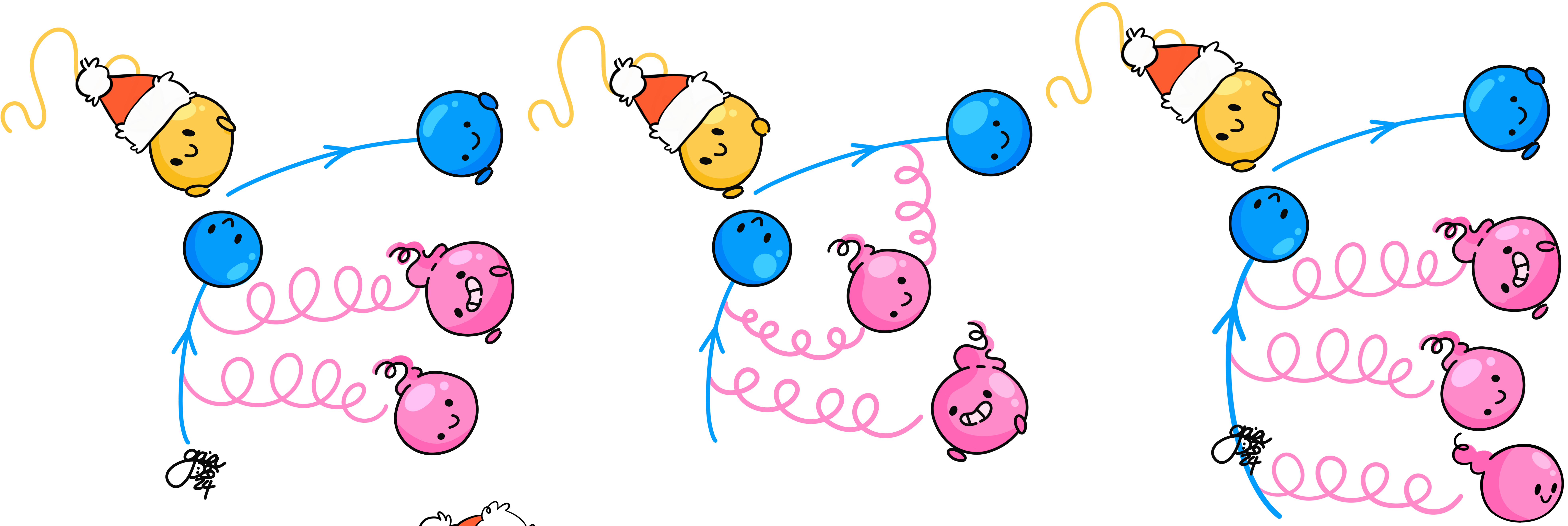
3 loops & 4 cuts

towards N3LO RRR antenna functions



Gaia Fontana (UZH)

In collaboration with Thomas Gehrmann & Kay Schönwald

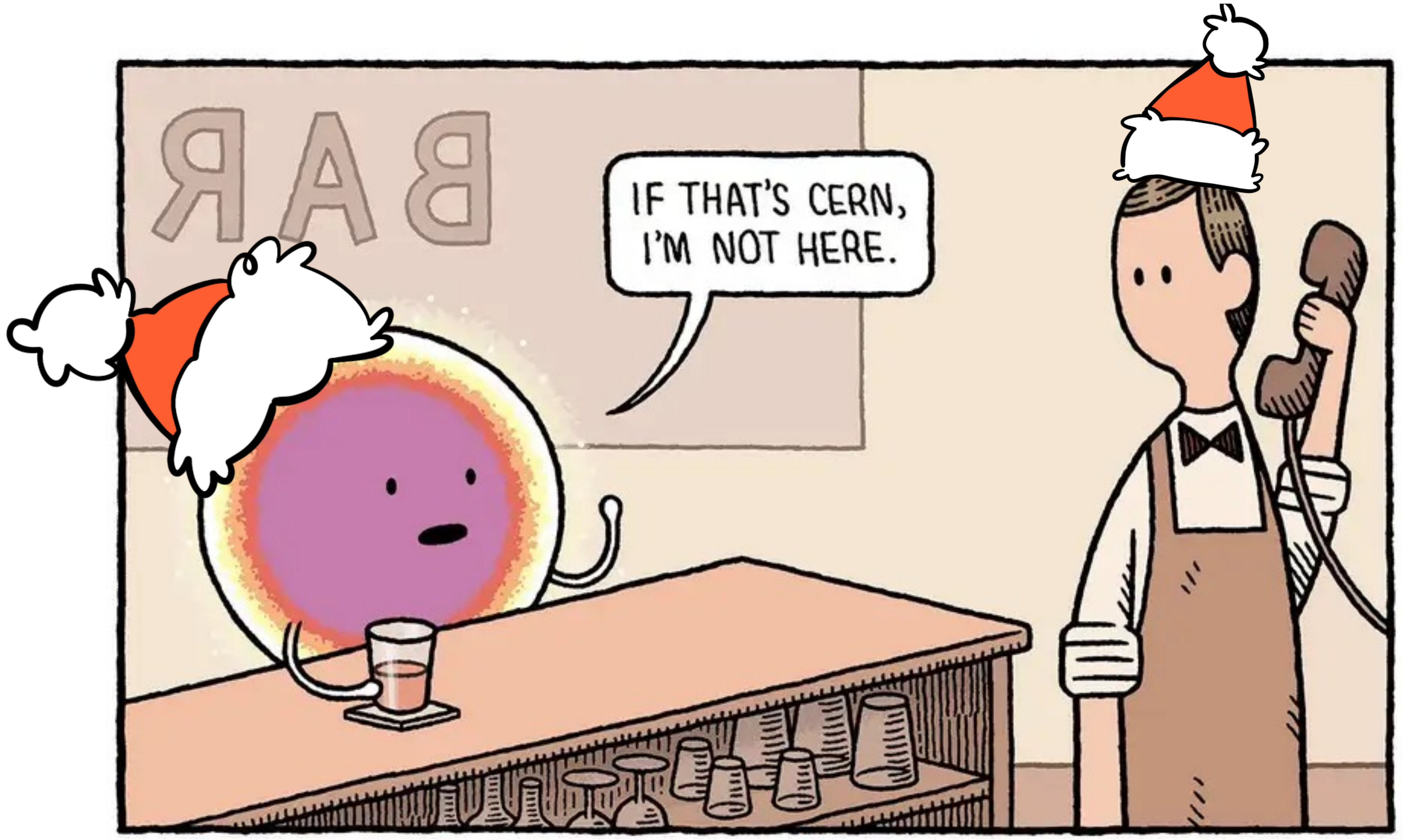
Based on JHEP 03 (2024) 159 & upcoming works

Milan Xmas meeting, 20/12/2024

Universität Zürich[UZH]

Why precision?

- **Precision physics** as
  - test of the Standard model
  - gate to new physics

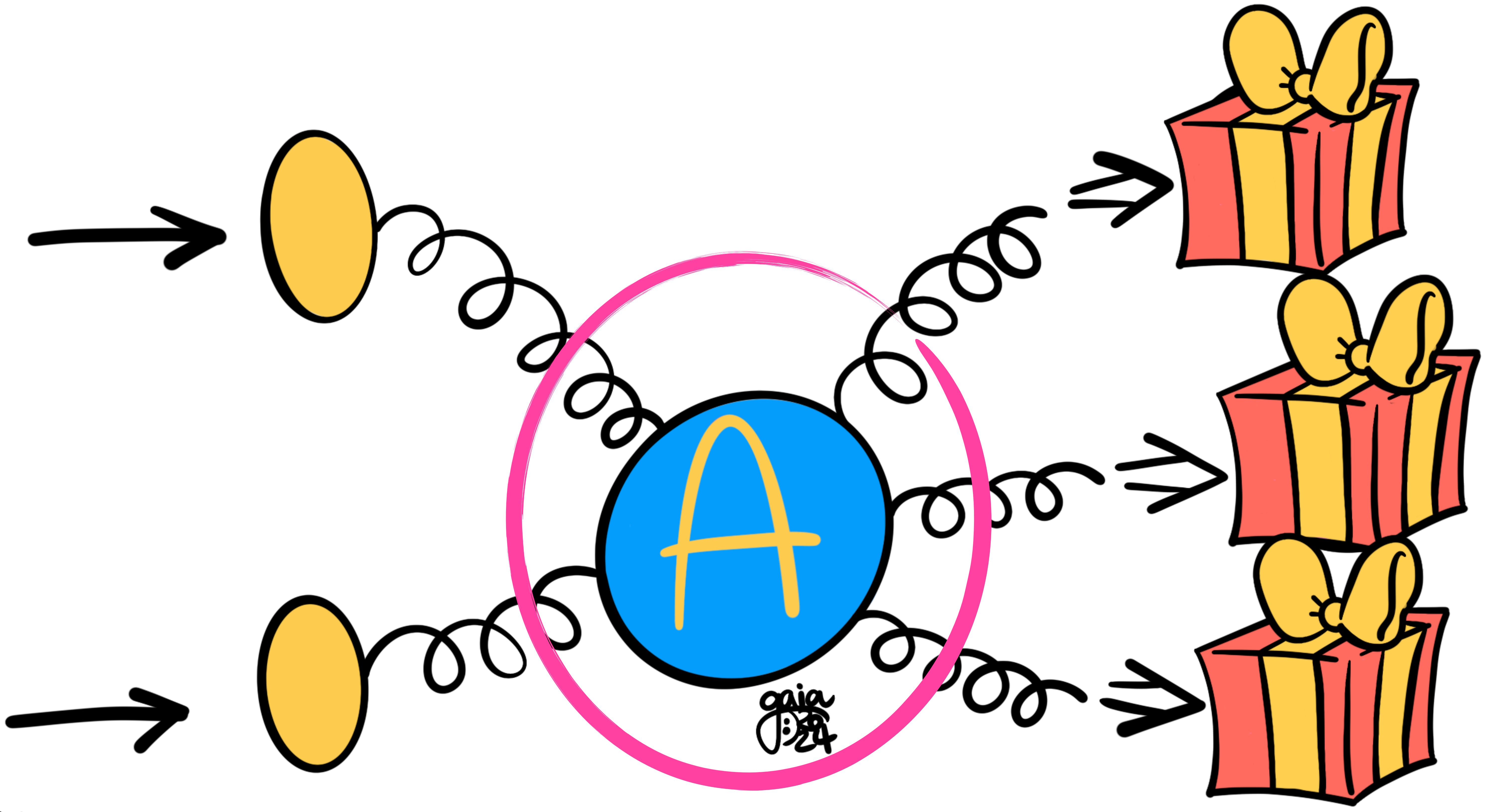


TOM GAULD for NEW SCIENTIST

- **High-Lumi upgrade of LHC :**
  - theory and experiments must have comparable uncertainties
  - needed: %-level accuracy:

perturbation theory @ **NNLO** and often **N3LO**

Recipe for a theoretical prediction



Many ingredients

- PDFs to describe the proton structure
- **Hard scattering**
- Radiation and evolution to hadronic states

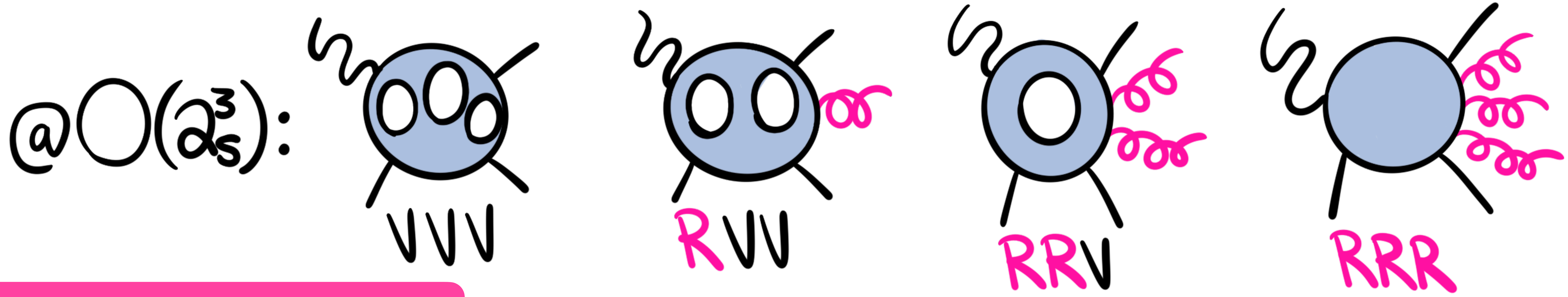
Hard Scattering

Looking @ QCD corrections:

$$d\sigma = d\sigma_{LO} + \alpha_S \, d\sigma_{NLO} + \alpha_S^2 \, d\sigma_{NNLO} + \alpha_S^3 \, d\sigma_{N3LO} + \ldots$$

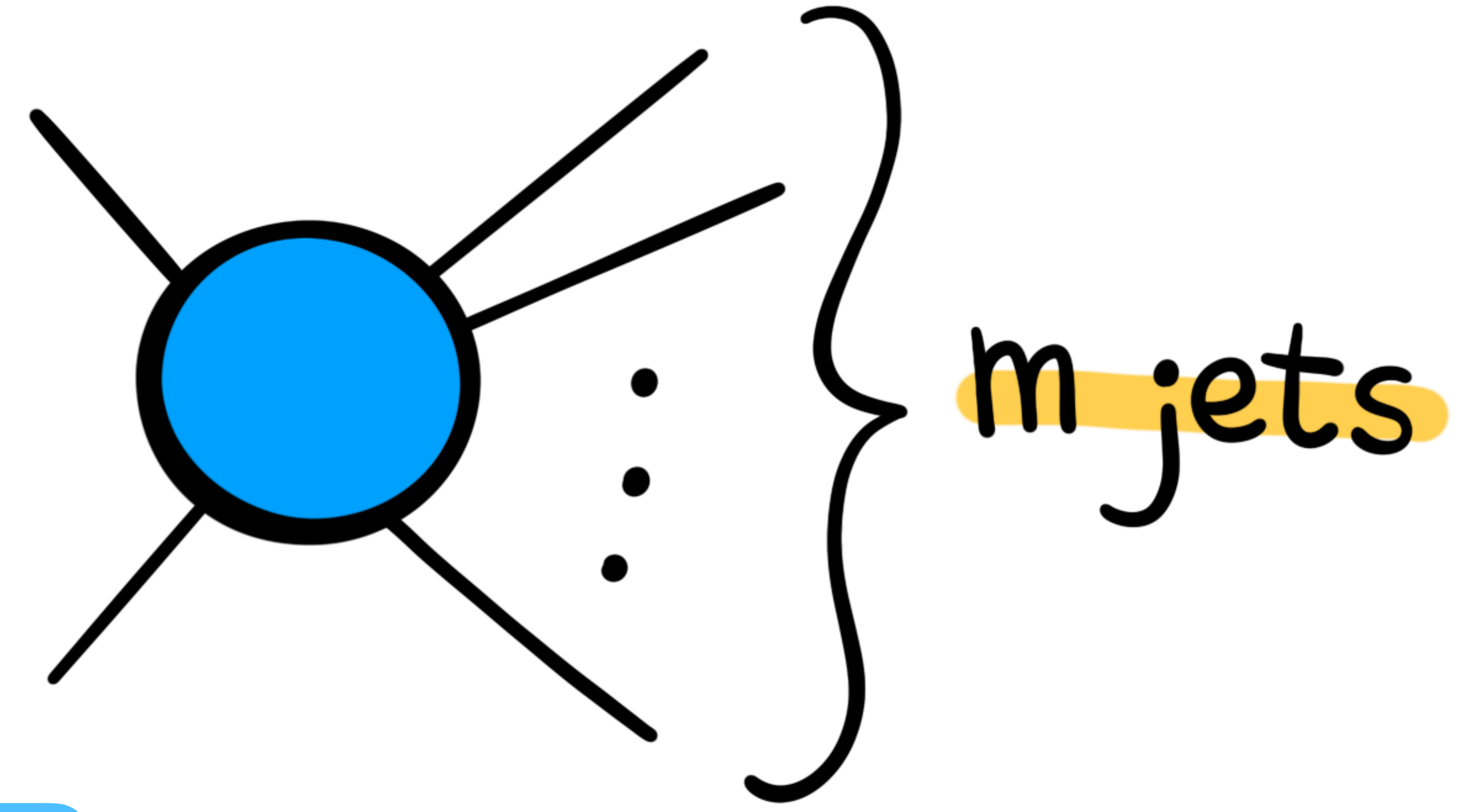
Perturbative series in the strong coupling

Beyond LO: contributions from diagrams with increasing loops and legs



Real corrections!

$$d\sigma = d\sigma_{LO} + \alpha_S\, d\sigma_{NLO} + \alpha_S^2\, d\sigma_{NNLO} + \alpha_S^3\, d\sigma_{N3LO} + \ldots$$



@ LO

$$d\sigma_{LO} = \int_{d\Phi_m} d\sigma_{Born}$$

@ NLO

$$d\sigma_{NLO} = \int_{d\Phi_{m+1}} d\sigma^{R}_{NLO} + \int_{d\Phi_m} d\sigma^{V}_{NLO}$$

@ NNLO

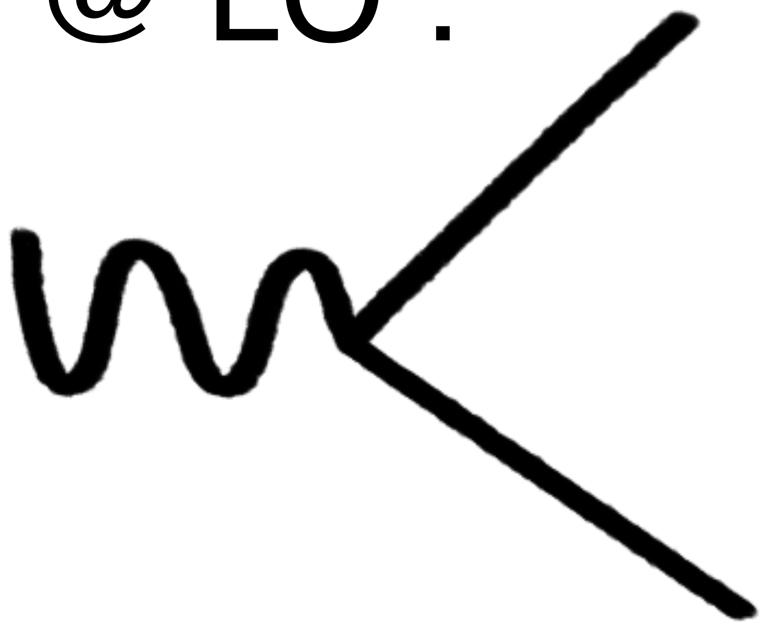
$$d\sigma_{NNLO} = \int_{d\Phi_{m+2}} d\sigma^{RR}_{NNLO} + \int_{d\Phi_{m+1}} d\sigma^{RV}_{NNLO} + \int_{d\Phi_m} d\sigma^{VV}_{NNLO}$$

@ N3LO

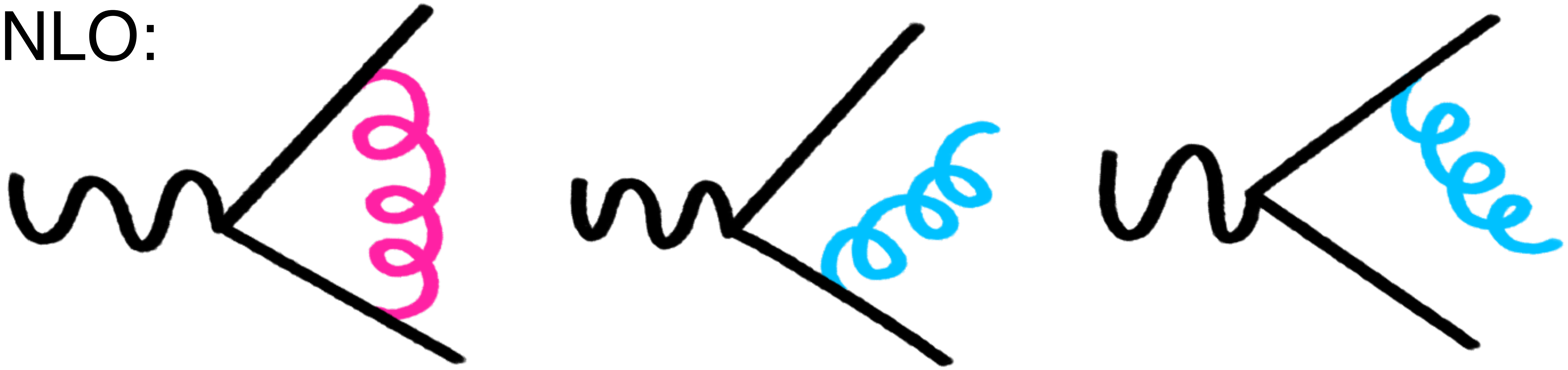
$$d\sigma_{N3LO} = \int_{d\Phi_{m+3}} d\sigma^{RRR}_{N3LO} + \int_{d\Phi_{m+2}} d\sigma^{RRV}_{N3LO} + \int_{d\Phi_{m+1}} d\sigma^{RVV}_{N3LO} + \int_{d\Phi_m} d\sigma^{VVV}_{N3LO}$$

$\gamma \to q\bar{q}$

@ LO :



@ NLO:



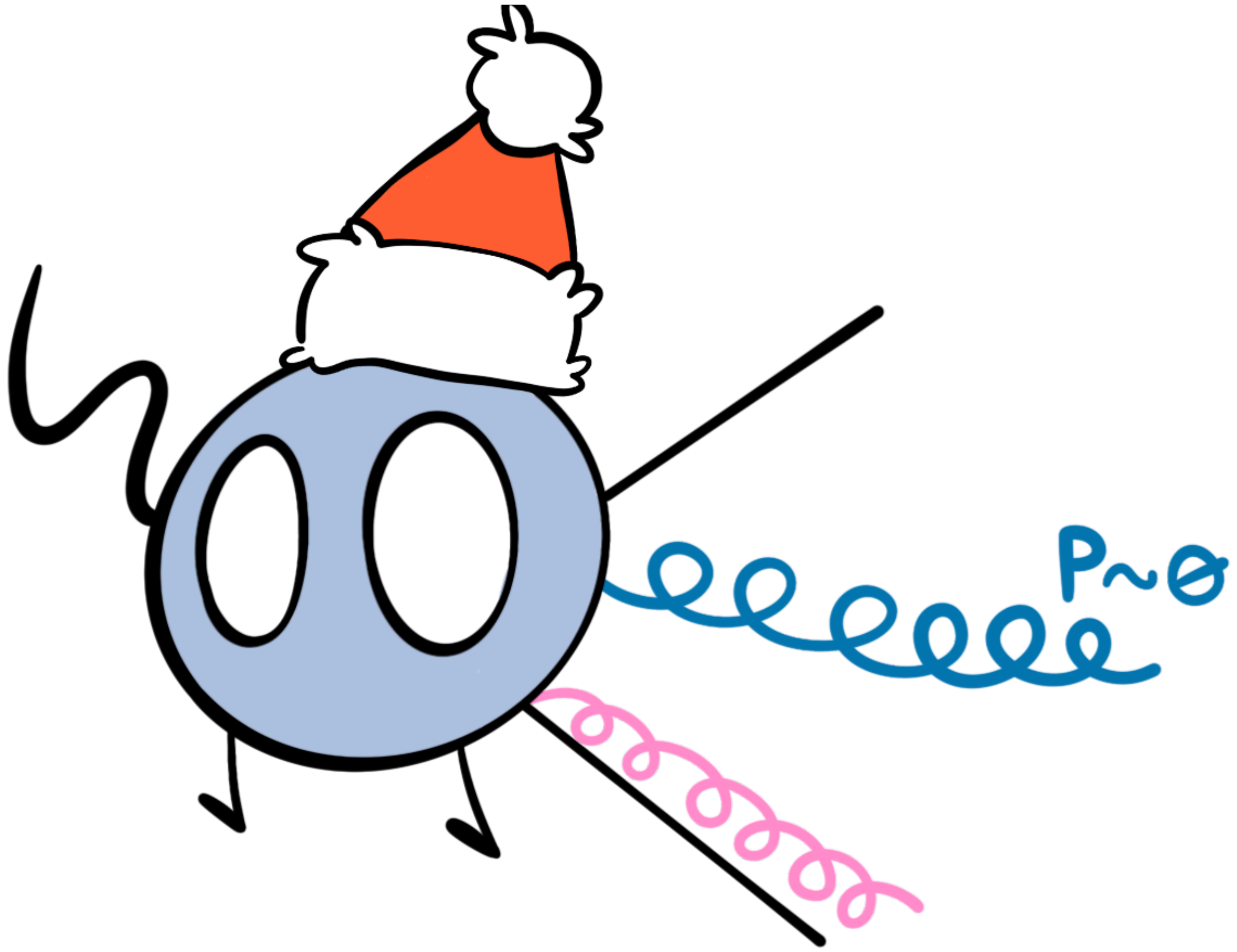
Virtual correction **Radiation of an extra gluon**

KLN thm **[Kinoshita 1962 ; Lee, Nauenberg 1964]**

finiteness when summing over all unresolved configurations

unresolved : soft or collinear

- Separate pieces are IR-divergent:
  - **Explicit** poles in ϵ after **loop** integration
  - **Implicit** divergencies from **real** radiation



How do we deal with these divergencies?

NLO example

⛔ Hard to solve analytically ⛔

$$d\sigma_{NLO} = \int_{d\Phi_{m+1}} d\sigma^R_{NLO} + \int_{d\Phi_m} d\sigma^V_{NLO}$$

- finite

$$\int_{d\Phi_{m+1}} d\sigma^R_{NLO} \quad + \quad \int_{d\Phi_m} d\sigma^V_{NLO}$$

∞ $\qquad$ ∞

NLO example

⛔ Hard to solve analytically ⛔

$$\mathrm{d}\sigma_{NLO} = \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{R}_{NLO} + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\sigma^{V}_{NLO}$$

- finite

$$\int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{R}_{NLO} - \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{S}_{NLO} \quad + \quad \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\sigma^{V}_{NLO} + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{S}_{NLO}$$

- finite

- finite

Subtraction Schemes

$$\mathrm{d}\sigma_{NLO} = \int_{\mathrm{d}\Phi_{m+1}} \left(\mathrm{d}\sigma^R_{NLO} - \mathrm{d}\sigma^S_{NLO}\right) + \int_{\mathrm{d}\Phi_m} \left[\mathrm{d}\sigma^V_{NLO} + \int_1 \mathrm{d}\sigma^S_{NLO} \right]$$

- finite
- finite

- Add and subtract the same quantity $\mathrm{d}\sigma^S$
  - Mimics singular behaviour in IR-limits of $\mathrm{d}\sigma^V, \mathrm{d}\sigma^R$
  - Makes the integrals individually finite
  - Simple enough to be analytically integrated over $\mathrm{d}\Phi_1$

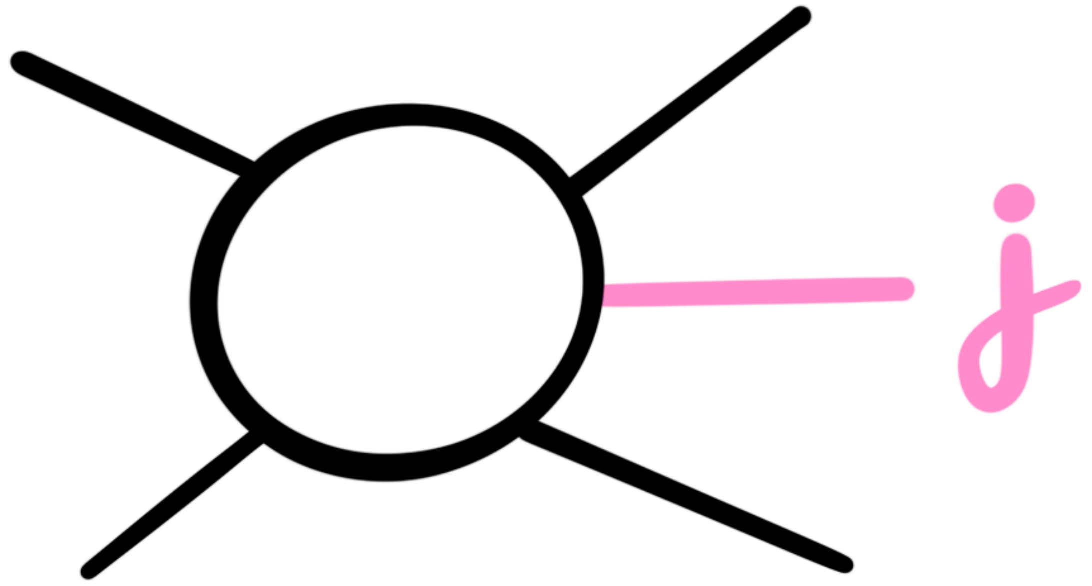
Antenna Subtraction scheme

[Gehrmann-De Ridder, Gehrmann, Glover (2007)]

- **Antenna functions**
  - Built from simple matrix elements
  - Mimic the divergent behaviour in singular limits
  - Can be easily integrated over phase space

$$d\sigma^{S}_{NLO} \sim X^{\ell}_{2+\,extra\,radiation}\,\tilde{M}^{\ell}_{hard\,partons}\,J_m$$

Exploit factorisation of matrix elements in IR limits



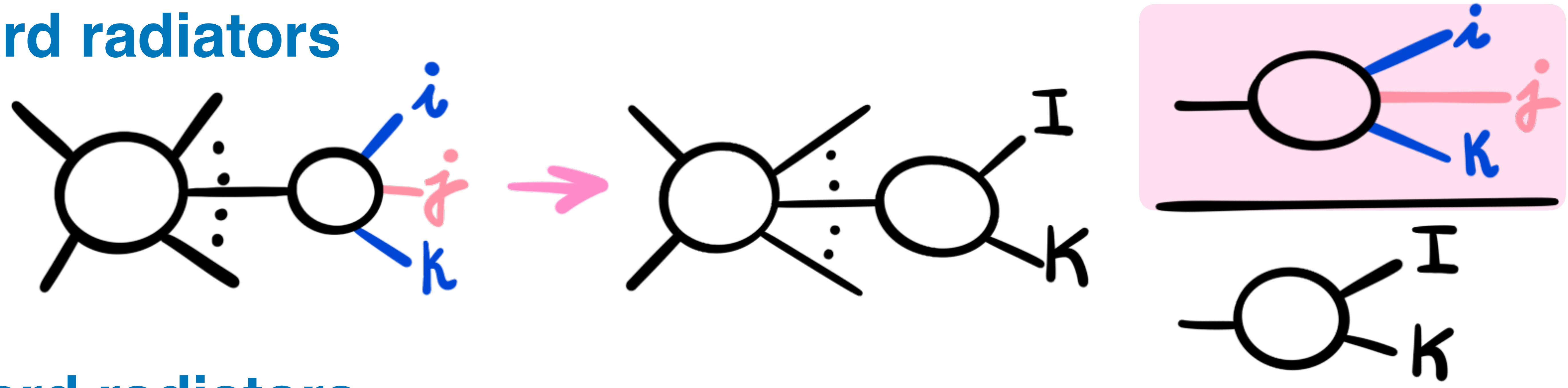
What can happen?

Matrix element with an extra radiation j

IR limit factorization

$j \parallel i, \quad j \parallel k, \quad j$ soft

Final state hard radiators

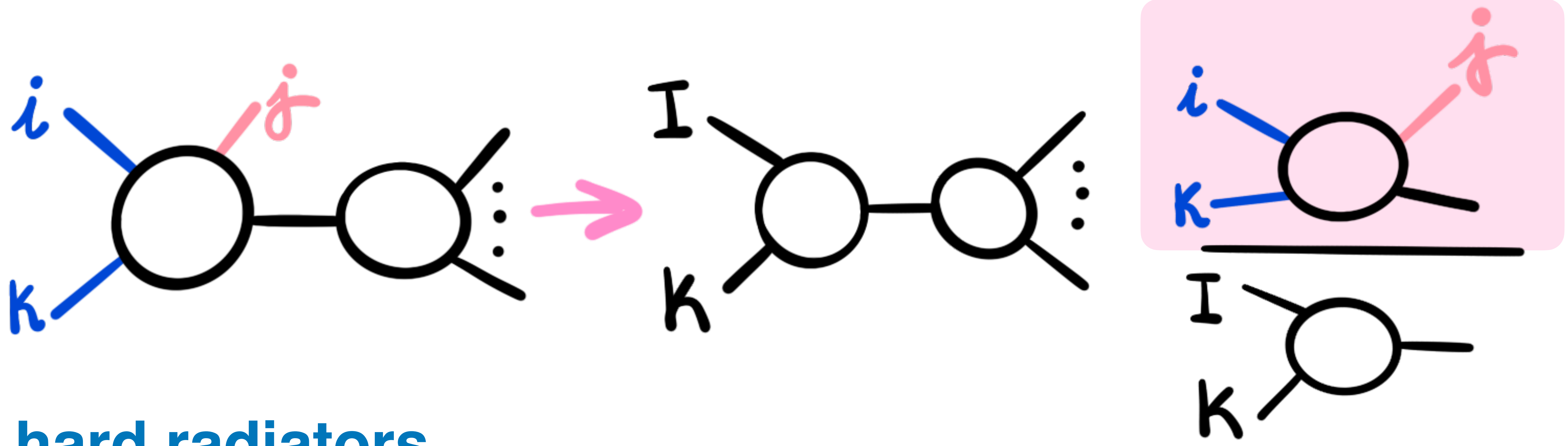


Annihilation into hadrons

Solved! :)

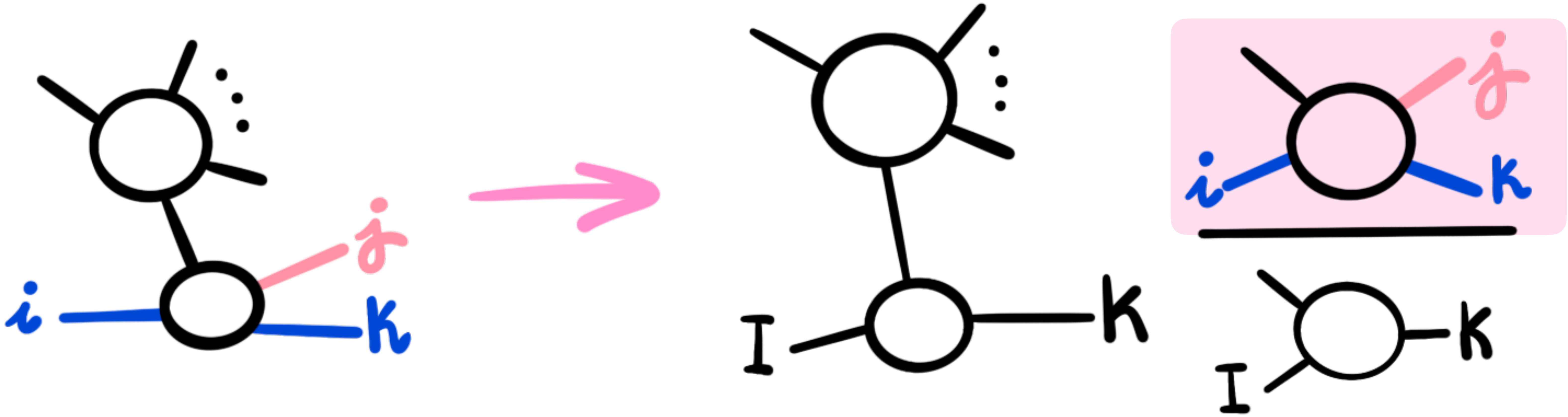
[Chen,Jakubčík,Marcoli, Stagnitto ’23]

Initial state hard radiators



Drell-Yan

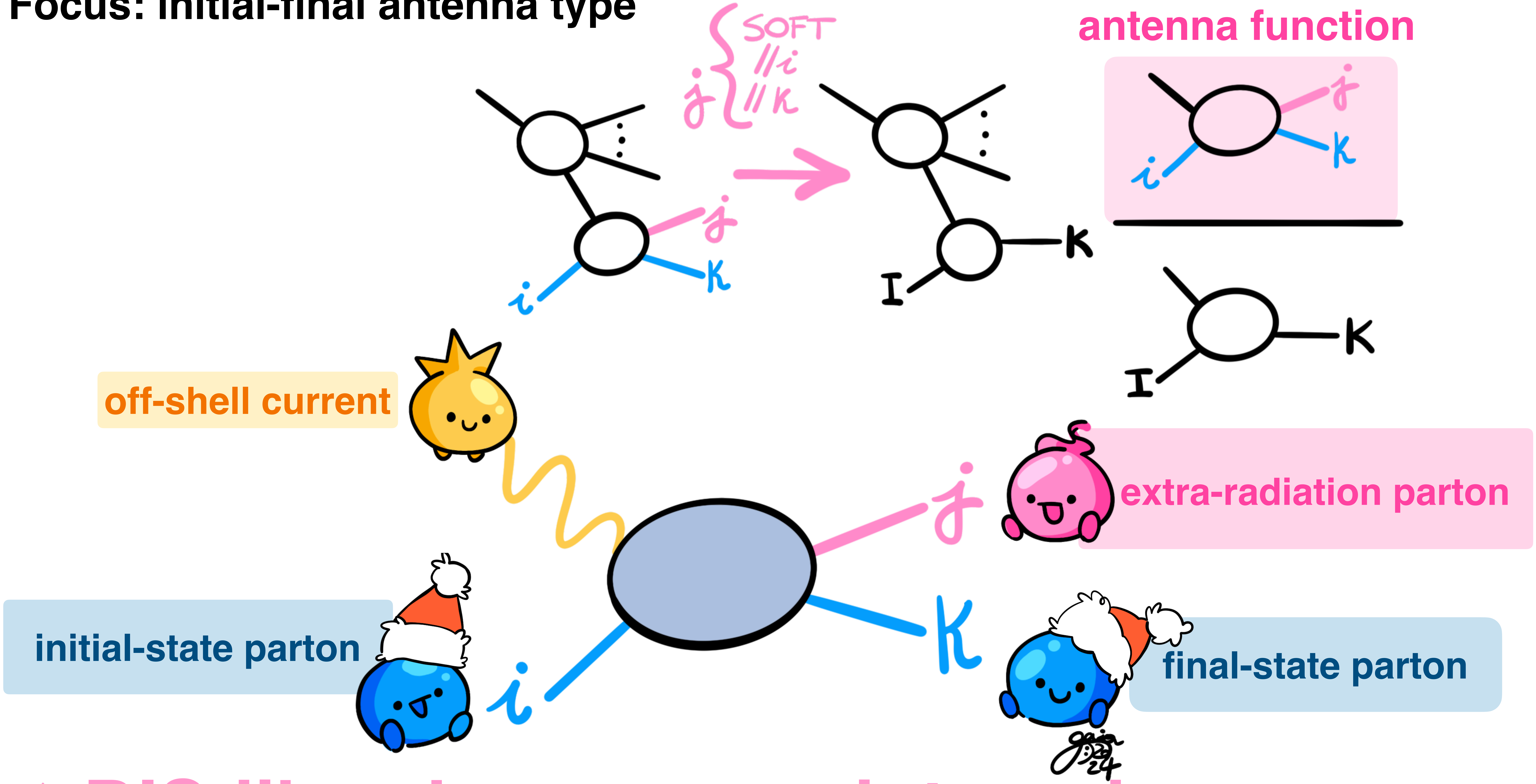
Initial-final state hard radiators



Deep Inelastic Scattering

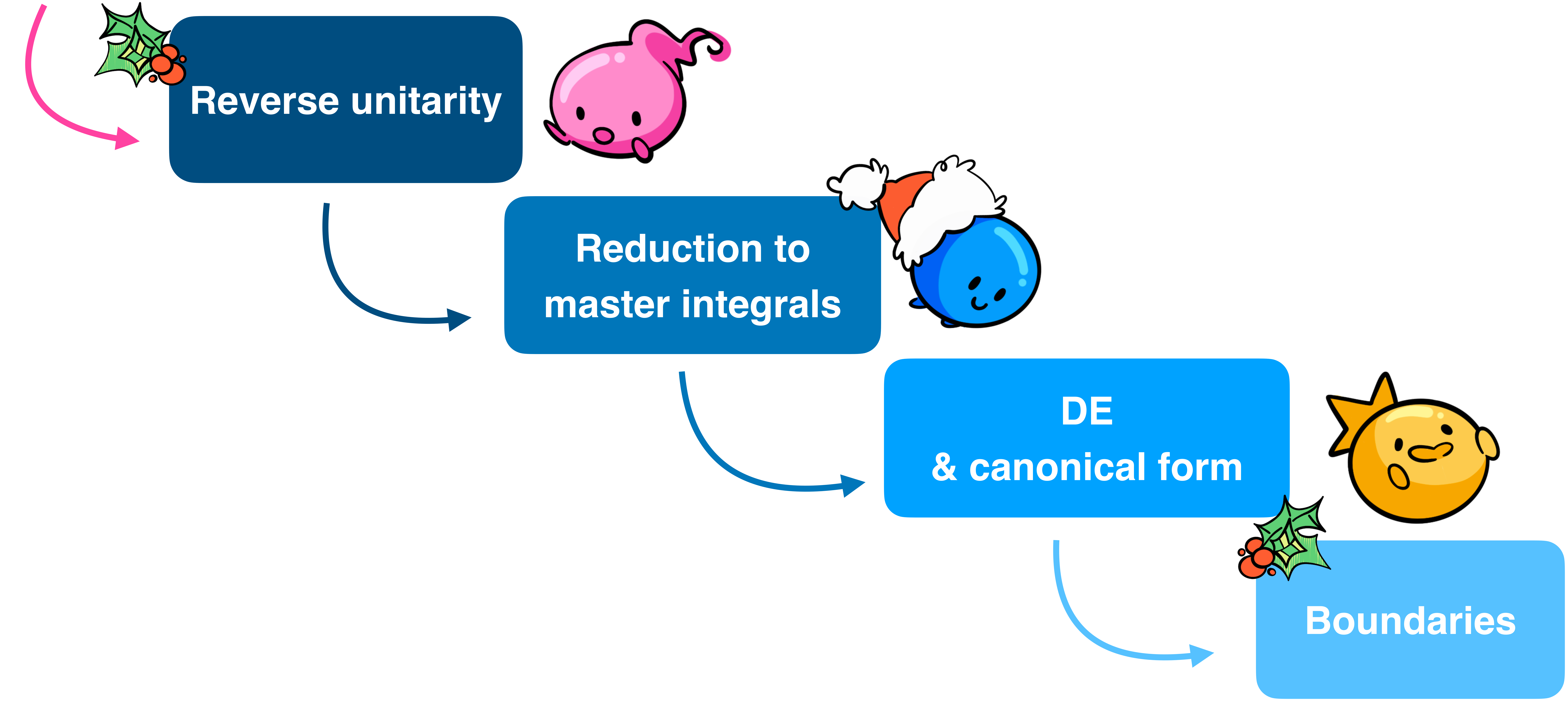
- Antenna functions

Focus: initial-final antenna type



⇒DIS-like phase space integrals

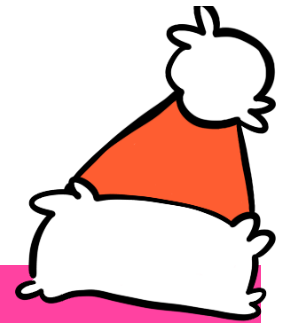
Workflow



Phase space integral
Reverse unitarity
Reduction to master integrals
DE & canonical form
Boundaries

Reverse Unitarity

[Anastasiou, Melnikov 2002]



phase space → (cut) loops

$$-2\pi i \delta^+(p_i^+) = \frac{1}{p_i^2 + i0^+} - \frac{1}{p_i^2 - i0^+} = \frac{1}{[p_i^2]_{cut}}$$

$$I_{RRR} = \int \mathrm{d}\Phi_4 \, (2\pi)^D \delta^D \left(q_1 + q_2 - \sum_{i=1}^{4} p_i \right) \prod_j \frac{1}{D_j^{\alpha_j}}$$

Notice!

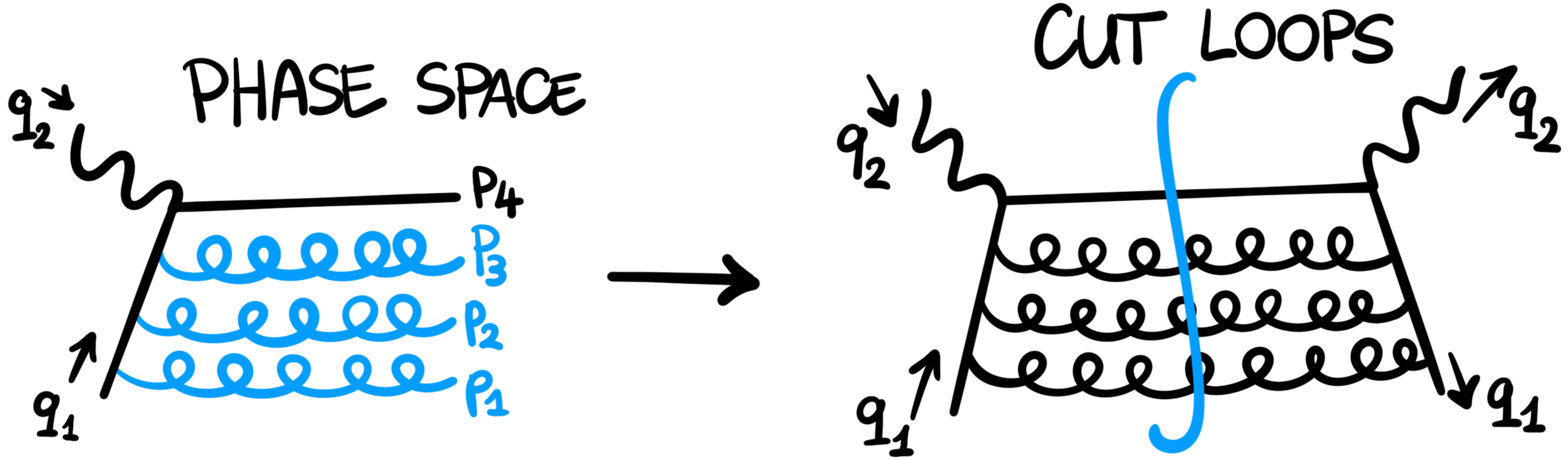
$$\mathrm{d}\Phi_n = \prod_{i=1}^{n} \frac{\mathrm{d}^d p_i}{(2\pi)^d} \delta^+ \left(p_i^2\right)$$

$$I_{RRR} = \int \frac{\mathrm{d}^D p_1}{(2\pi)^D} \int \frac{\mathrm{d}^D p_2}{(2\pi)^D} \int \frac{\mathrm{d}^D p_3}{(2\pi)^D} \frac{1}{[p_1^2]_{cut}} \frac{1}{[p_2^2]_{cut}} \frac{1}{[p_3^2]_{cut}} \frac{1}{[p_4^2]_{cut}} \prod_j \frac{1}{D_j^{\alpha_j}}$$

phase space → (cut) loops

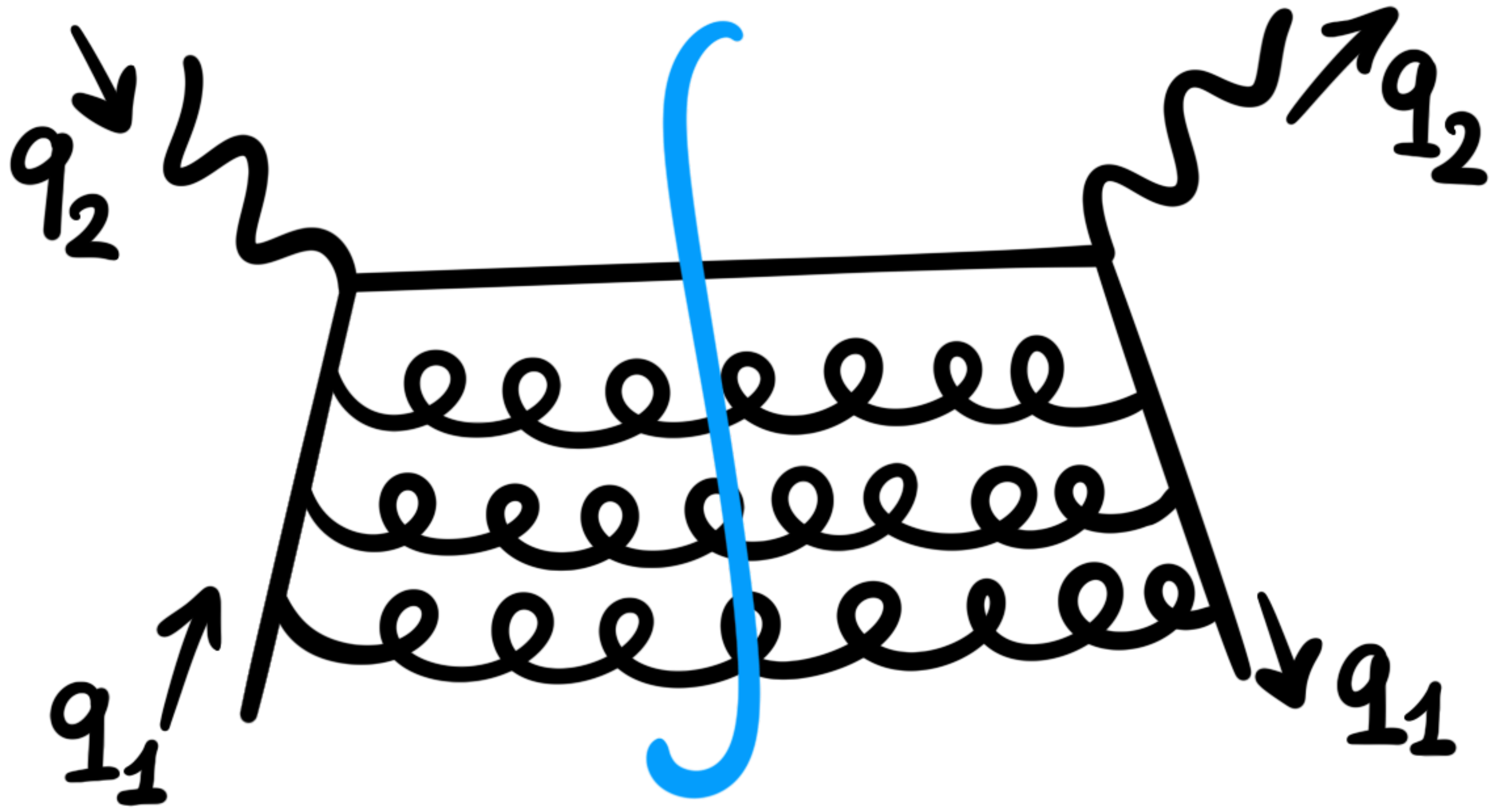
$$-2\pi i \delta^+(p_i^+) = \frac{1}{p_i^2 + i0^+} - \frac{1}{p_i^2 - i0^+} = \frac{1}{[p_i^2]_{cut}}$$

Diagrammatically



$$\int d\Phi_4 (2\pi)^D \delta^D\Big(P+q-\sum_{i=1}^{4} p_i\Big) \prod_j \frac{1}{D_j^{a_j}} \to \int \prod_{i=1}^{3} \frac{d^D p_i}{(2\pi)^D} \frac{1}{\cancel{p_1^2}\,\cancel{p_2^2}\,\cancel{p_3^2}\,\cancel{p_4^2}} \prod_j \frac{1}{D_j^{a_j}}$$

Forward kinematics



This DIS (squared) amplitude contains a lot of integrals $\{I_j\}$

$\rightarrow$ how to make things better?

Reduction to Master integrals

[Chetyrkin, Tkachov ’81; Laporta 2000]

Reduction into a basis of **linearly independent master integrals**

$\{g_j\} \subset \{I_j\}$

$$I_j = \sum_k c_{jk} \, g_k$$

rational coefficients (c_{jk})

master integrals (g_k)

modulo identities:

- Integration By Parts
- Lorentz Invariance
- symmetry relations

For phase-space integrals:

positive (integer) powers of cut denominators are zero

DE for Feynman integrals

[Barucchi, Ponzano ’73; Kotikov ’91; Bern, Dixon, Kosower ‘94; Gehrmann, Remiddi 2000]

Derivative of MI with respect to external invariants

$$\partial_z g_i = \sum_j a_{ij} I_j$$

Use IBP relations to rewrite RHS

$$I_j = \sum_k c_{jk}\, g_k$$

Obtain a system of first order DE for the MI!

$$\partial_z g_i = \sum_{jk} a_{ij} c_{jk}\, g_k$$

$$\partial_z \vec{g} = A(\epsilon, z) \cdot \vec{g}$$

How to solve a differential equation:

- Generic solution
- Boundary condition

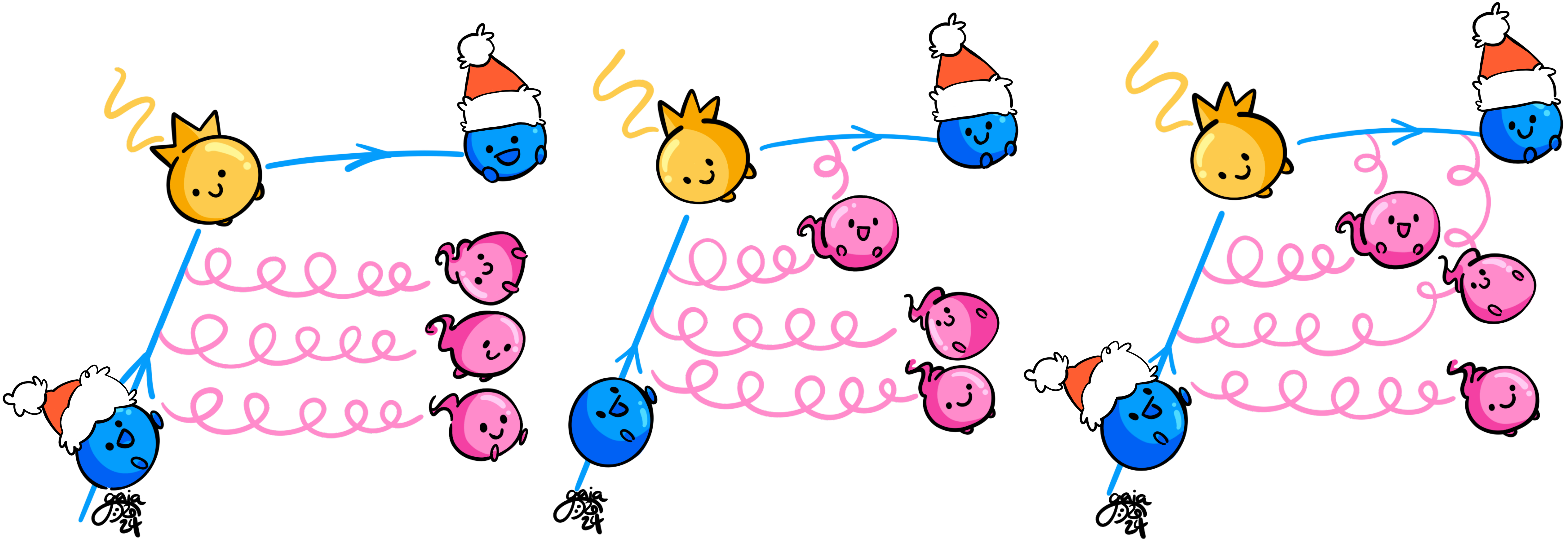
Rewrite the DE in **canonical form** [Henn 2013]: solution in terms of iterated integrals

$$\partial_z \vec{g} = \epsilon A^{\star}(z) \cdot \vec{g}$$

Boundary Conditions

- Consistency conditions
  - Finding relations between boundaries
- Evaluation in some kinematic limit
  - Fix the remaining ones

N3LO phase-space integrals

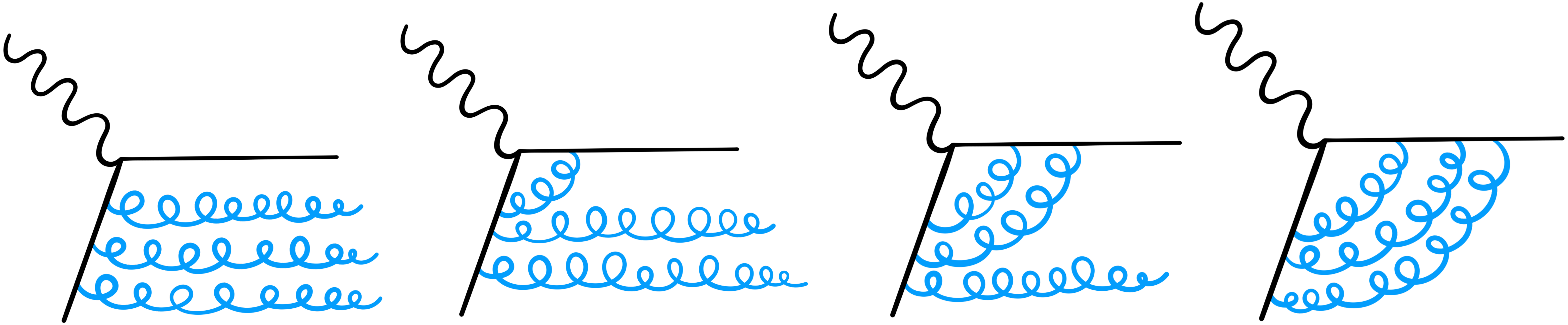


DIS @ N3LO

Representative contributions at order α_s^3

$$q_1 + q_2 \to p_1 + p_2 + (p_3) + (p_4)$$

$$q_2^2 = -Q^2 < 0,\ \ q_1^2 = 0,$$
$$p_i^2 = 0, \quad i = 1,2,3,4$$



RRR $2 \to 4$

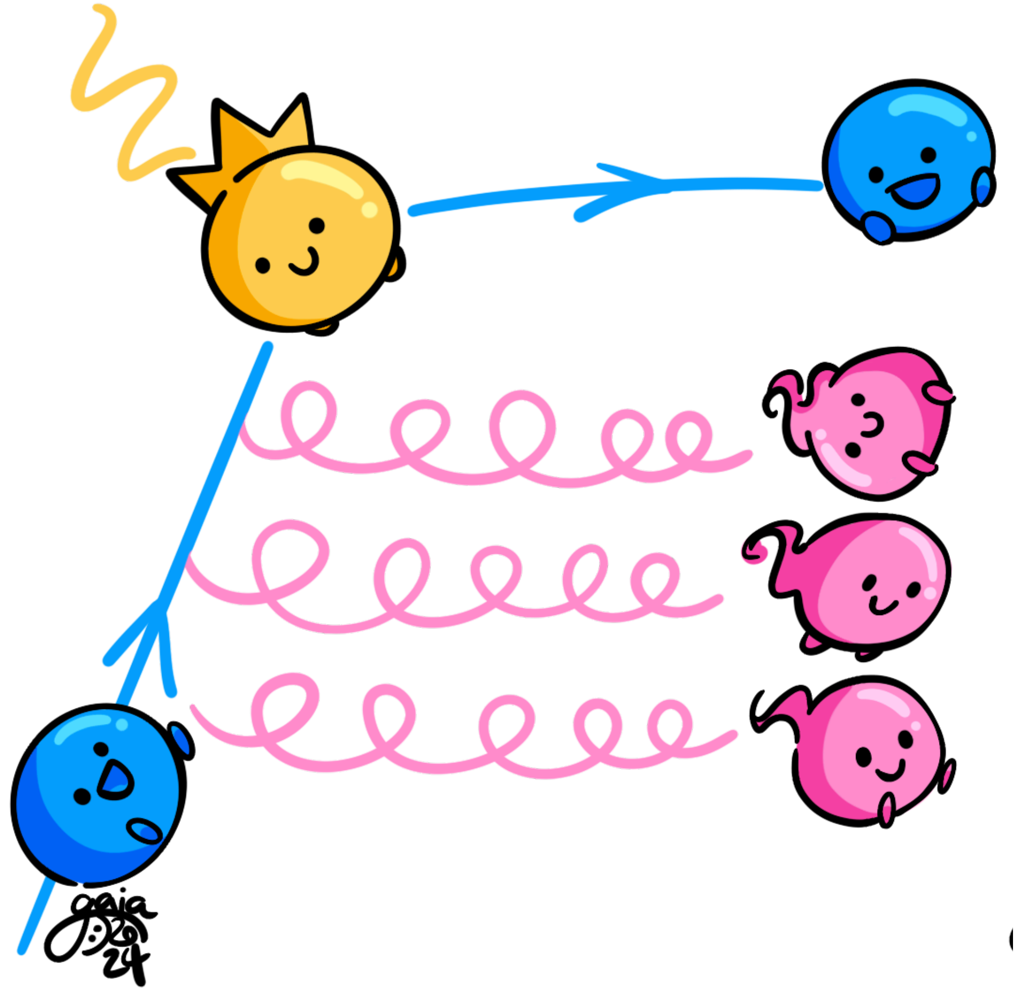
RRV $2 \to 3$

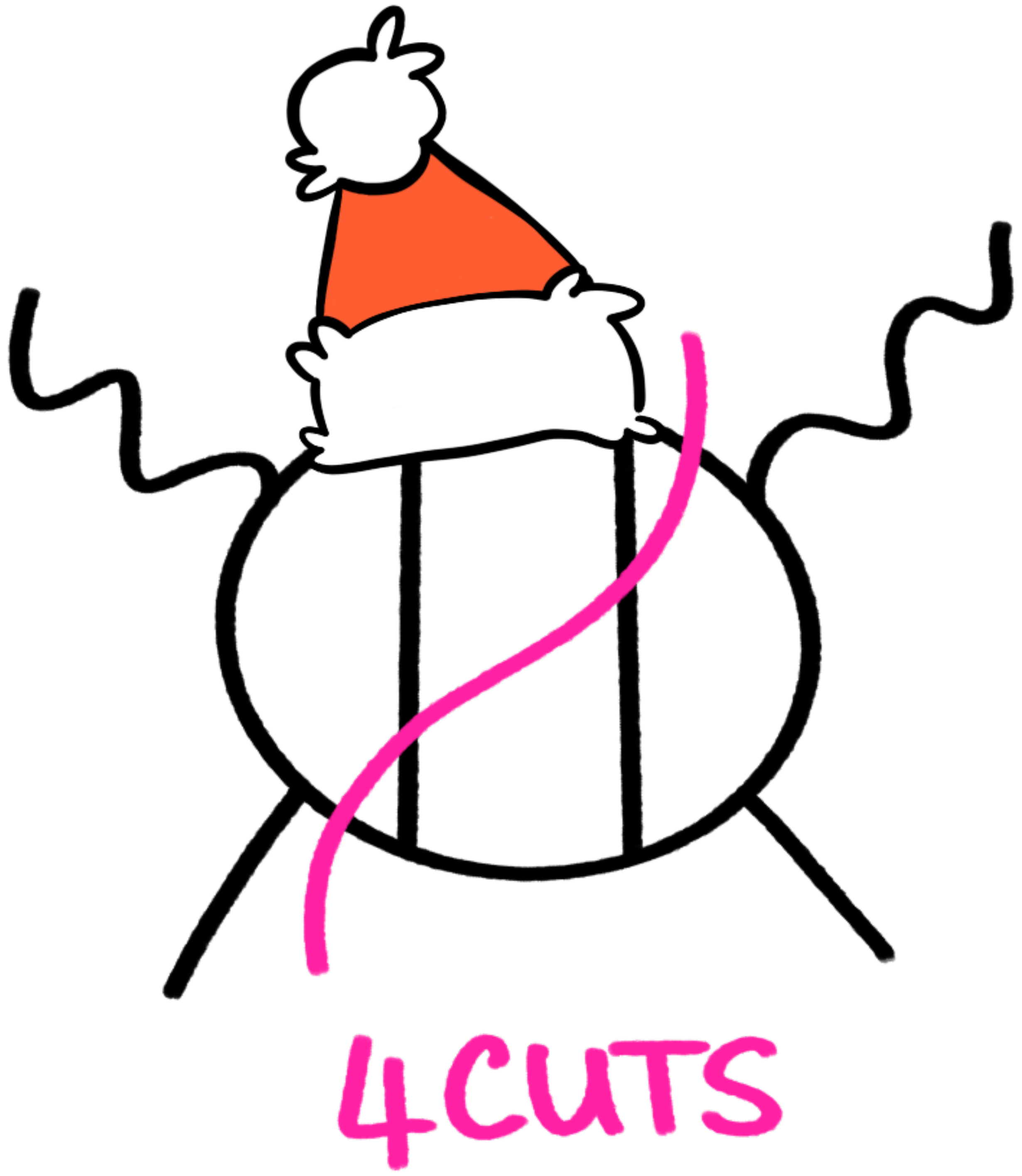
RVV $2 \to 2$

VVV $2 \to 1$

RRR layer

Physical 4-cuts of the 3 loop inclusive DIS amplitude

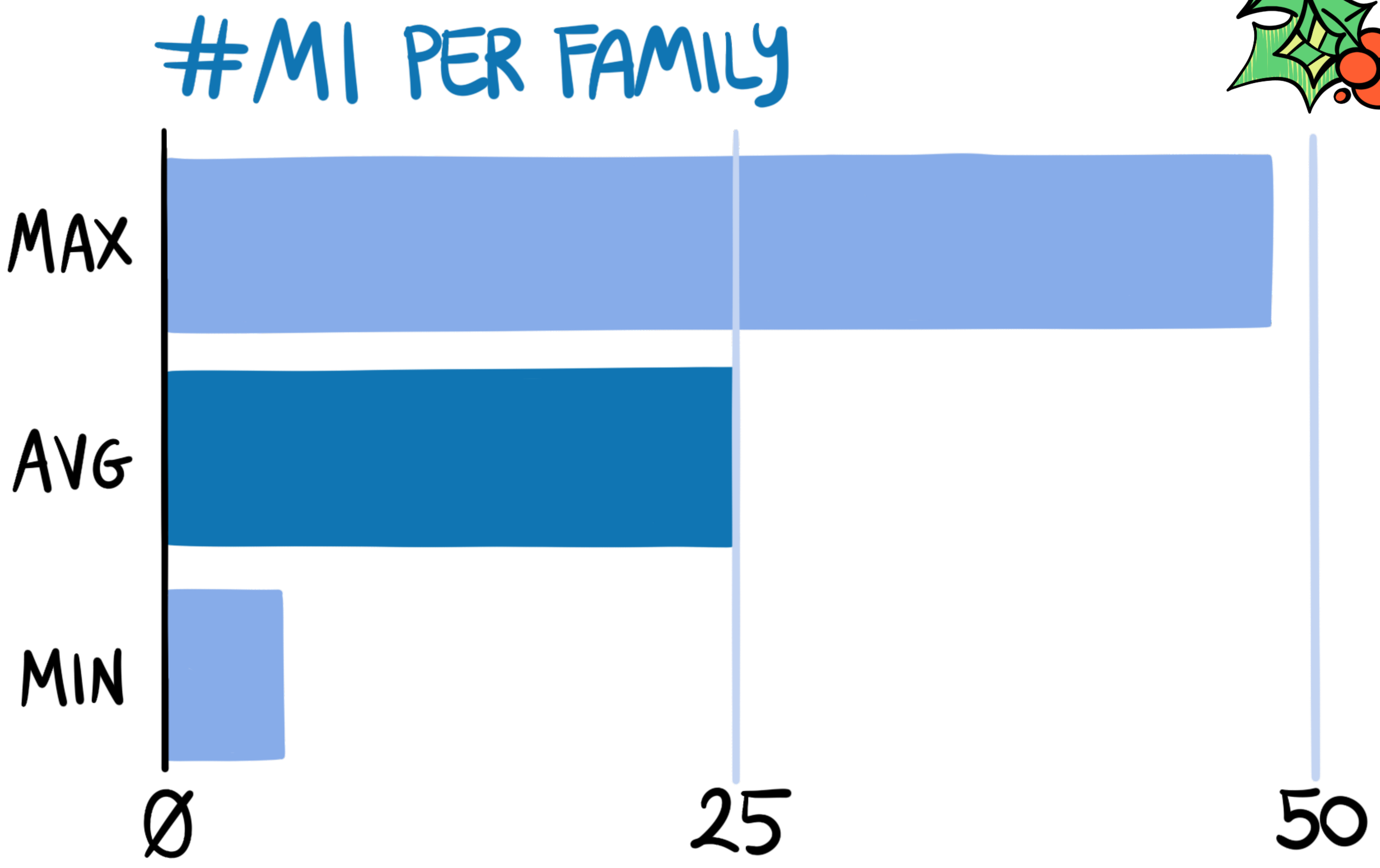




$$I_{RRR} = \int \frac{d^D p_1}{(2\pi)^D} \int \frac{d^D p_2}{(2\pi)^D} \int \frac{d^D p_3}{(2\pi)^D} \frac{1}{[p_1^2]_{cut}} \frac{1}{[p_2^2]_{cut}} \frac{1}{[p_3^2]_{cut}} \frac{1}{[p_4^2]_{cut}} \prod_j \frac{1}{D_j^{\alpha_j}}$$

Get to know the RRR families

- 65 families
- Small number of MI for each family →**4 cuts**
- **Total: 1620 MIs** (**No symmetries** between families included)



Strategy:

- DE matrix M is a function of $M(z, \epsilon)$

  ⇒ **playground for automatic tools!**

  eg LIBRA [Lee 2020]

- Analysis of their parametric representation to simplify the DE & get to a **canonical form**

  [Henn 2013]

Getting to a Canonical Basis (1): Balancing acts

For each family obeying a differential equation with associated matrix $M(z, \epsilon)$

LIBRA procedure:

- Build **elementary transformations** via graphical interface
    - Change pole order and eigenvalues of residue matrix around poles

$M(z, \epsilon)$

Fuchsification: no higher order poles

$$\sum_k \frac{M_k(z,\epsilon)}{z - z_k}$$

Normalizing eigenvalues $\propto \epsilon$

$$\sum_k \frac{M_k^\star(z,\epsilon)}{z - z_k}$$

$\epsilon-$**factorization**

$$\epsilon \sum_k \frac{A_k}{z - z_k}$$

Getting to a Canonical Basis (2): Strategy

- Strategy for multi loop calculations: exploit block triangular structure

Not acting on the full matrix at once

Diagonal blocks → **Global fuchsification** → **Factoring out ϵ**

Fuchsian & with eigenvalues proportional to ϵ

Global Fuchsian matrix

ϵ form

This point can fail!

What can we do?

Getting to a Canonical Basis (3): good candidates

- Sectors with higher number of propagators (top sector (TS), Next-to-TS, NNTS)
- **Baikov representation**

$$I = \int \left(\prod_{i=1}^{\ell} \mathrm{d}^d k_i \right) \frac{1}{z_1^{\nu_1} \cdots z_n^{\nu_n}} = K \int \mathrm{d}z_1 \cdots \mathrm{d}z_n \, B(\mathbf{z})^\gamma \frac{1}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}$$

- Numerator Ansatz $N(\vec{z})$

[Wasser (2020)]

- Check candidates with constant leading singularity with DLogBasis
- Keep only the linearly independent candidates for a new basis

$$\int dz_1 \ldots dz_n \, B^\gamma \frac{N(\vec{z})}{z_1^{\alpha_1} \ldots z_n^{\alpha_n}} \Bigg|_{\text{cut condition}}$$

Libra

N3LO Results

Canonical DE for all the RRR families ✓

We can find a generic solution

Letters: $\left\{ \frac{1}{x}, \frac{1}{1+x}, \frac{1}{1-x}, \frac{1}{1+2x}, \frac{1}{1-2x} \right\}$

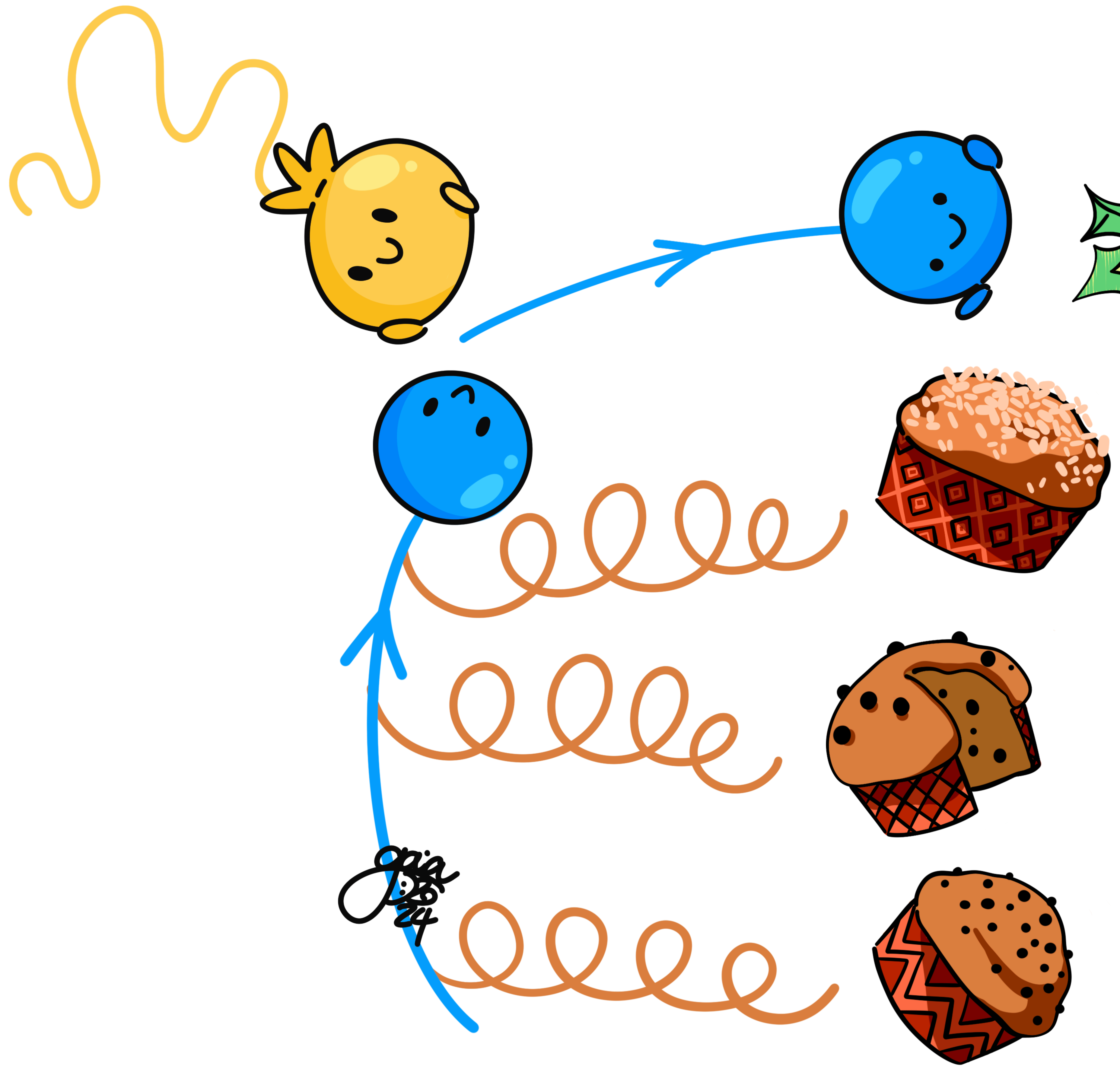
Boundary conditions ➡ SOON

[Liu, Ma 2022]

- Numerical evaluation with AMFlow @ **200 digits & PSLQ**
- Constraints from symmetry relations between the families
- Calculation of the amplitude → which boundaries are actually needed

Outlook:

- Finish calculation for all layers
- Ultimate goal: obtaining the full set of integrated initial-final antennae



Thank you for your attention!