RF Engineering

Introduction to the *Smith* Chart

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Outline and Objectives

- **Signals and reflections on transmission-lines**
  - **Reflections effects of pulse signals on transmission-lines due to characteristic-termination impedance mismatch**
  - **Standing waves on transmission-lines for continuous wave (CW) sinusoidal signals, definition of the reflection coefficient**
  - **Relation between reflection coefficient, standing wave ration and return loss**
- **The *Smith* chart**
  - **Refresher: Visualization of a complex impedance in the frequency domain**
  - **Definition of the *Smith* chart, mapping the complex impedance / admittance plane with the complex reflection coefficient**
  - **Basic facts and important points on the *Smith* chart**
  - **Examples for a RL and RC series circuit,**
    and for a transmission-line terminated with a RL series circuit.
  - **Operation of a $\lambda/4$ transformer based on a transmission-line as a (normalized) impedance inverter**
- ***Smith* chart exercise (if time permits…)**

Please note the references to the JUAS proceedings!

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II.2.x.y.z, p. nnn

Transmission-lines

- **TEM: coaxial cable, PCB stripline, micro-stripline, co-planar waveguide, etc.**
  - **but also, TE / TM: waveguides (rectangular, circular, elliptical)**
- **Transport RF energy (EM waves) from a RF source to a load.**
- **Physical length ℓ becomes relevant for**

$$\ell \gtrsim \frac{\lambda_g}{10}, \qquad \text{with guide wavelength: } \lambda_g = \frac{v_p}{f}$$

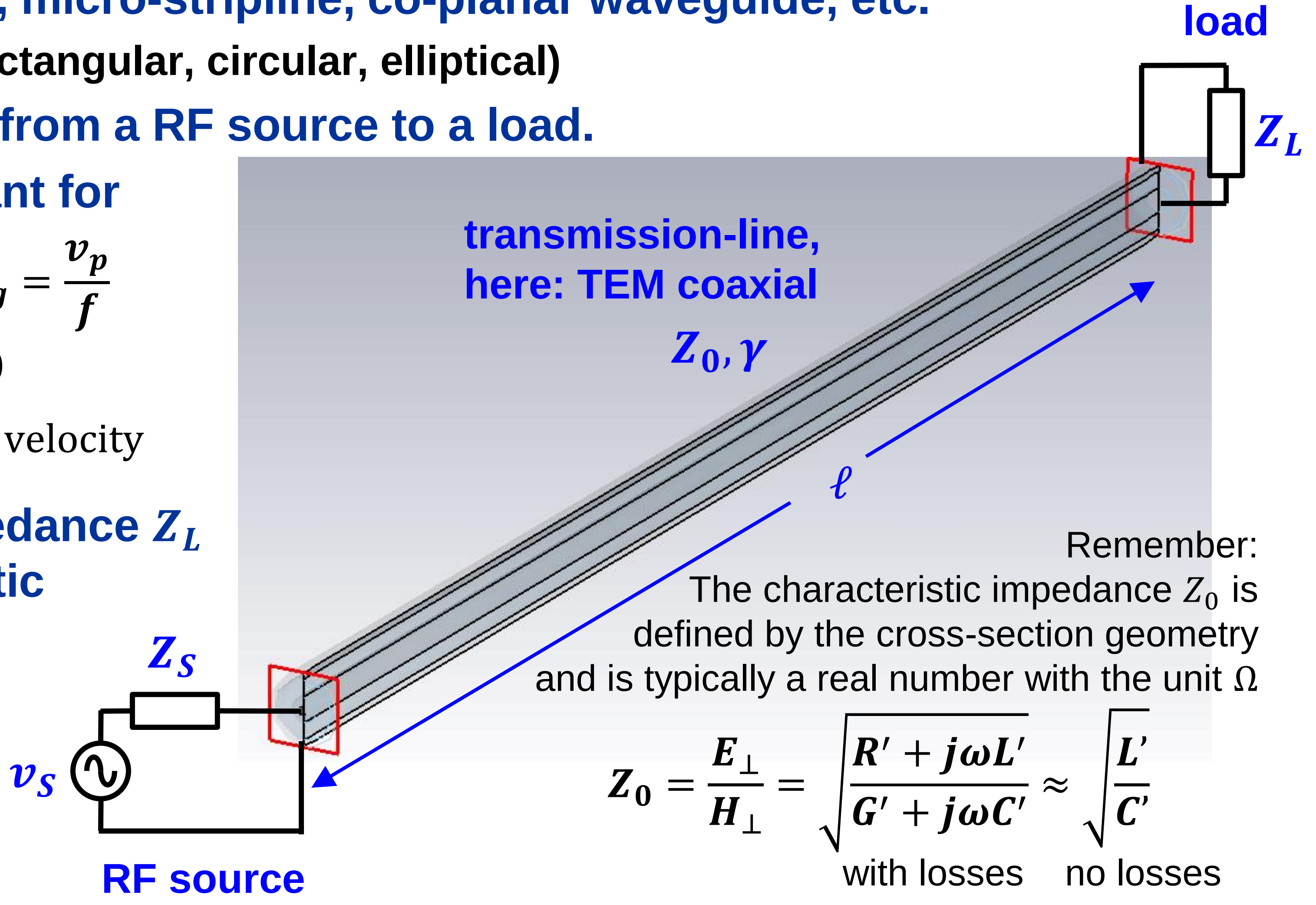
$$f\text{: operating frequency (max.)}$$

$$v_p = \frac{1}{\sqrt{L'C'}} = \frac{c}{\sqrt{\epsilon_r}} \text{ propagation velocity}$$

- **Reflections occur if the load impedance Z_L is not matched to the characteristic impedance Z_0 of the TL**

$$Z_L = Z_0 \;\Rightarrow \Gamma = 0\text{: no reflections}$$ ☺

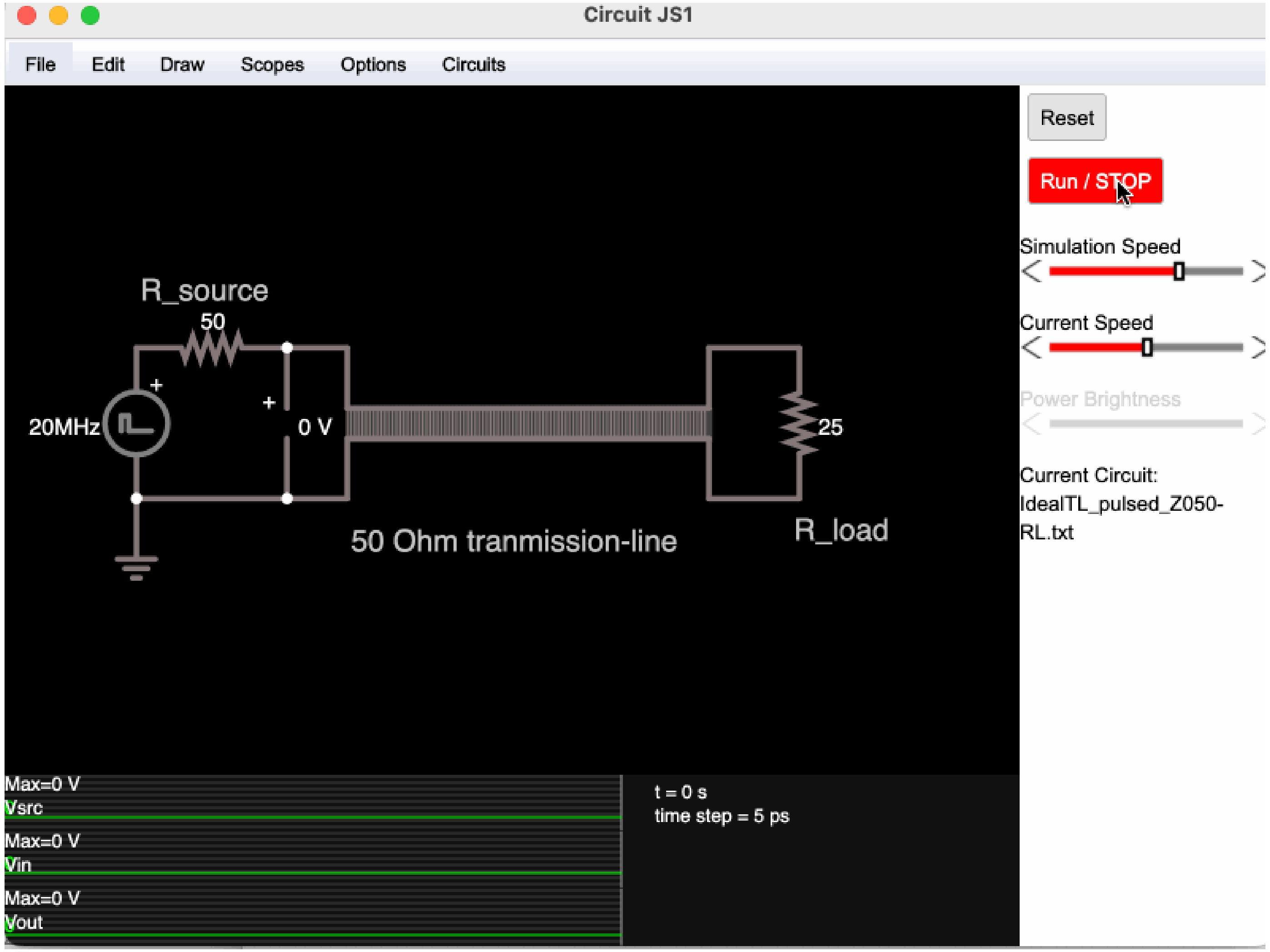
$$Z_L \neq Z_0 \;\Rightarrow \Gamma \neq 0\text{: reflections!}$$ ☹



Remember:
The characteristic impedance Z_0 is defined by the cross-section geometry and is typically a real number with the unit Ω

$$Z_0 = \frac{E_\perp}{H_\perp} = \underbrace{\sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}}_{\text{with losses}} \approx \underbrace{\sqrt{\frac{L'}{C'}}}_{\text{no losses}}$$

TL: Signal visualization in time-domain



Current Circuit: IdealTL_pulsed_Z050-RL.txt

t = 0 s
time step = 5 ps

- **Circuit simulator applet: https://www.falstad.com/circuit/**
  - **Load file: *IdealTL_DCswitched_Z050-RL.txt***
    - ➢ Change the load resistor value: *RL = 50, 100, 25 Ω*
    - ➢ Operate the switch and observe the signals at the beginning, and at the end of the transmission-line.
  - **Load file: *IdealTL_pulsed_Z050-RL.txt***
    - ➢ Change the load resistor value: *RL = 50, 100, 25 Ω*
    - ➢ Observe the signal waveforms! Can you predict the values?!
      - (Press *Run/STOP* and hover with the mouse over the waveform)

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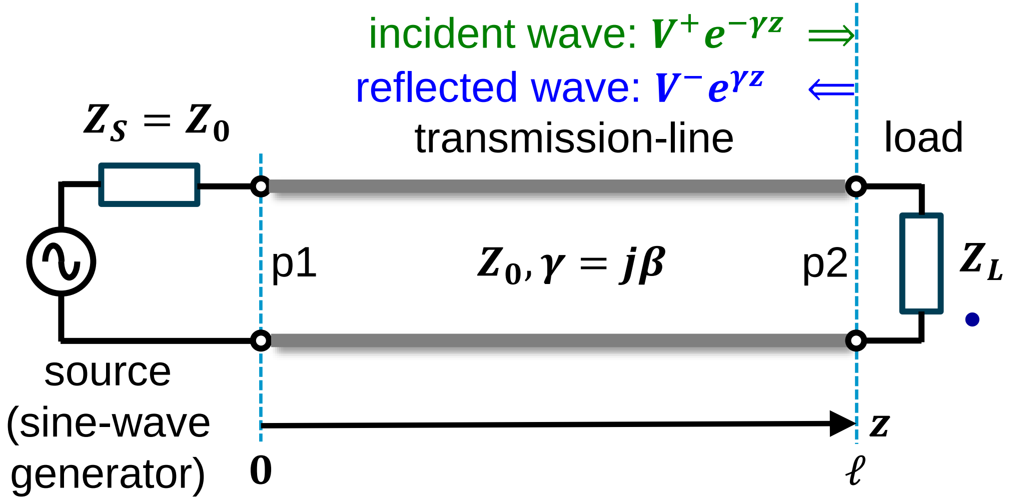
TL: Operating with sinusoidal signals (FD)


incident wave: $V^{+}e^{-\gamma z}$ $\Longrightarrow$

reflected wave: $V^{-}e^{\gamma z}$ $\Longleftarrow$

$Z_S = Z_0$

transmission-line

load

p1

$Z_0, \gamma = j\beta$

p2

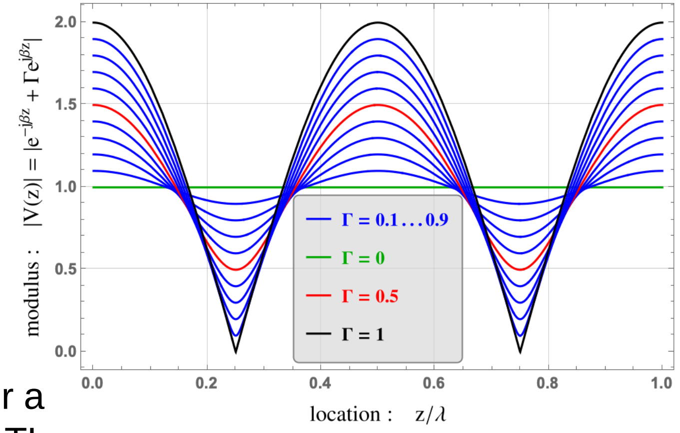
Z_L

source (sine-wave generator)

0

z

ℓ



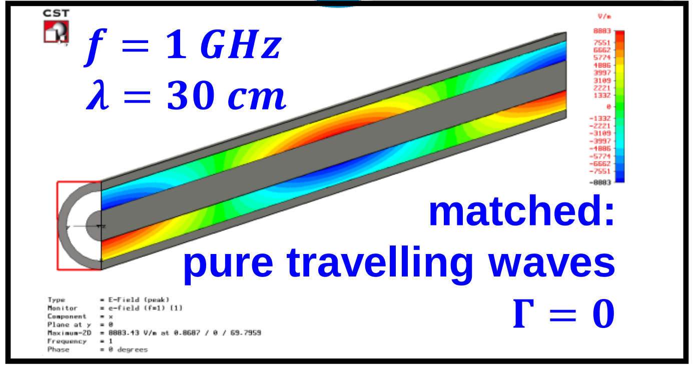
here: $|V(z)|$ for a lossless TL ($\gamma = j\beta$)

- **For a single frequency, continues wave (CW) signal on a transmission-line**
  - **Superposition of forward a, E^{inc}, V^{+} and backward b, E^{refl}, V^{-} traveling waves**
  - $\Rightarrow$ **standing waves**
- **Reflection coefficient Γ**

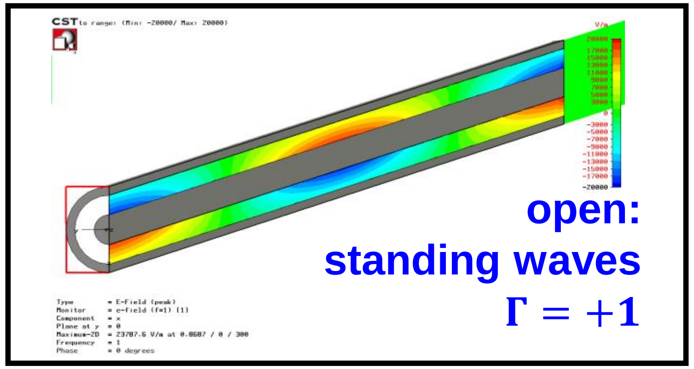
$$\boxed{\Gamma = \frac{b}{a} = \frac{E^{refl}}{E^{inc}} = \frac{V^{-}}{V^{+}} = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

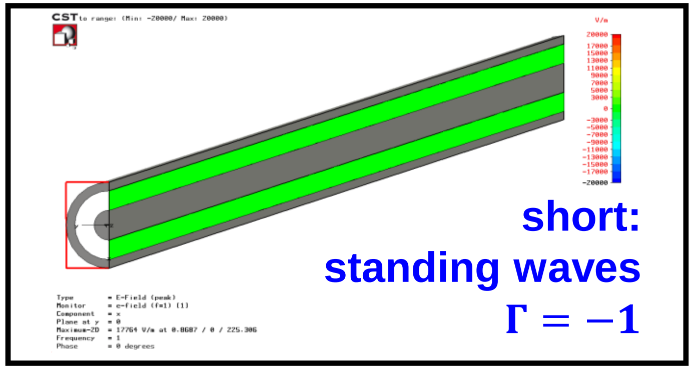
waves E-fields voltages impedances

  - **Γ is related to the impedance ratio: Z_L/Z_0**
    - The load impedance Z_L can be complex
  - **Γ is a complex number**
- **Reflections also originate due to discontinuities of the TL along z**
  - **Same Z_0 does not prevent reflections! EM-field effect!**


$f = 1\ GHz$
$\lambda = 30\ cm$

matched: pure travelling waves $\Gamma = 0$


open: standing waves $\Gamma = +1$


short: standing waves $\Gamma = -1$

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TL: Voltage Standing Wave Ratio (VSWR)

- **The voltage standing wave ratio (VSWR) expresses the ratio between the maximum and minimum voltage of a standing wave along a transmission-line**

$$\boxed{VSWR = \frac{|V_{max}|}{|V_{min}|} = \frac{|a|+|b|}{|a|-|b|} = \frac{1+|\Gamma|}{1-|\Gamma|} = \left|\frac{Z_L}{Z_0}\right|}$$

with: $\begin{cases} |V_{max}| = |V^+| + |V^-| \\ |V_{min}| = |V^+| - |V^-| \end{cases}$

symbols: incident (forward) wave: a, X^+
reflected (backward) wave: b, X^-

  - **The VSWR is a function of the frequency.**

- **The return loss (RL) is another way to express reflection effects**

$$\boxed{RL\,[dB] = 10\log_{10}\frac{P^+}{P^-} = -20\log_{10}|\Gamma|}$$

Γ	$VSWR = Z_L/Z_0$	Return Loss [dB]	Refl. Power $\lvert\Gamma\rvert^2$	Inc. Power $1-\lvert\Gamma\rvert^2$
0.0	1.00	∞	0.00	1.00
0.1	1.22	20.0	0.01	0.99
0.2	1.50	14.0	0.04	0.96
0.3	1.87	10.5	0.09	0.91
0.4	2.33	8.0	0.16	0.84
0.5	3.00	6.0	0.25	0.75
0.6	4.00	4.4	0.36	0.64
0.7	5.67	3.1	0.49	0.51
0.8	9.00	1.9	0.64	0.36
0.9	19.00	0.9	0.81	0.19
1.0	∞	0	1.00	0.00

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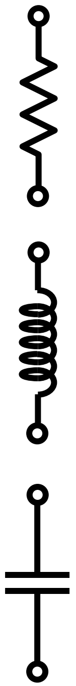
Reminder: Circuit Vocabulary



conductor, with:
conductance G $[S]$

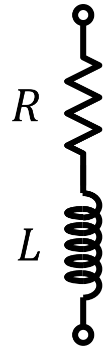
resistor, with:
resistance R $[\Omega]$

inductor, with:
inductance L $[H]$
reactance $X_L = \omega L$ $[\Omega]$

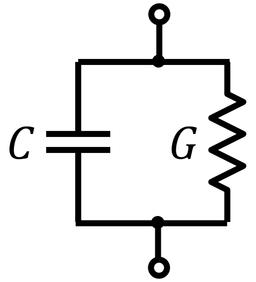
suceptance $B_L = 1/\omega L$ $[S]$

capacitor, with:
capacitance C $[F]$
susceptance $B_C = \omega C$ $[S]$

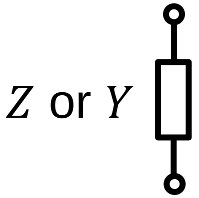
reactance $X_C = -1/\omega C$ $[\Omega]$



complex impedance example:
$Z = R + j\omega L$ $[\Omega]$



complex admittance example:
$Y = G + j\omega C$ $[S]$



complex impedance Z
or
complex admittance Y

- **Resistance, impedance, reactance are inverse proportional to conductance, susceptance, admittance**

$$\boxed{R = \frac{1}{G}} \qquad \boxed{Z = \frac{1}{Y}} \qquad \boxed{X = -\frac{1}{B}}$$

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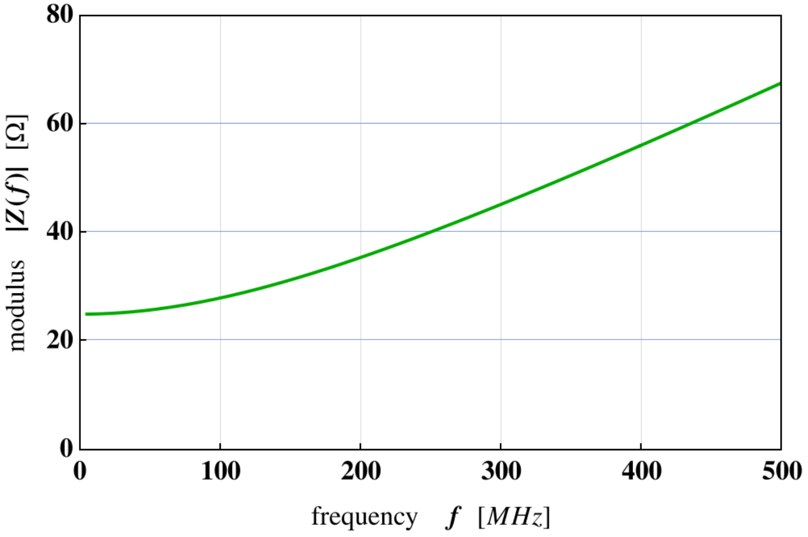
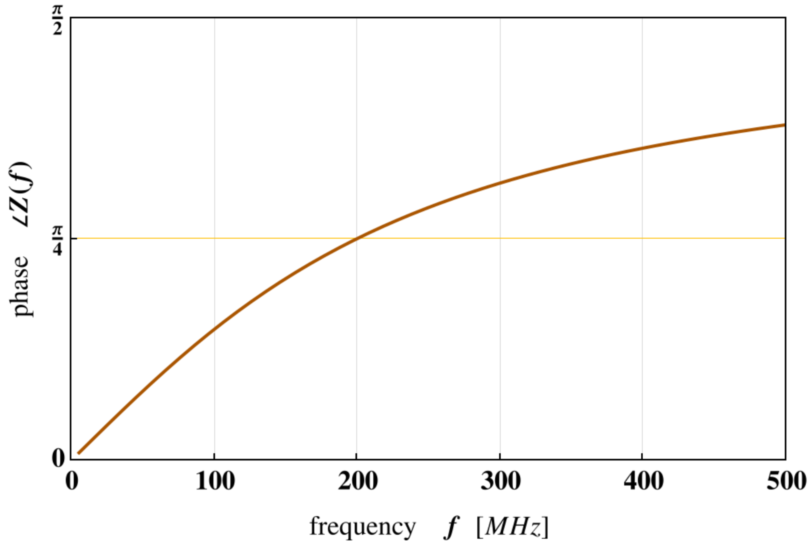
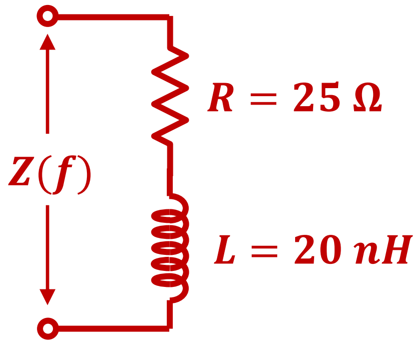
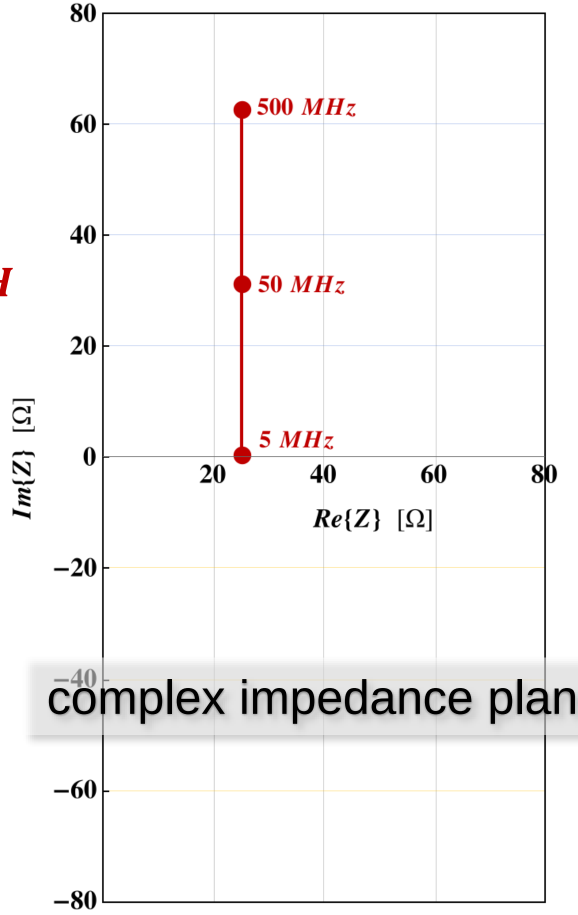
Reminder: The complex Z-Plane



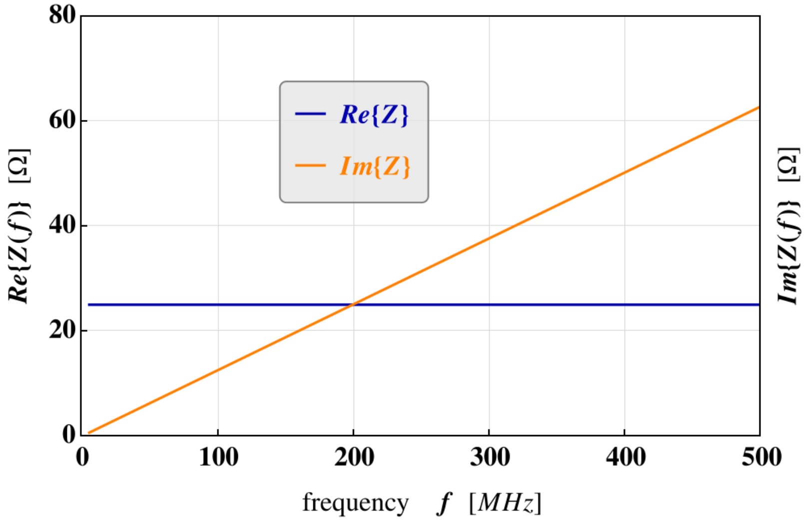


- **Different ways to visualize $Z(f)$**
  - **magnitude / phase**
  - **real / imaginary**
  - **complex Z-plane**
    - Plot in the complex plane with f as parameter





complex impedance plane



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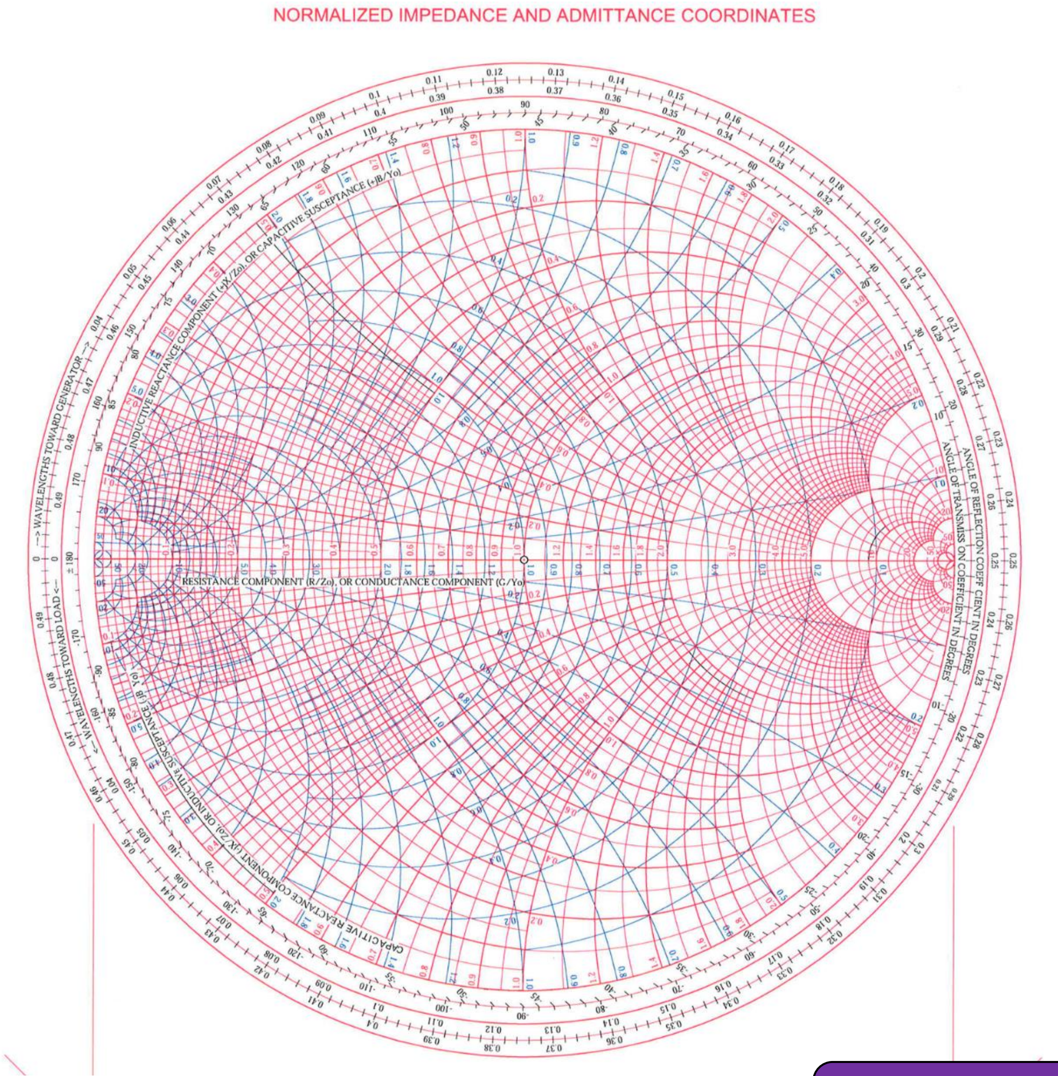
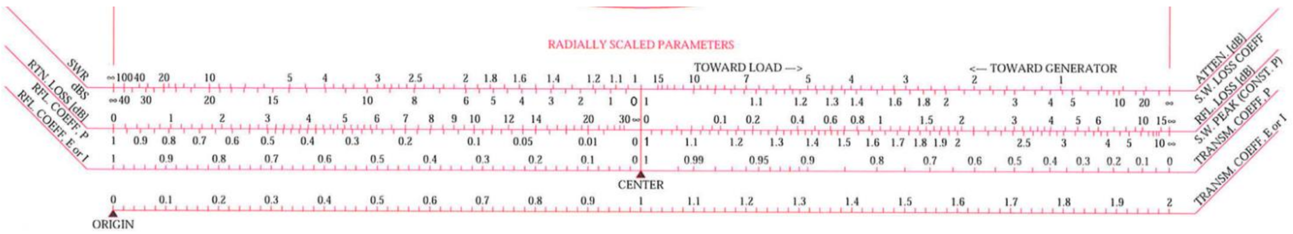
The *Smith* Chart (1)



- **The *Smith* chart is a calculation tool, divided in 2 parts:**
  - **A transformation of the complex, normalized impedance z and admittance y planes on a circle.**
  - **A set of “rulers” below, for additional computations**
    - VSWR, return and reflection loss, etc.



- **At a 1st look the Smith chart is quite overwhelming**
  - **In this introduction the focus is on the complex z-plane**

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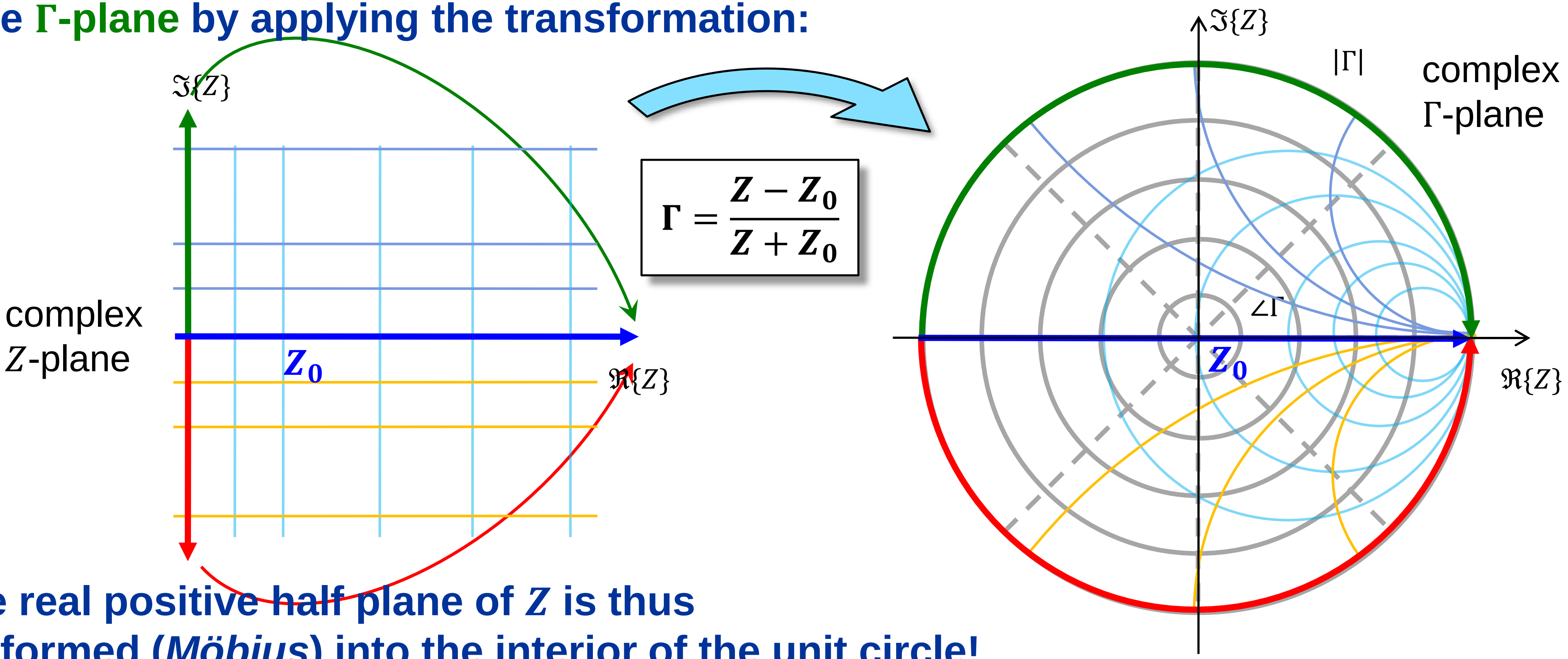
The *Smith* Chart (2)

- **The *Smith* chart (in impedance / admittance coordinates) represents the complex Γ-plane (in polar coordinates) within the unit circle.**
- **It is a conformal mapping of the complex Z-plane on the Γ-plane by applying the transformation:**



$$\Gamma = \frac{Z - Z_0}{Z + Z_0}$$

- **⇒ the real positive half plane of Z is thus transformed (*Möbius*) into the interior of the unit circle!**

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II.2.6.3, p. 815

The *Smith* Chart (3)

- In the classic paper *Smith* chart the impedance Z is normalized:

$$z = \frac{Z}{Z_0}$$

  to a reference impedance Z_0,
  typically, to the characteristic impedance of the coaxial cable transmission-lines used in RF / microwave engineering: $Z_0 = 50\ \Omega$.
- The normalized form of the transformation follows then as:

$$\Gamma = \frac{z-1}{z+1} \quad \Rightarrow \quad \frac{Z}{Z_0} = z = \frac{1+\Gamma}{1-\Gamma}$$

- The Smith chart is a parametric graph
  - with the frequency f as parameter
  - and the normalized, complex impedance z and complex reflection coefficient Γ as variables
    - also, the normalized, complex admittance $y = 1/z$ is mapped and can be used as variable.

- In the past
  - The *Smith* chart was used as a calculation tool for impedance matching, e.g., antennas to transmitters or receivers, amplifier input / output stages, couplers of accelerating cavities, etc.
- At presence, the *Smith* chart is still popular
  - for visualization purposes, e.g., vector network analyzer (VNA) measurement of input / output impedances (display of the Sii scattering parameters), in software tools, etc., usually using the un-normalized impedance Z
  - for the optimization of the coupling between RF source and a cavity resonator.

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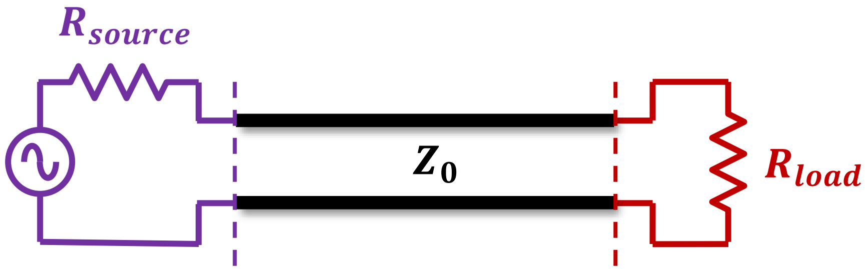
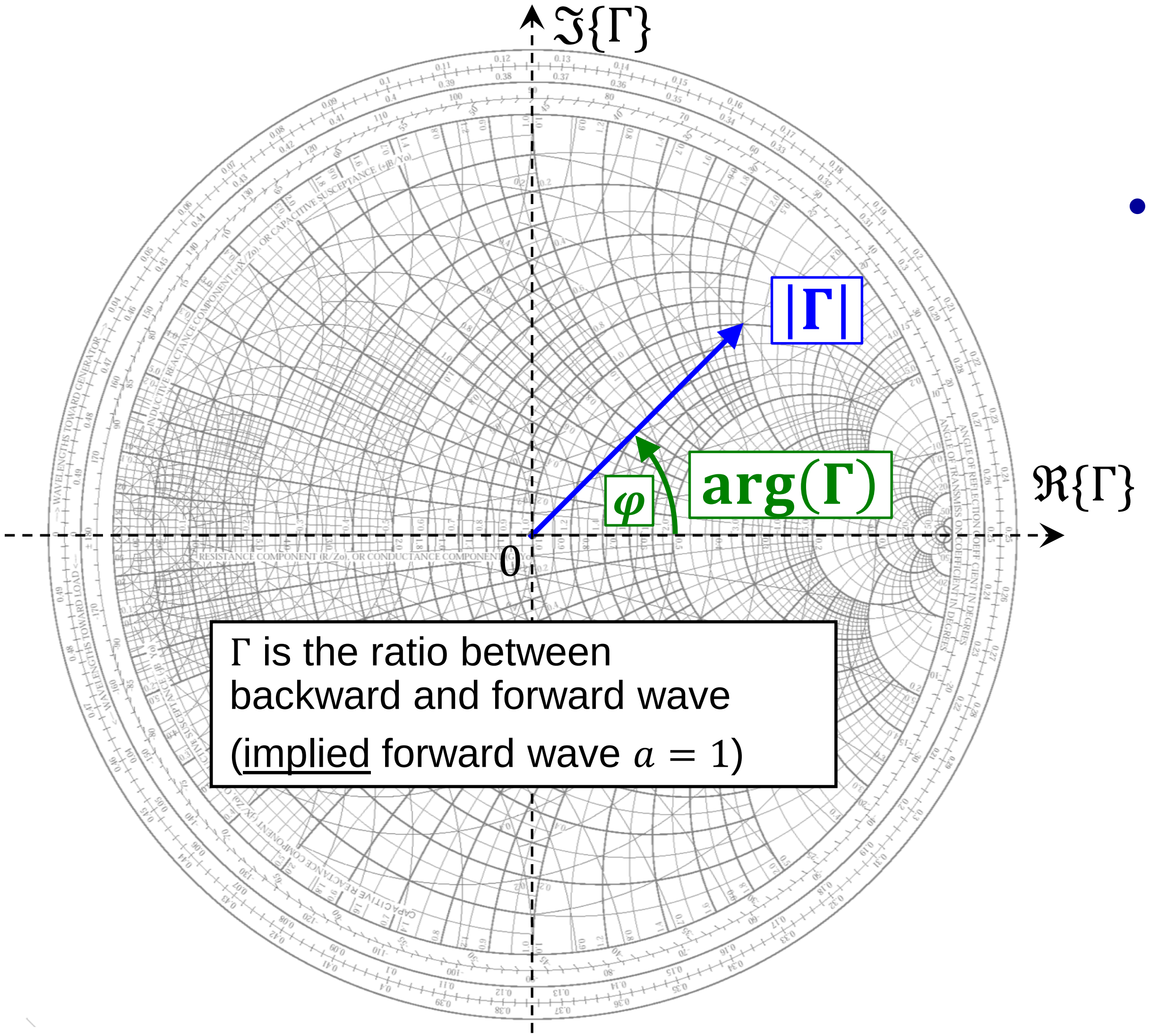
Quiz (1)



1. No signal reflections are observed at the end of a transmission-line, this means:
   - ○ $R_{source} = R_{load}$
   - ○ $R_{source} = Z_0$
   - ☒ $Z_0 = R_{load}$
   - ☒ $R_{source} = Z_0 = R_{load}$

2. The Smith chart transforms the   onto the complex Gamma (reflection coefficient) plane within the unit circle.

The *Smith* Chart (4)



Γ is the ratio between backward and forward wave (implied forward wave $a = 1$)

- **In the *Smith* chart, the complex reflection factor**

$$\boldsymbol{\Gamma = |\Gamma| e^{j\varphi} = \frac{b}{a}}$$

is expressed in linear polar coordinates, representing the ratio of backward b vs. forward a traveling waves.

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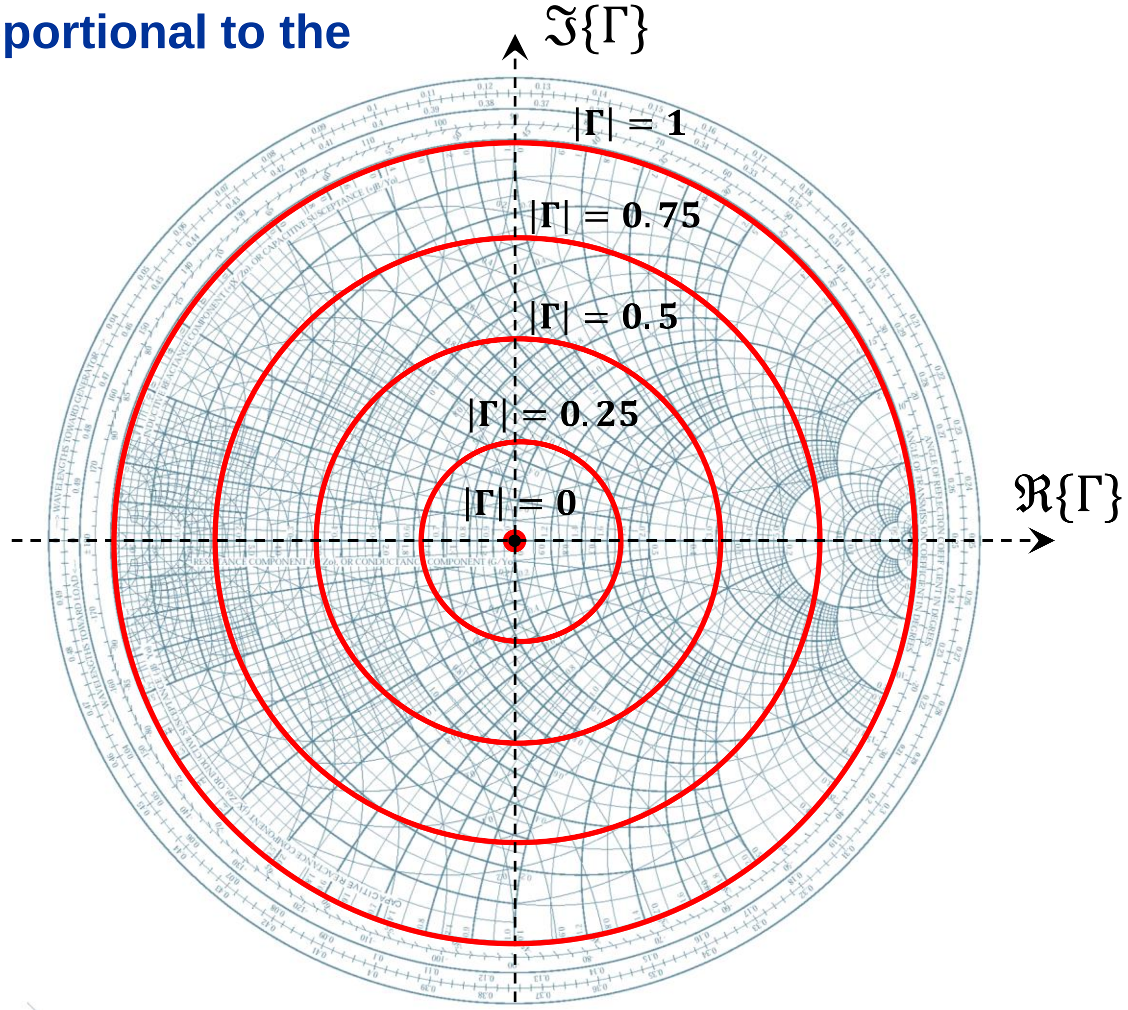
The *Smith* Chart (5)

- The distance from the center of the directly proportional to the magnitude of the reflection factor $|\Gamma|$ and permits an easy visualization of the matching performance.
  - In particular, the perimeter of the diagram represents total reflection: $|\Gamma| = 1$.
  - (power dissipated in the load) = (forward power) – (reflected power)

$$\mathbf{P} = |a|^2 - |b|^2$$
$$= |a|^2\left(1 - |\Gamma|^2\right)$$

available source power

mismatch losses



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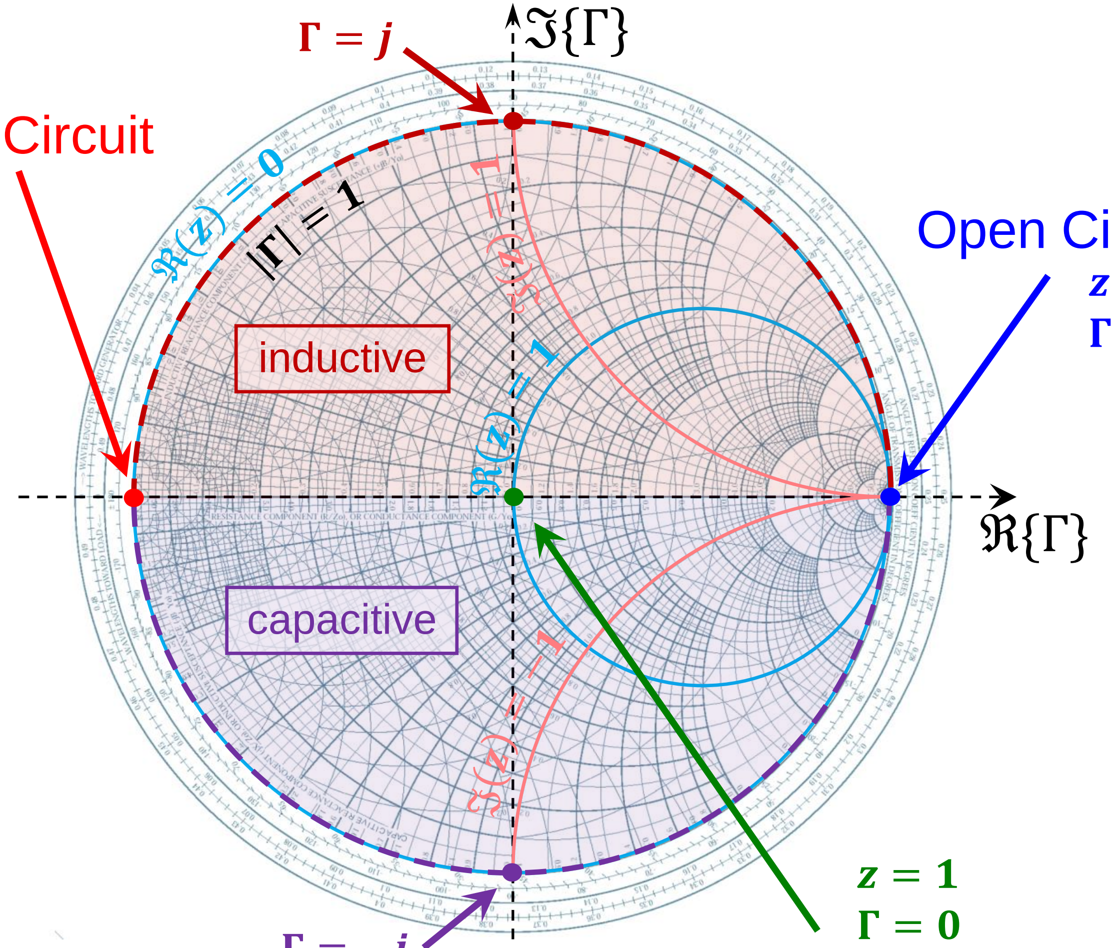
The *Smith* Chart – “Important Points”

Important Points:

- **Short Circuit**
  $\Gamma = -1, z = 0$
- **Open Circuit**
  $\Gamma = +1, z \rightarrow \infty$
- **Matched Load**
  $\Gamma = 0, z = 1$
- **On the circle $|\Gamma| = 1$:**
  lossless element
- **Upper half:**
  ”inductive” =
  positive imaginary part of Z
- **Lower half:**
  ”capacitive” =
  negative imaginary part of Z
  - **Outside the circle, $\Gamma > 1$:**
    active element, e.g., tunnel diode reflection amplifier



Short Circuit
$z = 0$
$\Gamma = -1$

Open Circuit
$z = \infty$
$\Gamma = +1$

$z = 1$
$\Gamma = 0$
Matched Load

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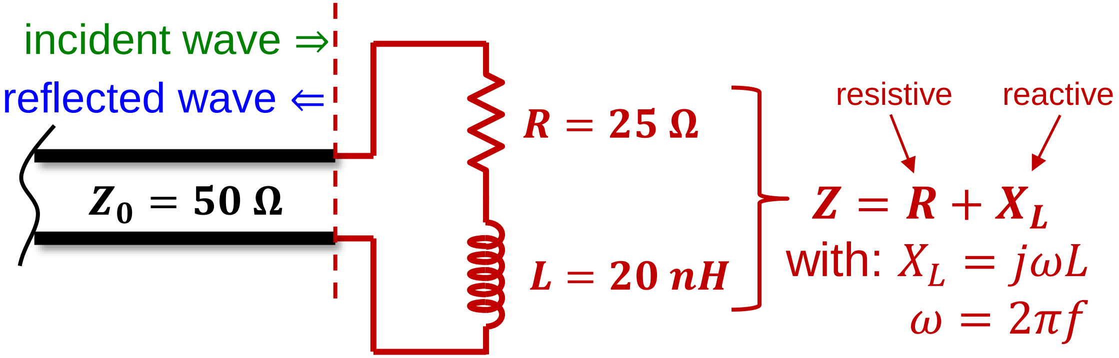
The *Smith* Chart – Basic Example (1)



incident wave ⇒

reflected wave ⇐

$Z_0 = 50\ \Omega$

$R = 25\ \Omega$

$L = 20\ nH$

resistive

reactive

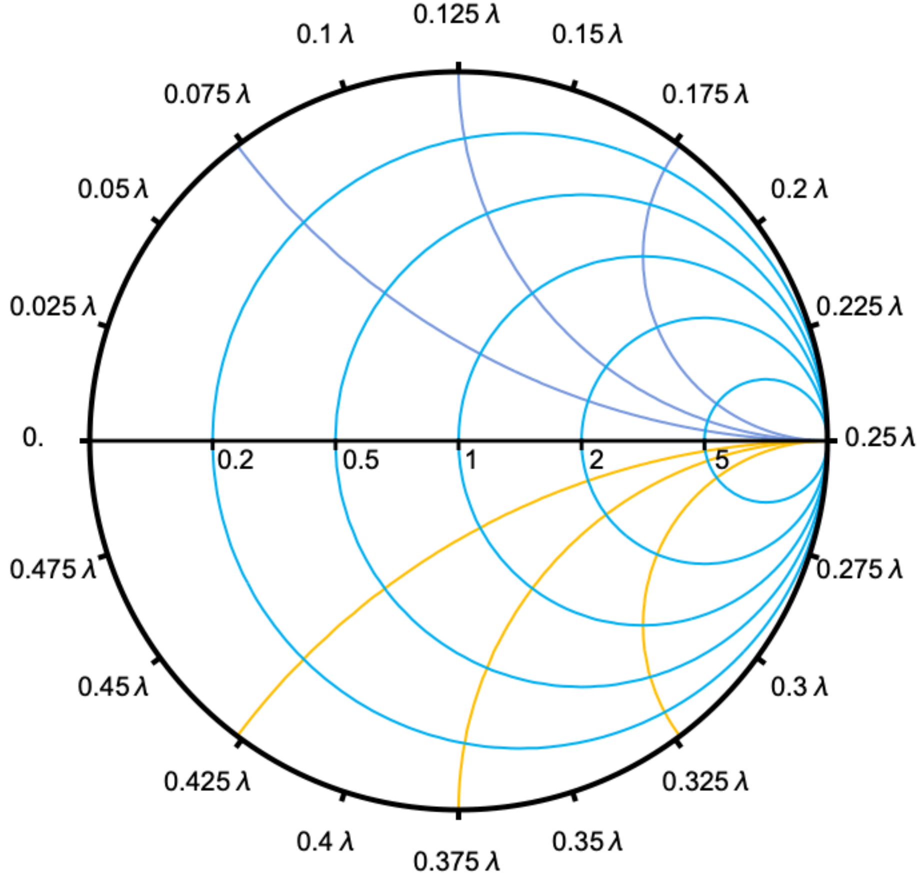
$Z = R + X_L$

with: $X_L = j\omega L$

$\omega = 2\pi f$

Complex impedance based on lumped element components

- **Calculate Z for a given frequency, e.g., $f = 50\ MHz$: $Z = (25 + j6.28)\ \Omega$**
- **Calculate the normalized impedance $z = Z/Z_0 = 0.5 + j0.126$**
  - **Locate z in the Smith chart**
  - **Retrieve $\Gamma = 0.34\angle 161° = 0.34e^{j2.81}$**
- **Repeat for other frequencies…**



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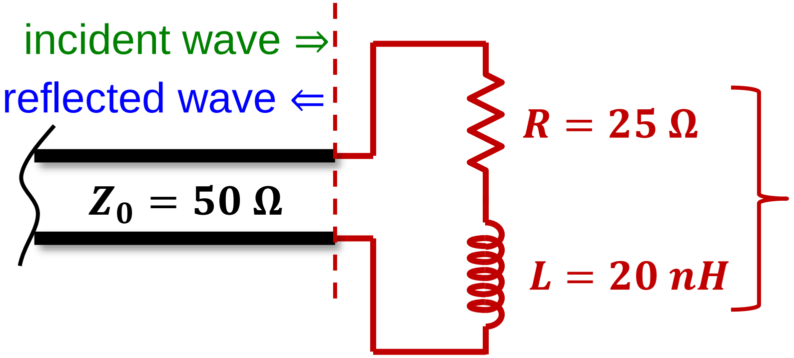
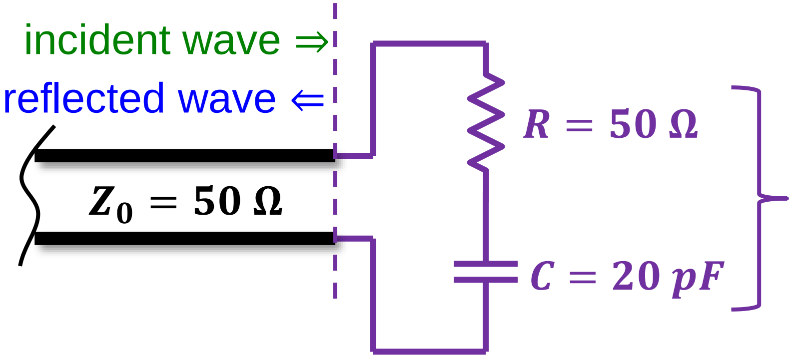
II.2.6.3, pp. 817/818

The *Smith* Chart – Basic Example (2)

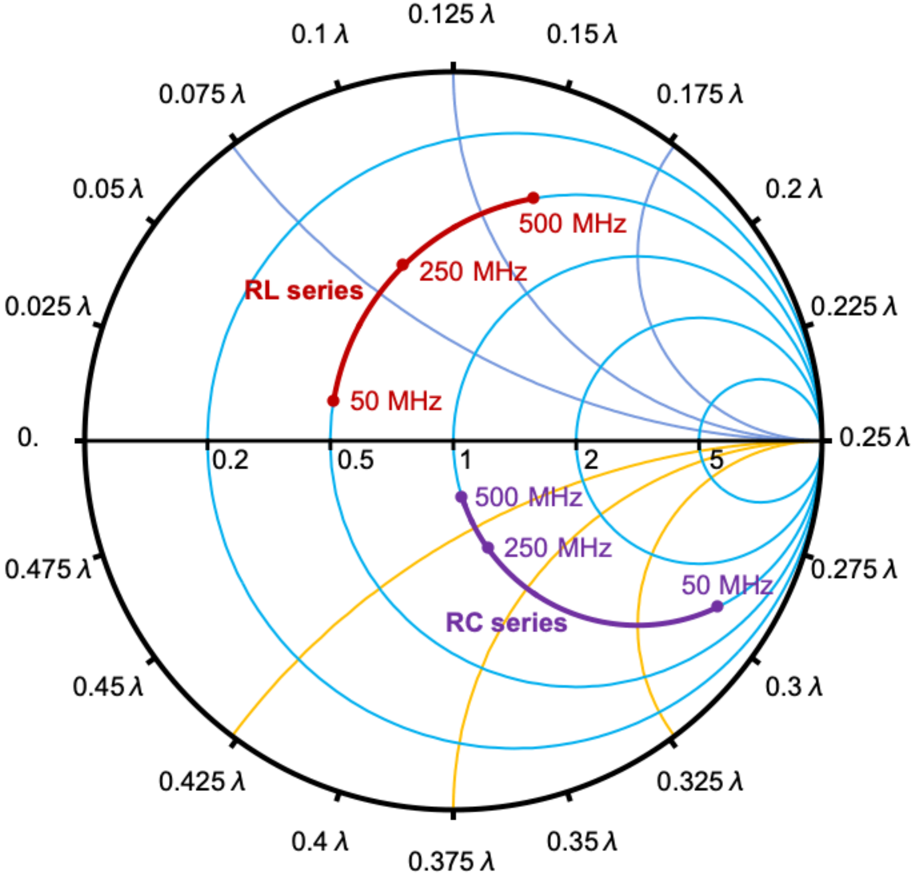


$$Z = R + jX_L$$
with: $X_L = \omega L$

- **…and for different component values and circuit combinations**



$$Z = R + jX_C$$
with: $X_C = \frac{-1}{\omega C}$



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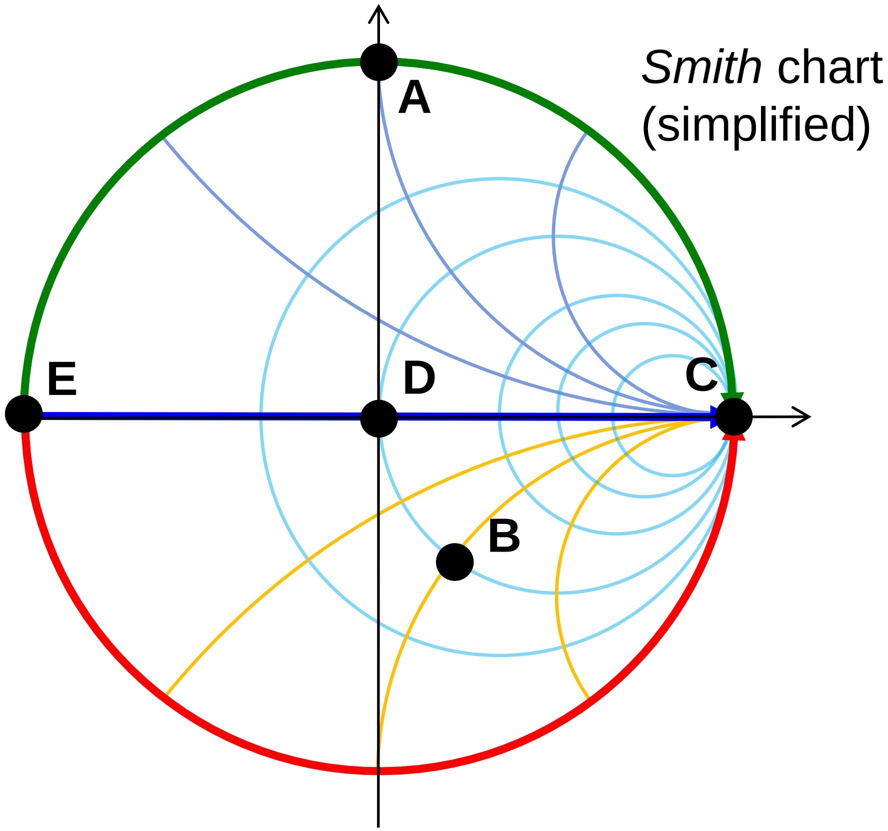
Quiz (2)

Prompts		Possible Answers
A. Point A		1. $\Gamma = +1, z \to \infty$
B. Point B		2. $\Gamma = -j$
C. Point C		3. $\Gamma = 0, z = 1$, match
D. Point D		4. Point in the capacitive half plane
E. Point E		5. $\Gamma = +j$
		6. $\Gamma = -1, z = 0$
		7. Point in the inductive half plane



Smith chart (simplified)

Quiz (3)

4. Trace with marker points in the simplified *Smith* chart for an RL series impedance

- ○ $f_B > f_A$
- ☒ $f_B < f_A$
- ○ **There is no frequency f related to points A and B**
- ○ $f_A = f_B$

Please try to complete
Exercise II.2.6.2
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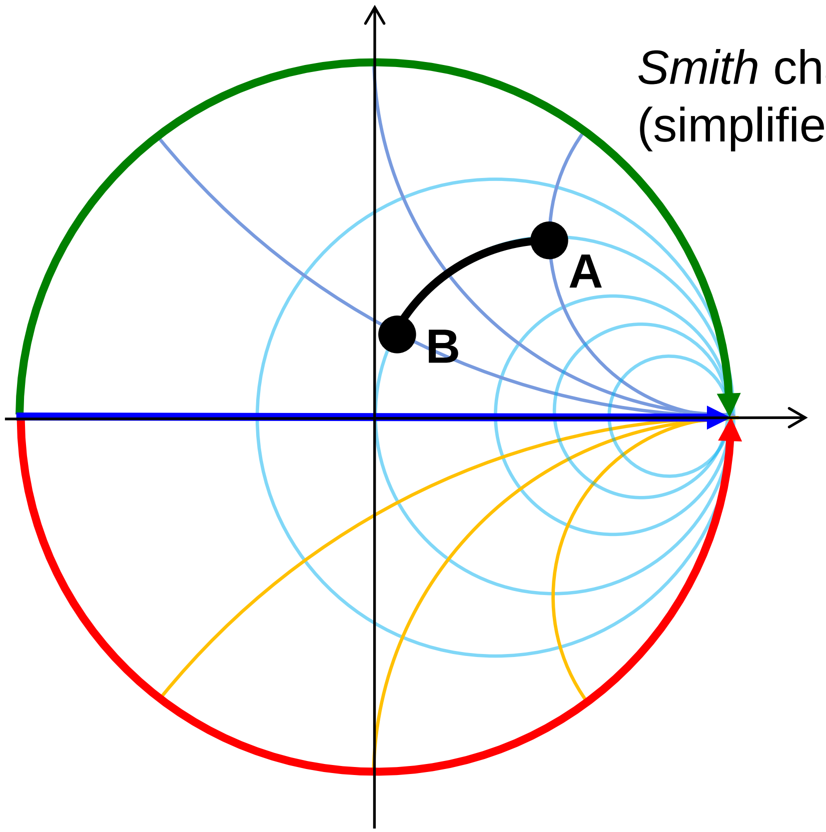
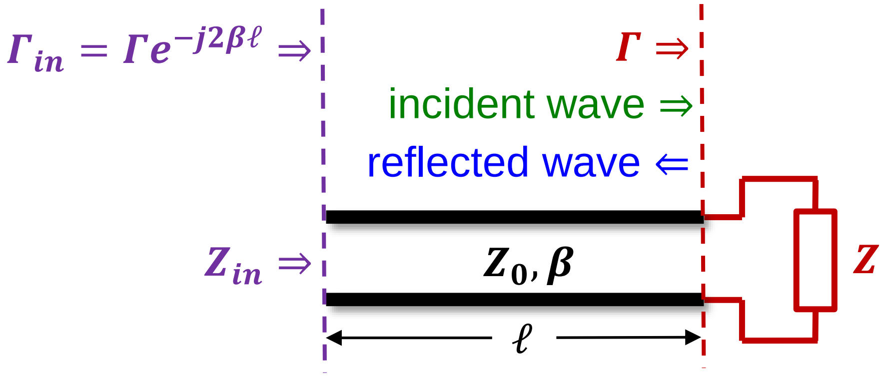
II.2.6.3, p. 820



Smith chart (simplified)

The *Smith* Chart – TL Transformer (1)



- S-parameter of a lossless transmission-line:

$$\boxed{S = \begin{bmatrix} 0 & e^{-j\beta\ell} \\ e^{-j\beta\ell} & 0 \end{bmatrix}}$$

backward transmission coefficient S12

forward transmission coefficient S21

  - Phase delay (electrical length) of

$$\boxed{\beta\ell = \theta}$$

with: $\beta = \frac{2\pi}{\lambda_g} = k$ (wave number)

- The lossless transmission-line adds a phase delay of $2\beta\ell$, seen at its input, Γ_{in}, to the reflection coefficient Γ at its output:

$$\boxed{\Gamma_{in} = \Gamma e^{-j2\beta\ell}}$$

- This results in a transformation of the impedance Z at the end of the line to a different impedance Z_{in} at the input of the line
  - The *Smith* chart offers an effective, simple graphical way to calculate this transmission-line based impedance transformation

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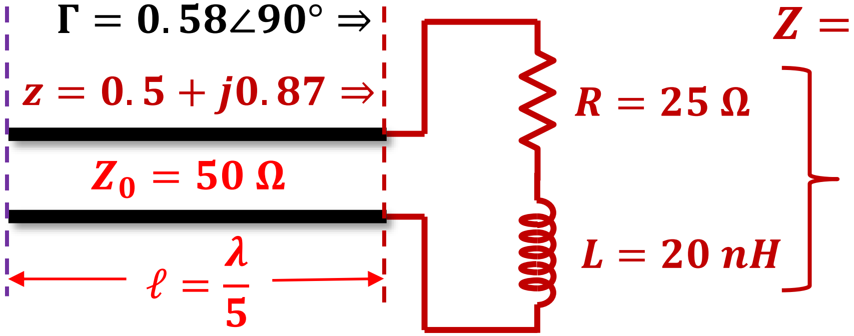
II.2.6.4.3, pp. 825/826

The *Smith* Chart – TL Transformer (2)



$\Gamma_{in} = 0.58\angle -54° \Rightarrow$ $\Gamma = 0.58\angle 90° \Rightarrow$

$z_{in} = 1.02 - j1.43 \Rightarrow$ $z = 0.5 + j0.87 \Rightarrow$

$Z_{in} = (50.8 - j71.3)\ \Omega \Rightarrow$ $Z_0 = 50\ \Omega$

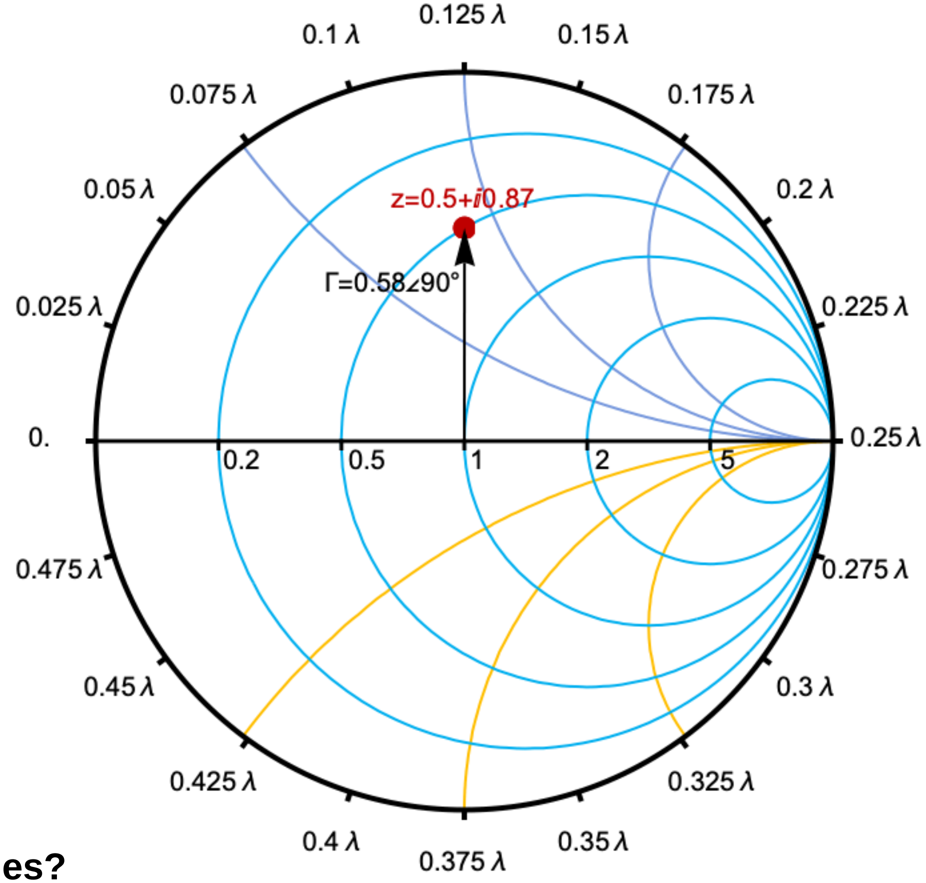
$\ell = \frac{\lambda}{5}$

$R = 25\ \Omega$

$L = 20\ nH$

$Z = (25 + j43.4)\ \Omega$

$@\ f = 345\ MHz$



Based on the previous example

- **Calculate the normalized impedance z for $f = 345\ MHz$ and locate the point in the Smith chart.**
  - **Notice the corresponding value of Γ, and read the λ-length value on the rim**
- **Add $\ell = \lambda/5$ by rotating Γ by $2\beta\ell = 4\pi/5 \equiv 144°$**
  - **From $0.125\ \lambda$ to $0.325\ \lambda$**
    - The phase is subtracted, therefore clockwise rotation!
- **Notice the value of Γ_{in} and read the corresponding value of the normalized impedance z_{in}**
  - **Calculate the transformed impedance $Z_{in}@f = 345\ MHz$**
  - **What is the equivalent circuit, and what are the component values?**
    - …and what is the physical length ℓ of the transmission-line?
      - assuming a coaxial cable as transmission-line with a dielectric constant of $\epsilon_r = 2.1$

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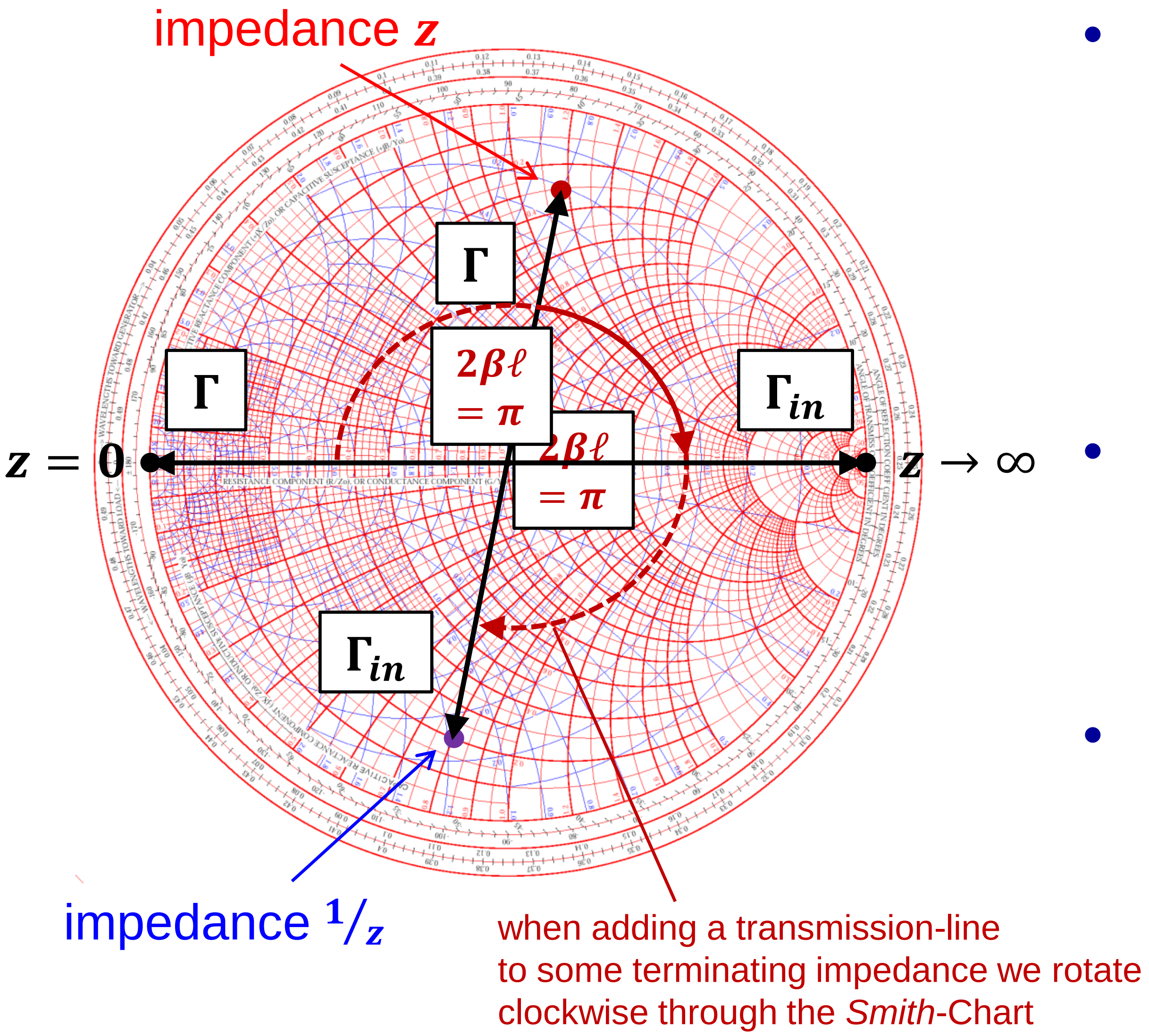
II.2.6.4.3, pp. 825/826

The *Smith* Chart – TL Transformer (3)



when adding a transmission-line to some terminating impedance we rotate clockwise through the *Smith*-Chart

- **A transmission-line of length**

$$\boxed{\ell = {}^{\lambda}/_{4} \equiv \beta\ell = \frac{\pi}{2}}$$

transforms a reflection Γ at the end of the line to its input as

$$\Gamma_{in} = \Gamma\, e^{-j2\beta\ell} = \Gamma\, e^{-j\pi} = -\Gamma$$

- **This results the unitless, normalized impedance z at the end of the line to be transformed into:**

$$\boxed{z_{in} = {}^{1}/_{z}}$$

at the beginning of the line

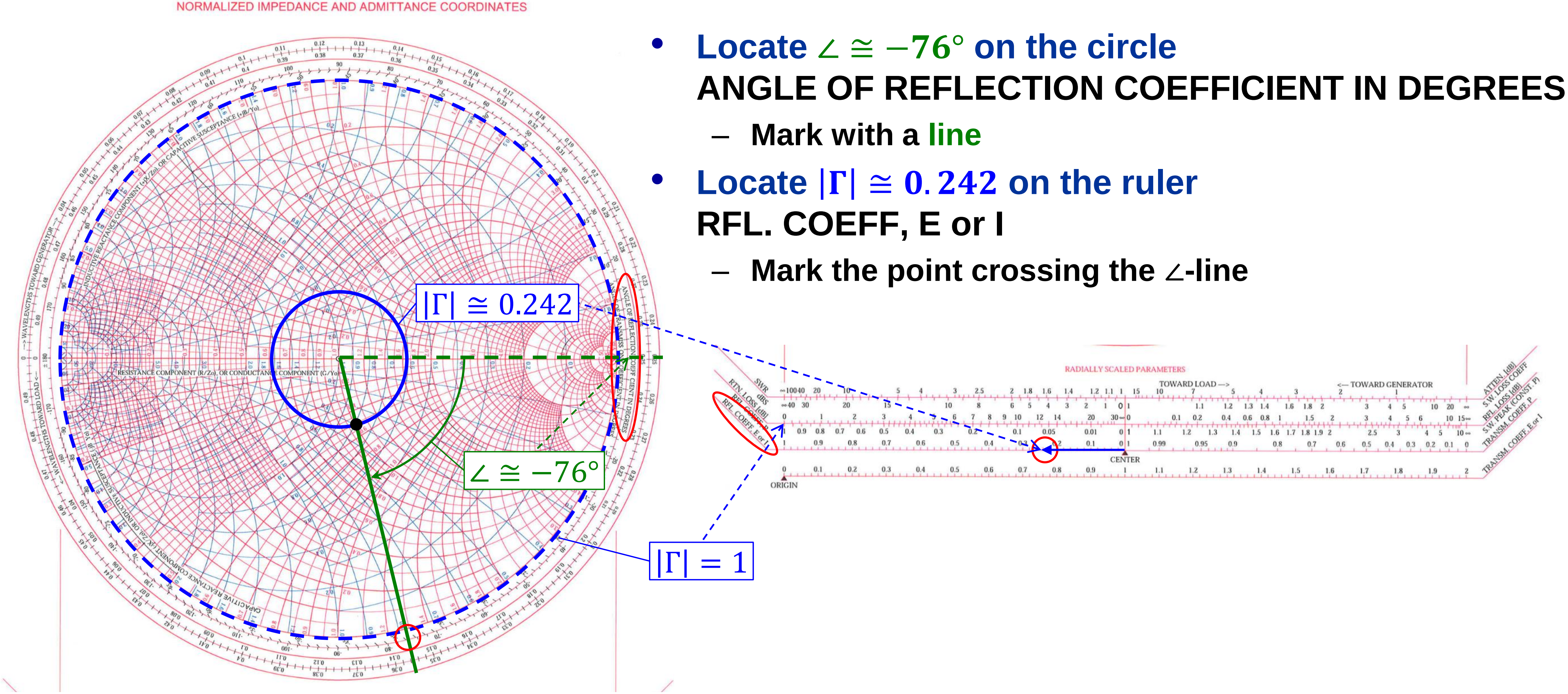
- **A short circuit at the end of the $\lambda/4$-transformer is transformed to an open, and vice versa**
  - **This is the principle of the $\lambda/4$-resonator.**

JUAS Proc. 2024
II.2.6.4.3, pp. 827

A *Smith* Chart matching Exercise

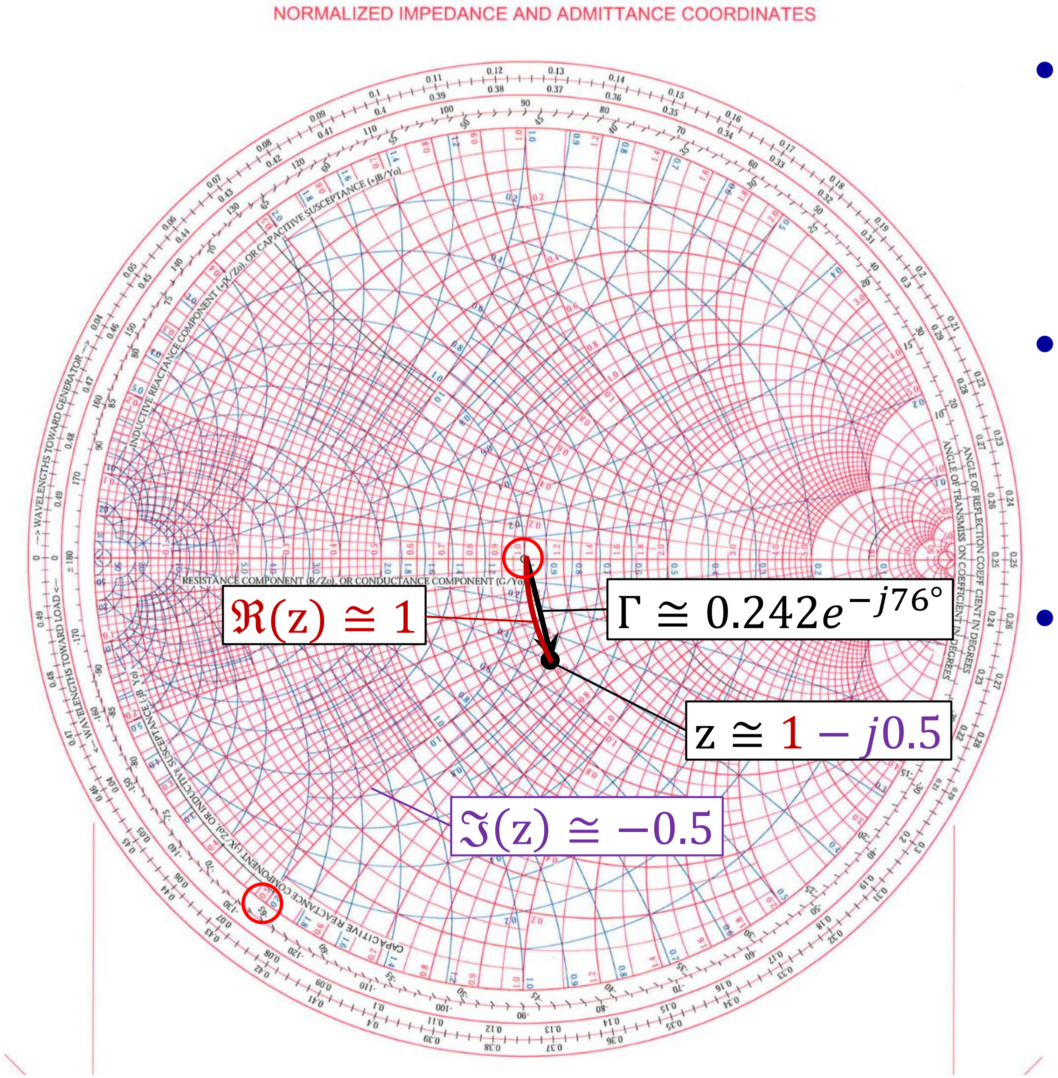
- **A network analyzer measurement of a RF amplifier, operated at $f = 500\ MHz$ shows a reflection coefficient of $\Gamma = 0.242e^{-j76^\circ}$**
  - **Locate Γ in the paper Smith chart**
  - **What is the un-normalized impedance Z ?**
  - **What is the lumped-element series equivalent circuit and what are the element values the make up Z ?**
  - **What circuit element , and its value, is required to match the impedance to $Z_0 = 50\ \Omega$?**
    - Please use the paper Smith chart and find the value of the normalized matching impedance
    - Then de-normalize the value and calculate the value of the lumped element.

$\Gamma = 0.242e^{-j76°}$



- **Locate $\angle \cong -76°$ on the circle**
  ANGLE OF REFLECTION COEFFICIENT IN DEGREES
  - **Mark with a line**
- **Locate $|\Gamma| \cong 0.242$ on the ruler**
  RFL. COEFF, E or I
  - **Mark the point crossing the ∠-line**

$z = 1 - j0.5$



- **Follow the real part circle to $\mathcal{R} \cong 1$ on the real axis RESISTANCE COMPONENT (R/Z_0)**
  - $\mathcal{R}(z) \cong 1$
- **Follow the imaginary part to $\Im \cong -0.5$ to the positive imaginary axis CAPACITIVE REACTANCE COMPONENT ($-jX/Z_0$)**
  - $\Im(z) = -0.5$
- **De-normalize the impedance for $Z_0 = 50\ \Omega$**

$$Z = z\,Z_0 = (1 - j0.5)50\ \Omega = (50 - j25)\Omega$$

  - $\Im(Z) < 0$ ⇒ capacitive

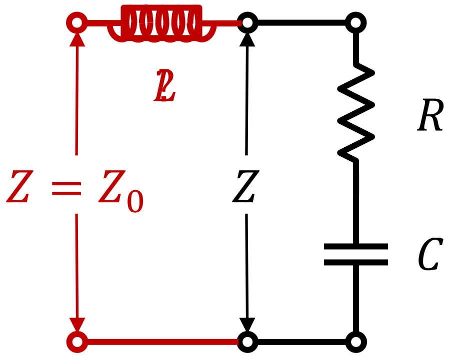
Impedance Matching

- **As the imaginary part of the impedance is negative, the lumped-element equivalent series-impedance circuit is a RC series circuit**
  - **We are in the lower, capacitive half plane of the Smith chart**

$$Z = R + jX = R + \frac{1}{j\omega C} = (50 - j25)\Omega$$

$$\Rightarrow\ R = 50\,\Omega\,,$$

$$\frac{1}{2\pi f C} = 25\,\Omega \ \Rightarrow\ C = \frac{1}{2\pi\,500 \cdot 10^6 s^{-1}\ 25\,V/A} \cong 12.7\,pF$$



- **We need to compensate the capacitive element by an inductive series element to get $\boldsymbol{Z(f = 500\,MHz) = Z_0 = 50\,\Omega}$**
  - **This is called a "dual" network:** $\omega L = \frac{1}{\omega C} \ \Rightarrow\ L = \frac{1}{\omega^2 C}$
  - **This is equivalent with the Smith chart operation:** $X_L = 2\pi f L = 25\,\Omega$

$$\Rightarrow\ L = \frac{25\,V/A}{2\pi\,500 \cdot 10^6 s^{-1}} = 7.96\,nH$$

Final Remarks

- **The *Smith* chart is a special type of a parametric plot for the complex impedance or admittance, mapped to the complex reflection coefficient.**
- **The use of the *Smith* chart can be viewed in two ways:**
  - **As a calculation and impedance matching tool as originally envisioned.**
    - However, today this will happen rather rarely!
    - Please notice: All the examples presented are based on the paper-style *Smith* chart, which is always based on an unitless, normalized impedance $z = Z/Z_0$
  - **As a visualization tool for the complex impedance, along with the reflection coefficient**
    - Still very popular and useful for displaying and analyzing S_{ii} on a vector network analyzer, also used in data-sheets, and RF simulation and education software.
    - Here the *Smith* chart utilizes the actual complex impedance Z in units of Ω! Markers on the parametric trace give all relevant information, including the element values of a selected equivalent circuit.
- **Old, but excellent information on transmission-lines and standing waves:**
  - **https://www.youtube.com/watch?v=I9m2w4DgeVk**
  - **https://www.youtube.com/watch?v=DovunOxlY1k&t=38s**
- ***Smith* chart education software (only for MS-Windows):**
  - **https://www.fritz.dellsperger.net/smith.html**

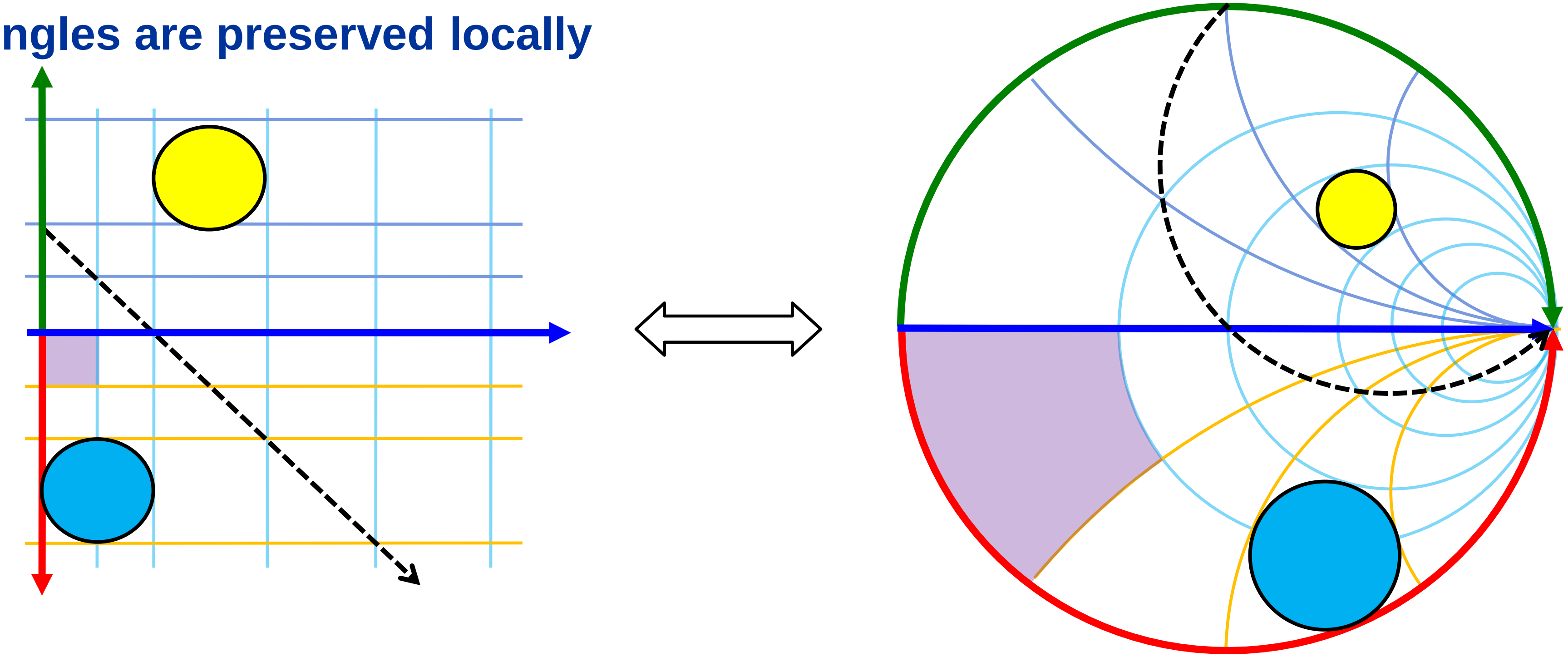
Backup Slides

Navigation in the *Smith* Chart (1)

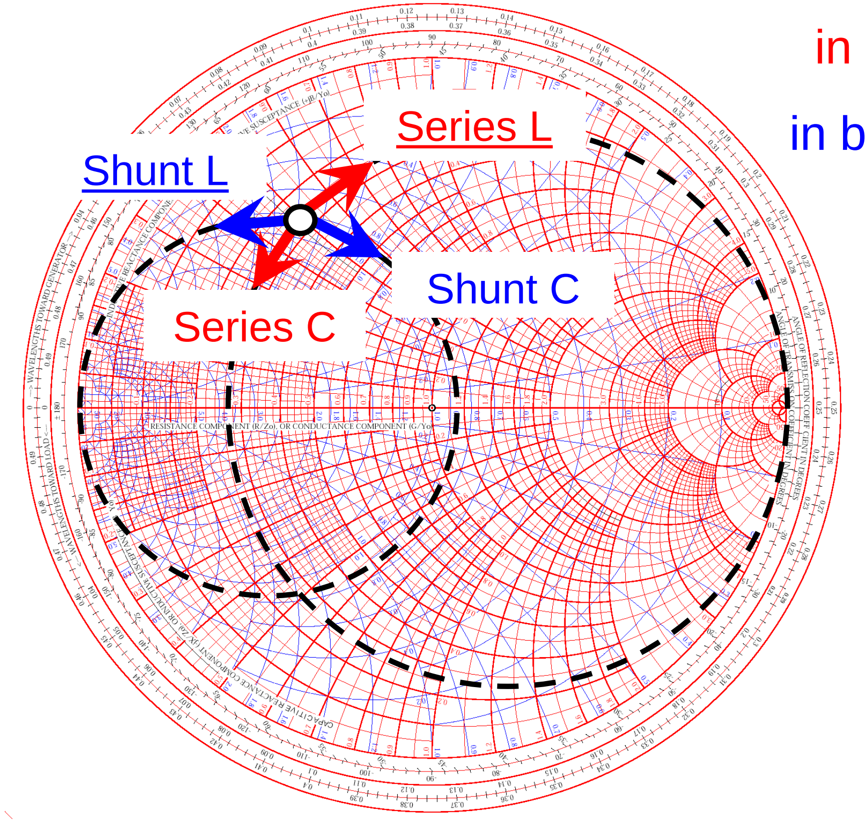
This is a "bilinear" transformation with the following properties:

- **Generalized circles are transformed into generalized circles**
  - **circle → circle**
  - **straight line → circle**
  - **circle → straight line**
  - **straight line → straight line**
- **Angles are preserved locally**

➢ a straight line is equivalent to a circle with infinite radius
➢ a circle is defined by 3 points
➢ a straight line is defined by 2 points



Navigation in the *Smith* Chart (2)

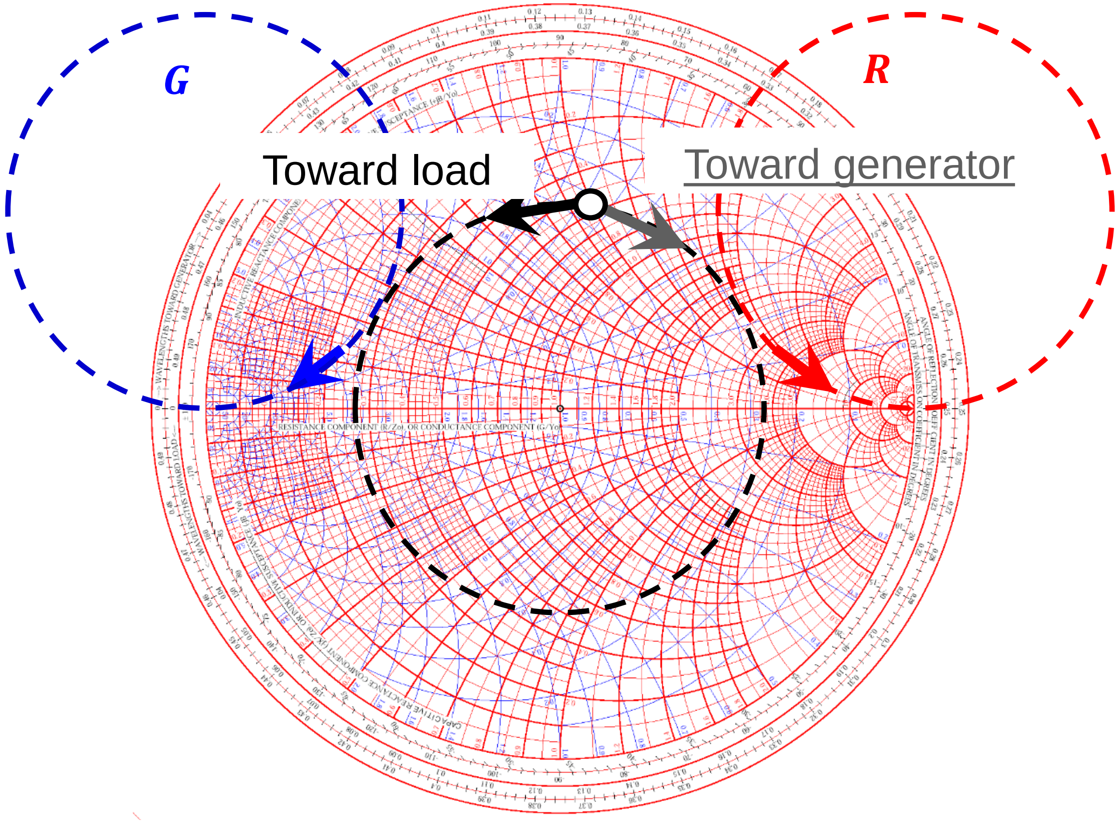


in red: impedance plane $(= \boldsymbol{z})$

in blue: admittance plane $(= \boldsymbol{y})$

	Up	Down
Red circles	**Series L**	**Series C**
Blue circles	**Shunt L**	**Shunt C**

Navigation in the *Smith* Chart (3)



Red arcs	**Resistance *R***
Blue arcs	**Conductance *G***
Con-centric circle	**Transmission line going Toward load Toward generator**