Three-loop master integrals for the production of two off-shell vector bosons

Mattia Pozzoli

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In collaboration with Dhimiter Canko

Theory and Phenomenology of Fundamental Interactions
UNIVERSITY AND INFN · BOLOGNA

INFN BOLOGNA
Istituto Nazionale di Fisica Nucleare
Sezione di Bologna

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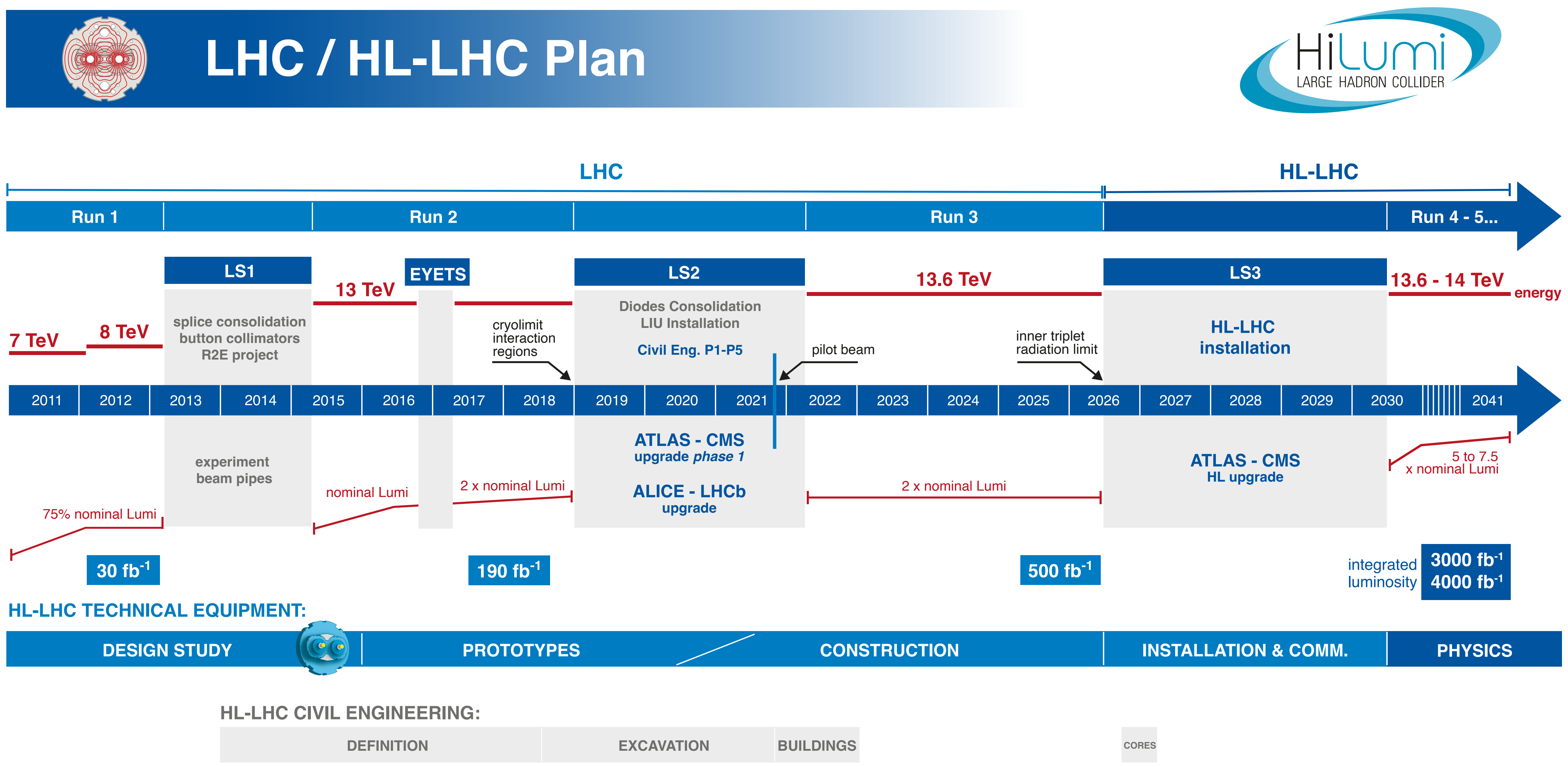
Outline

- Motivation
- Kinematics and integral families
- Feynman integrals and differential equations
- Results
- Outlook

Motivation

- High luminosity LHC plan



- Experimental precision for HL-LHC of $\mathcal{O}(1\%)$ for many observables
- Theoretical predictions at higher loops are required
- For di-boson production: NNLO QCD known, needed N3LO

Motivation

Cross sections for $h_1 h_2 \longrightarrow f$:

$$d\sigma_{h_1 h_2 \to f} = \sum_{i,j=q,\bar{q},g} \int\!\!\int dx_1 dx_2\, \mathcal{F}_{i/h_1}(x_1,\mu^2)\mathcal{F}_{j/h_2}(x_2,\mu^2)\, d\hat{\sigma}_{ij\to f}(\vec{s},\mu^2)$$

Partonic cross section:

$$d\hat{\sigma}_{ij\to f} \backsim \int d\Phi\, |\mathscr{A}|^2$$

Amplitude:

$$\mathscr{A} \backsim \sum_i F_i(\vec{s};\epsilon)\, G_i(\vec{s};\epsilon)$$

Feynman Integrals

Status of 3-loop calculations

Frontier: 4-point, one massive leg, all-massless propagators

Feynman integrals

- Ladder-box [Di Vita, Mastrolia, Schubert, Yundin 2014]
- Planars [Canko, Syrrakos 2020,2021 & 2023]
- Non- planars [Henn, Lim, Bobadilla 2023 & 2024; Gehrmann et al. 2024]

Amplitudes

- Planar V+ jet [Gehrmann, Jakubčík, Mella, Syrrakos, Tancredi 2023]
- Leading colour H + jet [Gehrmann et al. 2024]

Recent calculations

- Planar families for two massive legs of the same mass [Ming-Ming Long 2024]
- One planar family for 5-point all-massless [Liu et al. 2024]

Integral families

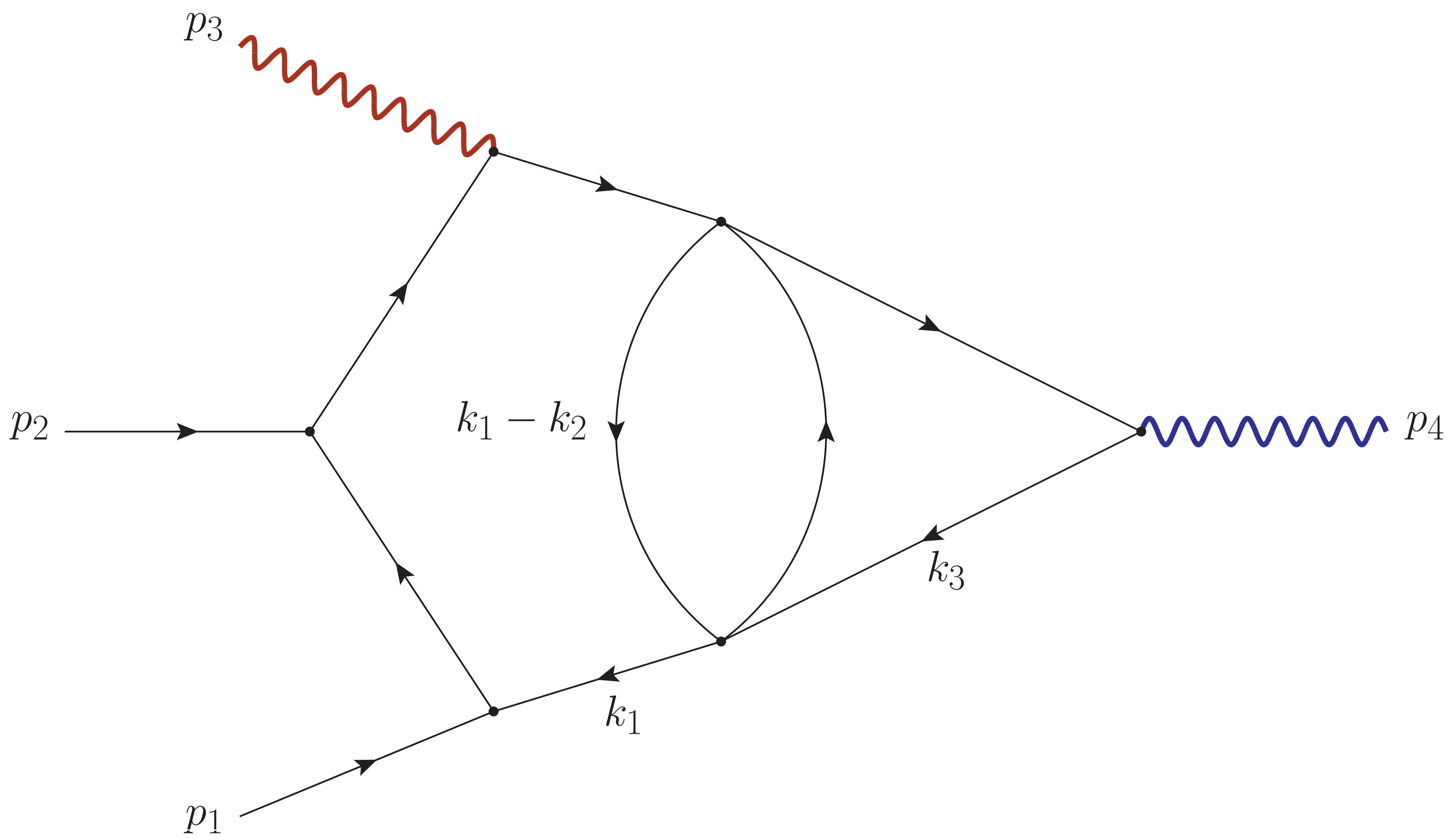
$$G_{a_1,\ldots,a_N} = e^{L\gamma_E\varepsilon} \int \frac{\prod_{j=1}^{L} \mathrm{d}^d k_j}{(i\pi)^{Ld/2}} \frac{1}{D_1^{a_1} \cdots D_N^{a_N}}, \qquad (a_1, \ldots, a_N) \in \mathbb{Z}^N$$

- Sectors: same non-negative exponents
- Top sector: maximum number of non-negative exponents
- Amplitude calculations: express $k_i \cdot p_j$ and $k_i \cdot k_j$ in terms of propagators

$\implies$ Beyond one-loop we need irreducible scalar products (ISPs)

Integral families: example

$$G_{a_1,\ldots,a_N} = e^{L\gamma_E\varepsilon}\int \frac{\prod_{j=1}^{L} \mathrm{d}^d k_j}{(i\pi)^{Ld/2}} \frac{1}{D_1^{a_1}\cdots D_N^{a_N}}, \qquad (a_1,\ldots,a_N) \in \mathbb{Z}^N$$



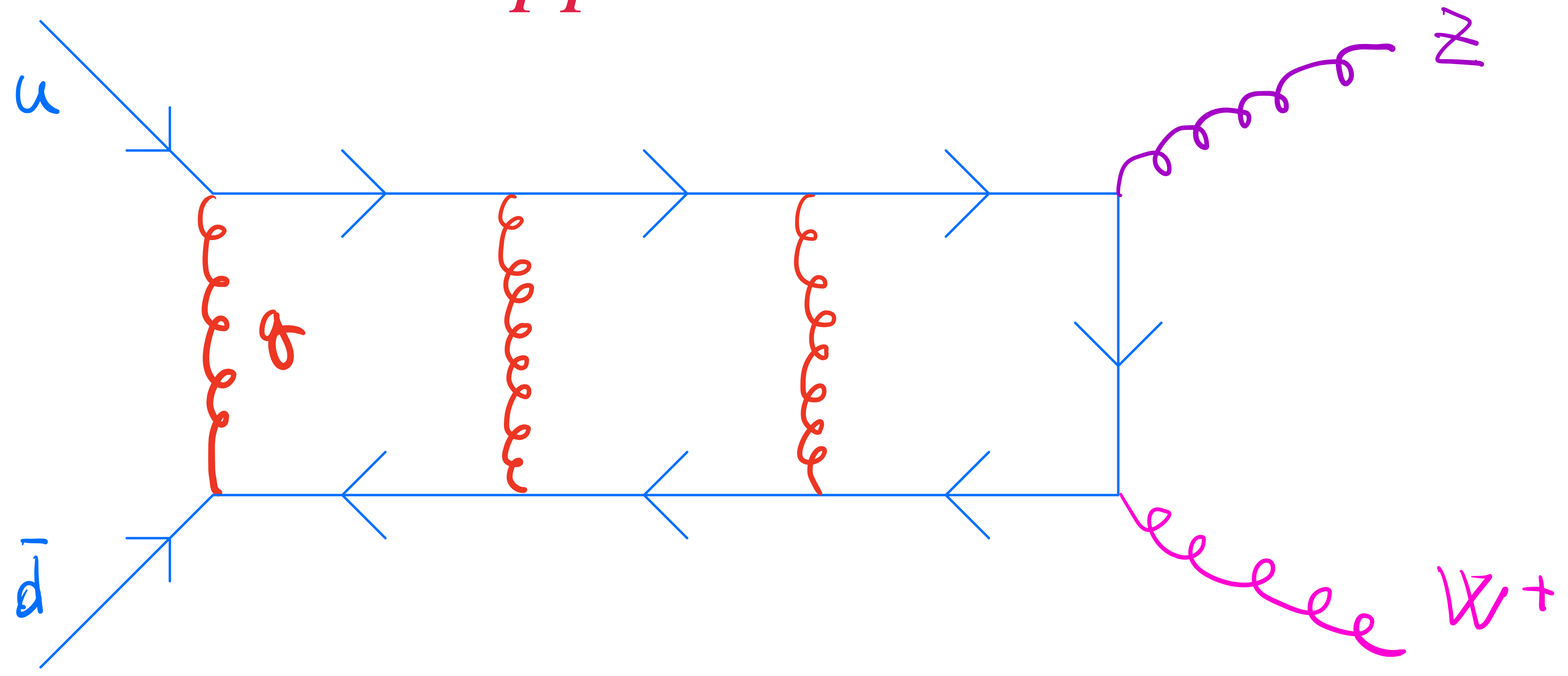
$$D_1 = k_1^2, \qquad D_2 = (k_1 + p_1)^2,$$
$$D_3 = (k_1 + p_{12})^2, \qquad D_4 = (k_1 + p_{123})^2,$$
$$D_5 = k_3^2, \qquad D_6 = (k_3 + p_{123})^2,$$
$$D_7 = (k_1 - k_2)^2, \qquad D_8 = (k_2 - k_3)^2$$

+7 ISPs

Kinematics

$pp \longrightarrow VV'$

$p_1^2 = 0$ $\qquad$ $p_4^2 = m_4^2$



$m_3 \neq m_4$

$p_2^2 = 0$ $\qquad$ $p_3^2 = m_3^2$

$$\vec{s} = \begin{cases} \left(s_{12}, s_{23}, m_3^2, m_4^2\right) \\ (x, y, z) \end{cases}$$

Second set of variables to rationalise $R := \sqrt{m_3^4 + (m_4^2 - s_{12})^2 - 2m_3^2(m_4^2 + s_{12})} = m_3^2 x(1-y)$

(Same as 2-loop [Henn, Melnikov, Smirnov 2014])

Propagators

We group the integral families into superfamilies: same propagators, different top sectors

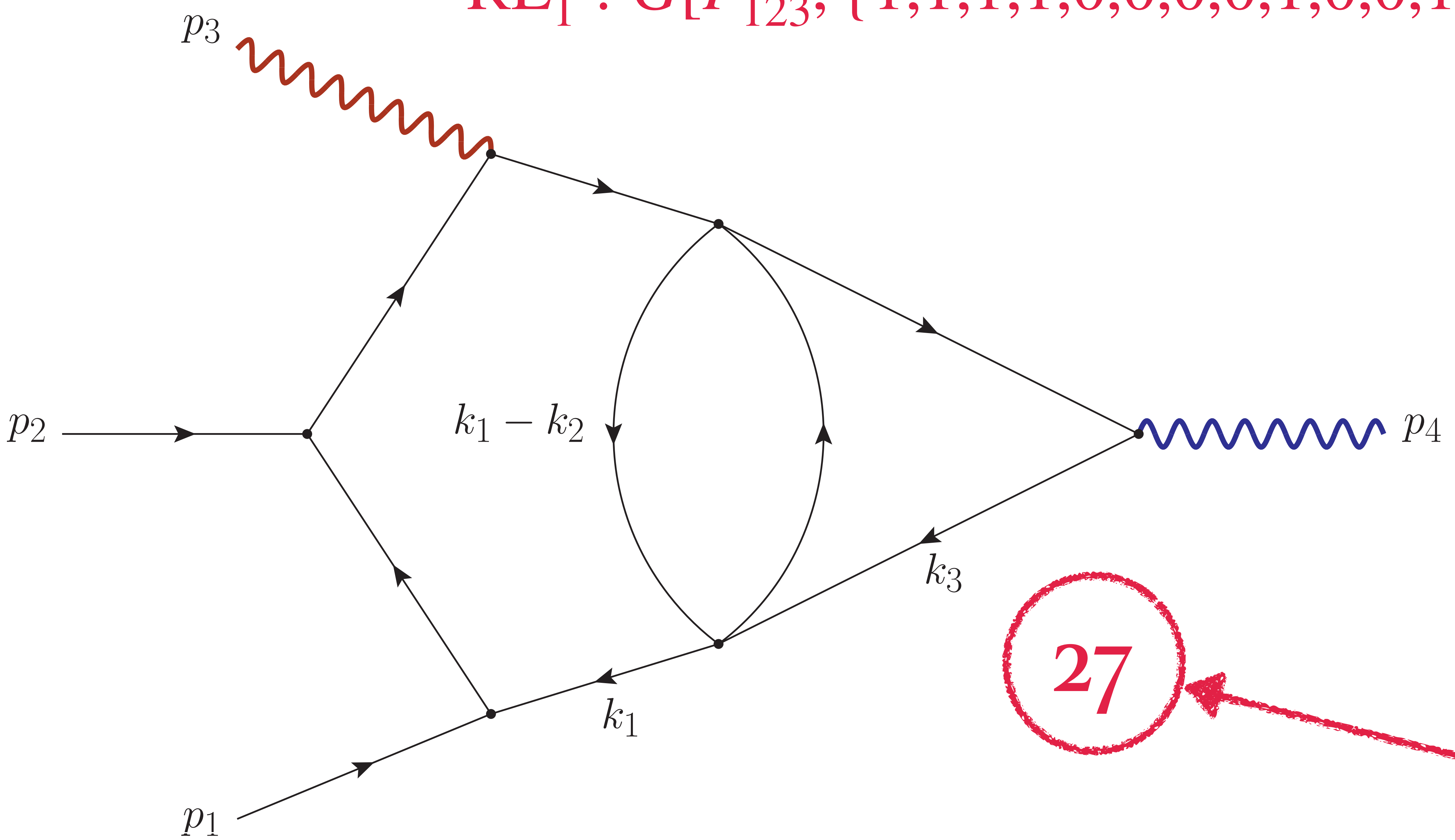
Superfamily F_{123}:

$$D_1 = k_1^2, \quad D_2 = (k_1 + p_1)^2, \quad D_3 = (k_1 + p_{12})^2, \quad D_4 = (k_1 + p_{123})^2,$$
$$D_5 = k_2^2, \quad D_6 = (k_2 + p_1)^2, \quad D_7 = (k_2 + p_{12})^2, \quad D_8 = (k_2 + p_{123})^2,$$
$$D_9 = k_3^2, \quad D_{10} = (k_3 + p_1)^2, \quad D_{11} = (k_3 + p_{12})^2, \quad D_{12} = (k_3 + p_{123})^2,$$
$$D_{13} = (k_1 - k_2)^2, \quad D_{14} = (k_1 - k_3)^2, \quad D_{15} = (k_2 - k_3)^2$$

Superfamily F_{132}: with the transformation $p_2 \longleftrightarrow p_3$

Integral families: RL_1 and RL_2

$\mathrm{RL}_1 : G[F_{123}, \{1,1,1,1,0,0,0,0,1,0,0,1,1,0,1\}]$



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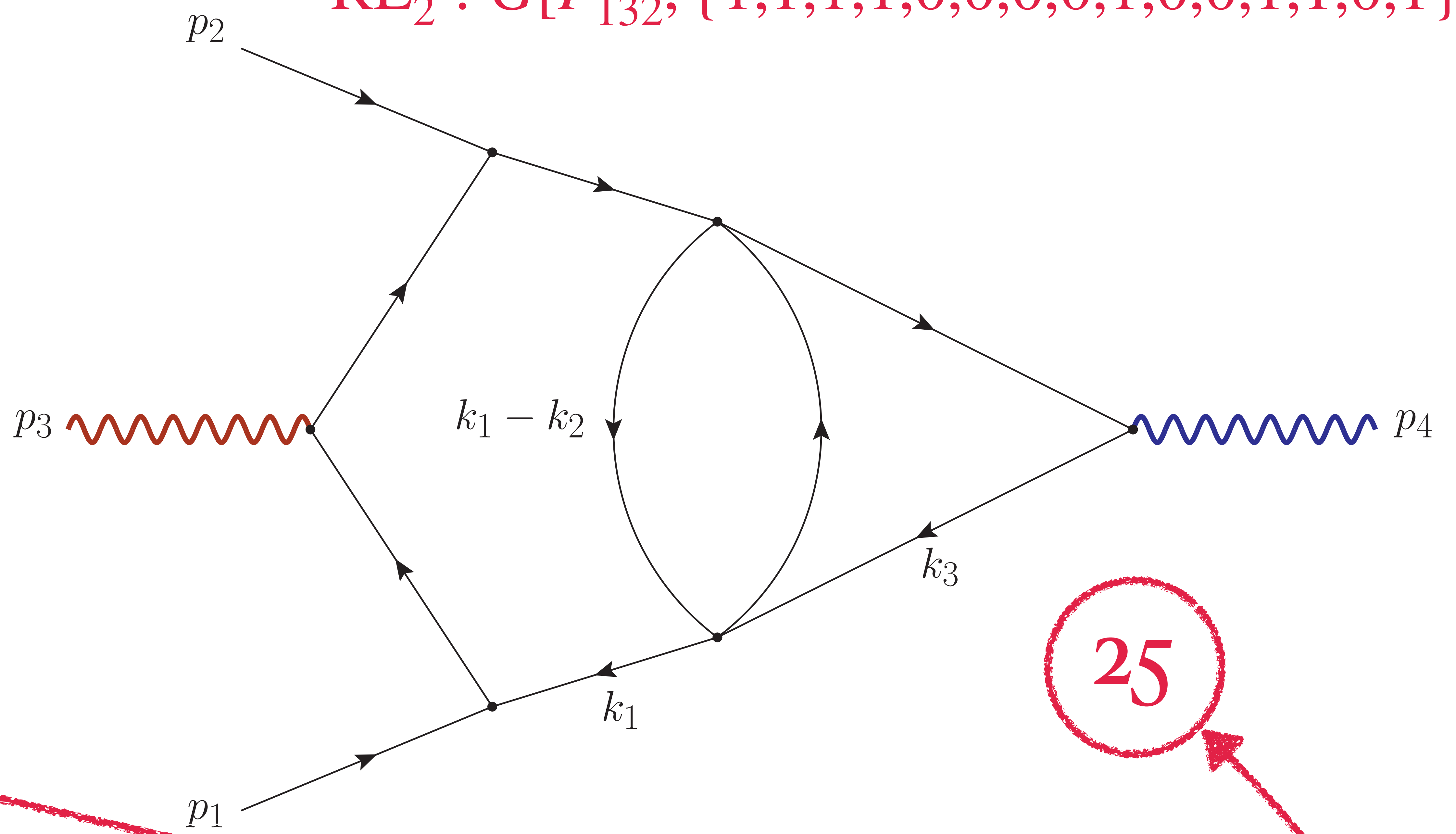
$\mathrm{RL}_2 : G[F_{132}, \{1,1,1,1,0,0,0,0,1,0,0,1,1,0,1\}]$



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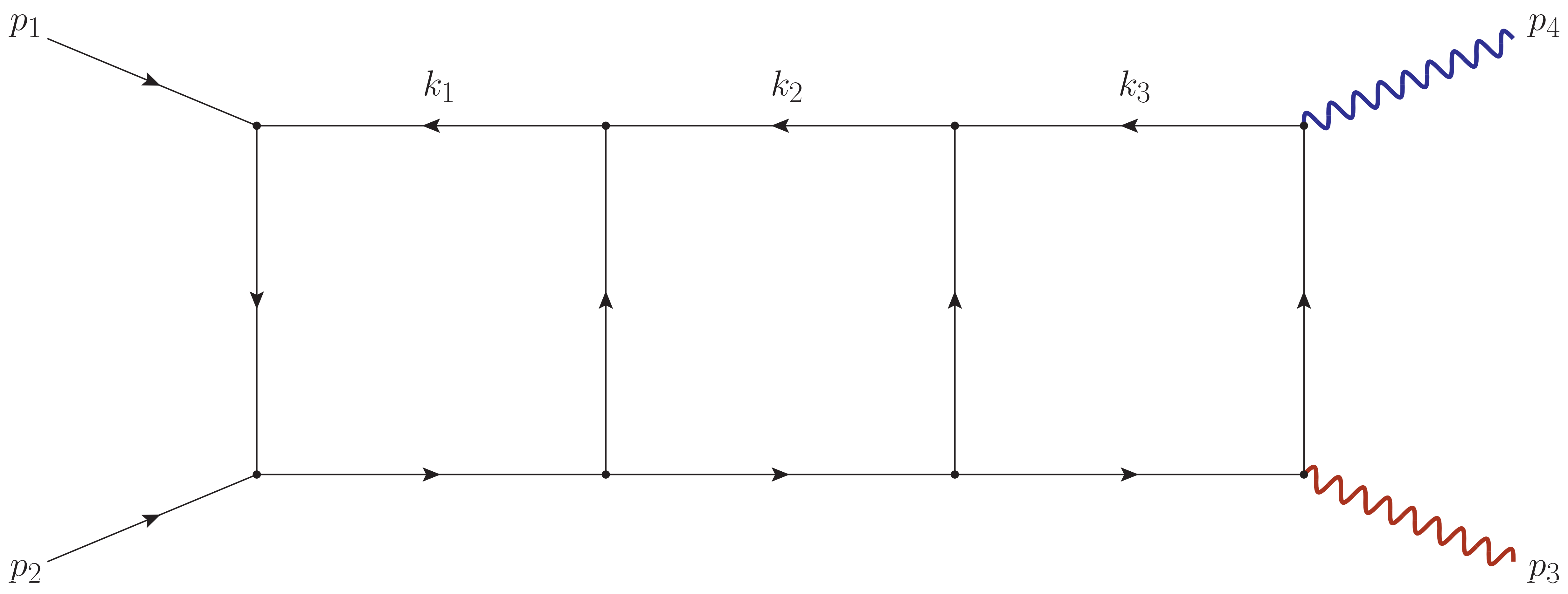
Feynman integrals satisfy linear relations: IBPs [Chetyrkin, Tkachov '81; Laporta 2000]. We generate and solve them with `FiniteFlow` [Peraro 2019]

$\Longrightarrow$ Basis of Master Integrals

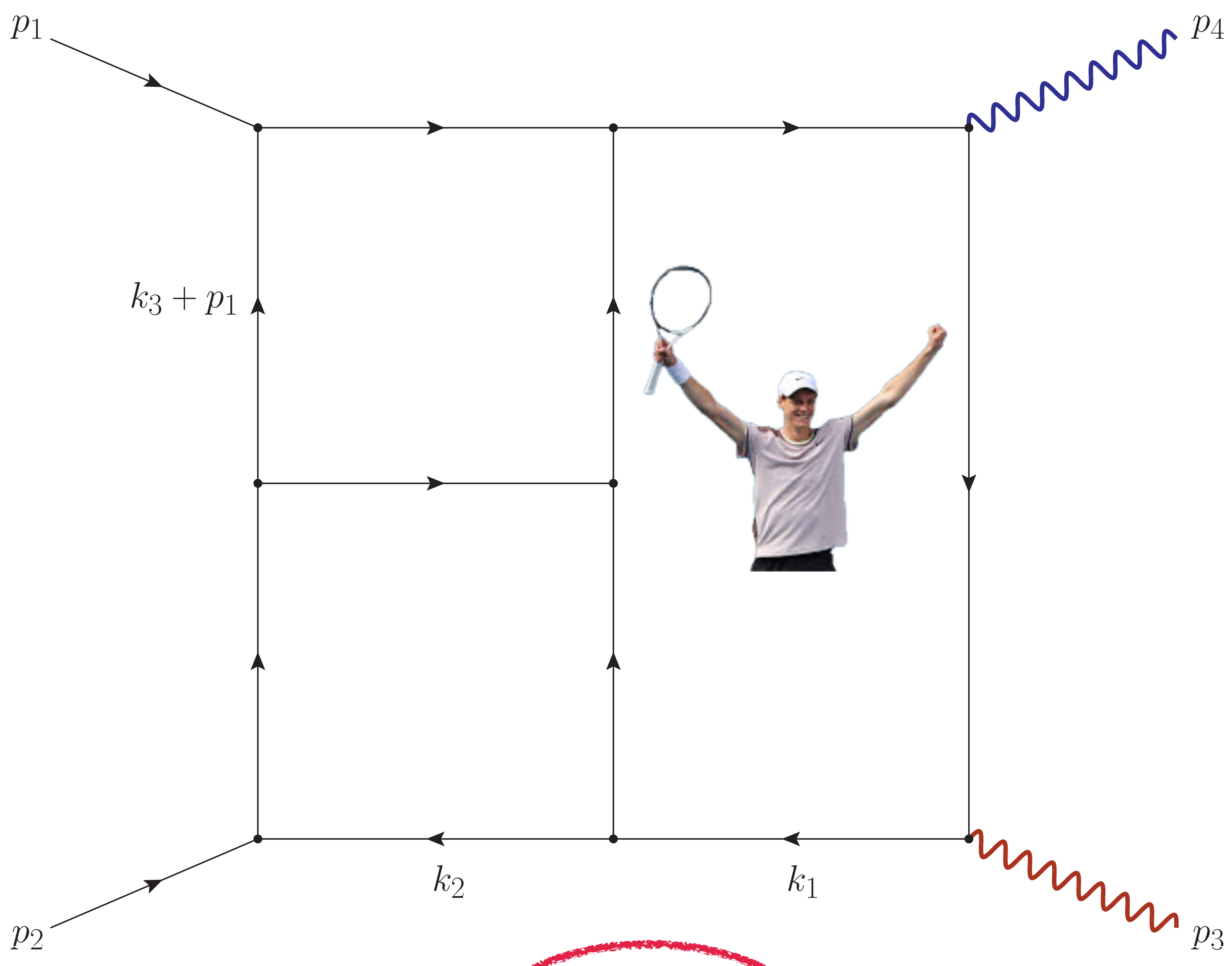
Integral families: PL_1 and PT_4

$PL_1 : G[F_{123}, \{1,1,1,0,1,0,1,0,1,0,1,1,1,0,1\}]$



150 MIs

$PT_4 : G[F_{123}, \{1,0,1,1,1,1,0,0,0,1,1,0,1,1,1\}]$



189 MIs

Method of differential equations

[Kotikov '91; Bern, Dixon, Kosower '94; Gehrmann, Remiddi 2000]

Using IBPs we can construct linear differential equations for the MIs

$$\partial_\xi \overrightarrow{G}(\vec{s}) = B_\xi(\vec{s};\epsilon)\ \overrightarrow{G}(\vec{s}) \qquad \forall\ \xi \in \vec{s}$$

[Henn 2014]: DEs in canonical form (no general algorithm)

$$\mathrm{d}\overrightarrow{G}(\vec{s}) = \epsilon\ \boxed{\mathrm{d}\tilde{A}(\vec{s})}\ \overrightarrow{G}(\vec{s})$$

one-forms with at most simple poles

Dlog-forms

In the best understood cases the one-forms are logarithmic

$$\mathrm{d}\vec{G}(\vec{s}) = \epsilon \ \mathrm{d}\tilde{A}(\vec{s}) \ \vec{G}(\vec{s}), \qquad \mathrm{d}\tilde{A}(\vec{s}) = \sum_i a_i \, \mathrm{d}\log w_i(\vec{s})$$

Letters

Usually integrate to multiple polylogarithms (not always! [Duhr, Brown 2020])

Construction of canonical basis

- Work sector-by-sector, first in the maximal cut (all sub-sectors set to zero)
- *Ansatz* for candidates $\vec{I}$ (squaring propagators and/or including ISPs, Baikov representation, package `DlogBasis`) such that:

$$\partial_\xi \tilde{T} = -\,\tilde{T}\, H_{0,\xi}$$

$$B_\xi(\vec{s};\epsilon) = H_{0,\xi}(\vec{s}) + \epsilon\, H_{1,\xi}(\vec{s})$$

- Look for matrix $\tilde{T}$ such that $\vec{G} = \tilde{T}\,\vec{I}$ has an ϵ-factorised DE
- For couplings to lower sectors: add or subtract linear combinations of basis integrals

Normalisation of the MIs, linear combinations of basis integrals

Multiple Polylogarithms

- Defined iteratively:

$$\mathcal{G}(a_1, \ldots, a_n; X) = \int_0^x \frac{\mathrm{d}t}{t - a_1} \mathcal{G}(a_2, \ldots, a_n; t), \qquad \mathcal{G}(; X) \equiv 1,$$

$$\mathcal{G}(\vec{0}_n; X) = \frac{1}{n!} \log^n(X), \qquad \vec{0}_n = (0, \ldots, 0)$$

- Satisfy a *shuffle algebra:*

$$\mathcal{G}(a_1, \ldots, a_{n_1}; X)\mathcal{G}(a_{n_1+1}, \ldots, a_{n_1+n_2}; X) = \sum_{\sigma \in \Sigma(n_1, n_2)} \mathcal{G}(a_{\sigma(1)}, \ldots, a_{\sigma(n_1+n_2)}; X)$$

Analytic solution in terms of MPLs

- Boundary point: $(x, y, z) = (0,0,0)$

- Choose path that minimises the number of MPLs:

$$F_{123} : (0,0,0) \xrightarrow{\gamma_1} (x,0,0) \xrightarrow{\gamma_2} (x,0,z) \xrightarrow{\gamma_3} (x, y, z)$$

$$F_{132} = (0,0,0) \xrightarrow{\gamma_1'} (0,0,z) \xrightarrow{\gamma_2'} (0,y, z) \xrightarrow{\gamma_3'} (x, y, z)$$

- Exploit shuffle algebra to further reduce the number of MPLs [Radford 1979]

Number of MPLs at each weight

	Weight 1	Weight 2	Weight 3	Weight 4	Weight 5	Weight 6	All weights
RL_1	12	27	137	492	1320	1631	3619
RL_2	12	29	168	996	4549	5219	10973
PL_1	14	35	188	690	1935	2554	5416
PT_4	16	51	312	1170	3032	3709	8290
All Families	21	78	478	2169	7609	8951	19306

Fixing the boundary conditions

At each weight we have

$$G_i^{(w)} = \epsilon^w \sum_{\vec{a}} c_{w,\vec{a}}\, \mathscr{G}(\vec{a}; X) + \ldots + \epsilon \sum_{a} c_{1,a}\, \mathscr{G}(a; X)\, b_{w-1} + b_w$$

Known from integration

Boundary constants: to be determined

Fact: b_w is a linear combination of transcendental constants

Ansatz: only products and powers of $i\pi$, ζ_3 and ζ_5 can appear

Boundary conditions with PSLQ

Evaluate the expression

$$G_i^{(w)} = \epsilon^w \sum_{\vec{a}} c_{w,\vec{a}}\, \mathcal{G}(\vec{a};X) + \ldots + \epsilon \sum_{a} c_{1,a}\, \mathcal{G}(a;X)\, b_{w-1} + b_w$$

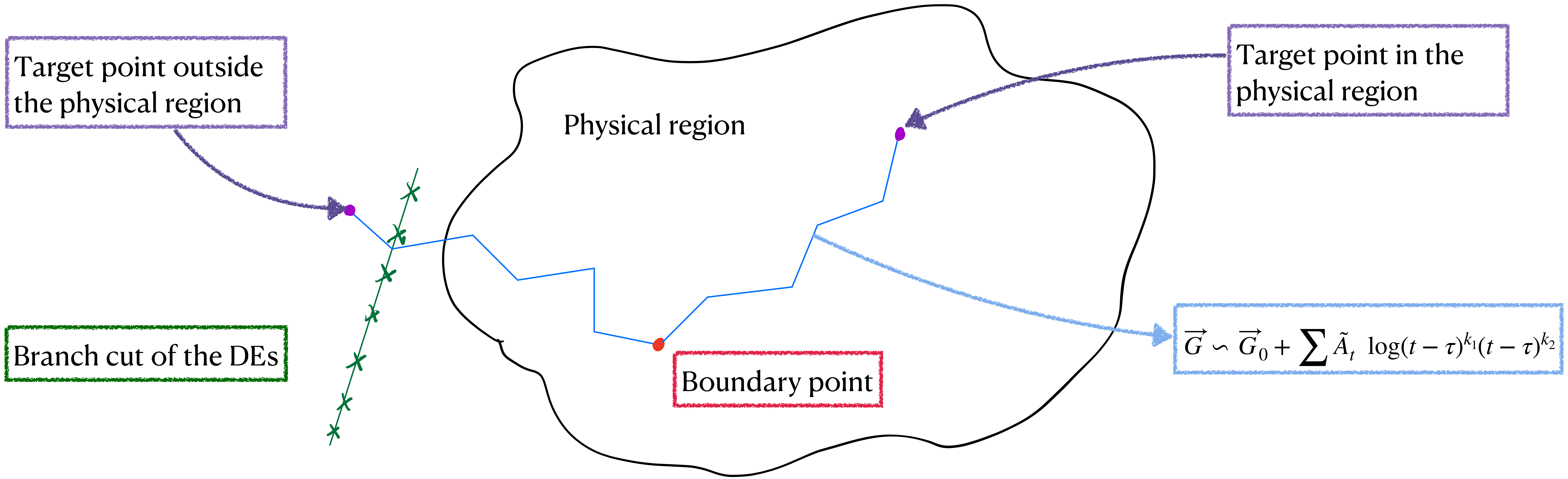
- $G_i^{(w)}$: Evaluate MIs with `AMFlow` [Liu, Ma 2022]
- $c_{w,\vec{a}}\, \mathcal{G}(\vec{a};X)$, $c_{1,a}\, \mathcal{G}(a;X)$: Evaluate MPLs with `DiffExp`
- b_{w-1}: Work recursively weight-by-weight $\Longrightarrow$ fixed at weight $w-1$
- b_w: Linear combination of transcendental constants with rational coefficients

We need integer coefficients for a linear combination of real numbers!

$\Longrightarrow$ PSLQ algorithm! [Ferguson, Bailey 1992]

Semi-numerical evaluation

- Generalised series expansion method [Moriello 2019] using `DiffExp` [Hidding 2020] (other implementations available, e.g. `SeaSyde` [Armadillo et al. 2022]).



- Work in the physical region: no analytic continuation needed!

Timings DiffExp

Average time per kinematic point

	$(s_{12}, s_{23}, m_3^2, m_4^2)$	(x, y, z)
RL_1	175 s	55 s
RL_2	169 s	64 s
PL_1	2765 s	818 s
PT_4	4987 s	1478 s

Average time per segment

	$(s_{12}, s_{23}, m_3^2, m_4^2)$	(x, y, z)
RL_1	8 s	9 s
RL_2	9 s	11 s
PL_1	126 s	136 s
PT_4	226 s	246 s

- Mandelstam variables: more complicated phase space, but less complicated DEs
- Smart choice of grid allows for fast evaluation of many kinematic points

Summary and Outlook

- Four integral families relevant for di-boson production
- Canonical DEs, integrating to MPLs
- Semi-numerical solution using `DiffExp`
- Room for improvement
    1. Minimal set of independent functions
    2. Evaluation using `GiNaC` or `HandyG`

Summary and Outlook

- Four integral families relevant for di-boson production
- Canonical DEs, integrating to MPLs
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  1. Minimal set of independent functions
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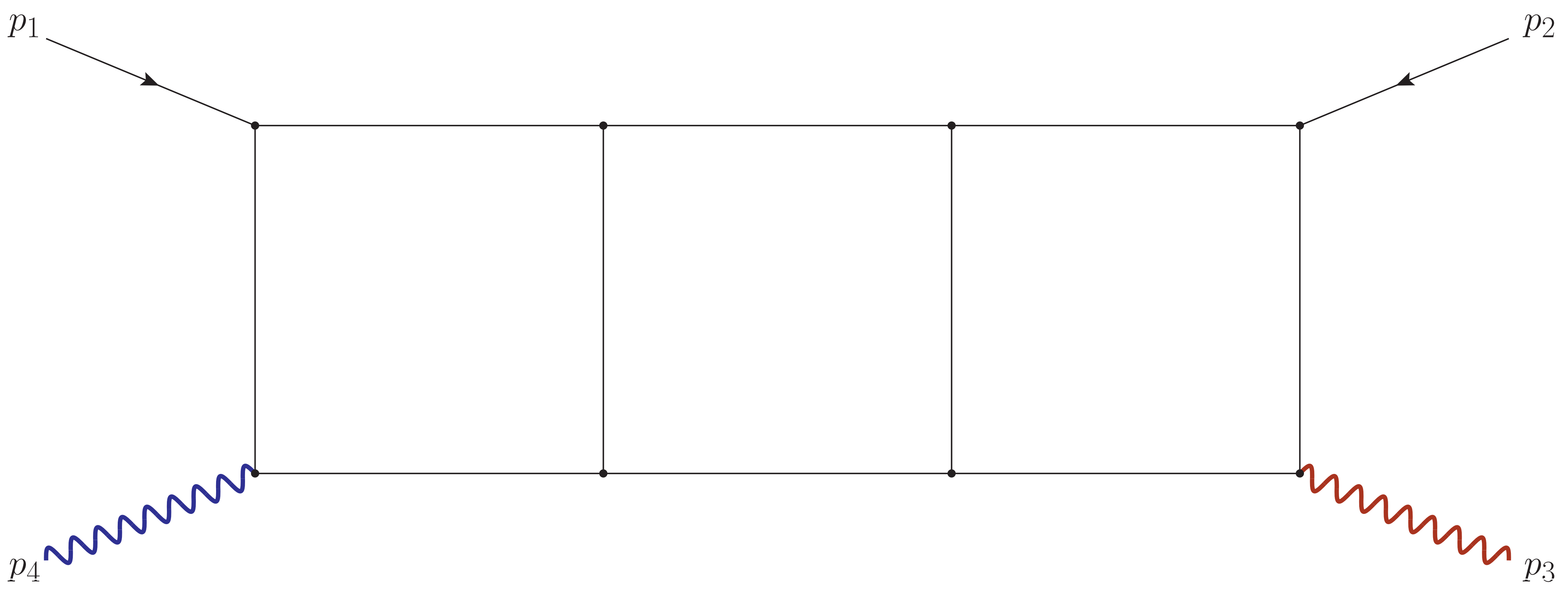
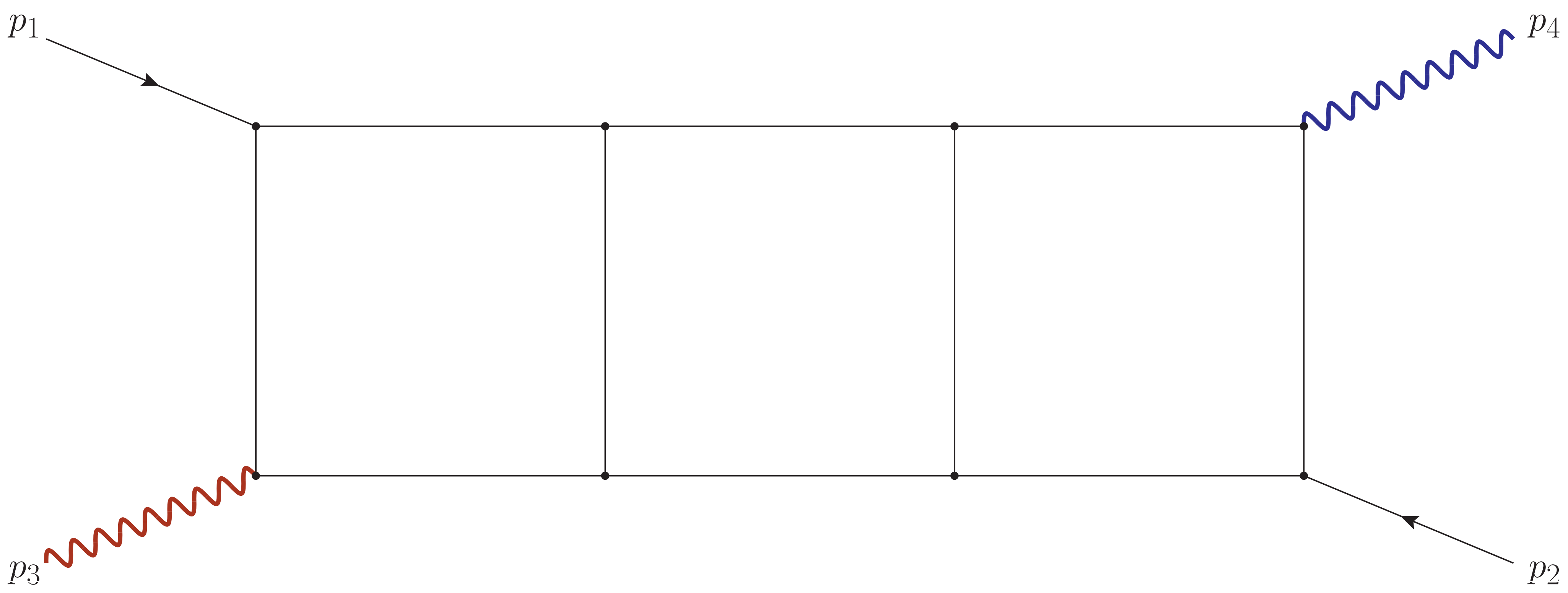
Thank you!

Backup: remaining families

PL_2

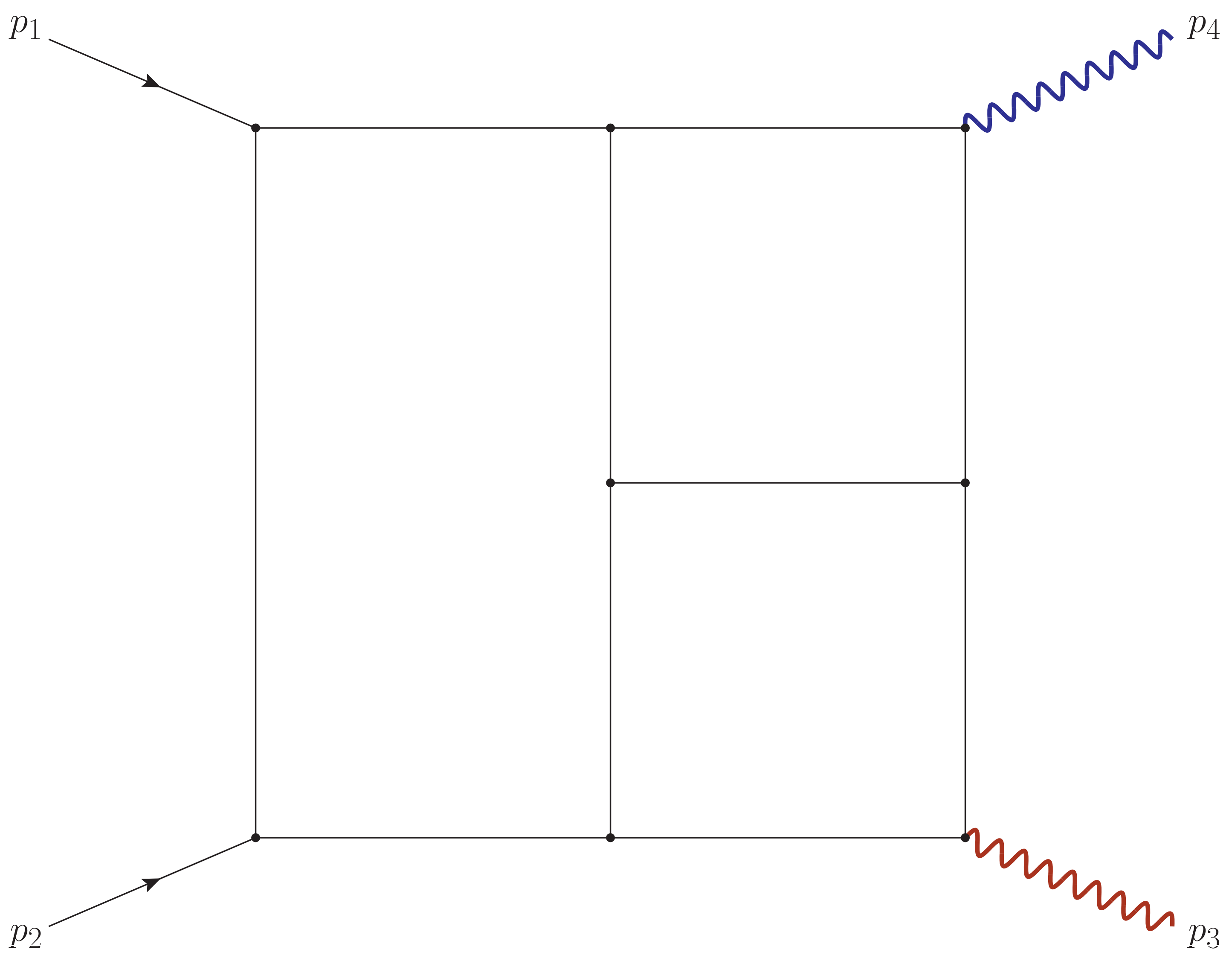


PL_3

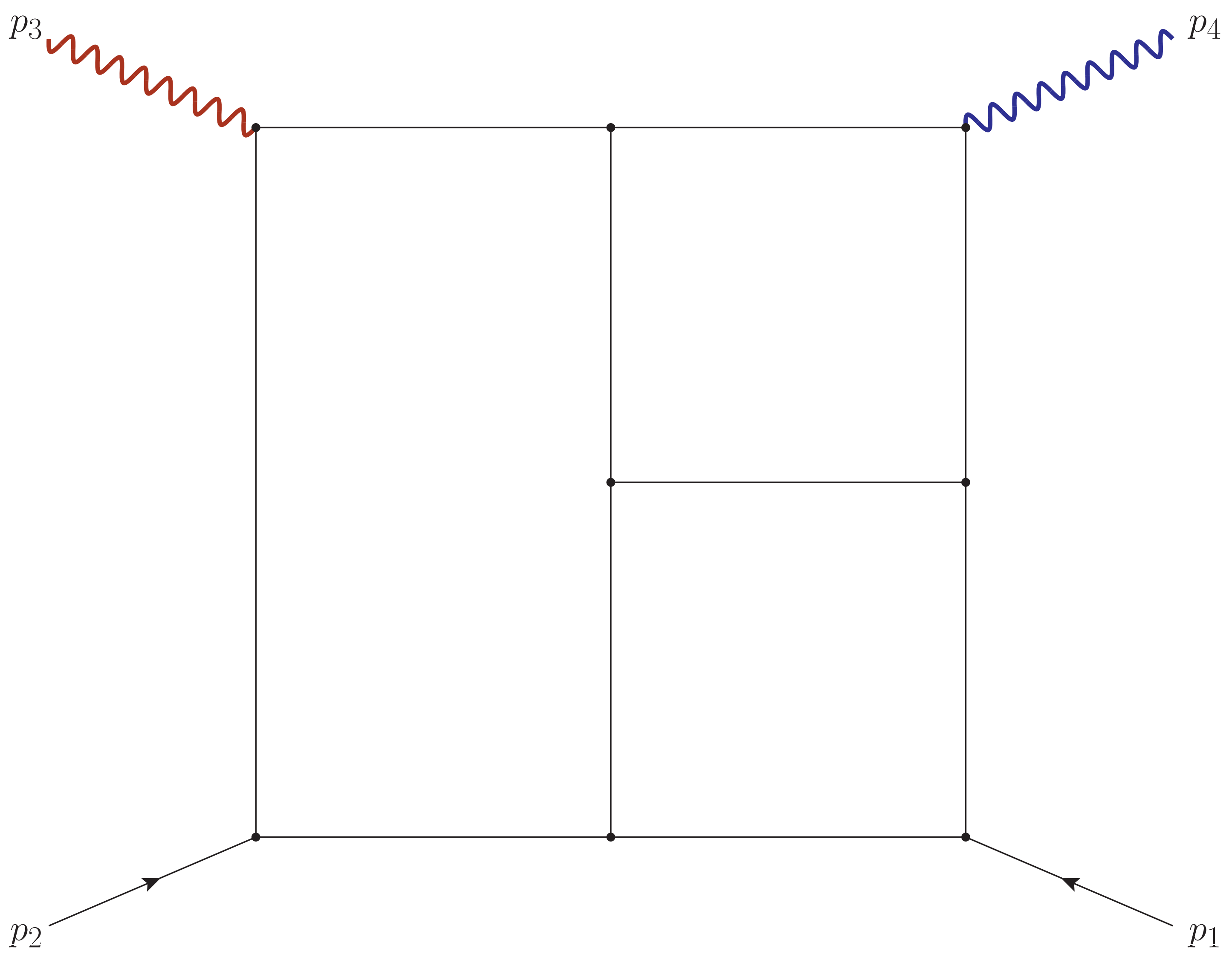


Backup: remaining families

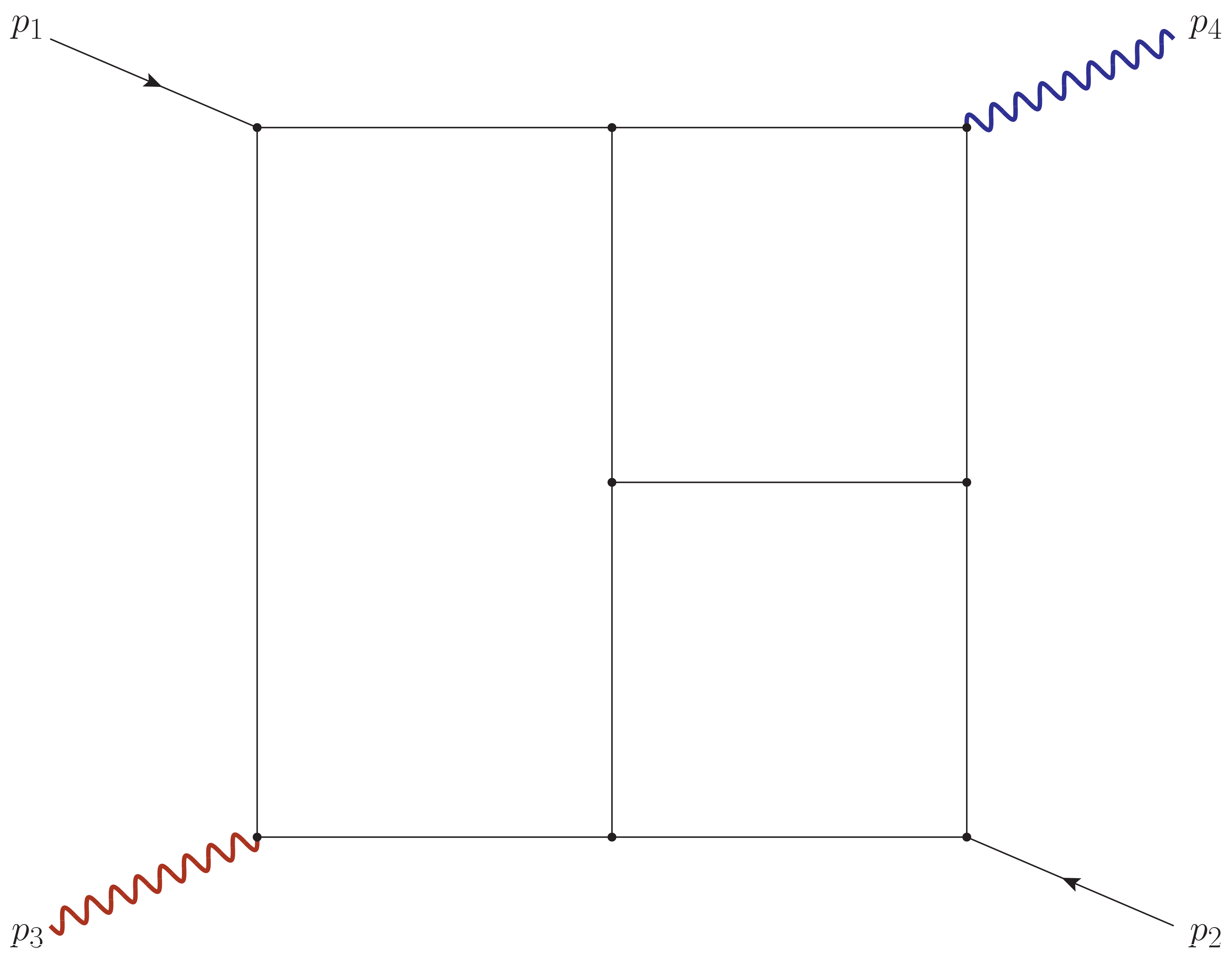
PT_1



PT_2



PT_3



Backup: alphabet in the Mandelstam invariants

$$
\begin{aligned}
\bar{W} = \{&m_3^2,\, m_4^2,\, s_{12},\, s_{23},\\
&m_3^2 - s_{23},\, m_4^2 - s_{23},\, s_{12} + s_{23} - m_3^2,\, s_{12} + s_{23} - m_4^2, s_{12} + s_{23} - m_3^2 - m_4^2,\\
&m_3^2(m_4^2 - s_{23}) + s_{12}s_{23},\, m_3^2 m_4^2 + (s_{12} - m_4^2)s_{23},\\
&m_3^4 + (m_4^2 - s_{12})^2 - 2m_3^2(m_4^2 + s_{12}),\, m_3^2(m_4^2 - s_{23}) + s_{23}(s_{12} + s_{23} - m_4^2),\\
&\frac{m_3^2 + m_4^2 - s_{12} - R}{m_3^2 + m_4^2 - s_{12} + R},\\
&\frac{m_3^2 - m_4^2 + s_{12} - R}{m_3^2 - m_4^2 + s_{12} + R},\\
&\frac{m_3^2 + m_4^2 - s_{12} - 2s_{23} - R}{m_3^2 + m_4^2 - s_{12} - 2s_{23} + R}\}
\end{aligned}
$$

Alphabet in (x, y, z)

$$\begin{aligned} W = \{&x, y, z, 1+x, 1-y, 1-z, z-y, 1+y-z, 1+xy, 1+xz, \\ &z+xy, 1-z+y(1+x), 1+x(1+y-z), \\ &z-y(1-z-xz), z-x(y-z-yz)\} \end{aligned}$$