RF Engineering
Scattering (S)-Parameters

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Outline and Objectives

- **A brief recap on electrical networks**
  - A simplified way to describe electrical, electronics and RF circuits
  - Electrical network composed out of lumped and distributed elements
  - Two-port RC-filter example using admittance (Y) parameters

- **Introduction to scattering (S) parameters**
  - General concept of incident / reflected waves scattered at the ports of an RF network
  - Reference impedance Z_0
  - 1-port and 2-port S-parameters
  - Properties of the S-matrix: reciprocity, symmetry, losses
  - A few S-matrix examples for RF networks with 1-, 2-, 3-, and 4-ports
  - S-parameters in practice: the SnP Touchstone file format
  - Cascading ("chaining") 2-port S-matrizes
  - General n-port networks

Please note the references to the JUAS proceedings!

JUAS Proc. 2024
II.2.x.y.z, p. nnn

Electrical Networks (1)

- **The electromagnetic behavior or RF circuits and systems, like any other electrical / electronics circuit or system can be described by Maxwell's equations**

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon}, \qquad \boldsymbol{\nabla} \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{c^2 \partial t} = \mu \boldsymbol{J}, \qquad \boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

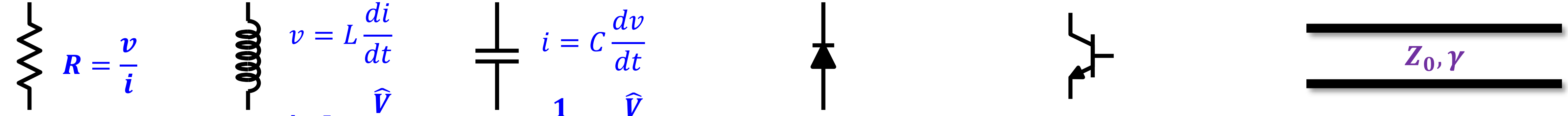
  - **These equations need to be solved, taking all the boundaries and materials into account**
- **However, this is far too complicated and inconvenient for most practical situations!**
  - **simplified electrical network description based on approximative lumped or distributed elements**
    - With given characteristics and values of each circuit element represented by a symbol in an electrical network, following the laws of *Ohm* and *Kirchhoff*. Here some examples:

lumped, passive, linear: resistor R, inductor L, capacitor C

lumped, passive, non-linear: diode D

lumped, active, non-linear: transistor T

distributed, passive, linear: transmission-line TL



$$\boldsymbol{R} = \frac{\boldsymbol{v}}{\boldsymbol{i}}$$

$$v = L\frac{di}{dt}, \qquad \boldsymbol{j\omega L} = \frac{\widehat{\boldsymbol{V}}}{\widehat{\boldsymbol{I}}}$$

$$i = C\frac{dv}{dt}, \qquad \frac{\boldsymbol{1}}{\boldsymbol{j\omega C}} = \frac{\widehat{\boldsymbol{V}}}{\widehat{\boldsymbol{I}}}$$

$$i = I_S\left(e^{v/V_T} - 1\right)$$

npn transistor

$$\boldsymbol{Z_0}, \boldsymbol{\gamma}$$

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Electrical Networks (2)

- **Complex electrical, electronics and RF systems are divided into functional blocks, i.e., networks, with n-ports, each port has two terminals**
  - **The two-port network is most popular, typical examples for two-port networks are filters, attenuators, amplifiers, etc.**

Y- parameters

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

I_1, I_2: dependent
V_1, V_2: independent

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \quad Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$
$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} \quad Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

- **The characteristic behavior of the n-port network is defined by a set of $2n$ parameters, linked to their ports.**
  - **A two-port network has four parameters as port voltages (V_1, V_2) and port currents (I_1, I_2), two are independent, the other two are dependent parameters.**
  - **Various combinations of dependent and independent port voltages and currents exist, accordingly various $n \times n$ matrix definitions for linear n-port networks exist, known as Y-, Z-, h- and g- parameters.**

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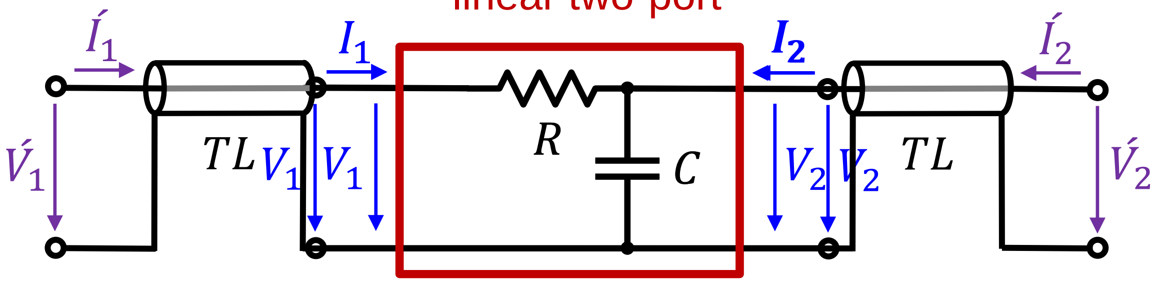
II.2.7.1, pp. 829/830

Electrical Networks (3)

- **It is more convenient to express the parameters for linear networks in the frequency domain**
  - **Avoids solving differential equations! Example: Simple RC two-port network**
    - ➢ Only for time-invariant and linear networks!



Y- parameters:

$$Y = \frac{1}{R}\begin{bmatrix} 1 & -1 \\ -1 & 1 + j\omega RC \end{bmatrix}$$

Z- parameters:

$$Z = \frac{1}{j\omega C}\begin{bmatrix} 1 + j\omega RC & 1 \\ 1 & 1 \end{bmatrix}$$

- **Voltage/current-based network parameters fail at high frequencies!**
  - **Now voltages and currents are a function of frequency (or time) AND space (location): $V(\omega, z)$, $I(\omega, z)$**
    - ➢ Originating from time/space varying EM-fields
  - **Example: RC two-port network embedded between transmission-lines**
    - ➢ While it is still possible to solve the network problem, it becomes complicated and cumbersome based on V and I
  - **The circuit may become unstable or might be damaged, when operating on a short or open end for characterizing the network parameters**
  - **Due to parasitic effects, a reliable measurement of V and I becomes almost impossible.**
- **Resolution: RF scattering parameters based on power-waves for linear networks which include distributed elements**
  - **The magnitude of a traveling wave is independent of the location z in a lossless transmission-line**

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Principle of Scattering (S)-Parameters

- **Analogy to optical waves**
  - **Light falls on a car window**
    - Some parts of the incident light is reflected (you see the mirror image)
    - Other parts of the light is transmitted through the window (you can still see objects inside the car)
  - **Optical reflection and transmission coefficients of the window glass define the ratio between reflected and transmitted light.**

- **Similar in RF networks:**
  The scattering (S)-parameters of an n-port RF network (DUT) is characterized by incident and reflected / transmitted (power) waves.

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II.2.7.1, p. 831

Generalized S-Parameters

- **for an arbitrary n-port microwave or RF network are defined by a set of normalized complex voltage waves:**

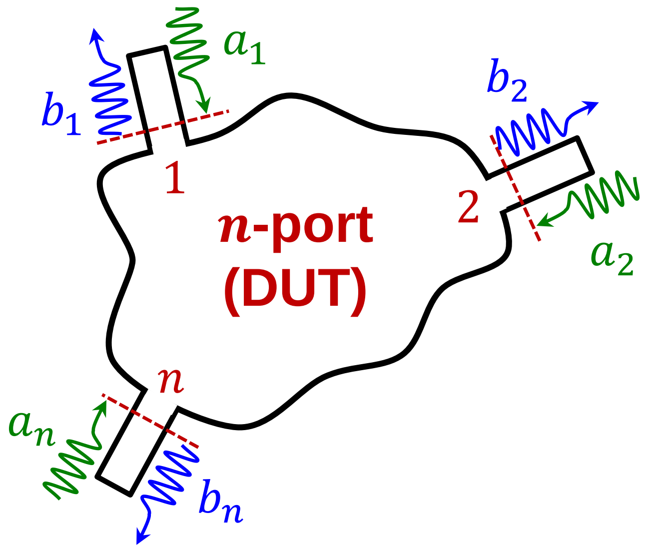
incident wave at port i:

$$a_i = \frac{V_i + Z_i I_i}{2\sqrt{\Re\{Z_i\}}} = \frac{V_i^{inc}}{\sqrt{\Re\{Z_i\}}}$$

reflected wave at port i:

$$b_i = \frac{V_i - Z_i^* I_i}{2\sqrt{\Re\{Z_i\}}} = \frac{V_i^{refl}}{\sqrt{\Re\{Z_i\}}}$$

Z_i^*: conjugate complex



  - **as incident a_i and reflected / transmitted b_i power waves at the i^{th} port of the network, defined by the terminal voltage V_i and current I_i, and an arbitrary reference impedance Z_i**
    - ➢ Please note the complex notation implies linear, time-invariant networks describe in the frequency-domain

- **Today, for most practical cases the RF network, also called "device under test" (DUT) is characterized by a vector network analyzer (VNA), connected with coaxial cables (transmission-lines) with a characteristic impedance of $Z_0 = 50\ \Omega$ to the ports.**
  - **Usually the S-parameters are defined for a port reference impedance: $Z_i = Z_0 = 50\ \Omega$**
  - **Some VNAs with a physical reference impedance of $Z_0 = 50\ \Omega$ allow a mathematical port impedance conversion to adapt to a port reference impedance $Z_i \neq 50\ \Omega$**

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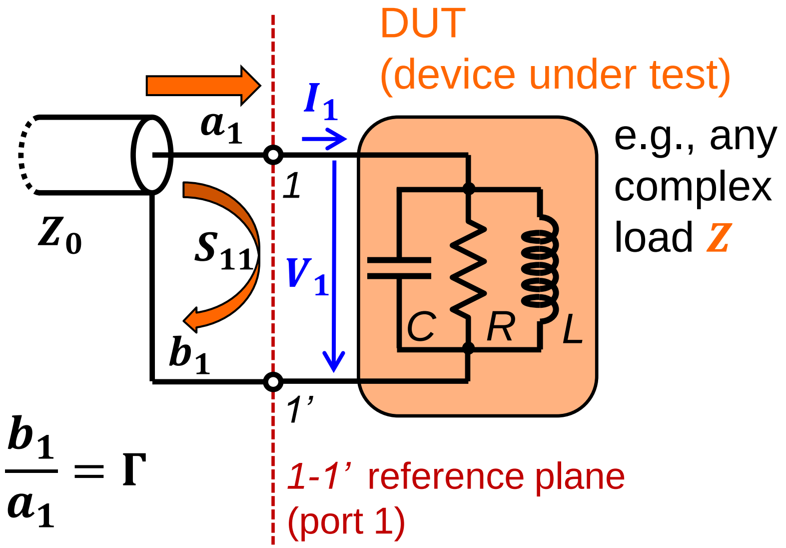
II.2.7.2, pp. 831/832

S-Parameters – 1-port

- **Electrical / electronics networks**
  - $1 \cdots n$-port electronics circuits
  - Defined by voltages $V_i(\omega)$ or $v_i(t)$ and currents $I_i(\omega)$ or $i_i(t)$ at the port terminals
  - Characterized by circuit matrices, e.g., Z, Y, h, etc.
- **RF / microwave networks**
  - $1 \cdots n$-port RF DUT circuit or subsystem, e.g., filter, amplifier, transmission-line, hybrid, circulator, resonator, etc.,
    which may include distributed elements
  - Defined by incident $a_i(\omega)$ and reflected / transmitted waves $b_i(\omega)$ at a reference plane s (physical position) at the ports.
  - Characterized by a scattering parameter (S-parameter) matrix of the reflected and transmitted power waves, typically as a function of the frequency $f = 2\pi/\omega$
  - Normalized to a reference impedance of typically $Z_0 = 50\ \Omega$



$$S_{11} = \frac{b_1}{a_1} = \Gamma$$

1-port RF network (DUT) example

- S-Parameters allow to characterize the DUT with the measurement equipment located at some physical distance
- All high frequency effects of distributed elements are included with respect to the reference plane

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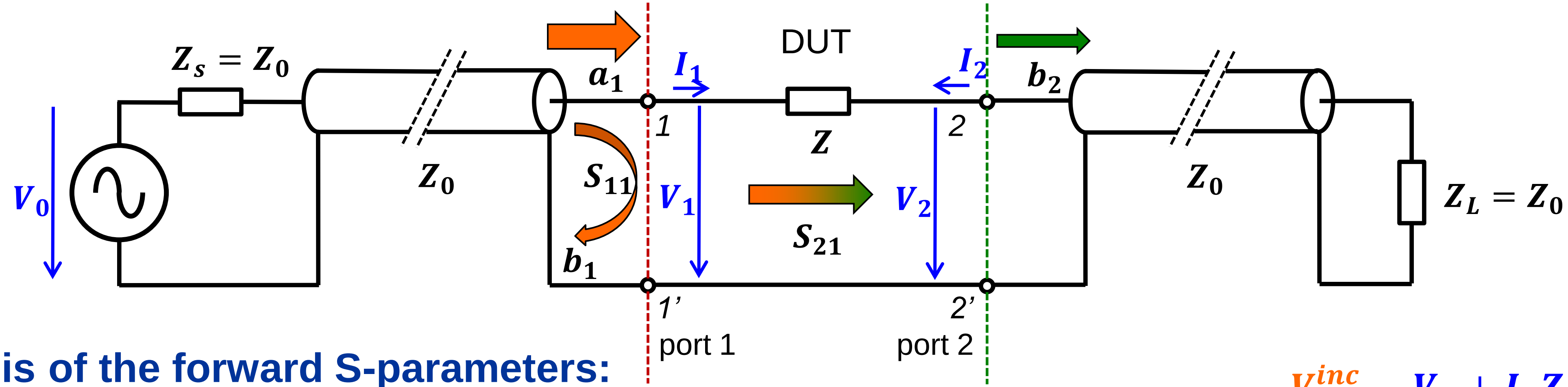
II.2.7.3, pp. 832/833

S-Parameters – 2-port (1)



- **Analysis of the forward S-parameters:**

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} \equiv \text{input reflection coefficient}$$

$$(Z_L = Z_0 \Rightarrow a_2 = 0)$$

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} \equiv \text{forward transmission gain}$$

  - 3 +-port networks still can be fully characterized with a 2-port VNA, but always remember:
  - **ALWAYS: Terminated unused ports in their characteristic impedance!**

- **Independent parameters:**

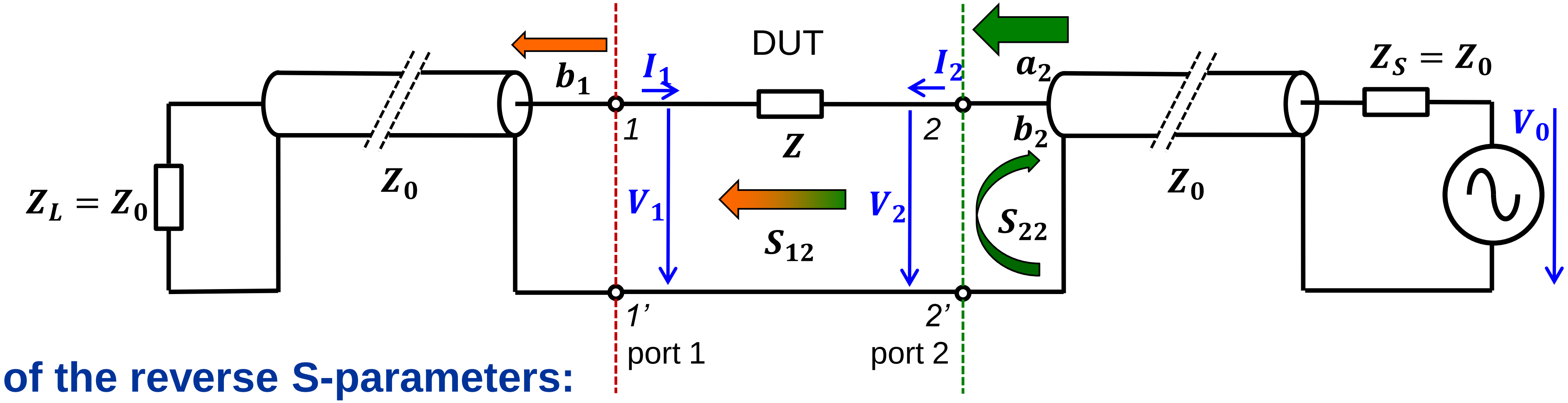
$$a_1 = \frac{V_1^{inc}}{\sqrt{Z_0}} = \frac{V_1 + I_1 Z_0}{2\sqrt{Z_0}}$$

- **Dependent parameters:**

$$b_1 = \frac{V_1^{refl}}{\sqrt{Z_0}} = \frac{V_1 - I_1 Z_0}{2\sqrt{Z_0}}$$

$$b_2 = \frac{V_2^{refl}}{\sqrt{Z_0}} = \frac{V_2 - I_2 Z_0}{2\sqrt{Z_0}}$$

S-Parameters – 2-port (2)



- **Analysis of the reverse S-parameters:**

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} \equiv \text{output reflection coefficient}$$

$$(Z_L = Z_0 \Rightarrow a_1 = 0)$$

$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0} \equiv \text{reverse transmission gain}$$

  - **Examples of 2-ports DUT: filters, amplifiers, attenuators, transmission-lines (cables), etc.**
  - **ALL ports ALWAYS need to be terminated in their characteristic impedance!**

- **Independent parameters:**

$$a_2 = \frac{V_2^{inc}}{\sqrt{Z_0}} = \frac{V_2 + I_2 Z_0}{2\sqrt{Z_0}}$$

- **Dependent parameters:**

$$b_1 = \frac{V_1^{refl}}{\sqrt{Z_0}} = \frac{V_1 - I_1 Z_0}{2\sqrt{Z_0}}$$

$$b_2 = \frac{V_2^{refl}}{\sqrt{Z_0}} = \frac{V_2 - I_2 Z_0}{2\sqrt{Z_0}}$$

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II.2.7.4, pp. 833-835

S-Parameters – 2-port (3)

- **Linear equations for a 2-port network (DUT):**

$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$

  – **with:**

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} \equiv \text{input reflection coefficient}$$

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} \equiv \text{output reflection coefficient}$$

impedance measurements

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} \equiv \text{forward transmission (insertion) gain}$$

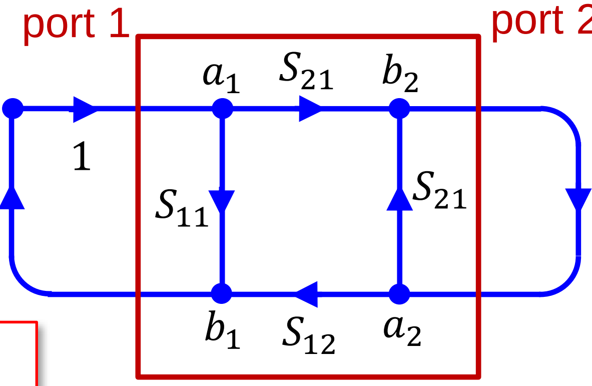
$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0} \equiv \text{reverse transmission (insertion) gain}$$

transmission (insertion) measurements

$$b_S = \frac{V_S\sqrt{Z_0}}{Z_S + Z_0}$$

$$\Gamma_S = \frac{Z_S - Z_0}{Z_S + Z_0}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$



signal flow graph (SFG) of a 2-port network

- port1: V_S with $Z_s \neq Z_0$
- port 2: $Z_L \neq Z_0$

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S-Parameters – n-port (1)

- **Reflection coefficient and impedance at the i^{th}-port of a RF network (DUT):**

$$S_{ii} = \frac{b_i}{a_i} = \frac{\frac{V_i}{I_i} - Z_0}{\frac{V_i}{I_i} + Z_0} = \frac{Z_i - Z_0}{Z_i + Z_0} = \Gamma_i$$

$$Z_i = Z_0 \frac{1 + S_{ii}}{1 - S_{ii}} \text{ with } Z_i = \frac{V_i}{I_i} \text{ being the input impedance at the } i^{th} \text{port}$$

- **Power reflection and transmission for a n-port network (DUT):**

$$|S_{ii}|^2 = \frac{\text{power reflected from port } i}{\text{power incident on port } i}$$

$$|S_{ij}|^2 = \text{transmitted power between ports } i \text{ and } j$$

with all ports terminated in their characteristic impedance Z_0 and $Z_s = Z_0$

Here the US notation is used, where power $= |a_i|^2$. European notation (often): power $= |a_1|^2/2$ These conventions have no impact on the S-parameters, they are only relevant for absolute power calculations

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The Scattering Matrix

- **Waves traveling towards the *n*-port:** $(\boldsymbol{a}) = (\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_2}, \dots \boldsymbol{a_n})$
- **Waves traveling away from the *n*-port:** $(\boldsymbol{b}) = (\boldsymbol{b_1}, \boldsymbol{b_2}, \boldsymbol{b_2}, \dots \boldsymbol{b_n})$

- **The relation between $\boldsymbol{a_i}$ and $\boldsymbol{b_i}$ $(\boldsymbol{i = 1 \dots n})$ can be written as a system of $\boldsymbol{n}$ linear equations**
  (a_i = the independent variable, b_i = the dependent variable)

one-port	$b_1 = S_{11}a_1 + S_{12}a_2 + S_{13}a_3 + S_{14}a_4 + \cdots$
two-port	$b_2 = S_{21}a_1 + S_{22}a_2 + S_{23}a_3 + S_{24}a_4 + \cdots$
three-port	$b_3 = S_{31}a_1 + S_{32}a_2 + S_{33}a_3 + S_{34}a_4 + \cdots$
four-port	$b_4 = S_{41}a_1 + S_{42}a_2 + S_{43}a_3 + S_{44}a_4 + \cdots$

  – **in compact matrix form follows**

$$(\boldsymbol{b}) = (\boldsymbol{S})(\boldsymbol{a})$$

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II.2.7.5, p. 837/838

Quiz (1)

1. Select all correct answers

- ○ Y- and Z-parameters of electrical networks require a reference impedance Z_0
- ✗ Scattering parameters of RF networks are based on
  normalized, complex power waves incident and reflected / transmitted at their ports
- ✗ DUT stands for "Device Under Test", as acronym for the RF network to be characterized
- ○ S-parameters are only defined for a reference impedance of $Z_0 = 50\Omega$.
- ✗ Unused ports in a S-parameter measurement setup
  always need to be terminated in their characteristic port impedance

S-Matrix Properties

- **A port of the network is matched if:** $\boldsymbol{S_{ii} = 0}$
  - **i.e., no reflections!**

- **A n-port is reciprocal if:** $(\boldsymbol{S})^{\boldsymbol{T}} = (\boldsymbol{S}) \Rightarrow \boldsymbol{S_{ij} = S_{ji}} \ \forall\, \boldsymbol{i, j}$ $(S)^T$: transpose

  Matrix symmetry

  - **Most passive components are reciprocal, e.g., resistor, capacitor, inductor, transformer, etc.**
    - But not components with inhomogeneous material properties, e.g., magnetized ferrites, plasma, etc.
  - **Active components, like amplifiers are non-reciprocal**

- **A n-port is symmetric if:** $\boldsymbol{S_{ij} = S_{ji}} \ \wedge \ \boldsymbol{S_{ii} = S_{jj}}$

  Matrix symmetry and electrical symmetry

  - **It needs to be reciprocal, and input and output reflection coefficient need to be equal.**

- **A n-port is passive and lossless if the matrix (S) is unitary:** $(\boldsymbol{S})^{\dagger}(\boldsymbol{S}) = (\boldsymbol{S})^{\boldsymbol{T}}(\boldsymbol{S})^{*} = (\boldsymbol{I})$

  $(S)^{\dagger} = (S^{*})^{T}$: conjugate transpose (*Hermitian*)
  (I): identity matrix

  - **Example: passive, lossless 2-port:**

$$(S^*)^T(S) = \begin{pmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{array}{c} \angle S_{11} - \angle S_{12} = \angle S_{21} - \angle S_{22} - \pi \\ |S_{11}| = |S_{22}|, \qquad |S_{12}| = |S_{21}| \\ |S_{11}| = \sqrt{1 - |S_{12}|^2} \end{array}$$

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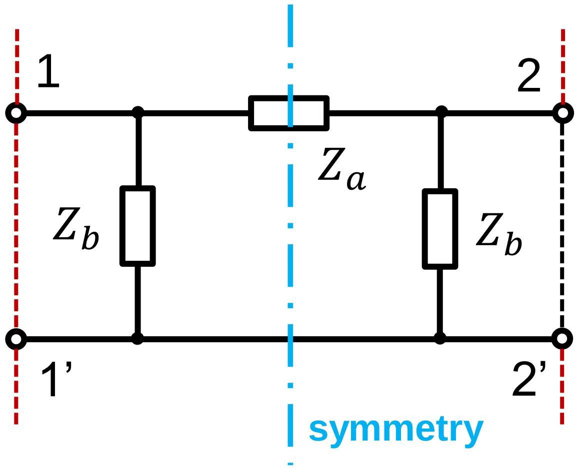
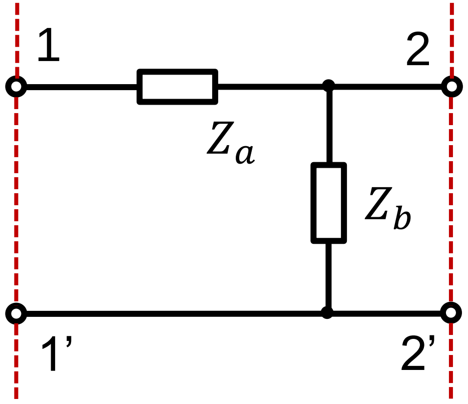
II.2.7.6, p. 838/839

Examples for Symmetry and Reciprocity

π-network



voltage-divider network



$$(S_\pi) = \frac{1}{\Delta}\begin{pmatrix} Z_a Z_b^2 - Z_0^2(Z_a + 2Z_b) & 2Z_0 Z_b^2 \\ 2Z_0 Z_b^2 & Z_a Z_b^2 - Z_0^2(Z_a + 2Z_b) \end{pmatrix}$$

with: $\Delta = (Z_a + Z_b)[Z_a Z_b + Z_0(Z_a + 2Z_b)]$

$S_{12} = S_{21} \wedge S_{11} = S_{22} \Rightarrow$ reciprocal and symmetric

$$(S_{div}) = \frac{1}{\Delta}\begin{pmatrix} Z_a Z_b - Z_0(Z_0 - Z_a) & 2Z_0 Z_b \\ 2Z_0 Z_b & Z_a Z_b - Z_0(Z_0 + Z_a) \end{pmatrix}$$

with: $\Delta = Z_0(Z_0 + Z_a) + Z_b(2Z_0 + Z_a)$

$S_{12} = S_{21} \wedge S_{11} \neq S_{22} \Rightarrow$ reciprocal, but not symmetric

- **Without prof: The S-matrix is always reciprocal for symmetric networks.**

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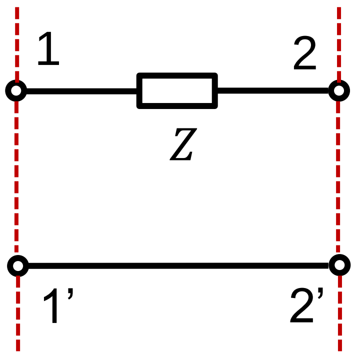
Numerical lossless / lossy Example

2-port series-network



$$(S) = \frac{1}{2Z_0 + Z}\begin{pmatrix} Z & 2Z_0 \\ 2Z_0 & Z \end{pmatrix}$$

$$S_{11} = S_{22} \wedge S_{12} = S_{21}$$

$\mathbf{Z = j\omega L = j10}$	$\mathbf{Z = R = 10}$
$\lvert S_{11}\rvert = \sqrt{1 - \lvert S_{12}\rvert^2}$	
$\frac{1}{\sqrt{101}} = \sqrt{1 - \left(\frac{10}{\sqrt{101}}\right)^2}$	$\frac{1}{11} \neq \sqrt{1 - \left(\frac{10}{11}\right)^2}$
$\angle S_{11} - \angle S_{12} = \angle S_{21} - \angle S_{22} - \pi$	
$\tan^{-1}(10) + \tan^{-1}\frac{1}{10} = -\tan\frac{1}{10} - \tan(10) - \pi$	$0 - 0 \neq 0 - 0 - \pi$
$\Rightarrow$ **lossless**	$\Rightarrow$ **lossy**

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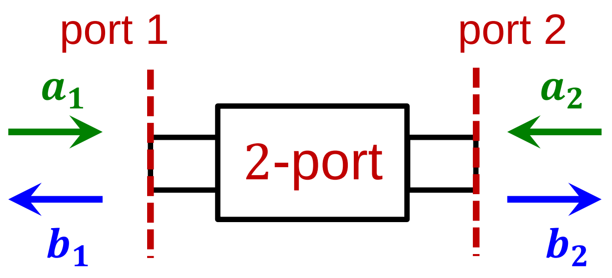
II.2.7.6, p. 841

Quiz (2)

Prompts		Possible Answers
A. matched		1. $S_{ii} = S_{ij}$
B. symmetric		2. $(S^*)^T(S) = (I)$
C. reciprocal		3. $S_{ij} = S_{ji} \wedge S_{ii} = S_{jj}$
D. passive and lossless		4. $S_{ii} = 0$
		5. $\Gamma = +j$
		6. $S_{ij} = S_{ji}$

Quiz (3)



3. Mark all correct answers for the S-parameters of a 2-port RF network

- ○ **a_1 and b_1 are independent parameters**
- ✗ **S_{11} is the input reflection coefficient**
- ✗ **a_1 and a_2 are the incident waves at port 1 and port 2, respectively.**
- ○ **b_1 and b_2 are the transmitted waves between port 1 and port 2, and vice versa.**
- ✗ **S_{21} and S_{12} are the forward and reverse transmission gains.**
- ✗ **To characterize the S-parameters at port 2, port 1 needs to be terminated in its characteristic port impedance.**

S-Matrix Examples – 1-port

- **Examples for 1-port S-matrices are any simple, passive (complex) impedances Z**
  - **Any R, L, C, RL, RC, LC and RLC circuit or any combinations of those elements leading to a single port network, which of course also my include distributed (transmission-line) elements**
  - **“Special” cases are:**
    - $Z = Z_0 \Rightarrow S_{11} = 0$ (matched, ideal termination)
    - $Z = 0 \Rightarrow S_{11} = -1$ (ideal short)
    - $Z = \infty \Rightarrow S_{11} = +1$ (ideal open)
  - **If $|S_{11}| > 1$ an active element is involved, e.g., a reflection amplifier**

$$(\boldsymbol{S}) = \boldsymbol{S_{11}} = \boldsymbol{\Gamma}$$

- **Strictly speaking, a simple RF resonator, e.g., a “pill-box” cavity, is a 3-port**
  - **One coaxial or waveguide port as RF power coupler, plus two beam (waveguide) ports.**
- **However, for many practical cases it can be treated as 1-port**
  - **The mode of interest,**
    e.g., TM010, is trapped with no or negligible fields contribution near the beam-ports
  - **We consider only a single coupler to characterize,**
    e.g., the TM010 mode in terms of a 1-port S-parameter measurement
    - Typically applying an RLC-parallel equivalent circuit

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II.2.7.7.1, pp. 842/843

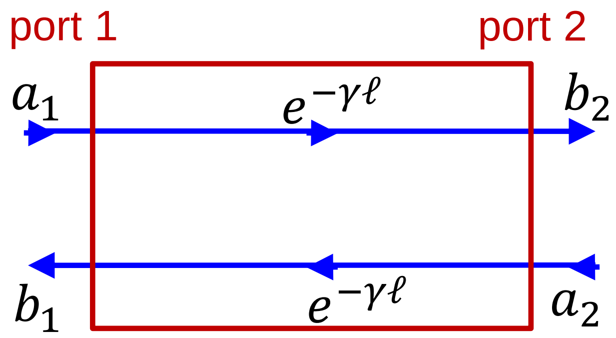
S-Matrix Examples – 2-port

- **Ideal (matched: $Z = Z_0$) transmission-line of length ℓ**

$$(S) = \begin{pmatrix} 0 & e^{-\gamma\ell} \\ e^{-\gamma\ell} & 0 \end{pmatrix}$$

$\gamma = \alpha + j\beta$: propagation constant
α: attenuation constant in [*Np/m*]
$\beta = 2\pi/\lambda$: phase constant [*rad/m*]

signal flow graph (SFG):



  – **For a lossless transmission-line:** $\alpha = 0 \;\Rightarrow\; |S_{21}| = |S_{12}| = 1$

  – **For a lossless line of length $\ell = \lambda/4$:** $(S) = \begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix}$

- **Ideal attenuator**

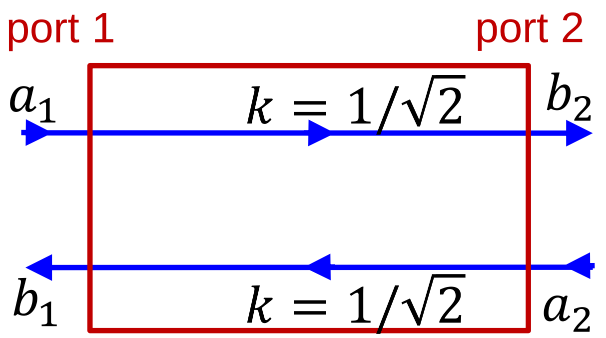
$$(S) = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$$

$k = V_2/V_1 = 10^{-(\Delta dB/20)}$: attenuation $k < 1, k \in \mathbb{R}$
$\Delta dB = 20\log_{10} V_1/V_2$: attenuation in *dB*
$\alpha = -\ln k$: attenuation in *neper*

SFG example: *3 dB* attenuator

$S_{21} = \dfrac{b_2}{a_1} = \dfrac{1}{\sqrt{2}}$

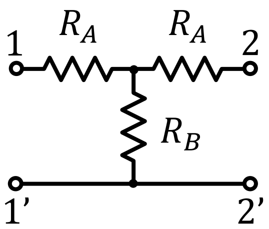
$S_{12} = \dfrac{a_2}{b_1} = \dfrac{1}{\sqrt{2}}$



T-attenuator:

$R_A = \dfrac{1-k}{1+k} Z_0$

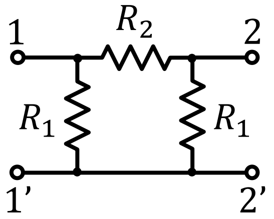
$R_B = \dfrac{2k}{1-k^2} Z_0$



π-attenuator:

$R_1 = \dfrac{1-k}{1+k} Z_0$

$R_2 = \dfrac{1-k^2}{2k} Z_0$



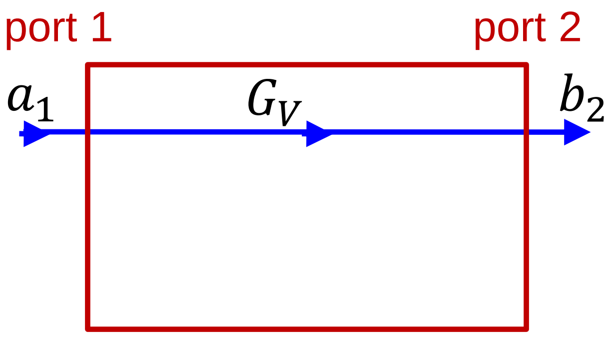
JUAS Proc. 2024
II.2.7.7.2, pp. 843/844

S-Matrix Examples – 2-port

- **Ideal amplifier (gain stage)**

$$(S) = \begin{pmatrix} 0 & 0 \\ G_V & 0 \end{pmatrix}$$

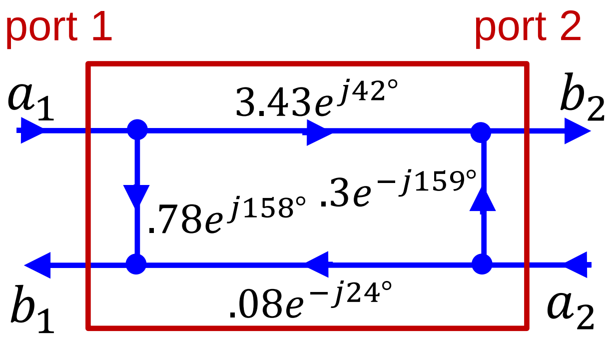
$G_V = V_{out}/V_{in} = 10^{g/20}$: voltage gain $G_V > 1$
$g = 20\log_{10} V_{out}/V_{in}$: voltage gain in *dB*



- **Low-noise RF transistor**

$$(S) = \begin{pmatrix} 0.78e^{j158°} & 0.08e^{-j24°} \\ 3.43e^{j42°} & 0.3e^{-j159°} \end{pmatrix}$$

Datasheet *Avago* VMMK-1218:
$f = 10\ GHz, Z_0 = 50\Omega, T_A = 25°C,$
$V_{ds} = 2V, I_{ds} = 20mA$



- ***Avago* VMMK-1218**
- **E-pHEMT GaAs FET**
  - The S-parameters are different at other frequencies and operational conditions
  - The transistor requires impedance matching networks at in- and output

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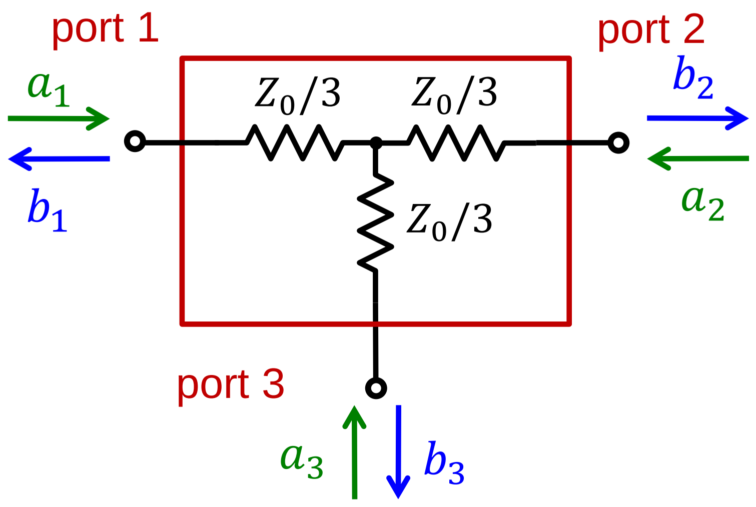
II.2.7.7.2, pp. 844/845

S-Matrix Examples – 3-port

- **3-port resistive power divider**



$$(S) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$b_1 = \frac{1}{2}(a_2 + a_3)$$
$$b_2 = \frac{1}{2}(a_1 + a_3)$$
$$b_3 = \frac{1}{2}(a_1 + a_2)$$

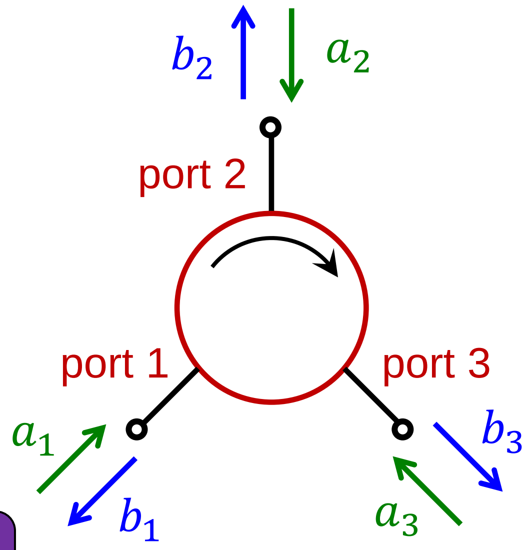
  – **The transfer-loss between *ij*-ports is 6 dB.**

- **Ideal circulator**

$$(S) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

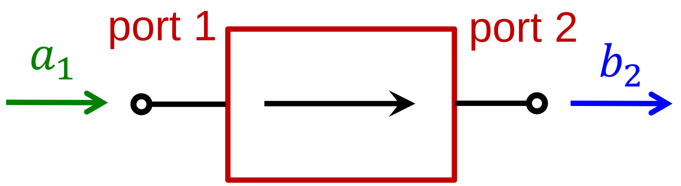
$$b_1 = a_3$$
$$b_2 = a_1$$
$$b_3 = a_2$$

  – **Matched, but not reciprocal**



- **Isolator, based on the circulator**
  – **Terminating, e.g., port 3 internally results in a 2-port, called isolator**

$$(S) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad b_2 = a_1$$



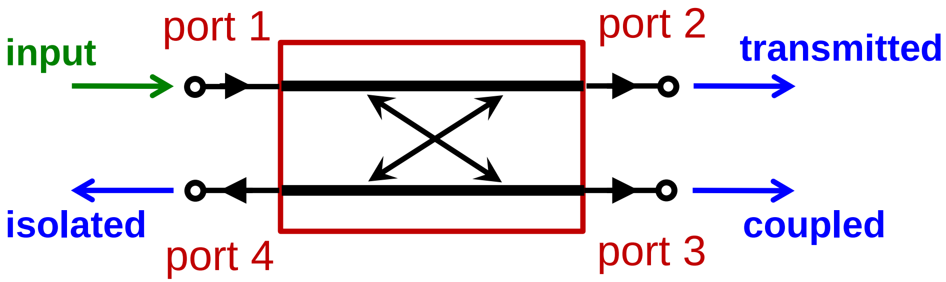
JUAS Proc. 2024
II.2.7.7.3, pp. 845/846

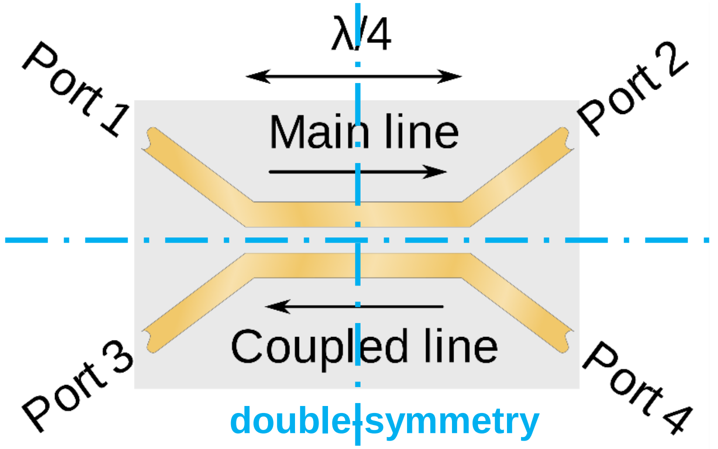
S-Matrix Example – 4-port

- **Ideal directional coupler**

$$(S) = \begin{pmatrix} 0 & \tau & \kappa & 0 \\ \tau & 0 & 0 & \kappa \\ \kappa & 0 & 0 & \tau \\ 0 & \kappa & \tau & 0 \end{pmatrix}$$

τ: transmission coefficient
κ: coupling coeffient





https://en.wikipedia.org/wiki/Power_dividers_and_directional_couplers

- **Matched, symmetric, infinite isolation**
- **Distinguishes between forward and reflected signal power on the main line port 1-2**
  - Forward power at coupled port 3
  - Reflected power at coupled port 4

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II.2.7.7.4, pp. 846/847

S-Matrix Example – 4-port

- **Ideal, lossless TEM directional coupler**
  - **The coupled port 3 is in-phase with the input port 1**
  - **The main line output port 2 is in quadrature phase ($\pi/2$ phase delay) with the input port 1**

$$(\boldsymbol{S}) = \begin{pmatrix} 0 & -jt & k & 0 \\ -jt & 0 & 0 & k \\ k & 0 & 0 & -jt \\ 0 & k & -jt & 0 \end{pmatrix}$$

Lossless conditions:

$$(S)^{\dagger}(S) = (I) \;\Rightarrow \sum_{k=1}^{N} S_{ki}S_{kj}^{*} = 0 \;\forall\; i \neq j \;\wedge \sum_{k=1}^{N} S_{ki}S_{ki}^{*} = \sum_{k=1}^{N} |S_{ki}|^2 = 1$$

$$\sum_{k=1}^{N} |S_{ki}|^2 = 1 \;\Rightarrow\; k^2 + t^2 = 1 \;\Rightarrow\; t = \sqrt{1-k^2}$$

  - **Lossless TEM direction coupler:**

$$(\boldsymbol{S}) = \begin{pmatrix} 0 & -j\sqrt{1-k^2} & k & 0 \\ -j\sqrt{1-k^2} & 0 & 0 & k \\ k & 0 & 0 & -j\sqrt{1-k^2} \\ 0 & k & -j\sqrt{1-k^2} & 0 \end{pmatrix}$$

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II.2.7.7.4, pp. 847/848

S-Parameters in Practice

- In practice, S-parameters are a function of the frequency: $S(f)$
  - Some instruments or applications can also provide time-domain S-parameters

- In most real-world practical situations, S-parameters are acquired by a measurement, e.g., characterization of a RF component or sub-system by a VNA.
  - By characterizing the DUT over a range of frequencies, $f_{min} < f < f_{max}$ in steps of Δf

- Also, numerical RF analysis tools (Qucs, ADS, Microwave Office, etc.) generate S-parameters through linear RF circuit / systems simulations.
  - Numerical EM software tools (CST, HFSS, etc.) and PCB tools (Cadence Allegro) can also generate S-parameters

- Both application types, VNA measurements and RF/EM simulation software exchange S-parameters on a file basis
  - The *SnP Touchstone* ASCII file format is de-facto the industry standard for S-parameters
  - Example *Touchstone s2p* file:

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II.2.7.8, pp. 848/849

SnP Touchstone S-Parameter Files

BRSTM_refline_2-4GHz_20001.s2p

```
!Keysight Technologies,P5024A,MY58100247,A.15.20.07
!Date: Wednesday, October 06, 2021 16:19:17
!Correction: S11(C  2-Port  )
!S21(C  2-Port  )
!S12(C  2-Port  )
!S22(C  2-Port  )
!S2P File: Measurements: S11, S21, S12, S22:
# Hz S  dB   R 50
2000000000 -2.5430779 -88.497566 -18.274168 38.763039 -18.26178 38.742687 -2.4251425 -85.792152
2000100000 -2.5365531 -88.473915 -18.266272 38.606499 -18.269154 38.716461 -2.4215624 -85.861908
2000200000 -2.5314419 -88.471634 -18.280306 38.559021 -18.258684 38.624985 -2.4253747 -85.806236
2000300000 -2.5216722 -88.617905 -18.269596 38.352692 -18.266785 38.342678 -2.4210978 -85.8368
2000400000 -2.5178108 -88.521271 -18.257862 38.298622 -18.275055 38.266792 -2.4244151 -85.914787
2000500000 -2.5327342 -88.484985 -18.263821 38.31945 -18.27046 38.20409 -2.4263382 -85.85775
2000600000 -2.5191193 -88.462044 -18.262426 38.195797 -18.246525 38.250874 -2.3989511 -85.885635
2000700000 -2.5219827 -88.44445 -18.253748 38.216946 -18.245417 38.138355 -2.4155877 -85.859711
2000800000 -2.5198817 -88.588783 -18.255999 37.902744 -18.257757 38.035915 -2.415241 -85.887199
2000900000 -2.5370498 -88.496101 -18.256392 38.004234 -18.258446 37.872906 -2.4170656 -85.854294
2001000000 -2.5363033 -88.544846 -18.252661 37.830475 -18.259445 37.925297 -2.4102871 -85.901245
2001100000 -2.5417418 -88.47406 -18.270754 37.731613 -18.252403 37.74762 -2.4158244 -85.822517
```

! header

format

2-port VNA file

Touchstone v1.1 example file

- v2.0 is different, file ext. **.ts*

frequency f $|S_{11}|$[dB] $\angle S_{11}$[deg]

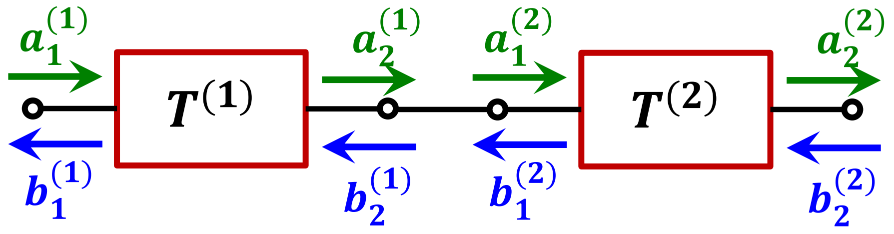
JUAS Proc. 2024
II.2.7.8, p. 849

- **The file name extension specifies the number n of ports**
  - **Attention: NOT equal to the number of columns! The carriage return (CR) is different between s1p, s2p and s3p, s4p files!**
- **The comment header (!) includes general information, e.g., type of instrument, measurement time, etc.**
- **The format line (#) defines the format (mag[dB]/angle[deg], mag/angle, real/imag), stimulus units and the reference impedance**
- **The column delimiter varies, e.g., space, comma, semicolon, etc.**
  - **Column order in the example file: *f_Hz S11_dB S11_deg S21_dB S21_deg S12_dB S12_deg S22_dB S22_deg***

T-Parameters

- **Cascading e.g., 2-port S-parameter networks is important to characterize a larger RF system.**
  - **Solution: Scattering transfer (T) parameters, which directly relates the waves at input and output**

$$(\boldsymbol{T}) = \begin{pmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{12} \\ \boldsymbol{T}_{12} & \boldsymbol{T}_{22} \end{pmatrix} \Rightarrow \begin{aligned} \boldsymbol{b}_1 &= \boldsymbol{T}_{11}\boldsymbol{a}_2 + \boldsymbol{T}_{12}\boldsymbol{b}_2 \\ \boldsymbol{a}_1 &= \boldsymbol{T}_{21}\boldsymbol{a}_2 + \boldsymbol{T}_{22}\boldsymbol{b}_2 \end{aligned}$$



  - **T-parameters enable cascaded 2-port networks by simply multiplying their matrices:**

$$(\boldsymbol{T}) = \left(\boldsymbol{T}^{(1)}\right)\left(\boldsymbol{T}^{(2)}\right) \cdots \left(\boldsymbol{T}^{(N)}\right)$$

  - **Relation between 2-port T-parameters and S-parameters:**

$$(\mathrm{T}) = \frac{1}{S_{21}} \begin{pmatrix} -\det(S) & S_{11} \\ -S_{22} & 1 \end{pmatrix} \qquad (\mathrm{S}) = \frac{1}{T_{22}} \begin{pmatrix} T_{12} & \det(T) \\ 1 & -T_{21} \end{pmatrix}$$

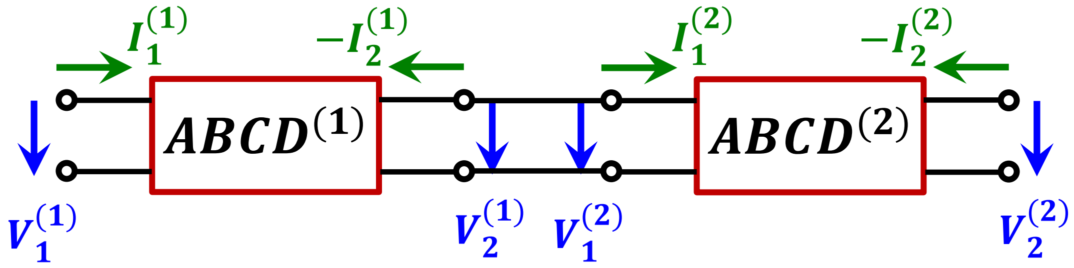
with: $\det(S) = S_{11}S_{22} - S_{12}S_{21}$ with: $\det(T) = T_{11}T_{22} - T_{12}T_{21}$

JUAS Proc. 2024 II.2.7.9, pp. 850/851

ABCD-Parameters

- **Also called "chain" parameters, used for cascading networks based on V and I**
  - **Useful for chaining a mix of lumped elements and transmission-lines**

$$(ABCD) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow \begin{matrix} V_1 = AV_2 - BI_2 \\ I_1 = CV_2 - DI_2 \end{matrix}$$



  - **Chaining 2-port ABCD-parameter networks:**

$$(ABCD) = \left(ABCD^{(1)}\right)\left(ABCD^{(2)}\right)\cdots\left(ABCD^{(N)}\right)$$

Please try to follow Example II.2.7.3 JUAS Proc. 2024 II.2.7.10, p. 852/853

  - **Relation between 2-port ABCD-parameters**

$$(\mathrm{T}) = \frac{1}{2S_{21}}\begin{pmatrix} (1+S_{11})(1-S_{22})+S_{12}S_{21} & [(1+S_{11})(1+S_{22})-S_{12}S_{21}]Z_0 \\ [(1-S_{11})(1-S_{22})-S_{12}S_{21}]/Z_0 & (1-S_{11})(1+S_{22})+S_{12}S_{21} \end{pmatrix}$$

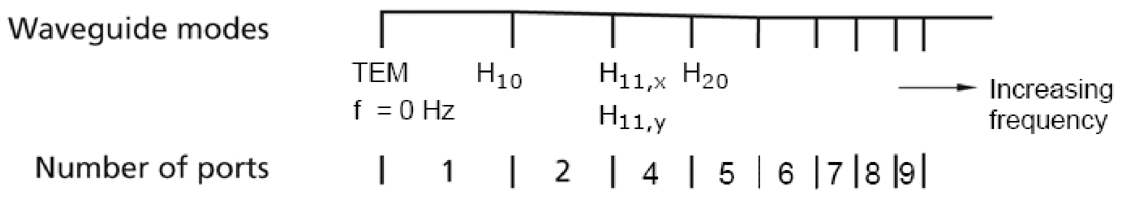
and S-parameters:

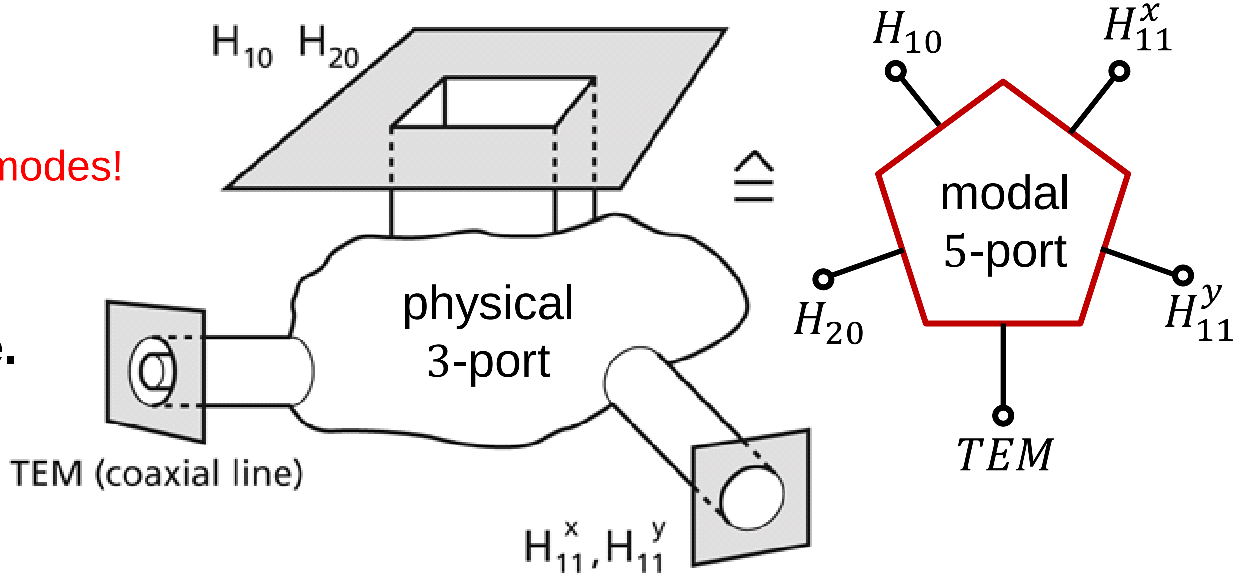
$$(\mathrm{S}) = \frac{1}{A+B/Z_0+CZ_0+D}\begin{pmatrix} A+B/Z_0-CZ_0-D & 2(AD-BC) \\ 2 & -A+B/Z_0-CZ_0+D \end{pmatrix}$$

JUAS Proc. 2024 II.2.7.10, pp. 851/852

General n-Port Networks

- **A general n-port may include ports of different technologies, i.e., waveguides, as well as TEM transmission-lines, such as coaxial lines, microstrip lines etc.**
  - **In the frequency range of interest different modes may propagate at each physical port, e.g., several waveguide modes in a rectangular waveguide and/or higher order modes in a coaxial line..**
  - **Each EM-mode must then be represented by a distinct modal port.**
    - This is very important in EM-simulation to ensure the absorption of the energy for all modes!
  - **The number of modal ports needed generally, increases with frequency, as more waveguide modes can propagate.**





H_{11}^{x}, H_{11}^{y}: x, y-polarization of the E_{11} circular mode

JUAS Proc. 2024 II.2.7.11, pp. 853-855

Summary

- **The scattering (S) parameters are based on incident and reflected normalized complex voltage waves (power waves), defined at the ports of a RF network.**
- **S-Parameters are used to characterize a linear, time-invariant RF component, circuit or sub-system as function of frequency under realistic operational conditions**
  - **The S-parameters are given in a matrix notation, and have complex values**
- **The characteristic of the S-matrix may provide additional details about the network, such as reciprocity, symmetry, losses.**
- **Typically, the S-parameters matrix of a RF network is acquired by measurement characterization with a vector network analyzer (VNA), or by a numerical analysis, e.g., circuit analysis or electromagnetic simulation software**
- **The S-parameter matrices of a set of networks can be converted to transfer (T) parameter matrices to enable a simple cascading of those networks**
- **The number of logical, modal ports might be higher than the number of physical ports for a general RF network utilizing various transmission-line technologies.**

Backup Slides

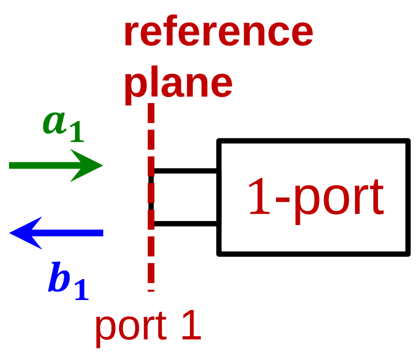
The Scattering Matrix (2)

- **Its simplest form is for a passive 1-port network:**

$$(\mathbf{S}) = S_{11} \quad \Rightarrow \quad b_1 = S_{11} a_1$$

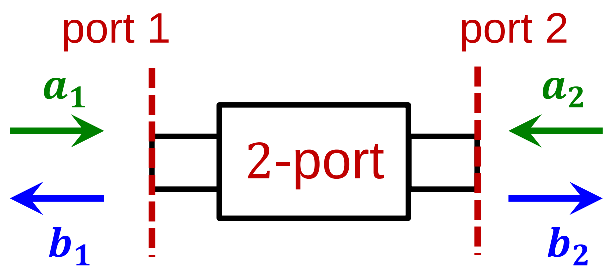
  – with the **reflection coefficient:**

$$\Gamma = S_{11} = \frac{b_1}{a_1}$$



- **Most popular is the 2-port network:**

$$(\mathbf{S}) = \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} b_1 &= S_{11} a_1 + S_{12} a_2 \\ b_2 &= S_{21} a_1 + S_{22} a_2 \end{aligned}$$

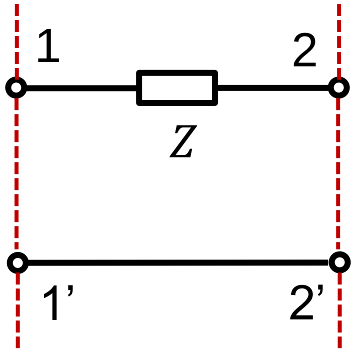


  – An unmatched load, present at port 2 with a **reflection coefficient** Γ_{load} transfers to the input port as:

$$\Gamma_{in} = S_{11} + \frac{S_{21} \Gamma_{load} S_{12}}{1 - S_{22} \Gamma_{load}}$$

Numerical lossless / lossy Examples

2-port series-network



$$(S) = \frac{1}{2Z_0 + Z}\begin{pmatrix} Z & 2Z_0 \\ 2Z_0 & Z \end{pmatrix}$$

$$S_{11} = S_{22} \wedge S_{12} = S_{21}$$

$\mathbf{Z = j\omega L = j10}$	$\mathbf{Z = R = 10}$
$\lvert S_{11}\rvert = \sqrt{1 - \lvert S_{12}\rvert^2}$	
$\frac{1}{\sqrt{101}} = \sqrt{1 - \left(\frac{10}{\sqrt{101}}\right)^2}$	$\frac{1}{11} \neq \sqrt{1 - \left(\frac{10}{11}\right)^2}$
$\angle S_{11} - \angle S_{12} = \angle S_{21} - \angle S_{22} - \pi$	
$\tan^{-1}(10) + \tan^{-1}\frac{1}{10} = -\tan\frac{1}{10} - \tan(10) - \pi$	$0 - 0 \neq 0 - 0 - \pi$
⇒ **lossless**	⇒ **lossy**

4-port ideal directional coupler

$$(S) = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & j & 0 \\ \sqrt{3} & 0 & 0 & j \\ j & 0 & 0 & \sqrt{3} \\ 0 & j & \sqrt{3} & 0 \end{pmatrix}$$

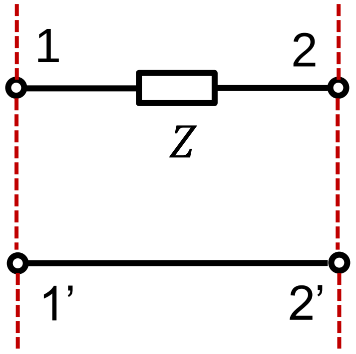
- **It is evident, this ideal 4-port coupler is symmetric and reciprocal**

  $$S_{ij} = S_{ji} \wedge S_{ii} = S_{jj}$$

- **It also is matched:** $S_{ii} = 0$
- **But is it lossless or lossy?**

Numerical lossless / lossy Examples

2-port series-network



$$(S_{ser}) = \frac{1}{2Z_0 + Z}\begin{pmatrix} Z & 2Z_0 \\ 2Z_0 & Z \end{pmatrix}$$

$$S_{11} = S_{22} \wedge S_{12} = S_{21}$$

$Z = j\omega L = j10$	$Z = R = 10$
$\lvert S_{11}\rvert = \sqrt{1 - \lvert S_{12}\rvert^2}$	
$\frac{1}{\sqrt{101}} = \sqrt{1 - \left(\frac{10}{\sqrt{101}}\right)^2}$	$\frac{1}{11} \neq \sqrt{1 - \left(\frac{10}{11}\right)^2}$
$\angle S_{11} - \angle S_{12} = \angle S_{21} - \angle S_{22} - \pi$	
$\tan^{-1}(10) + \tan^{-1}\frac{1}{10} = -\tan\frac{1}{10} - \tan(10) - \pi$	$0 - 0 \neq 0 - 0 - \pi$
⇒ **lossless**	⇒ **lossy**

4-port ideal directional coupler

$$(S_{dc}) = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & j & 0 \\ \sqrt{3} & 0 & 0 & j \\ j & 0 & 0 & \sqrt{3} \\ 0 & j & \sqrt{3} & 0 \end{pmatrix}$$

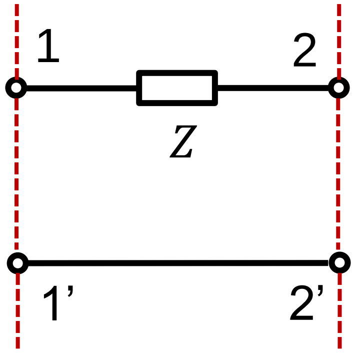
$$(S)^{\dagger}(S) = (I) \;\Rightarrow \sum_{k=1}^{N} S_{ki}S_{ki}^{*} = 1 \;\wedge\; \sum_{k=1}^{N} S_{ki}S_{kj}^{*} = 0 \;\forall\; i \neq j$$

- **Multiply matrix columns by itself with the conjugate complex**
  - **and test for = 1**

$$S_{11}S_{11}^{*} + S_{21}S_{21}^{*} + S_{31}S_{31}^{*} + S_{41}S_{41}^{*} = (0 \cdot 0 + \sqrt{3} \cdot \sqrt{3} + j \cdot (-j) + 0 \cdot 0)/2^2 = 1$$
$$S_{12}S_{12}^{*} + S_{22}S_{22}^{*} + S_{32}S_{32}^{*} + S_{42}S_{42}^{*} = (\sqrt{3} \cdot \sqrt{3} + 0 \cdot 0 + 0 \cdot 0 + j \cdot (-j))/2^2 = 1$$
$$S_{13}S_{13}^{*} + S_{23}S_{23}^{*} + S_{33}S_{33}^{*} + S_{43}S_{43}^{*} = (j \cdot (-j) + 0 \cdot 0 + 0 \cdot 0 + \sqrt{3} \cdot \sqrt{3})/2^2 = 1$$
$$S_{14}S_{14}^{*} + S_{24}S_{24}^{*} + S_{34}S_{34}^{*} + S_{44}S_{44}^{*} = (0 \cdot 0 + j \cdot (-j) + \sqrt{3} \cdot \sqrt{3} + 0 \cdot 0)/2^2 = 1$$

Numerical lossless / lossy Examples

2-port series-network



$$(S_{ser}) = \frac{1}{2Z_0 + Z}\begin{pmatrix} Z & 2Z_0 \\ 2Z_0 & Z \end{pmatrix}$$

$$S_{11} = S_{22} \wedge S_{12} = S_{21}$$

$Z = j\omega L = j10$	$Z = R = 10$
$\lvert S_{11}\rvert = \sqrt{1 - \lvert S_{12}\rvert^2}$	
$\frac{1}{\sqrt{101}} = \sqrt{1 - \left(\frac{10}{\sqrt{101}}\right)^2}$	$\frac{1}{11} \neq \sqrt{1 - \left(\frac{10}{11}\right)^2}$
$\angle S_{11} - \angle S_{12} = \angle S_{21} - \angle S_{22} - \pi$	
$\tan^{-1}(10) + \tan^{-1}\frac{1}{10} = -\tan\frac{1}{10} - \tan(10) - \pi$	$0 - 0 \neq 0 - 0 - \pi$
⇒ **lossless**	⇒ **lossy**

4-port ideal directional coupler

$$(S_{dc}) = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & j & 0 \\ \sqrt{3} & 0 & 0 & j \\ j & 0 & 0 & \sqrt{3} \\ 0 & j & \sqrt{3} & 0 \end{pmatrix}$$

$$(S)^{\dagger}(S) = (I) \;\Rightarrow \sum_{k=1}^{N} S_{ki}S_{ki}^{*} = 1 \;\wedge\; \sum_{k=1}^{N} S_{ki}S_{kj}^{*} = 0 \;\forall\; i \neq j$$

- **Multiply all different matrix columns with the conjugate complex**
  - **and test for = 0**

$$S_{11}S_{12}^* + S_{21}S_{22}^* + S_{31}S_{32}^* + S_{41}S_{42}^* = (0 \cdot \sqrt{3} + \sqrt{3} \cdot 0 + j \cdot 0 + 0 \cdot (-j))/2^2 = 0$$
$$S_{11}S_{13}^* + S_{21}S_{23}^* + S_{31}S_{33}^* + S_{41}S_{43}^* = (0 \cdot (-j) + \sqrt{3} \cdot 0 + j \cdot 0 + 0 \cdot \sqrt{3})/2^2 = 0$$
$$S_{11}S_{14}^* + S_{21}S_{24}^* + S_{31}S_{34}^* + S_{41}S_{44}^* = (0 \cdot 0 + \sqrt{3} \cdot (-j) + j \cdot \sqrt{3} + 0 \cdot 0)/2^2 = 0$$
$$S_{12}S_{13}^* + S_{22}S_{23}^* + S_{32}S_{33}^* + S_{42}S_{43}^* = (\sqrt{3} \cdot (-j) + 0 \cdot 0 + 0 \cdot 0 + j \cdot \sqrt{3})/2^2 = 0$$
$$S_{12}S_{14}^* + S_{22}S_{24}^* + S_{32}S_{34}^* + S_{42}S_{44}^* = (\sqrt{3} \cdot 0 + 0 \cdot (-j) + 0 \cdot \sqrt{3} + j \cdot 0)/2^2 = 0$$
$$S_{13}S_{14}^* + S_{23}S_{24}^* + S_{33}S_{34}^* + S_{43}S_{44}^* = (j \cdot 0 + 0 \cdot (-j) + 0 \cdot \sqrt{3} + \sqrt{3} \cdot 0)/2^2 = 0$$