Two-loop integrals for $pp \to t\bar{t}j$ in the leading colour approximation

Matteo Becchetti

Università di Bologna

Based on works with Badger, Brancaccio, Chaubey, Dlapa, Giraudo, Marzucca, Zoia

Five-point scattering amplitudes with internal massive propagators

- Massive propagators increase substantially the complexity of the computation

  IBPs — Analytic structure — Numerical Evaluation

- Processes with internal massive propagators not on the same level of massless ones

- First results for processes involving massive propagators appeared recently:

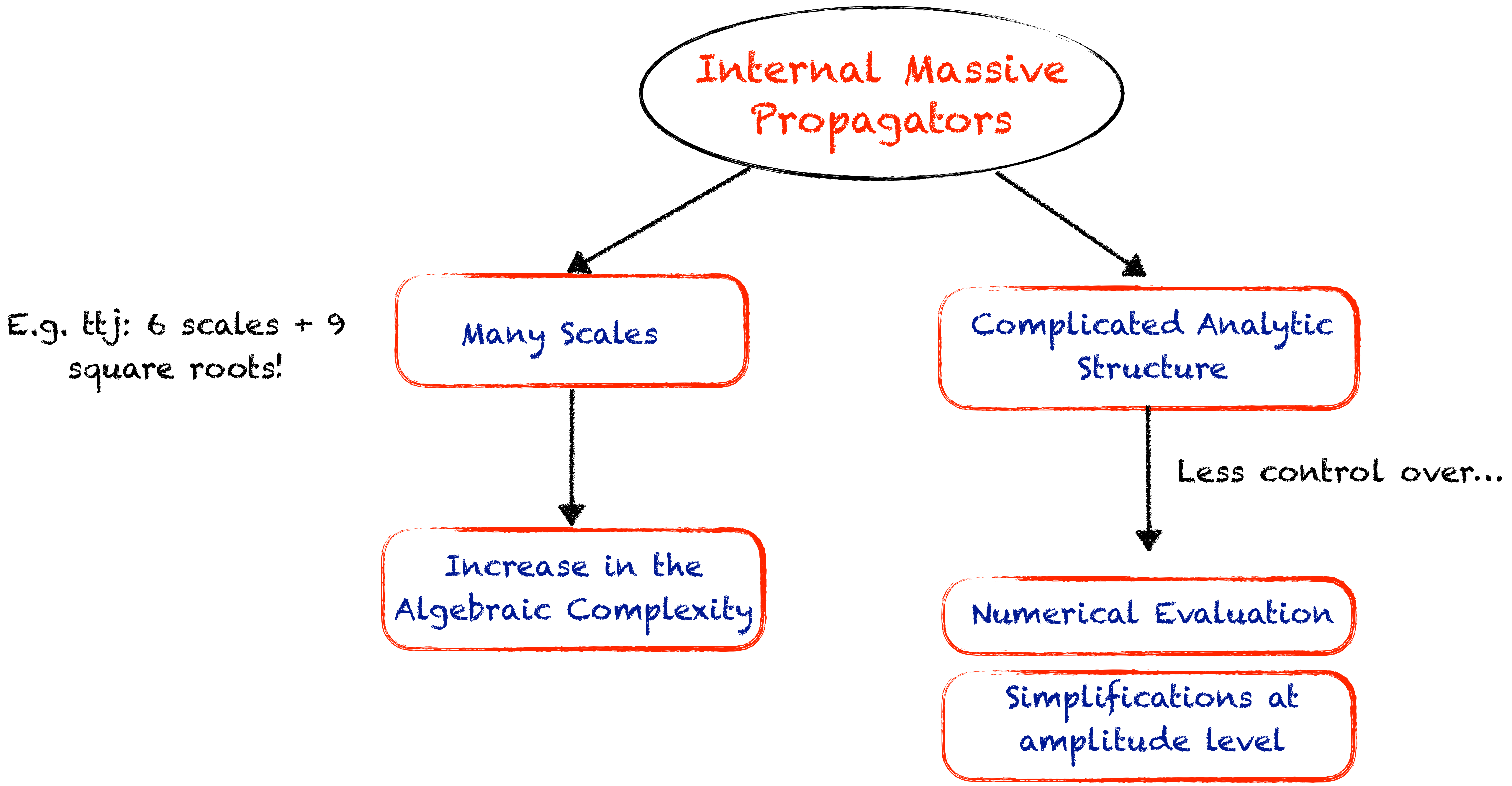
  One-loop helicity amplitudes for $pp \to t\bar{t}j$ up to $O(\epsilon^2)$ [Badger,**MB**,Chaubey,Marzucca,Sarandrea '22]

  Two-loop MIs for leading colour to $pp \to t\bar{t}j$ [Badger, **MB**, Giraudo, Zoia 2404.12325]

  Two-loop MIs for leading colour $pp \to t\bar{t}H$ with a light-quark loop [Febres Cordero, Figueiredo, Kraus, Page, Reina '23]

  One-loop QCD correction to $gg \to t\bar{t}H$ to $O(\epsilon^2)$ [Buccioni, Kreer, Liu, Tancredi '23]

Five-point scattering amplitudes with internal massive propagators



Five-point scattering amplitudes: NO internal massive propagators

Canonical DEQs:

$$\mathrm{d}\vec{g}_{\mathrm{F}}(X;\epsilon) = \epsilon \sum_i A_i \, \mathrm{d}\log W_i(X) \cdot \vec{g}_{\mathrm{F}}(X;\epsilon)$$

Pentagon Functions:

[Gehrmann, Henn, Lo Presti; Chicherin, Sotnikov; Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, **Zoia**]

MIs expressed in terms of algebraically independent basis of special functions

Algorithmic!

- Control over special functions relations: simplifications at amplitude level
- Efficient and stable numerical evaluation: Phenomenology applications!

Five-point scattering amplitudes with internal massive propagators

Picture generated with ChatGPT

How can we improve the current methods?

First Path:

Construction of a over-complete basis of special functions

Current implementation for ttj

See Colomba's Talk!

Simplifications at amplitude level

More efficient method for numerical evaluation

Second Path:

Construction of canonical DEQs for the elliptic topology

First step towards extension of pentagon functions method to the elliptic case

Two-Loop Planar Integral Topologies for Leading Colour ttj

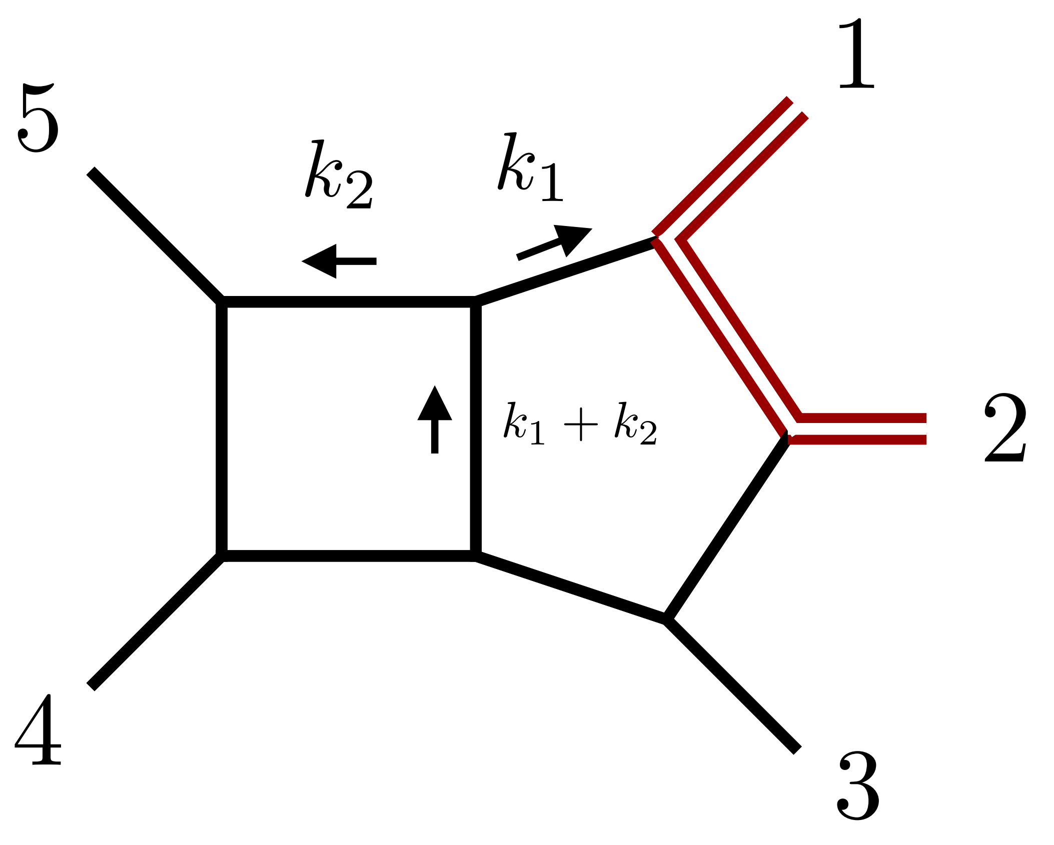
Current Amplitude Implementation

[Badger,MB,Chaubey,Marzucca,Sarandrea ’22] [Badger,MB,Chaubey,Marzucca 2210.17477]

[Badger, MB, Giraudo, Zoia 2404.12325] [Badger, MB, Brancaccio, Hartanto, Zoia 2412.13876]

Two-Loop Planar Integral Topologies for Leading Colour (LC) ttj

PB_A



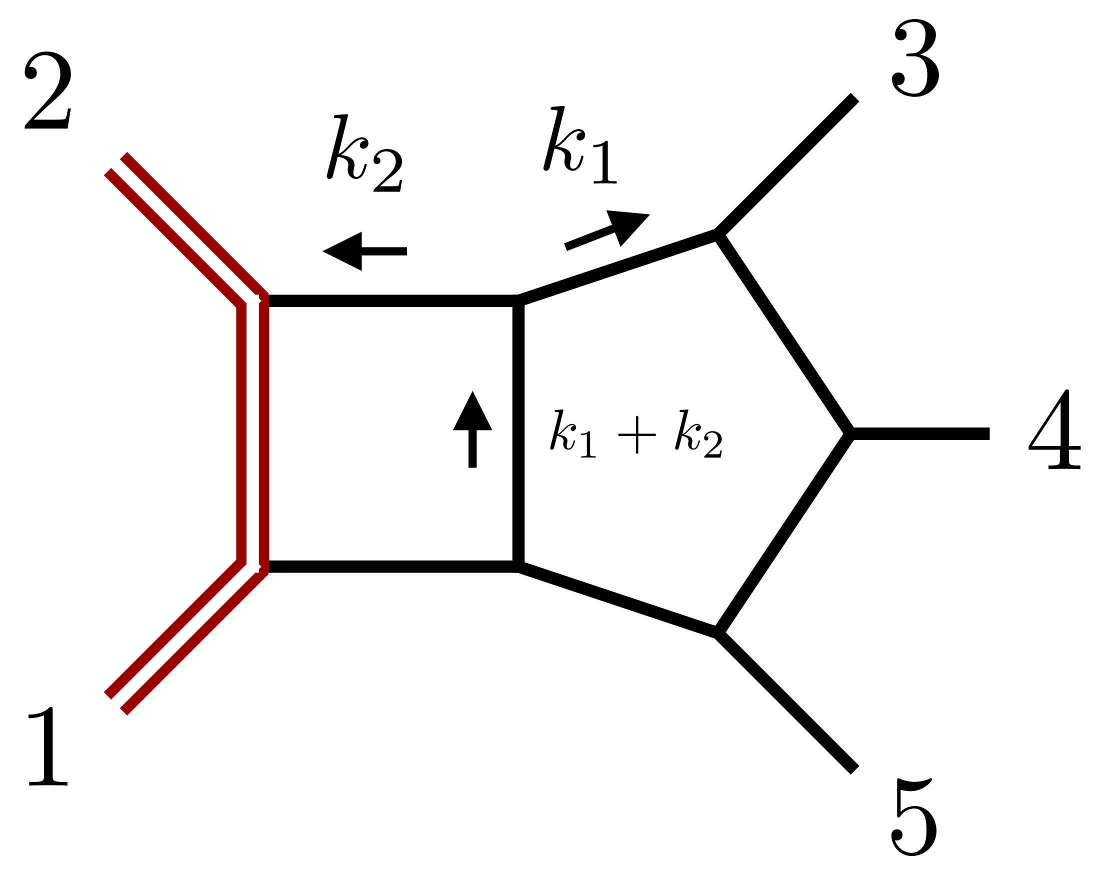
[Badger, MB, Chaubey, Marzucca 2210.17477]

88 MIs

Canonical basis ✓

Alphabet ✓

PB_C

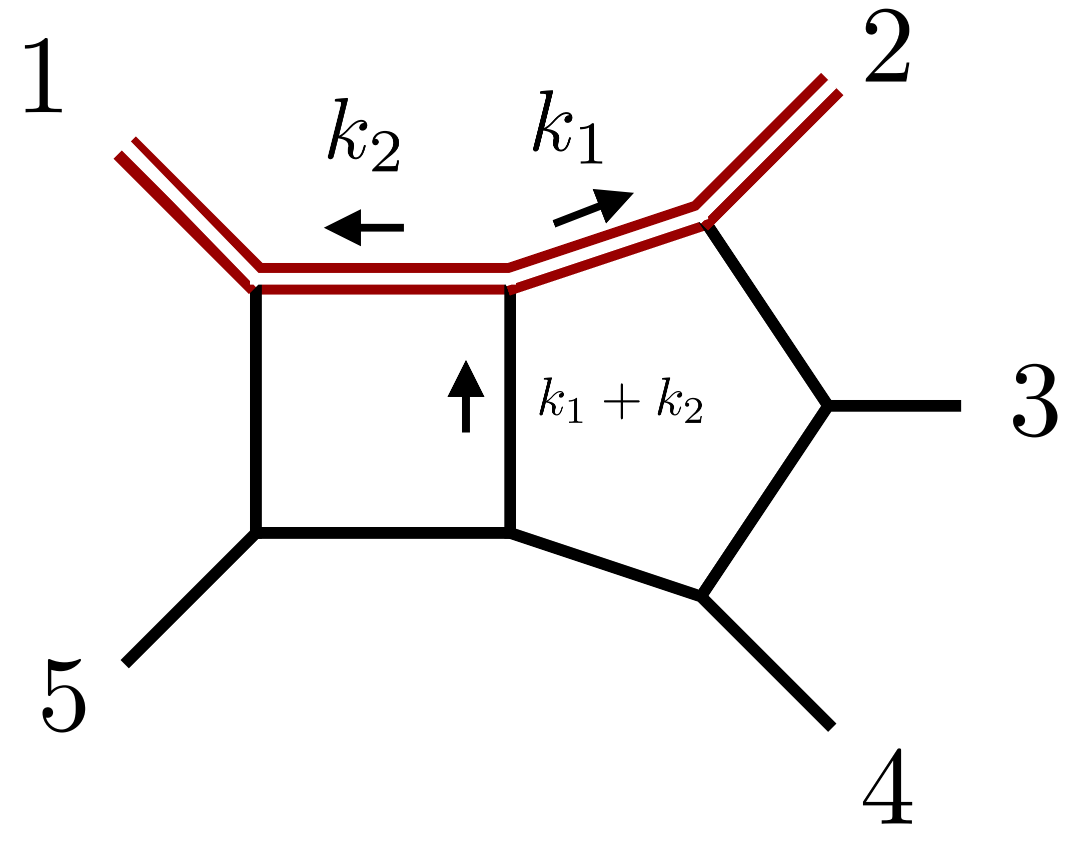


109 MIs

Canonical basis ✓

Alphabet ✓

PB_B



[Badger, MB, Giraudo, Zoia 2404.12325]

121 MIs

Canonical basis ✗

Elliptic Sector

Nested square-roots

Differential Equations Structure

General Structure DEs:

$$\mathrm{d}\vec{g}_{\mathrm{F}}(X;\epsilon) = \Omega_{\mathrm{F}}(X;\epsilon)\cdot\vec{g}_{\mathrm{F}}(X;\epsilon)$$

Canonical DEs $\mathrm{PB}_A\,\mathrm{PB}_C$

$$\mathrm{d}\vec{g}_{\mathrm{F}}(X;\epsilon) = \epsilon\sum_i A_i\,\mathrm{d}\log W_i(X)\cdot\vec{g}_{\mathrm{F}}(X;\epsilon)$$

ϵ-polynomial DEs PB_B

$$\Omega_{\mathrm{F}}(X;\epsilon) = \sum_{k=0}^{2}\epsilon^k\,\Omega_{\mathrm{F}}^{(k)}(X)$$

$$\Omega_{\mathrm{F}}^{(k)}(X) = \sum_i A_{k,i}^{(\mathrm{F})}\,\mathrm{d}\log W_i(X) + \sum_j B_{k,j}^{(\mathrm{F})}\,\omega_j(X)$$

- $X = \{d_{12}, d_{23}, d_{34}, d_{45}, d_{15}, m_t^2\},\quad d_{ij} = p_i\cdot p_j$
- $W_i(X)$ algebraic functions of X called letters
- $\omega_j(X)$ Linearly independent non-logarithmic one-forms

Construction of Special Functions

- Extension of the Pentagon Functions method in the polylogarithmic case

[Gehrmann, Henn, Lo Presti; Chicherin, Sotnikov; Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, **Zoia**]

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- Designed to achieve the following goals:

Analytic subtraction of UV/IR poles

Simplification of finite remainder

More efficient numerical evaluation

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Connection matrix ϵ-polynomial. Expressed in terms of dlogs and linearly independent non-polylogarithmic one-forms

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Numerical values for the MIs to check relations and establish vanishing conditions

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Numerical values for the MIs to check relations and establish vanishing conditions

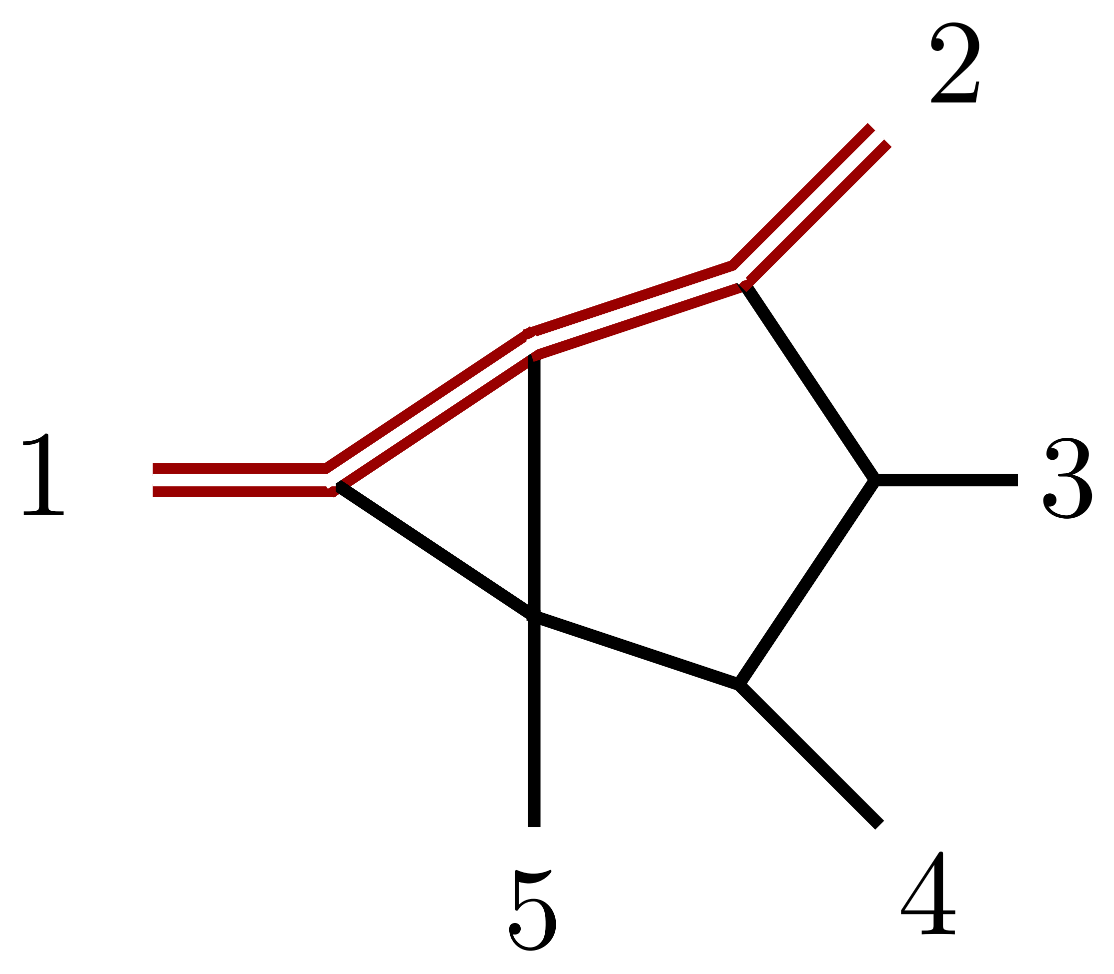
Non-canonical MIs non-zero only at the highest order in ϵ needed for the amplitude

Two-Loop Planar Integral Topologies for Leading Colour $t\bar{t}j$

Canonical DEQs for Elliptic Topology

[MB, Dlapa, Zoia 25XX.XXXXX]

Nested Square root sector



- **3 MIs:** $\mathscr{I}_{421B} = \left\{ I_{\mathrm{PB}_B,18}, I_{\mathrm{PB}_B,19}, I_{\mathrm{PB}_B,20} \right\}$

$$\mathcal{I}_{\mathrm{PB}_B,18} = \epsilon^4 \ \mathrm{tr}_5 \ I^{(\mathrm{PB}_B),[11],0,0,0}_{1,1,1,1,1,1,0,1} ,$$
$$\mathcal{I}_{\mathrm{PB}_B,19} = \varepsilon^4 \, d_{45} \, I^{(\mathrm{PB}_B),0,0,0}_{1,1,1,1,1,1,0,1} \left[\mathrm{tr}\left(\gamma_5 \not{p}_3 (\not{k}_1 - \not{p}_2 - \not{p}_3) \not{p}_4 \not{p}_2\right) \right] ,$$
$$\mathcal{I}_{\mathrm{PB}_B,20} = \varepsilon^4 \, d_{45} \, I^{(\mathrm{PB}_B),0,0,0}_{1,1,1,1,1,1,0,1} \left[\mathrm{tr}\left(\not{p}_3 (\not{k}_1 - \not{p}_2 - \not{p}_3) \not{p}_4 \not{p}_2\right) \right] ,$$

$$\mathrm{d}\,\mathcal{I}_{421B} = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & X & Y \\ 0 & Y & X \end{pmatrix} + \mathcal{O}(\varepsilon) \right] \cdot \mathcal{I}_{421B} + (\text{sub-sectors})$$

- Choice of the numerators made in order to cancel potential singularities

- DEs in ϵ-factorised form by means of the rotation

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{n_+}}{d_{45}\, r_2} & 0 \\ 0 & 0 & \frac{\sqrt{n_-}}{d_{45}\, r_2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

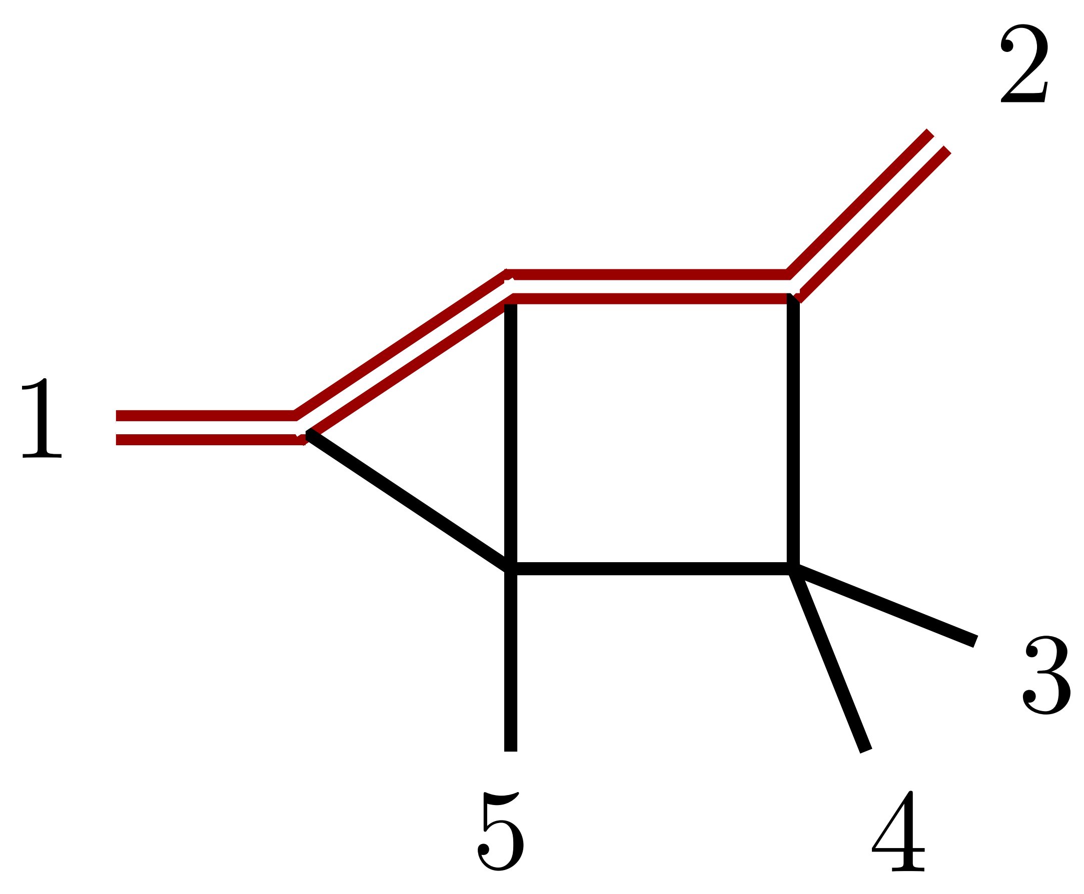
$$\sqrt{n_\pm} = \sqrt{\frac{1}{2} d_{23}^2 \, \mathrm{tr}_5^2 - 4 r_2 r_4 \pm 2 d_{23} r_3 \, \mathrm{tr}_5}$$

- $\sqrt{n_\pm}$ nested square roots as contain tr_5

- DEs with nested square roots cannot be handled by Diffexp

- For the numerical evaluation we use the basis $\mathscr{I}_{421B}$

Elliptic sector



MIs basis:

$$\mathcal{I}_{\mathrm{PB}_B,35} = \varepsilon^4\, d_{15}\,(d_{12}+m_t^2)\, I^{(\mathrm{PB}_B),0,0,0}_{1,1,0,1,1,1,0,1}\,,$$
$$\mathcal{I}_{\mathrm{PB}_B,36} = \varepsilon^4 \sqrt{(d_{15}-d_{34})^2 - 2d_{34}m_t^2}\, I^{(\mathrm{PB}_B),0,0,0}_{1,1,0,1,1,1,0,1}\,[2k_1\cdot p_1]$$
$$\mathcal{I}_{\mathrm{PB}_B,37} = \varepsilon^4\,(-1+2\varepsilon)\, d_{15}\, I^{(\mathrm{PB}_B),0,0,0}_{1,1,0,1,1,1,0,1}\,[2k_2\cdot p_2]\,,$$

DEs are quadratic in ϵ. We use this basis for numerical evaluation

$$\mathrm{d}\,\mathcal{I}_{321B} = \begin{pmatrix} *+\varepsilon * & \varepsilon * & * \\ *+\varepsilon * & \varepsilon * & * \\ (1-2\varepsilon)(*+\varepsilon *) & (1-2\varepsilon)\varepsilon * & *+\varepsilon * \end{pmatrix} \cdot \mathcal{I}_{321B} + (\text{sub-sectors})$$

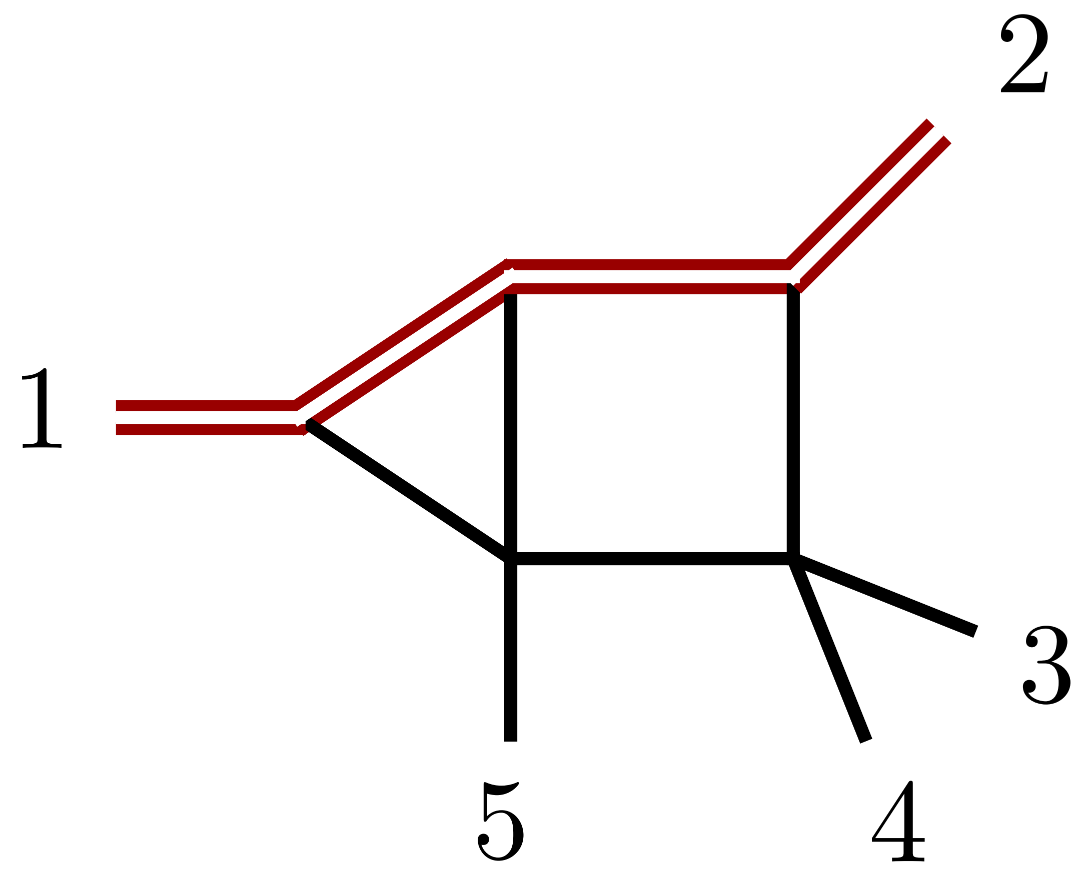
Picard-Fuchs operator

$$L_i\, \mathrm{MaxCut}\,[\mathcal{I}_{\mathrm{PB}_B,i}]_{\varepsilon=0} = 0\,, \qquad L_i = \sum_{k=0}^{r_i} c_{i,k}(\lambda)\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}$$

$$r_{35} = r_{37} = 2 \text{ and } r_{36} = 3$$

- Operators L_{35}, L_{37} are second order irreducible
- Solution to L_{35} involves elliptic integrals of the first kind
- Solution to L_{37} involves elliptic integrals of the first kind and derivatives
- Operators L_{36} is third order factorisable into the product of a second- and a first-order operator.

Elliptic sector



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Elliptic curve from Maximal Cut

$$\mathrm{MaxCut}\,[J]\,\big|_{\varepsilon=0} \propto \int \mathrm{d}\log\left[\alpha(z_7,z_9)\right]\, \frac{\mathrm{d}z_9}{\sqrt{\mathcal{P}(z_9)}}$$

$$\mathcal{P}(z_9) = (z_9+m_t^2)(z_9-3m_t^2)\big(\mathcal{P}_0+\mathcal{P}_1\, z_9+\mathcal{P}_2\, z_9^2\big)$$

Periods of the elliptic curve

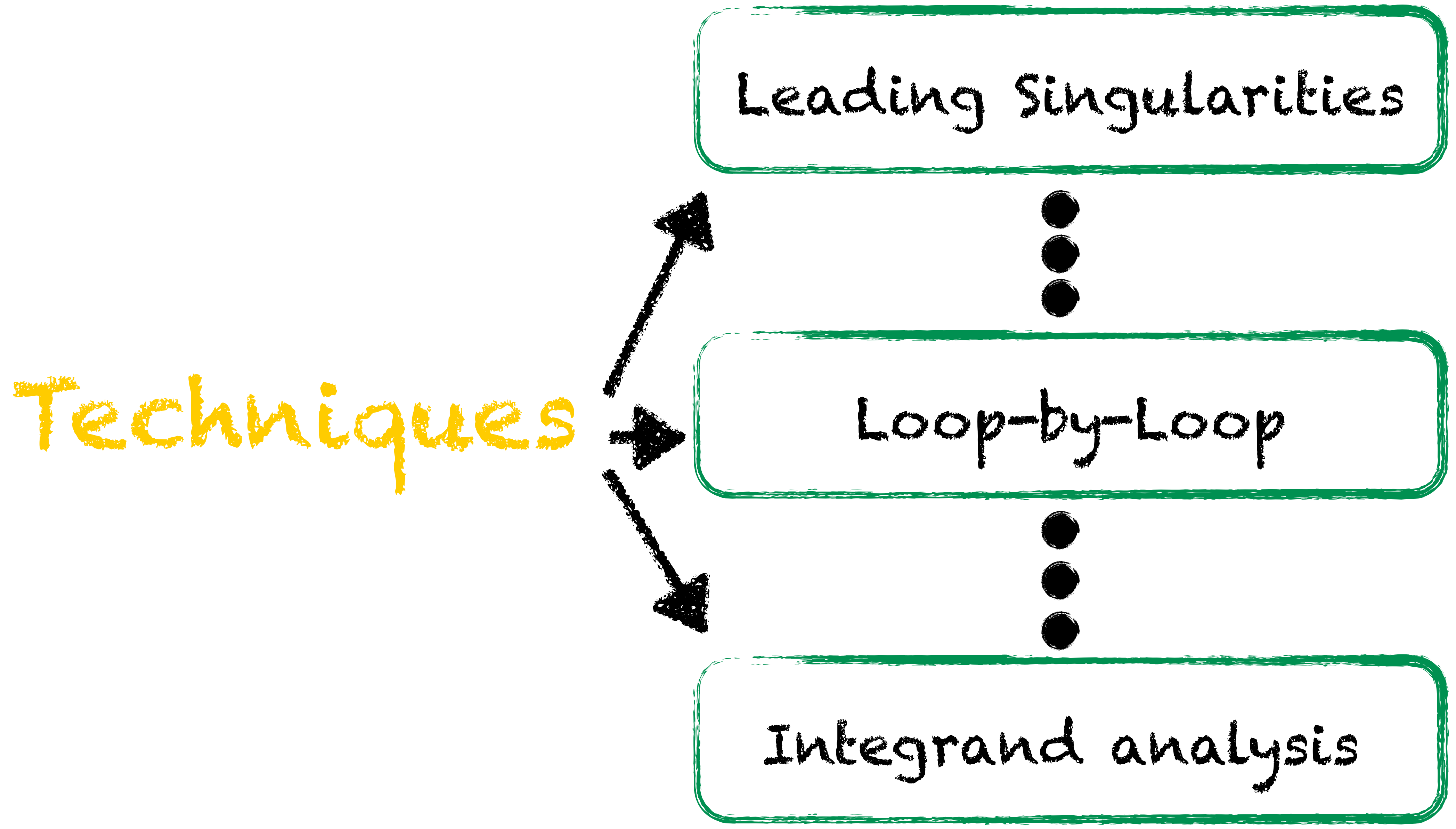
$$\psi_1 = \frac{2}{\pi}\int_{e_2}^{e_3}\frac{\mathrm{d}z_8}{y} = \frac{4\mathrm{K}(\kappa^2)}{\pi\sqrt{(e_3-e_1)(e_4-e_2)}},$$
$$\psi_2 = 4i\int_{e_1}^{e_2}\frac{\mathrm{d}z_8}{y} = \frac{-8\mathrm{K}(1-\kappa^2)}{\sqrt{(e_3-e_1)(e_4-e_2)}},$$

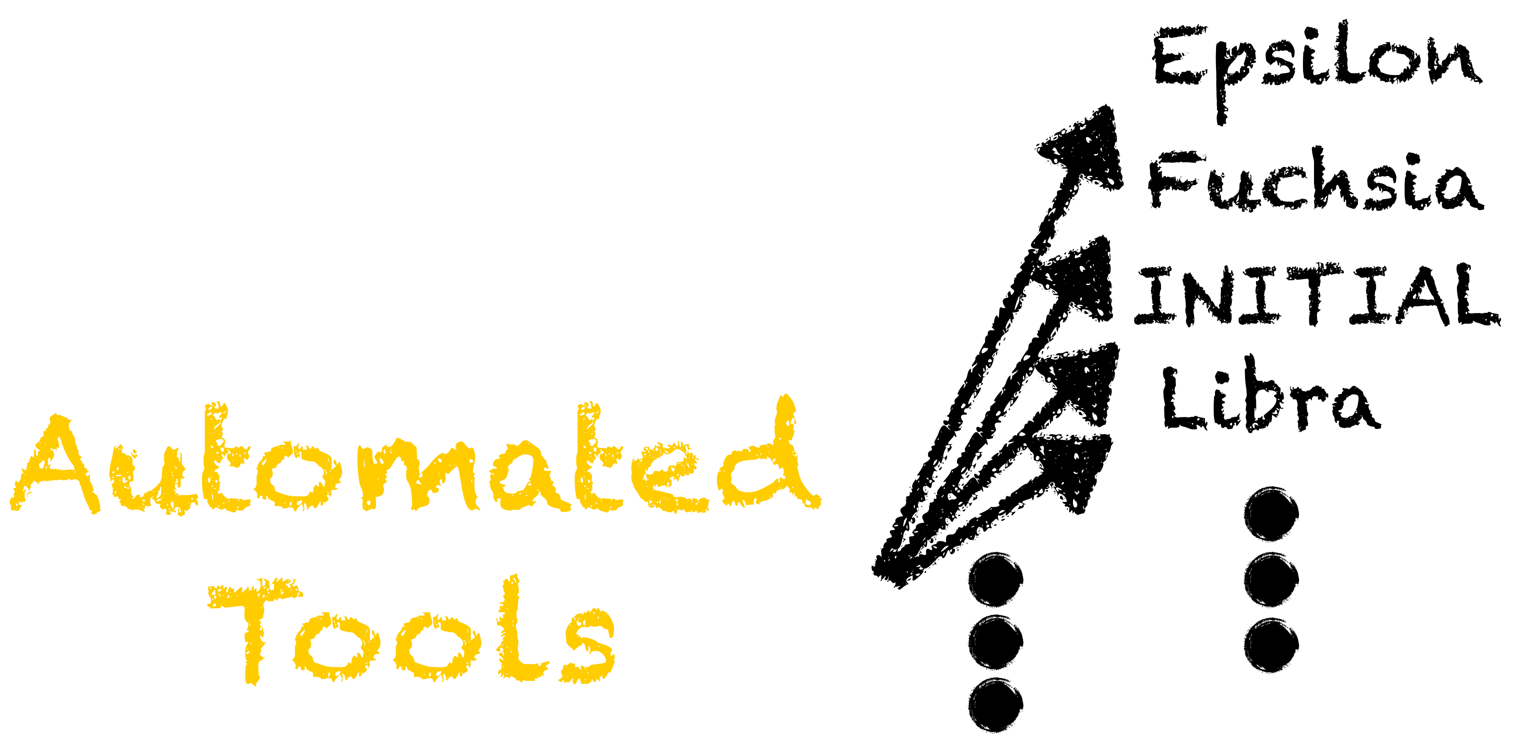
$$e_1 = -m_t^2\,,$$
$$e_2 = 4\,\frac{2d_{12}d_{15}(d_{34}-d_{15}) - (d_{15}^2+d_{34}^2-2d_{12}d_{34})m_t^2 + d_{34}\sqrt{\delta}}{\det G(p_2,p_1+p_5)}$$
$$e_3 = e_2\big|_{\sqrt{\delta}\to-\sqrt{\delta}}\,,$$
$$e_4 = 3m_t^2\,,$$

$$\kappa^2 = \frac{(e_3-e_2)(e_4-e_1)}{(e_3-e_1)(e_4-e_2)}$$

Building the Basis of MIs

- ✪ Canonical Basis greatly improves effectiveness of DEQs for Feynman Integrals
- ✪ The construction of Canonical Bases is in general a hard problem





[Prausa ’17]
[Gituliar,Magerya ’17]
[Dlapa,Henn,Yan ’20]
[Lee ’21]

- ✪ Canonical DEs are studied also in the elliptic case [Frellesvig, Weinzierl ’23] [Görges, Nega, Tancredi, Wagner ’23]
- ✪ Given the complexity of the kinematics previous approaches are difficult to apply

However….

Building the Basis of MIs

- Application of the procedure [Görges, Nega, Tancredi, Wagner '23]
- We start by looking at the DEQs in the top mass

Step 0: We study the homogeneous DEQs at $\epsilon = 0$

Choose of basis: Decoupling of elliptic integrals

$$A_1|_{\epsilon=0} = \begin{pmatrix} \star & \star & 0 \\ \star & \star & 0 \\ \star & \star & 0 \end{pmatrix} \equiv \begin{pmatrix} B & & 0 \\ & & 0 \\ \star & \star & 0 \end{pmatrix}$$

Step 1: Compute fundamental matrix of solutions at $\epsilon = 0$

We split it into its nilpotent and semi-simple parts

$$W = LDU, \qquad W_{\text{ss}} = LD, \qquad W_{\text{u}} = U,$$

$$W_{\text{ss}} = \begin{pmatrix} \psi_1 & 0 \\ \psi_1' & \frac{\psi_1\psi_2' - \psi_1'\psi_2}{\psi_1} \end{pmatrix} = \begin{pmatrix} \psi_1 & 0 \\ \psi_1' & \frac{\det W}{\psi_1} \end{pmatrix}, \qquad W_{\text{u}} = \begin{pmatrix} 1 & \frac{\psi_2}{\psi_1} \\ 0 & 1 \end{pmatrix}$$

Step 2: We rotate the basis using the split in the previous point to reach an ϵ triangular form

$$A_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \star & \star & 0 \end{pmatrix} + \epsilon \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix}$$

Step 3: We integrate out the ϵ^0 terms in the whole matrix

Step 4: We iterate the procedure for the other variables

Building the Basis of MIs

D-Logarithmic case: rotation to canonical basis involves only algebraic functions of the kinematic invariant

Elliptic case: we need to introduce functions that can be written as complete elliptic integrals

$$\psi_1 = \frac{2}{\pi}\int_{e_2}^{e_3} \frac{\mathrm{d}z_8}{y} = \frac{4\mathrm{K}(\kappa^2)}{\pi\sqrt{(e_3-e_1)(e_4-e_2)}} \qquad G_1 = \frac{4}{\pi}(e_2-e_1)\frac{\Pi\left(\frac{e_3-e_2}{e_3-e_1}, k^2\right)}{\sqrt{(e_3-e_1)(e_4-e_2)}} - 4m_t^2\psi_1, \dots$$

New analytic structure: Need to understand relations among these new set of functions

Numerical Evaluation: Currently not on the same level of the logarithmic case…

Analytic Structure of Canonical DEQs

Symmetries of Connection Matrix: First step towards classification of relations for special functions

$\mathbb{Z}_2$ charge: entries of connection matrix have charge under sign flip of all square roots, except tr_5, i.e. they are either odd or even

tr_5 charge: entries of connection matrix form doublets under sign flip of tr_5

$$\begin{aligned} \sqrt{n_+} &\longleftrightarrow \sqrt{n_-}\,, \\ W_{n_+} &\longleftrightarrow W_{n_-}\,, \\ a_{22} &\longleftrightarrow a_{33}\,, \\ \mathcal{I}'_{421B,19} &\longleftrightarrow \mathcal{I}'_{421B,20}\,, \\ G_3^+ &\longleftrightarrow G_3^-\,, \end{aligned}$$

Elliptic grading: entries of connection matrix depend on elliptic functions $\psi_1\,,\quad G_1\,,\quad G_2\,,\quad G_3^+\,,\quad G_3^-$

By assigning a grade +1 to those objects, all the quantities in the connection matrix have uniform grade from -2 to +2

Final form of DEQs: $\mathrm{d}\mathcal{I} = \epsilon \sum_i A_i\,\omega_i \cdot \mathcal{I}$

Well defined transformation rules w.r.t previous automorphisms

Closed one-forms ω_i written in terms of dlogs if possible

Summary

- We completed the computation of two-loop planar integral topologies for $pp \to t\bar{t}j$ at leading colour
- MIs computation for two-loop planar topology represents the first ingredient for a NNLO QCD corrections to ttj
- Canonical DEQs also for the Topology involving elliptic integrals

Outlook

- Simplification at amplitude level including canonical MIs for elliptic Topology
- Extension of Pentagon Functions method to elliptic case
- Efficient numerical evaluation strategy for the elliptic case