New Frontiers of the Nested Soft-Collinear Subtraction Scheme

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1 The problem of subtracting IR singularities at NLO is solved. But what does "solved" means?

1.1 We can take papers on Catani-Seymour and FKS subtractions and find general procedures to cancel IR poles at NLO

1.2 These formulas are transparent and can be used to cancel $\mathcal{O}(\epsilon^{-2})$ and $\mathcal{O}(\epsilon^{-1})$ poles explicitly

1.3 These formulas, at least in principle, can be applied to compute NLO cross sections of every process at LHC

2 *Question*: Do we have the same kind of generality at NNLO as well?

Answer: NO

2.1 Numerous collaborations are actively working on this challenge (highly topical issue)

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2 *Question*: Do we have the same kind of generality at NNLO as well?

Answer: NO

2.1 Numerous collaborations are actively working on this challenge (highly topical issue)

2.2 Currently, the community knows how to compute NNLO corrections to an arbitrary process with a colorless initial state

2.3 Remarkable calculations have been performed, for instance, the NNLO corrections to the process $pp \to X + 3$ Jets, but the big picture is still missing

3 Even though the problem was solved in a general way at NLO more than two decades ago, there are reasons why the solution is still lacking at NNLO

3.1 At NNLO, we face the plague of overlapping singularities, which implies

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3.2 Sectoring the phase space allows us to perform the integrals but, at the same time, smashes the physics transparency of the calculation

4 We have to identify the building blocks of the class of QCD processes $pp \to X + N$ Jets. A good starting point is the process $\mathcal{A}_0 : \; q\bar{q} \to X + N\, g$ (see [2310.17598])

4.1 From the subtraction point of view, it is the most complicated channel

4.2 From a combinatorial perspective, it is the simplest channel, as it is symmetric in both the initial and final states

Main Features of NSC at NNLO

$$\langle \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\mathrm{LM}}^{ab}[\,\ldots\,|\,\mathfrak{m},\mathfrak{n}]\rangle$$

Main Features of NSC at NNLO

Double-Soft Counterterm

Can be integrated over the phase spaces $[\mathrm{d}p_{\mathfrak{m}}]$ and $[\mathrm{d}p_{\mathfrak{n}}]$

[Caola, Delto, Frellesvig, Melnikov '18]

Soft-Regulated Term

It contains TRIPLE- and SINGLE-COLLINEAR singularities

$$\langle \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\mathrm{LM}}^{ab}[\ldots|\,\mathfrak{m},\mathfrak{n}]\rangle = \langle S_{\mathfrak{m}\mathfrak{n}} F_{\mathrm{LM}}^{ab}[\ldots|\,\mathfrak{m},\mathfrak{n}]\rangle + \langle \bar{S}_{\mathfrak{m}\mathfrak{n}} S_{\mathfrak{n}} \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\mathrm{LM}}^{ab}[\ldots|\,\mathfrak{m},\mathfrak{n}]\rangle + \langle \bar{S}_{\mathfrak{m}\mathfrak{n}} \bar{S}_{\mathfrak{n}} \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\mathrm{LM}}^{ab}[\ldots|\,\mathfrak{m},\mathfrak{n}]\rangle$$

Single-Soft Counterterm

It contains SINGLE-COLLINEAR singularities

Main Features of NSC at NNLO

$$\langle \Delta^{(\mathfrak{mn})} F_{\rm LM}^{ab}[\ldots|\mathfrak{m},\mathfrak{n}]\rangle = \langle S_{\mathfrak{mn}} F_{\rm LM}^{ab}[\ldots|\mathfrak{m},\mathfrak{n}]\rangle + \langle \overline{S}_{\mathfrak{mn}} S_{\mathfrak{n}} \Delta^{(\mathfrak{mn})} F_{\rm LM}^{ab}[\ldots|\mathfrak{m},\mathfrak{n}]\rangle + \langle \overline{S}_{\mathfrak{mn}} \overline{S}_{\mathfrak{n}} \Delta^{(\mathfrak{mn})} F_{\rm LM}^{ab}[\ldots|\mathfrak{m},\mathfrak{n}]\rangle$$

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Soft-Regulated Term

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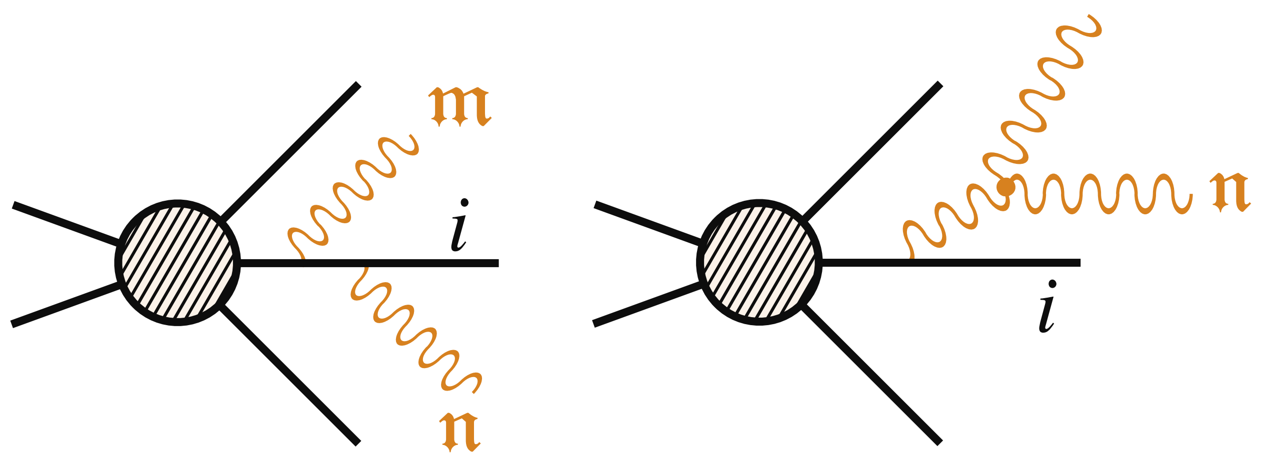
Problem of Overlapping Singularities

$$1 = \sum_{i,j\in\mathscr{H}_{\rm f}} \omega^{\mathfrak{m}i,\mathfrak{n}j}$$

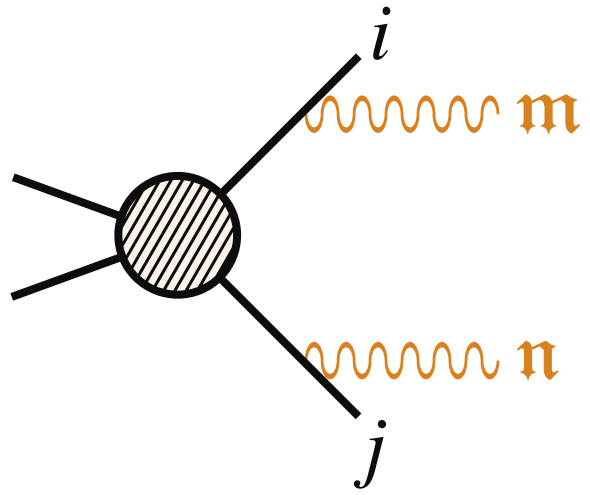
Triple-Collinear Sector

$$C_{i\mathfrak{m}} - C_{i\mathfrak{n}} - C_{\mathfrak{mn}} - C_{\mathfrak{mn},i}$$



Double-Collinear Sector

$$C_{i\mathfrak{m}} - C_{j\mathfrak{n}}$$



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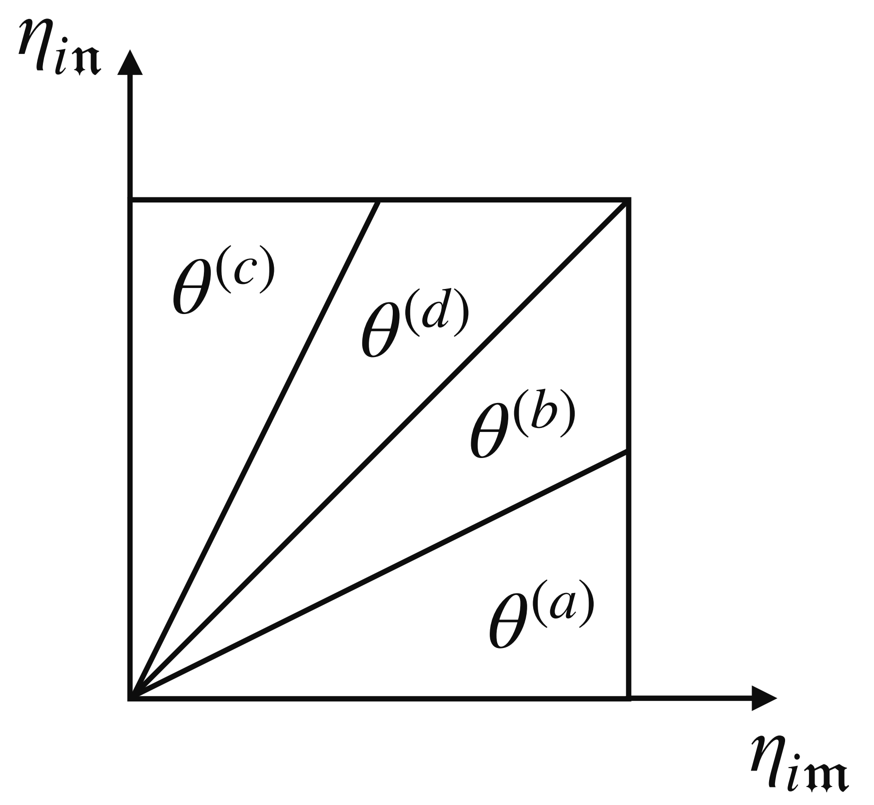
Sector Decomposition

[Czakon '10]

$$\eta_{ij} = (1-\cos\theta_{ij})/2$$



Angular Ordering

$$\theta^{(a)} = \Theta\big(\eta_{i\mathfrak{n}} < \eta_{i\mathfrak{m}}/2\big)$$

$$\theta^{(c)} = \Theta\big(\eta_{i\mathfrak{m}} < \eta_{i\mathfrak{n}}/2\big)$$

$$\theta^{(b)} = \Theta\big(\eta_{i\mathfrak{m}}/2 < \eta_{i\mathfrak{n}}/2 < \eta_{i\mathfrak{m}}\big)$$

$$\theta^{(d)} = \Theta\big(\eta_{i\mathfrak{n}}/2 < \eta_{i\mathfrak{m}}/2 < \eta_{i\mathfrak{n}}\big)$$

Problem of Overlapping Singularities

$$1 = \sum_{i,j\in\mathcal{H}_f} \omega^{\mathfrak{m}i,\mathfrak{n}j}$$

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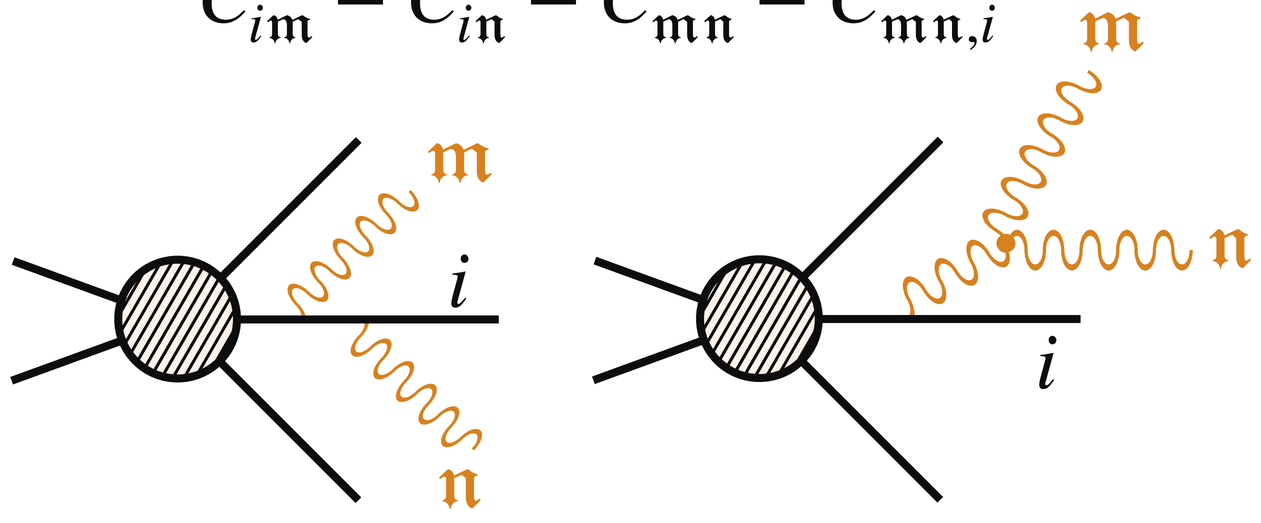
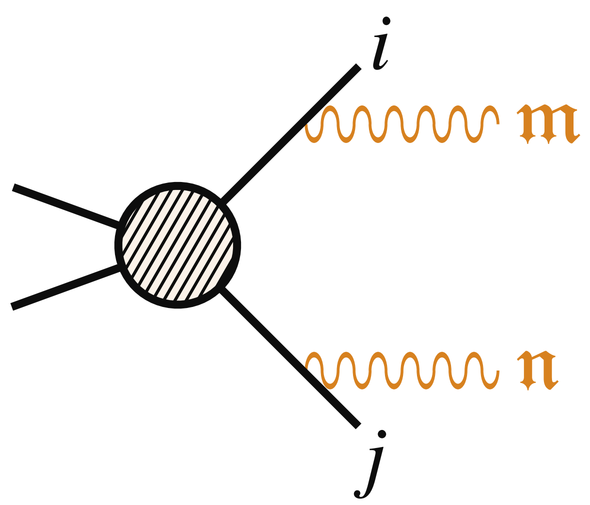


Double-Collinear Sector

$$C_{i\mathfrak{m}} - C_{j\mathfrak{n}}$$



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In principle, this formula can be applied to any process at the LHC.

In practice, identifying structures that can be combined with the VV and RV contributions becomes nearly impossible, rendering the calculation heavily process-dependent.

$$\Omega_1 = \sum_{(ij)} \overline{C}_{i\mathfrak{m}}\overline{C}_{j\mathfrak{n}}[dp_\mathfrak{m}][dp_\mathfrak{n}]\,\omega^{\mathfrak{m}i,\mathfrak{n}j} + \sum_{i\in\mathcal{H}} \left[\overline{C}_{i\mathfrak{n}}\theta^{(a)} + \overline{C}_{\mathfrak{m}\mathfrak{n}}\theta^{(b)} + \overline{C}_{i\mathfrak{m}}\theta^{(c)} + \overline{C}_{\mathfrak{m}\mathfrak{n}}\theta^{(d)}\right][dp_\mathfrak{m}][dp_\mathfrak{n}]\,\overline{C}_{\mathfrak{m}\mathfrak{n},i}\,\omega^{\mathfrak{m}i,\mathfrak{n}i}$$

$$\Omega_2 = \sum_{i\in\mathcal{H}} \left[\overline{C}_{i\mathfrak{n}}\theta^{(a)} + \overline{C}_{\mathfrak{m}\mathfrak{n}}\theta^{(b)} + \overline{C}_{i\mathfrak{m}}\theta^{(c)} + \overline{C}_{\mathfrak{m}\mathfrak{n}}\theta^{(d)}\right][dp_\mathfrak{m}][dp_\mathfrak{n}]\,C_{\mathfrak{m}\mathfrak{n},i}\,\omega^{\mathfrak{m}i,\mathfrak{n}i}$$

$$\Omega_3 = -\sum_{(ij)} C_{j\mathfrak{n}}C_{i\mathfrak{m}}[dp_\mathfrak{m}][dp_\mathfrak{n}]\,\omega^{\mathfrak{m}i,\mathfrak{n}j}$$

$$\Omega_4 = \sum_{(ij)} \left[C_{i\mathfrak{m}}[dp_\mathfrak{m}] + C_{j\mathfrak{n}}[dp_\mathfrak{n}]\right]\omega^{\mathfrak{m}i,\mathfrak{n}j} + \sum_{i\in\mathcal{H}} \left[C_{i\mathfrak{n}}\theta^{(a)} + C_{\mathfrak{m}\mathfrak{n}}\theta^{(b)} + C_{i\mathfrak{m}}\theta^{(c)} + C_{\mathfrak{m}\mathfrak{n}}\theta^{(d)}\right][dp_\mathfrak{m}][dp_\mathfrak{n}]\,\omega^{\mathfrak{m}i,\mathfrak{n}i}$$

$$\sum_{i=1}^{4} \langle \overline{S}_{\mathfrak{m}\mathfrak{n}} \overline{S}_\mathfrak{n}\, \Omega_i\, \Delta^{(\mathfrak{m}\mathfrak{n})} F_{\rm LM}^{ab}[\ldots|\mathfrak{m},\mathfrak{n}]\rangle$$

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Soft and Collinear Regularizations in FKS/NSC

Soft Counterterm

Hard-Collinear Counterterm

$$\langle \Delta^{(\mathfrak{m})} F_{\rm LM}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle = \langle S_{\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\rm LM}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle + \sum_{i \in \mathscr{H}} \langle \bar{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\rm LM}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle$$

$$+ \langle \mathscr{O}_{\rm NLO}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\rm LM}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle$$

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The Soft Operator

$$\langle S_\mathfrak{m}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_\mathfrak{g}] \rangle \sim \langle I_S(\epsilon) \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \rangle$$

$$I_S(\epsilon) = -\frac{(2E_{\max}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{i \neq j} \eta_{ij}^{-\epsilon} K_{ij} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)$$

Color - Correlations

Soft and Collinear Regularizations in FKS/NSC

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Color - Correlations

+

The Virtual Operator

$$2s_{ab}\, \mathrm{d}\hat{\sigma}^{\rm V}_{\mathscr{A}_0} \equiv \langle F_{\rm LV}^{\mathscr{A}_0}\rangle \sim \langle I_{\rm V}(\epsilon) \cdot F_{\rm LM}^{\mathscr{A}_0}\rangle$$

$$I_{\rm V}(\epsilon) = \bar{I}_1(\epsilon) + \bar{I}_1^{\dagger}(\epsilon)$$

$$\bar{I}_1(\epsilon) = \frac{1}{2}\sum_{i\neq j}\left(\frac{1}{\epsilon^2} + \frac{\gamma_i}{\epsilon\, \boldsymbol{T}_i^2}\right)\left(\frac{\mu^2 e^{i\pi\lambda_{ij}}}{s_{ij}}\right)^{\epsilon} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j)$$

Color - Correlations

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$$+ \langle \mathscr{O}_{\mathrm{NLO}}^{(\mathfrak{m})} \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \rangle$$

$$I_{\mathrm{S}}(\epsilon) + I_{\mathrm{V}}(\epsilon) = -\sum_{i \in \mathscr{H}} \frac{1}{\epsilon} \left(2 \boldsymbol{T}_i^2 L_i + \gamma_i \right) + \mathscr{O}(\epsilon^0)$$

- The pole of $\mathscr{O}(\epsilon^{-2})$ vanishes
- No color - correlations at $\mathscr{O}(\epsilon^{-1})$
- Trivially dependent on the number of hard partons N

Soft and Collinear Regularizations in FKS/NSC

Soft Counterterm | Hard-Collinear Counterterm

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$$\sum_{i\in\mathscr{H}} \langle \bar{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{m})} F_{\rm LM}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}] \sim \langle I_{\rm C}(\epsilon) \cdot F_{\rm LM}^{\mathscr{A}_0} \rangle$$

$$I_{\rm C}(\epsilon) = \frac{\Gamma_{a,g} + \Gamma_{b,q}}{\epsilon} + \sum_{i\in\mathscr{H}_{\mathfrak{f}_{\mathfrak{g}}}} \frac{\Gamma_{i,g\to gg}}{\epsilon}$$

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$$\langle \Delta^{(\mathfrak{m})} F_{\text{LM}}^{\mathscr{A}_1}[\mathfrak{m}_g] \rangle = \langle S_{\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\text{LM}}^{\mathscr{A}_1}[\mathfrak{m}_g] \rangle + \sum_{i \in \mathscr{H}} \langle \bar{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\, \Delta^{(\mathfrak{m})} F_{\text{LM}}^{\mathscr{A}_1}[\mathfrak{m}_g] \rangle$$

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$$+ \langle \mathscr{O}_{\mathrm{NLO}}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\mathrm{LM}}^{\mathscr{A}_1}[\mathfrak{m}_g] \rangle$$

$$I_{\mathrm{T}}(\epsilon) = I_{\mathrm{S}}(\epsilon) + I_{\mathrm{V}}(\epsilon) + I_{\mathrm{C}}(\epsilon) = \mathscr{O}(\epsilon^0)$$

The Total Operator

- It does not contain poles
- General procedure for color - correlations
- Trivially dependent on the number of hard partons

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$$+\langle \mathscr{O}_{\rm NLO}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\rm LM}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}]\rangle$$

The Final Result at NLO

$$2s_{ab}\, {\rm d}\hat{\sigma}_{\mathscr{A}_0}^{\rm NLO} = [\alpha_{\rm s}]\langle I_{\rm T}^{(0)} \cdot F_{\rm LM}^{\mathscr{A}_0}\rangle + [\alpha_{\rm s}]\big[\langle \mathscr{P}_{aa}^{\rm NLO} \otimes F_{\rm LM}^{\mathscr{A}_0}\rangle + \langle F_{\rm LM}^{\mathscr{A}_0} \otimes \mathscr{P}_{bb}^{\rm NLO}\rangle\big] + \langle \mathscr{O}_{\rm NLO}^{(\mathfrak{m})}\, \Delta^{(\mathfrak{m})} F_{\rm LM}^{\mathscr{A}_1}[\mathfrak{m}_{\mathfrak{g}}]\rangle + \langle F_{\rm LV,fin}^{\mathscr{A}_0}\rangle$$

Total Operator at $\mathscr{O}(\epsilon^0)$

$$I_{\rm T}^{(0)} = -\sum_{i\neq j} (\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\left[\left(2L_{\rm max} + \frac{1}{2}\log\eta_{ij}\right)\log\eta_{ij} - \frac{1}{2}L_{ij}\left(L_{ij} + \frac{2\gamma_i}{\boldsymbol{T}_i^2}\right)\right.$$
$$\left. + {\rm Li}_2(1-\eta_{ij}) + \frac{\pi^2}{2}\lambda_{ij}\right] + \sum_{i\in\mathscr{H}} \boldsymbol{T}_i^2\left[2L_{\rm max}^2 - \frac{\pi^2}{6} - (2\widetilde{L}_i\tilde{\gamma}_i^{(0)} - \tilde{\gamma}_i^{(1)})\theta_{\mathscr{H}_{\rm f}}\right.$$
$$\left. -2\left(L_i^2 + 2L_i\widetilde{L}_i + \widetilde{L}_i\frac{\gamma_i}{\boldsymbol{T}_i^2}\right)\bar{\theta}_{\mathscr{H}_{\rm f}}\right]$$

NB

$$\beta_0 \equiv \frac{11}{6}C_{\rm A}$$

These operators also describe the NNLO

$$\mathrm{d}\sigma^{\mathrm{NNLO}} \sim \left\langle \frac{1}{2} I_{\mathrm{T}}^{2} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle + \cdots$$

$$Y_{\mathrm{RR}}^{(\mathrm{ss})} = \left\langle \frac{1}{2} I_{\mathrm{S}}^{2} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle + \cdots$$

$$Y_{\mathrm{VV}} = \left\langle \frac{1}{2} I_{\mathrm{V}}^{2} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle + \cdots$$

$$Y_{\mathrm{RR}}^{(\mathrm{cc})} = \left\langle \frac{1}{2} I_{\mathrm{C}}^{2} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle + \cdots$$

$$Y_{\mathrm{RV}}^{(\mathrm{s})} = \left\langle I_{\mathrm{V}}\, I_{\mathrm{C}} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle + \cdots$$

$$Y_{\mathrm{RR}}^{(\mathrm{shc})} = \left\langle I_{\mathrm{S}}\, I_{\mathrm{C}} \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle + \cdots$$

$$Y_{\mathrm{RV}}^{(\mathrm{shc})} = \left\langle \frac{1}{2} \left(I_{\mathrm{S}}\, I_{\mathrm{V}} + I_{\mathrm{V}}\, I_{\mathrm{S}} \right) \cdot F_{\mathrm{LM}}^{\mathscr{A}_0} \right\rangle + \cdots$$

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6 We add the corrections proportional to n_f to the processes $\mathscr{A}_0$ and $\mathscr{B}_0$

5.1 We complete β_0

5.2 We can see how the final-state gluon anomalous dimension arises

5.3 We complete the operator I_T for the processes $\mathscr{A}_0$ and $\mathscr{B}_0$

We start from the following Born Process:

$$\mathscr{B}_0 : \; a_g b_q \to X + (N-1)g + q$$

$$2 s_{ab}\, \mathrm{d}\hat{\sigma}^{\mathrm{LO}}_{\mathscr{B}_0} = \langle F^{gq}_{\mathrm{LM}}[\{g\}_{N-1}, q] \rangle = \langle F^{\mathscr{B}_0}_{\mathrm{LM}} \rangle$$

Main Channel at NLO

It contains the soft singularities that will combine with the virtual ones

$$\mathscr{B}_1 : \; a_g b_q \to X + N g + q$$

$$2 s_{ab}\, \mathrm{d}\hat{\sigma}^{\mathrm{R}}_{\mathscr{B}_1} = \langle F^{gq}_{\mathrm{LM}}[\{g\}_N, q] \rangle$$

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$$\mathscr{B}_0: \ a_g b_q \to X + (N-1)g + q$$

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Damping Factors

They select the final-state parton that is potentially unresolved

$$2s_{ab}\,\mathrm{d}\hat{\sigma}^{\mathrm{R}}_{\mathscr{B}_1} = \langle F^{gq}_{\mathrm{LM}}[\{g\}_N, q]\rangle = \sum_{i\in\mathscr{H}_{\mathrm{f}}} \langle \Delta^{(i)} F^{gq}_{\mathrm{LM}}[\{g\}_N, q]\rangle$$

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$$= \left\langle \Delta^{(\mathfrak{m})}\left(F^{gq}_{\mathrm{LM}}[\{g\}_{N-1}, q \,|\, \mathfrak{m}_{\mathfrak{g}}] + F^{gq}_{\mathrm{LM}}[\{g\}_N \,|\, \mathfrak{m}_{\mathfrak{q}}]\right)\right\rangle$$

Rename the Damping Factors

Two contributions identified

- Unresolved gluon
- Unresolved quark

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$$I_{\mathrm{S}}(\epsilon) + I_{\mathrm{V}}(\epsilon) = -\sum_{i\in\mathscr{H}} \frac{1}{\epsilon}\left(2\boldsymbol{T}_i^2 L_i + \gamma_i\right) + \mathscr{O}(\epsilon^0)$$

- The pole of $\mathscr{O}(\epsilon^{-2})$ vanishes
- No color - correlations at $\mathscr{O}(\epsilon^{-1})$
- Trivially dependent on the number of hard partons N

$$\sum_{i\in\mathscr{H}} \langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\,\Delta^{(\mathfrak{m})} F^{\mathscr{B}_1}_{\mathrm{LM}}[\mathfrak{m}_{\mathfrak{g}}] \sim \langle I^{\mathrm{inc}}_{\mathrm{C}}(\epsilon) \cdot F^{\mathscr{B}_0}_{\mathrm{LM}}\rangle$$

$$\sum_{i\in\mathscr{H}} \langle C_{i\mathfrak{m}}\,\Delta^{(\mathfrak{m})} F^{\mathscr{B}_1}_{\mathrm{LM}}[\mathfrak{m}_{\mathfrak{q}}] \sim \left\langle \frac{\Gamma_{q\to gq}}{\epsilon} \cdot F^{\mathscr{B}_0}_{\mathrm{LM}}\right\rangle$$

$$I^{\mathrm{inc}}_{\mathrm{C}}(\epsilon) = \frac{\Gamma_{a,g} + \Gamma_{b,q}}{\epsilon} + \sum_{i\in\mathscr{H}_{\mathrm{f_g}}} \frac{\Gamma_{i,g\to gg}}{\epsilon} + \frac{\Gamma_{q\to qg}}{\epsilon}$$

$$I_{\mathrm{C}}(\epsilon)$$

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$$I_{\mathrm{T}}(\epsilon) = I_{\mathrm{S}}(\epsilon) + I_{\mathrm{V}}(\epsilon) + I_{\mathrm{C}}(\epsilon) = \mathcal{O}(\epsilon^0)$$

NB

$$\beta_0 \equiv \frac{11}{6} C_{\mathrm{A}}$$

symmetric in both the initial and final states

4.3 For NLO and NNLO corrections, we add only gluons

5 Now we want to break the initial- and final-state symmetry of process $\mathscr{A}_0$. Thus we consider $\mathscr{B}_0 : \; gq \to X + (N-1)g + q$

5.1 It contains all the IR divergences of the process $\mathscr{A}_0$

5.2 From a combinatorial perspective, it is more complex than $\mathscr{A}_0$

5.3 We can see how the final-state quark anomalous dimension arises

6 We add the corrections proportional to n_{f} to the processes $\mathscr{A}_0$ and $\mathscr{B}_0$

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7 At this point, generalizing the calculation for $pp \to X + N$ Jets becomes merely a matter of combinatorics

7.1 We expect $I_{\rm C}$ to work precisely as in processes $\mathscr{A}_0$ and $\mathscr{B}_0$

7.2 Our final result will be a generla formula analogous to that we currently have for NLO corrections

How to complete the Collinear Operator

$$I_{\mathrm{C}}^{\mathrm{inc}}(\epsilon) = \frac{\Gamma_{a,g} + \Gamma_{b,q}}{\epsilon} + \sum_{i \in \mathcal{H}_{\mathrm{f}_g}} \frac{\Gamma_{i,g\to gg}}{\epsilon} + \sum_{i \in \mathcal{H}_{\mathrm{f}_q}} \frac{\Gamma_{i,q\to qg}}{\epsilon}$$

$$gq \to X + (N-1)g + q + gg$$

$\oplus$

$$gq \to X + (N-1)g + q' + q''\bar{q}''$$

$$gq \to X + (N-3)g + q' + q''\bar{q}'' + q'''\bar{q}'''$$

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When we select the unresolved pair

$$S_{\mathfrak{m}\mathfrak{n}} \Rightarrow \Theta_{\mathfrak{m}\mathfrak{n}} F_{\mathrm{LM}}^{ab}[\ldots \mid \mathfrak{m}, \mathfrak{n}] \qquad \cancel{S_{\mathfrak{m}\mathfrak{n}}} \Rightarrow F_{\mathrm{LM}}^{ab}[\ldots \mid \mathfrak{m}, \mathfrak{n}]$$

We complete the anomalous dimensions

Gluon Leg

$$\sum_{i \in \mathcal{H}_{\mathrm{f}_g}} \frac{\Gamma_{i,g\to q\bar{q}}}{\epsilon}$$

Quark Leg

$$\sum_{i \in \mathcal{H}_{\mathrm{f}_q}} \frac{\Gamma_{i,q\to gq}}{\epsilon}$$

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$$I_{\mathrm{C}}(\epsilon) = \frac{\Gamma_{a,g} + \Gamma_{b,q}}{\epsilon} + \sum_{i \in \mathcal{H}_{\mathrm{f}_g}} \frac{\Gamma_{i,g}}{\epsilon} + \sum_{i \in \mathcal{H}_{\mathrm{f}_q}} \frac{\Gamma_{i,q}}{\epsilon}$$

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THANK YOU FOR YOUR ATTENTION