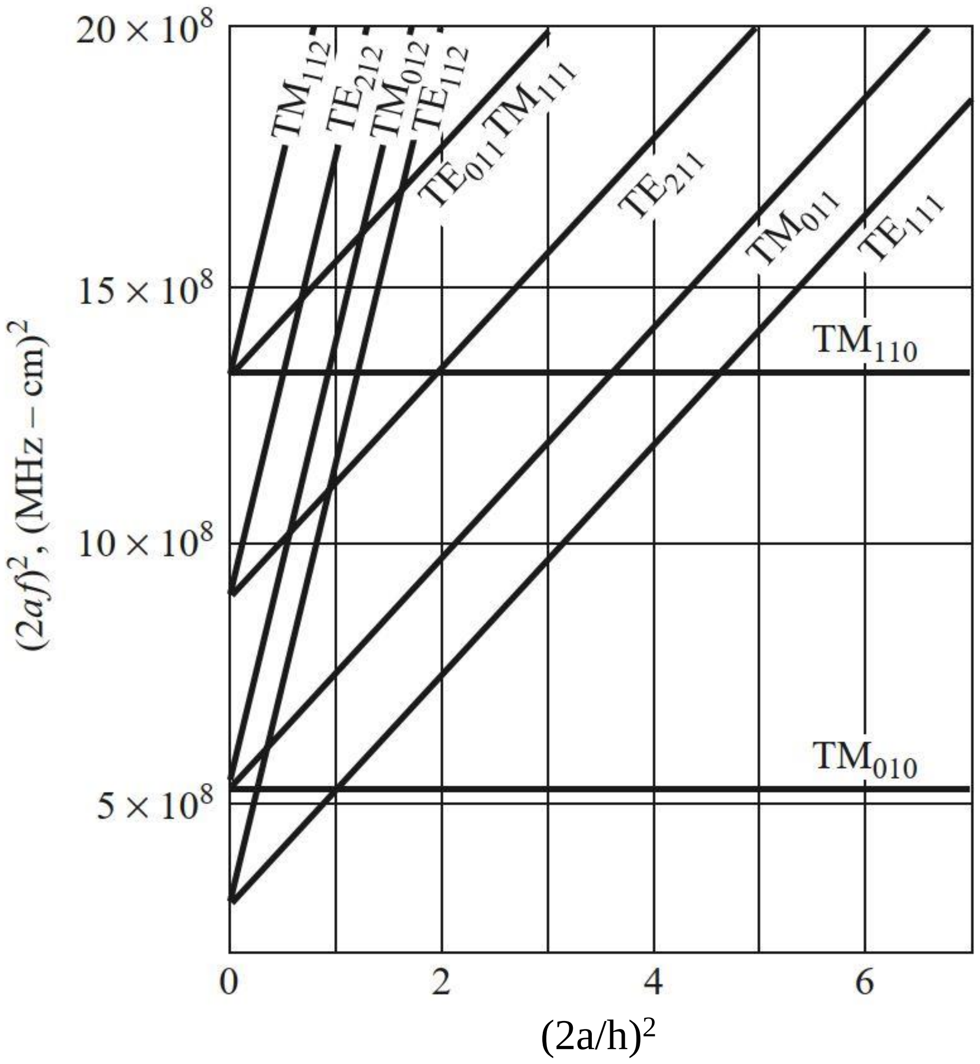
$$0.383\,\lambda_{\mathrm{TM},010} = a$$

$$Q = \left(0.383\frac{\lambda_{\mathrm{TM},010}}{\delta}\right)\left[1 + \left(0.383\frac{\lambda_{\mathrm{TM},010}}{h}\right)\right]^{-1}$$

$$Q = \frac{0.383\,\lambda_{\mathrm{TM},010}}{\delta}\left[1 + \frac{a}{h}\right]^{-1} = \frac{a}{\delta}\left[1 + \frac{a}{h}\right]^{-1}$$

Remember? Skindepth is: $\delta = \sqrt{\frac{2}{\omega\sigma\mu}}$

Pillbox Cavity resonators (6/6)



Resonant chart for a general cylindrical cavity showing the excited modes as a function of cavity dimensions.

→ 1st TE-mode is TE111

→ 1st TM-mode is TM010, and shows up for ratios $(2a/h)^2 > 1$

Source: Pozar, *Microwave engineering*, 4th ed., Wiley

Thank you for your attention.

Let's have a break!

Further reading…

1. POZAR, David M., *"Microwave Engineering"*, 4th edition, Wiley and sons.

2. ZHANG, Keqian, "*Electromagnetic Theory for Microwaves and Optoelectronics*", 2nd edition, Springer.

3. BHAT, Shiban, *"Stripline-like transmission Lines for Microwave Integrated Circuits"*, New Age International Publishers.

4. HOFFMANN, *"Integrierte Mikrowellenschaltungen"*, Springer.

5. SAAD, *"Microwave Engineer's Handbook",* Vol.1, Artech House Inc.