RF Engineering
Cylindrical Resonator
– Summary “Pillbox” Cavity –

Christine Völlinger & Manfred Wendt – CERN

Cylindrical Resonator Equations (1)

- **Eigenfrequencies**
  - **For the TE_{nml}-modes:**

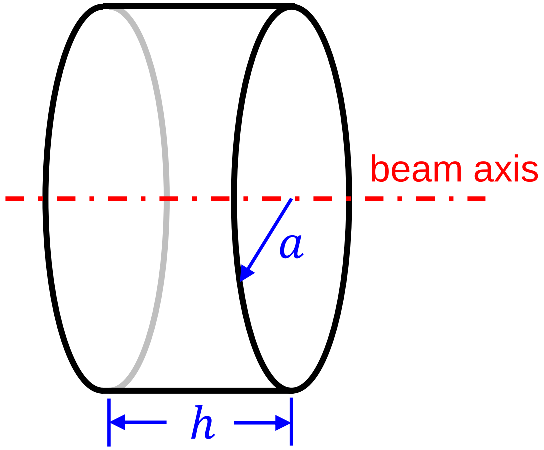
$$f_{TEnml} = \frac{c}{2\pi}\sqrt{\left(\frac{p'_{nm}}{a}\right)^2 + \left(\frac{\pi l}{h}\right)^2} \quad [Hz]$$

  - **For the TM_{nml}-modes:**

$$f_{TMnml} = \frac{c}{2\pi}\sqrt{\left(\frac{p_{nm}}{a}\right)^2 + \left(\frac{\pi l}{h}\right)^2} \quad [Hz]$$

➢ p_{nm} and p'_{nm} are the zeros of the Bessel function of 1st kind J_0, respectively the zeros of the derivative of the Bessel function of 1st kind J'_0

➢ $c \cong 2.998 \times 10^8\ m/s$ speed of light

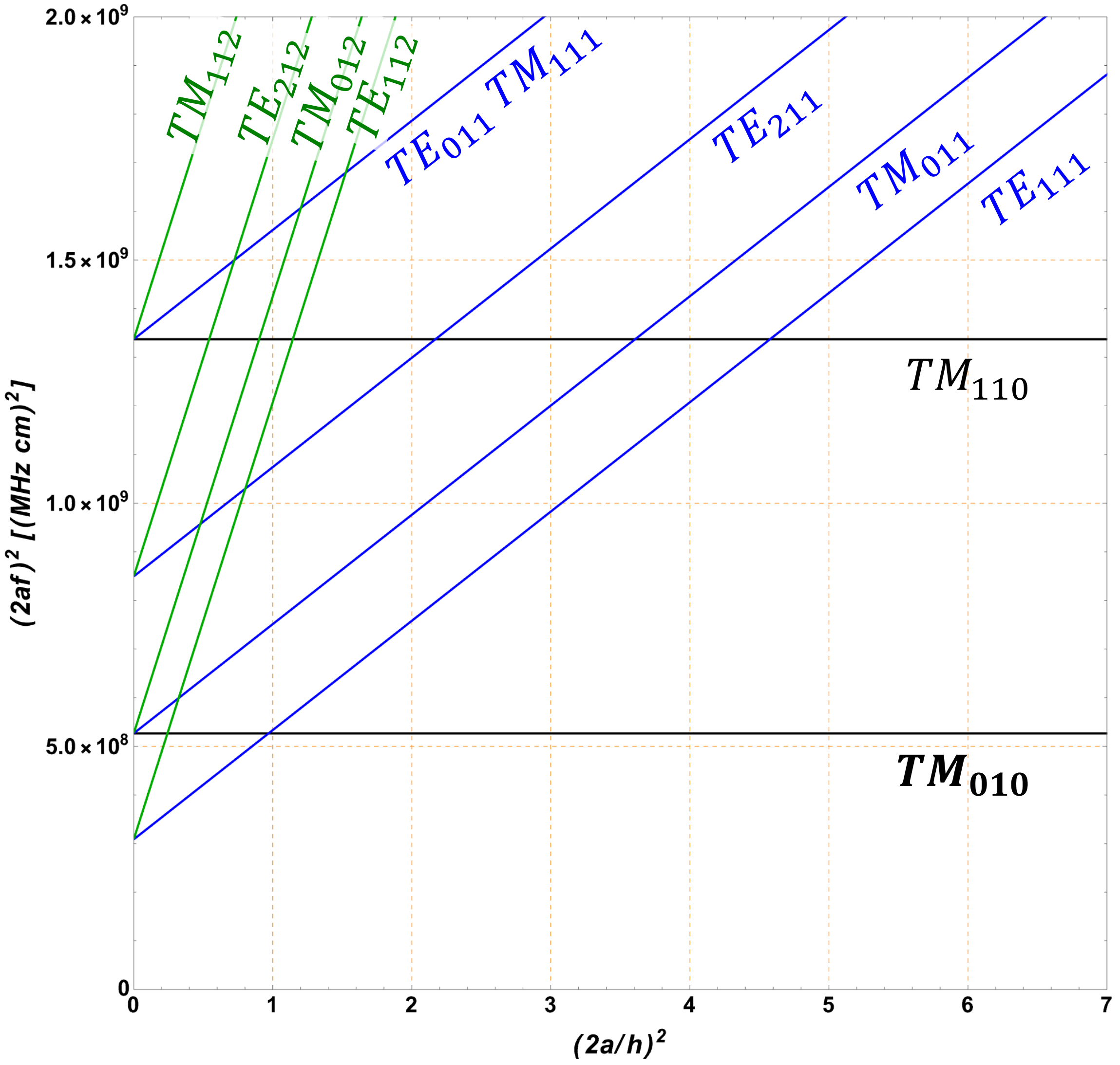


- **"Ideal" cylindrical resonator**
  - **No beam ports**
  - **Diameter: $2a$**
  - **Height: h**
  - **Vacuum: $\varepsilon_r = 1$**

n	p'_{n1}	p'_{n2}	p'_{n3}
0	3.832	7.016	10.174
1	1.841	5.331	8.536
2	3.054	6.706	9.970

n	p_{n1}	p_{n2}	p_{n3}
0	**2.405**	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620

Cylindrical Resonator Equations (2)



Mode chart for cylindrical resonators

- **Analytical equations for the TM_{010} accelerating mode**
  - **TM_{010}-mode related wavelength:**

$$\lambda_{TM010} = \frac{c}{f_{TM010}} = \frac{2\pi}{p_{01}} a \cong 2.61274\, a \;\; [m]$$

$$a = \frac{p_{01}}{2\pi} \lambda_{TM010} \cong 0.38274\, \lambda_{TM010} \;\; [m]$$

  - **TM_{010}-mode resonance frequency:**

$$f_{TM010} = \frac{c}{\lambda_{TM010}} \;\; [Hz]$$

Cylindrical Resonator Equations (3)

- **Quality factor – unloaded Q_0 (Q-factor or Q-value)**

$$\boldsymbol{Q = \frac{a}{\delta}\left[1 + \frac{a}{h}\right]^{-1}}$$

  – **with the skin-depth:**

$$\delta = \sqrt{\frac{2}{\omega_{TM010}\sigma\mu}} \quad [m]$$

  - with:

$$\omega_{TM010} = 2\pi f_{TM010}$$

$$\mu = \mu_r \mu_0 \cong 1 \times 4\pi \cdot 10^{-7} \ H/m$$

(non-magnetic media)

$$\sigma \ [S/m]$$

(conductivity of the material of the cavity walls)

- **"Geometry factor" R/Q (R-over-Q)**

$$\boldsymbol{R/Q = \frac{4\,\eta_0}{\pi p_{01}^3 J_1^2(p_{01})} \frac{\sin^2\left(\frac{p_{01}}{2}\frac{h}{a}\right)}{h/a}} \quad [\Omega]$$

$$R/Q \cong 128\,\Omega \frac{\sin^2\left(\frac{p_{01}}{2}\frac{h}{a}\right)}{h/a} [\Omega]$$

$$\approx 185\,\Omega\, h/a\ [\Omega] \quad \text{for: } \frac{p_{01}}{2}\frac{h}{a} \ll 1$$

  – **With:** $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi \ [\Omega] \cong 377 \ [\Omega]$

$$p_{01} = 2.40483$$

$$J_1(p_{01}) = 0.519147$$

Cylindrical Resonator Equations (4)

- **Transit time factor**

$$\boldsymbol{T} = \frac{\sin(\boldsymbol{\pi g/\beta\lambda})}{\boldsymbol{\pi g/\beta\lambda}}$$

  - **With:**

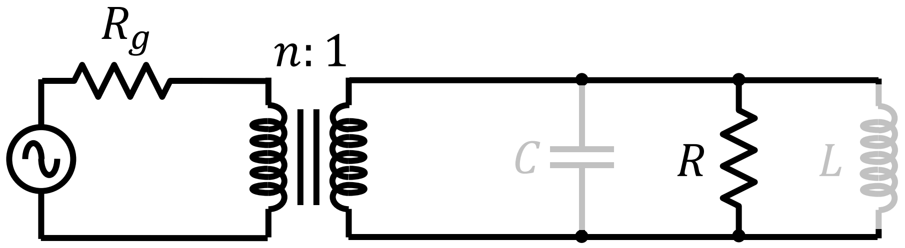
$$\beta = \frac{v}{c} \quad \text{(relative beam velocity)}$$

$$g = h \ \ [m] \text{ (gap length)}$$

$$\lambda = \lambda_{TM010} \ \ [m] \text{ (wavelength)}$$

Equivalent Circuit, without Beam

- **Resonant frequency:** $$\omega_0 = \frac{1}{\sqrt{LC}} \quad [s^{-1}]$$
  - with: $\omega = 2\pi f$
- **Characteristic impedance "R over Q"** $$X = \frac{R}{Q} = \omega_0 L = \frac{1}{\omega_0 C} = \sqrt{L/C} \quad [\Omega]$$
- **Stored energy at resonance:** $$U = \underbrace{\frac{1}{4}|V|^2 C}_{= U_e} + \underbrace{\frac{1}{4}|V|^2 \frac{1}{\omega_0^2 L}}_{= U_m = \frac{1}{4}|I_L|^2 L} \quad [Ws]$$
- **Dissipated power** $$P = \frac{V^2}{2R} \quad [W]$$
- **Q-factor (unloaded Q_0):** $$Q = \frac{R}{X} = \frac{\omega_0 U}{P}$$
- **Shunt impedance: (circuit definition)** $$R = \frac{V^2}{2P} \quad [\Omega]$$



- **Coupling parameter: (in critical coupling)** $$k^2 = n^2 = \frac{R}{R_g}$$
- **Tuning sensitivity:** $$\frac{\Delta f}{f} = \frac{1}{2}\frac{\Delta C}{C} = -\frac{1}{2}\frac{\Delta L}{C}$$
- **More on the Q-factor:**

$$Q = \frac{f_0}{f_{BW,3dB}} \qquad \frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{ext}}$$

  - **Critical coupling:** $Q_0 = Q_{ext}$ $\Rightarrow Q_0 = 2\,Q_L$