Towards two-loop QCD corrections to $pp \to t\bar{t}j$

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In collaboration with: Simon Badger, Matteo Becchetti, Heribertus Bayu Hartanto, Simone Zoia [arXiv: 2412.13876]

(Thanks also to Gaia Fontana for the wonderful drawings)

Expectations for this meeting from ChatGPT

Outline

1. Introduction $\rightarrow$ What and why?

2. Scattering amplitudes workflow $\rightarrow$ How?

3. Two-loop $gg \to t\bar{t}g$ amplitudes $\rightarrow$ What are our results?

4. Conclusions and outlook $\rightarrow$ What's next?

Precision physics

Monte Carlo generators

Cross-section predictions

Data from colliders

σ = ∫ PDFs x |A|^2 x dPS

⇒ Current frontier NNLO/N^{3}LO
⇒ ***Amplitudes*** are key ingredients for cross-section predictions

Run 1+2+3 + HL-LHC

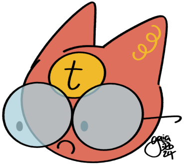
⇒ Huge amount of data
⇒ Small uncertainties on experimental measurements (***% level accuracy***)
⇒ Observe rare processes

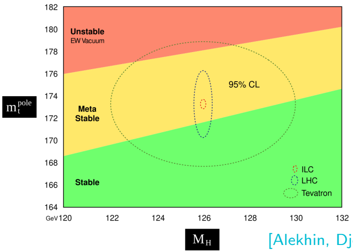
Images generated with PromeAI

Relevance of the top quark

Unique properties of the top quark

- To-date **heaviest** fundamental particle
- Decays before forming hadrons
- Information about its spin state preserved in the decay product distributions





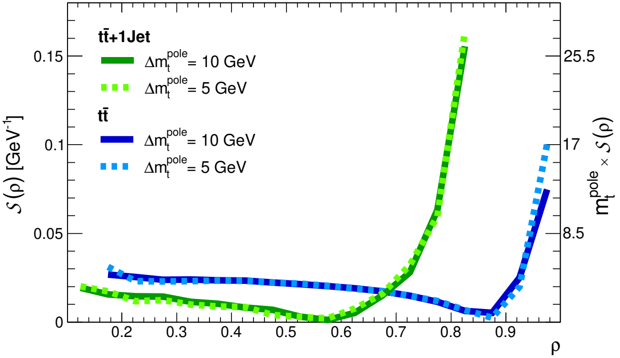
[Alekhin, Djouadi, Moch '12]

Role in the Standard Model

- Largest coupling to the Higgs boson
- Affects the **EW vacuum stability**

Motivations for $t\bar{t}j$ production

- 50% of $t\bar{t}$ events produced at LHC are associated with a jet
- $\boldsymbol{t\bar{t}j}$ normalised differential cross-section w.r.t. invariant mass of final state particles is **highly sensitive to m_t**

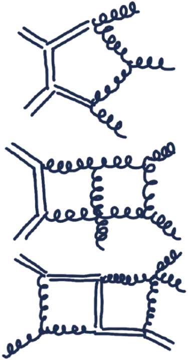


[Alioli, Fernandez, Fuster, Irles, Moch, Uwer '13]

Theory status

What do we know about $t\bar{t}j$?

- NLO QCD corrections [Dittmaier, Uwer, Weinzierl, '07]
- Full off-shell decays and interfaces with parton shower [Melnikov and Schulze '10]
  [Alioli, Moch, Uwer '12]
  [Bevilacqua, Czakon, Hartanto, Kraus, Worek '15-'16]
- Mixed QCD and EW corrections
  [Gütschow, Lindert, Schönherr '18]
- **NNLO QCD corrections needed**
  $\rightarrow$ initial steps toward this challenge
  [Badger, Becchetti, Chaubey, Marzucca, Sarandrea '22]
  [Badger, Becchetti, Chaubey, Marzucca '23]
  [Badger, Becchetti, Giraudo, Zoia '24]



Current frontier: $2 \to 3$ two-loop scattering amplitudes

Massless external particles:

- $pp \to \gamma\gamma\gamma$
  [Abreu, Page, Pascual, Sotnikov '20]
  [Chawdhry, Czakon, Mitov, Poncelet '21]
  [Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov '23]
- $pp \to \gamma\gamma j$
  [Agarwal, Buccioni, von Manteuffel, Tancredi '21]
  [Chawdhry, Czakon, Mitov, Poncelet '21]
  [Badger, Brönnum-Hansen, Chicherin, [Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21]
- $pp \to \gamma jj$
  [Badger, Czakon, Hartanto, Moodie, Peraro, Poncelet, Zoia '23]
- $pp \to jjj$
  [Abreu, Febres Cordero, Ita, Page, Sotnikov '21]
  [De Laurentis, Ita, Klinkert, Sotnikov '23]
  [Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi '23]
  [De Laurentis, Ita, Sotnikov '23]

One massive external particle: (leading colour)

- $pp \to Wbb$
  [Badger, Hartanto, Zoia '21]
  [Hartanto, Poncelet, Popescu, Zoia '22]
- $pp \to Wjj$
  [Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov '22]
- $pp \to Hbb$
  [Badger, Hartanto, Krys, Zoia '21]
- $pp \to W\gamma j$
  [Badger, Hartanto, Krys, Zoia '22]
- $pp \to W/Z + bb$
  [Buonocore, Devoto, Kallweit, Mazzitelli, Rottoli, Savoini '22]
  [Mazzitelli, Sotnikov, Wiesemann '24]

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  [De Laurentis, Ita, Sotnikov '23]

One massive external particle: (leading colour)

Full colour:

- $pp \to W\gamma\gamma$
  [Badger, Hartanto, Poncelet, Wu, Zhang, Zoia '24]
- $pp \to Hb\bar{b}$
  [Badger, Hartanto, Wu, Zhang, Zoia '24]

More masses:

- $pp \to t\bar{t}H$
  [Agarwal, Heinrich, Jones, Kerner, Klein, Lang, Magerya, Olsson, '24]


INTERNAL MASSES

Colour decomposition

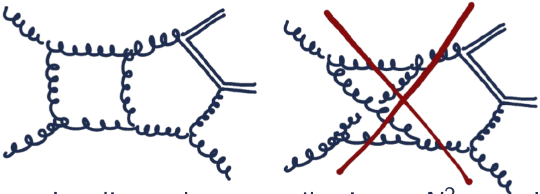
- Consider all diagrams contributing to the process

$$A^{(L)}(\vec{x}, \epsilon) = \sum \Big(\text{Feynman diagrams}\Big)$$

Qgraf, Nogueira '93

- **Colour expansion** $\to$ take the **leading colour** limit
  $\to$ reduce the complexity of the loop integrals

Example: @2L



diagrams	tot	LC
LO	16	6
NLO	384	77
NNLO	11370	1357

Leading colour contribution $\propto N_c^2 \to$ only planar diagrams

Helicity amplitudes

Amplitude written in terms of tensor structure and form factors

$$A^{(L)} = \sum_{i=1}^{N} \mathcal{T}_i \, \mathcal{F}_i^{(L)}$$

Ex. $\mathcal{T}_1 = m_t^2 \; \bar{u}(p_2) v(p_1) \varepsilon(p_3, q_3) \cdot p_1 \; \varepsilon(p_4, q_4) \cdot p_1 \; \varepsilon(p_5, q_5) \cdot p_1$

The form factors can be derived as

$$\mathcal{F}_i^{(L)} = \sum_{j=1}^{N} \left(\Theta^{-1}\right)_{ij} \underbrace{\sum_{\text{pol.}} \mathcal{T}_j^{\dagger} A^{(L)}}_{A^{(L),\text{proj}}} \, ,$$

with $\Theta_{ij} = \mathcal{T}_i^{\dagger} \mathcal{T}_j$. We compute the **helicity amplitudes**:

$$A^{(L),h_1 h_2 h_3 h_4 h_5} = \sum_{i=1}^{N} \mathcal{T}_i^{h_1 h_2 h_3 h_4 h_5} \; \mathcal{F}_i^{(L)} .$$

Massive spinors

- Helicity amplitudes encode spin correlation information

$\rightarrow$ **inclusion of top-quark decay** in narrow-width approximation

- **Helicity**: projection of the spin along the direction of momentum

- For massive particles, define the **massless projection**:

$$p^{\flat,\mu} = p^{\mu} - \frac{m^2}{2p\cdot n} n^{\mu}$$

with n an arbitrary light-like momentum. The **massive fermion spinor** is:

$$u_+(p,m) = \frac{(\not{p}+m)|n\rangle}{\langle p^{\flat}\, n\rangle}, \quad u_-(p,m) = \frac{(\not{p}+m)|n]}{[p^{\flat}\, n]}$$

see [Kleiss, Stirling '85] [Arkani-Hamed, Huang, Huang '17] [Badger, Chaubey, Hartanto, Marzucca '21] [Badger, Becchetti, Chaubey, Marzucca, Sarandrea '22]

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$$u_{-}(p, m) = \frac{\langle p^{\flat}\, n\rangle}{m} \left(u_{+}(p, m)\big|_{p^{\flat} \leftrightarrow n} \right),$$

see [Kleiss, Stirling '85] [Arkani-Hamed, Huang, Huang '17] [Badger, Chaubey, Hartanto, Marzucca '21] [Badger, Becchetti, Chaubey, Marzucca, Sarandrea '22]

Reduction to MIs

- The amplitude is a linear combination of Feynman integrals:

$$A^{(L),\text{proj}}(\vec{x},\epsilon) = \sum_i c_i(\vec{x},\epsilon) I_i(\vec{x},\epsilon),$$

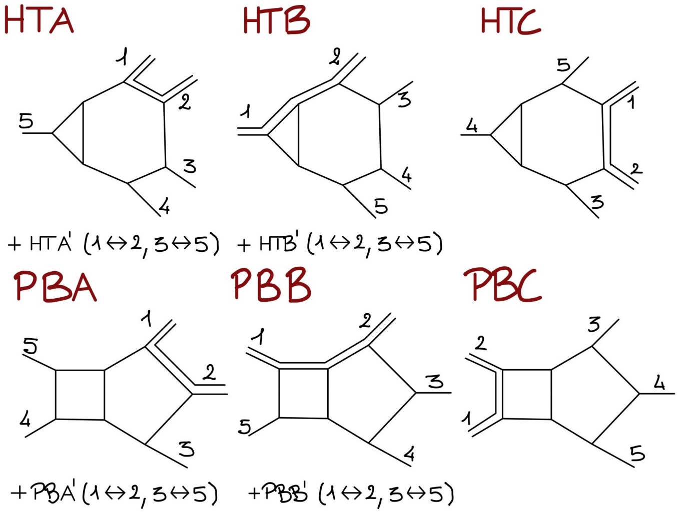
i.e. $$I(\vec{x},\epsilon) = \int \frac{d^{\mathrm{D}}k_1 d^{\mathrm{D}}k_2}{k_1^2(k_1+p_1)^2(k_1+p_1+p_2)^2\ldots} \quad \text{and} \quad \mathrm{D} = 4-2\epsilon$$

- $I_i(\vec{x},\epsilon)$ written as linear combination of **MIs** using:
  Integration by Parts Identities (IBPs) [Chetyrkin, Kataev, Tkachov, '80]

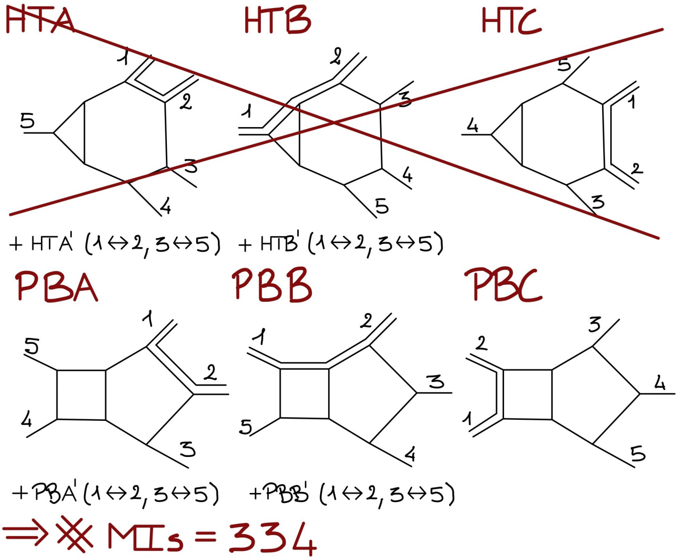
i.e. $$\int d^{\mathrm{D}}k_1 d^{\mathrm{D}}k_2 \frac{\partial}{\partial k_1^{\mu}} \left(p_1^{\mu} \frac{1}{k_1^2(k_1+p_1)^2(k_1+p_1+p_2)^2\ldots} \right) = 0$$

- IBPs generated with `NeatIBP` [Wu, Boehm, Ma, Xu, Zhang '23]

Integral families



Integral families



Algebraic complexity

- Amplitude in terms of MIs

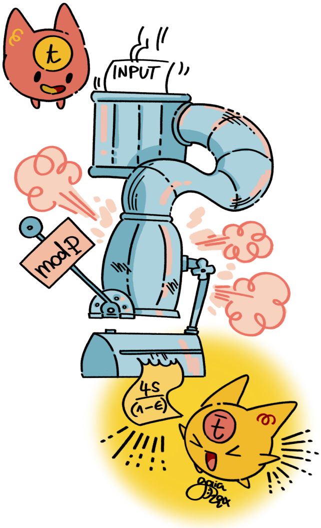
$$A^{(2),\text{proj}}(\vec{x},\epsilon) = \sum_i d_i(\vec{x},\epsilon)\, \text{MI}_i(\vec{x},\epsilon)$$

- Replace symbolic operations with numerical evaluations in a **finite field** (integers mod prime P)

[von Manteuffel, Schabinger '14] [Peraro '16]

- Numerical framework: `FiniteFlow`

[Peraro '19]



DEs for MIs

MIs satisfy the following **differential equation**:

$$\boxed{\mathrm{d}\vec{f}(\vec{x},\epsilon) = \mathrm{d}A(\vec{x},\epsilon)\ \vec{f}(\vec{x},\epsilon)},$$

where $\vec{x}$ are the kinematic invariants $\to$ **6 variables**

Stay tuned for Matteo's talk!

For **PBA**[1] and **PBC**[2]:

$$\mathrm{d}A(\vec{x},\epsilon) = \boxed{\epsilon} \sum_j c_j \boxed{\mathrm{dlog}(\alpha_j(\vec{x}))}$$

ϵ**-factorises** | **dlog form**

For **PBB**[2]:

$$\mathrm{d}A(\vec{x},\epsilon) = \sum_{k=0}^{2} \boxed{\epsilon^k} \sum_j c_{kj} \boxed{\omega_j(\vec{x})}$$

DEs quadratic in ϵ | **One-form**

[1] Badger, Becchetti, Chaubey, Marzucca '23 [2] Badger, Becchetti, Giraudo, Zoia '24

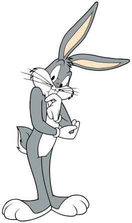
Laurent expansion of the MIs

- **Expand the MIs around $\epsilon = 0$:**

$$f(\vec{x}, \epsilon) = \sum_{k=0}^{4} \epsilon^k f^{(k)}(\vec{x})$$

$\rightarrow$ amplitudes in terms of a set of **special functions**

- Advantages:
  - **Analytic UV/IR pole subtraction**
  - **Simplification of finite remainders**
  - **Improved numerical evaluation**

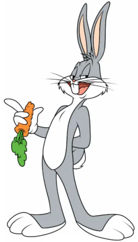


- Pentagon functions method applies to DEs:

$$\mathrm{d}\vec{f}(\vec{x}, \epsilon) = \epsilon \sum_j c_j \,\mathrm{dlog}(\alpha_j(\vec{x}))\ \vec{f}(\vec{x}, \epsilon)$$

see [Gehrmann, Henn, Lo Presti '18] [Chicherin, Sotnikov '20] [Chicherin, Sotnikov, Zoia '22] [Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia '24]

A basis of special functions for tt̄j



Pentagon functions method applies straightforward to PB_A and PB_C $\to$ for PB_B **non-canonical DEs**

$\downarrow$

MIs of non-canonical sectors are:

non-zero only starting from $\mathcal{O}(\epsilon^4)$!

$\downarrow$

Construction of an over-complete basis with

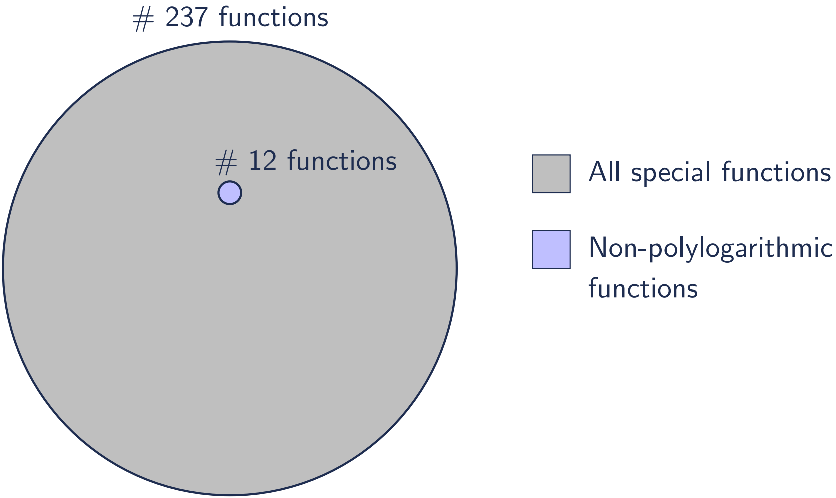
non-polylogarithmic only in the finite reminder!

$\downarrow$

Finally, the amplitude takes the form

$$A^{(2L),\mathrm{proj}}(\epsilon, \vec{x}) = \sum_i \sum_{k=-4}^{0} \epsilon^k r_{ki}(\vec{x}) \, \mathrm{F}_i(\vec{x})$$

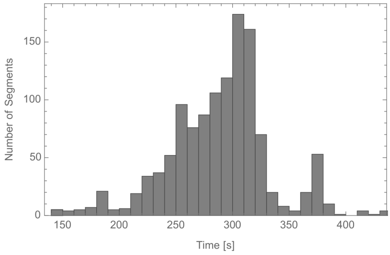
Numerical evaluation of special functions

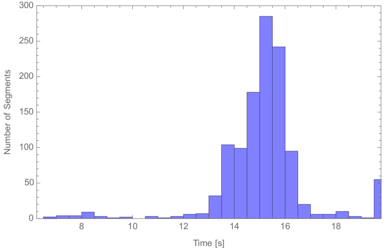


Numerical evaluation of special functions

$\mathsf{F}_i(\vec{x})$ evaluated using generalised power series [Moriello '20]

method as implemented in `DiffExp` [Hidding '21]



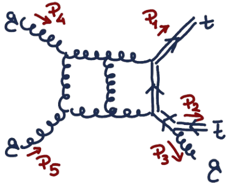


	PB_A	PB_B	PB_C	all MIs	special func. (all)	special func. (non-polylog.)
$\langle T \rangle$	$43\,s$	$77\,s$	$66\,s$	$309\,s$	$297\,s$	$16\,s$
σ	$7\,s$	$17\,s$	$14\,s$	$27\,s$	$65\,s$	$3\,s$

Notation and kinematics

Evaluated process:

$$g(-p_4) + g(-p_5) \to \bar{t}(p_1) + t(p_2) + g(p_3)$$



Kinematics:

We have six scalar invariants

$$\vec{d} = \left(d_{12}, d_{23}, d_{34}, d_{45}, d_{15}, m_t^2\right)$$

Spin Structure Basis for Helicity States:

$$A^{(L)}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5}; n_t, n_{\bar{t}}) = m_t \Phi^{h_3 h_4 h_5} \times \sum_{i=1}^{4} \Theta_i(n_t, n_{\bar{t}})\, A^{(L),[i]}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5})$$

Finite remainder reconstruction

- Mass-renormalised amplitudes are gauge invariant

  $\to$ **Gauge invariance check** ✓

- UV/IR poles identified analytically and finite remainder computed directly

  $\to$ **Pole check** ✓

- Results crosschecked against independent computation of helicity amplitudes in terms of momentum-twistor variables

  $\to$ **simplication of the amplitude**

helicity	max degrees MIs recon.	max degrees SF recon.
$+++++$	294	131
$+++-+$	384	269
$++++-$	395	264

Results

Numerical evaluation of finite remainders up to two loops in the leading color limit for $gg \to t\bar{t}g$ production

Helicity	$R^{(0),[1]}$	$R^{(0),[2]}$	$R^{(0),[3]}$	$R^{(0),[4]}$
$+++$	$0.26326 - 0.0097514\,i$	0	0	0
$+-+$	$5.9619 - 0.16047\,i$	0	0	$-0.31659 - 0.097935\,i$
$++-$	$-5.9575 + 0.0089231\,i$	$-12.606 - 0.067440\,i$	$4.6564 + 0.024911\,i$	$-1.9692 - 0.010535\,i$
Helicity	$R^{(1),[1]}/R^{(0),[1]}$	$R^{(1),[2]}/R^{(0),[1]}$	$R^{(1),[3]}/R^{(0),[1]}$	$R^{(1),[4]}/R^{(0),[1]}$
$+++$	$38.396 - 5.8002\,i$	$71.982 - 4.0653\,i$	$-14.289 + 0.70866\,i$	$17.909 - 0.39528\,i$
$+-+$	$19.221 - 8.4151\,i$	$-4.8506 + 4.8015\,i$	$0.67096 - 0.09959\,i$	$-1.2201 + 2.1594\,i$
$++-$	$20.369 - 19.991\,i$	$41.522 - 41.969\,i$	$-15.990 + 15.739\,i$	$6.2964 - 6.4584\,i$
Helicity	$R^{(2),[1]}/R^{(0),[1]}$	$R^{(2),[2]}/R^{(0),[1]}$	$R^{(2),[3]}/R^{(0),[1]}$	$R^{(2),[4]}/R^{(0),[1]}$
$+++$	$882.48 - 91.619\,i$	$2489.7 - 266.72\,i$	$-492.28 + 8.1003\,i$	$593.35 - 87.569\,i$
$+-+$	$414.16 - 206.87\,i$	$-171.78 + 189.69\,i$	$25.226 - 1.5639\,i$	$-54.820 + 95.716\,i$
$++-$	$332.97 - 646.02\,i$	$623.01 - 1325.1\,i$	$-259.14 + 512.33\,i$	$89.185 - 198.65\,i$

Check out our paper on arXiv :)

Summary

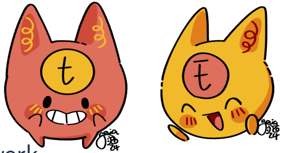
What and why?

- **Two-loop scattering amplitude for $pp \to t\bar{t}j$** $\to$ bottleneck for $t\bar{t}j$ **precise theoretical predictions**

What are our results?

- Analytic pole check $\to$ direct determination of the **finite remainder !**
- **Numerical evaluation of the two-loop amplitudes**

4 key questions



How?

- **Finite fields** framework
- Express MIs in terms of basis of **Special functions**

What's next?

- Deliver **phenomenological viable** results
- Explore analytical **reconstruction** viability

Thank you for your attention!

Merry Christmas!

Backup

Physical point

The point chosen in the **physical region** for the sub-amplitudes evaluation is given by

$$d_{12} = \frac{1617782845110651539}{1506833389797120000}\,, \quad d_{23} = \frac{335}{1232}\,, \quad d_{34} = -\frac{5}{32}\,,$$
$$d_{45} = \frac{3665}{7328}\,, \quad d_{15} = -\frac{45}{1408}\,, \quad m_t^2 = \frac{376940175237098461}{1506833389797120000}\,,$$

with

$$\mathrm{tr}_5 = \mathrm{i}\,\frac{\sqrt{582950030096630501}}{426229309440}\,.$$

In terms of **momentum twistor variables**

$$s_{34} = -\frac{5}{16}\,, \quad t_{45} = -\frac{733}{229}\,, \quad t_{12} = -\frac{61}{72}\,,$$
$$t_{23} = -\frac{134}{77}\,, \quad t_{51} = \frac{9}{44}\,, \quad x_{5123} = \frac{11}{51} + \frac{1}{125}\mathrm{i}\,.$$