# Dispersive calculation of isospin–breaking corrections for $\tau$ data

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CVC between EM and weak form factors:

$$\sigma^{(0)}_{e^+e^-\to 2\pi}(s) = \frac{1}{\mathcal{N}(s)\Gamma_e^{(0)}} \frac{\mathrm{d}\Gamma(\tau^- \to \pi^-\pi^0\nu_\tau)}{\mathrm{d}s} \frac{R_{\mathrm{IB}}(s)}{S_{\mathrm{EW}}}$$

where $\mathcal{N}(s) = \frac{3|V_{ud}|^2}{2\pi\alpha m_\tau^2} s \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau}\right)$, $\Gamma_e^{(0)} = \frac{G_F^2 m_\tau^5}{192\pi^3}$ and

$$R_{\mathrm{IB}}(s) = \frac{1}{G_{\mathrm{EM}}(s)} \frac{\beta^3_{\pi^+\pi^-}(s)}{\beta^3_{\pi^0\pi^-}(s)} \left|\frac{F_\pi^V(s)}{f_+(s)}\right|^2$$

and $S_{\mathsf{EW}}$ $\to$ dominant short-distance electroweak corrections

# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

## Other works

Form [Castro et al, '24]:

| Source | $\Delta a_\mu^{\text{had, LO}}[\pi\pi,\tau]$ $(10^{-10})$ | | | |
|---|---|---|---|---|
| | GS | KS | GP | |
| | Davier *et al.* | | FF1 | FF2 |
| $S_{\text{EW}}$ | $-12.21(0.15)$ | | $-11.96(0.15)$ | |
| $G_{\text{EM}}$ | $-1.92(0.90)$ | | $-1.71^{+0.61}_{-1.48}$ | |
| FSR | $+4.67(0.47)$ | | $+4.56(0.46)$ | |
| $m_{\pi^\pm} - m_{\pi^0}$ effect on $\sigma$ | $-7.88$ | | $-7.47$ | |
| $m_{\pi^\pm} - m_{\pi^0}$ effect on $\Gamma$ | $+4.09$ | $+4.02$ | $+4.07$ | |
| $m_{K^\pm} - m_{K^0}$ effect on $\Gamma$ | $-$ | $-$ | $+0.37$ | |
| $m_{\rho^\pm} - m_{\rho^0}$ | $+0.20^{+0.27}_{-0.19}$ | $+0.11^{+0.19}_{-0.11}$ | $+1.27^{+1.51}_{-1.45}$ | |
| $\rho - \omega$ interference | $+2.80(0.19)$ | $+2.80(0.15)$ | $+3.56^{+0.84}_{-0.80}$ | |
| $\pi\pi\gamma$ | $-5.91(0.59)$ | $-6.39(0.64)$ | $-5.14(4.45)$ | $-1.54(1.54)$ |
| TOTAL | $-16.07(1.22)$ | $-16.70(1.23)$ | $-12.45^{+4.84}_{-5.00}$ | $-8.85^{+2.44}_{-2.75}$ |

- Q1: Model dependence of the assumed form factor parametrizations
- Q2: Model assumptions for the $\rho$ parameters
- Q3: Simplified assumptions for long-range corrections $\to$ this talk
- Q4: Simplified assumptions for RG corrections and matching

# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

## General considerations

For $\tau^-(l_1) \to \pi^-(q_1)\pi^0(q_2)\nu_\tau(l_2)$:

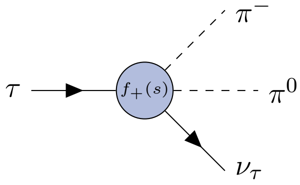
$$\begin{aligned} i\mathcal{M} = & - iG_F V_{ud}^* \bar{u}(l_2,\nu_\tau)\gamma^\mu(1-\gamma_5)u(l_1,\tau) \\ & \times \left[(q_1-q_2)_\mu f_+(s,t) + (q_1+q_2)_\mu f_-(s,t)\right] \end{aligned}$$

where $f_+ \to J^P = 1^-$ is the weak current component, $f_- \to J^P = 0^+$ and

$$\begin{aligned} s &= (l_1-l_2)^2 = (q_1+q_2)^2 \ , \\ t &= (l_1-q_1)^2 = (q_2+l_2)^2 \ . \end{aligned}$$

At tree level:

$f_+^{\text{tree}}(s,t) = f_+(s)$ , $\quad f_-^{\text{tree}}(s,t) = 0.$



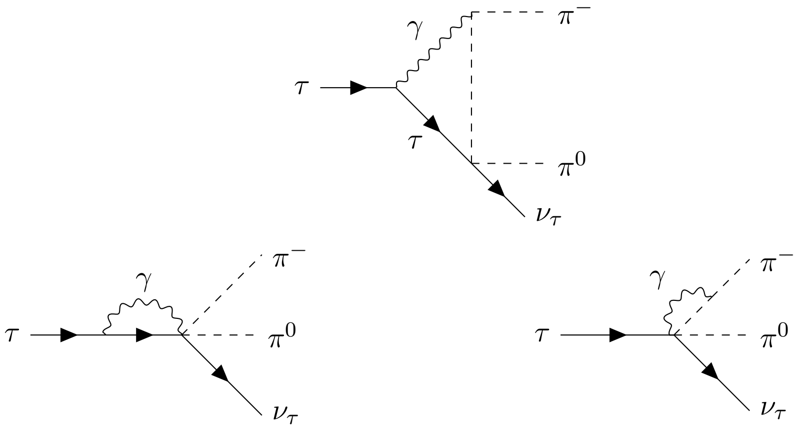
# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

Previous work: $\mathcal{O}\left(p^4\right)$ and $\mathcal{O}(e^2p^2)$ corrections

- [Cirigliano et al, '01 & '02]: $\mathcal{O}(e^2p^2)$ corrections in $\chi$PT + FF

1. $\chi$PT diagrams:

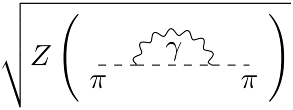
One–loop radiative corrections



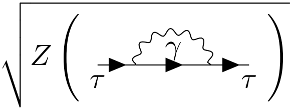
# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

Previous work: $\mathcal{O}\left(p^4\right)$ and $\mathcal{O}(e^2p^2)$ corrections

External leg corrections:

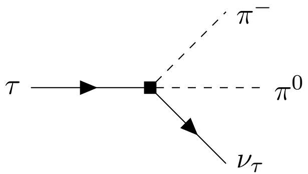



$\times$ tree level




$\times$ tree level

Counterterms:




2. Form factor:

$$f_+(s,t) = f_+(s)\left[1 + f_{\text{loop}}^{\text{elm}}(t, M_\gamma)\right]$$

where

$$f_+(s) = \frac{M_\rho^2}{M_\rho^2 - s - iM_\rho\Gamma_\rho(s)}\exp\left[2\tilde{H}_{\pi^0\pi^-}(s) + \tilde{H}_{K^0K^-}(s)\right] + f_{\text{local}}^{\text{elm}}$$

# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

Previous work: $\mathcal{O}\left(p^4\right)$ and $\mathcal{O}(e^2p^2)$ corrections

3. Short-distance corrections:

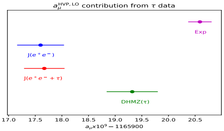
$$e^2 X_6^{\mathsf{SD}} = 1 - S_{\mathsf{EW}} - \frac{e^2}{4\pi^2}\log\frac{m_\tau^2}{M_\rho^2}$$

5. Bremsstrahlung: full photon-energy spectrum
6. Result:

$$\Delta a_\mu^{\mathsf{VP}} = (-120 \pm 26 \pm 3) \times 10^{-11}$$

$\to$ $\tau$-decay data-driven approach abandoned because isospin-breaking corrections model-dependent $\to$ disagreement with $e^+e^-$: reliability?

- [Davier et al, '10]: reconsidered the $\tau^- \to \pi^-\pi^0\nu_\tau$
  - meson dominance model + photon corrections according to [Flores-Tlalpa et al, '06]: $a_\mu^{\mathsf{HVP,LO}} = (7053 \pm 45) \times 10^{-11}$



# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

## LECs

$$f^{\text{elm}}_{\text{local}} \supset e^2 \left( 2K^r_{12}(\mu) - \frac{2}{3}X_1 - \frac{1}{2}X^r_6(\mu) \right)$$

Defining

$$X_6^{\text{phys}}(\mu) \equiv X_6^r(\mu) - 4K_{12}^r(\mu) \Rightarrow X_6^{\text{phys}}(\mu) \equiv X_6^{\text{SD}} + \bar{X}_6^r(\mu)$$

- [Cirigliano et al, '01 & '02]:

$$K^r_{12}(M_\rho) = -(3 \pm 1) \times 10^{-3}, \; |X_1| \leq \frac{1}{(4\pi)^2}, \; |\bar{X}^r_6(M_\rho)| \leq \frac{5}{(4\pi)^2}$$

- Lattice QCD [Peng-Xiang et al, '21]: exactly needed LECs due to LFU

$$\frac{4}{3}X_1 + \bar{X}^r_6(M_\rho) = -\frac{1}{2\pi\alpha}\left( \Box^{V\,A}_{\gamma W}\Big|_\pi - \frac{\alpha}{8\pi}\log\frac{M_W^2}{M_\rho^2} \right) + \frac{1}{8\pi^2}\left( \frac{5}{4} - \tilde{a}_g \right)$$

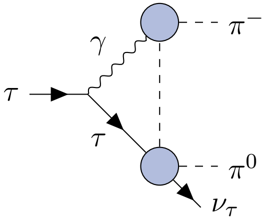
  - $\Box^{V\,A}_{\gamma W}\Big|_\pi$: box contribution for $\pi_{\ell 3}$ decay [Feng et al, '20 & Yoo et al, '23]
  - $\tilde{a}_g$: $\mathcal{O}(\alpha_s)$ QCD correction
  - $\alpha_s$ corrections for $\bar{X}^r_6(M_\rho)$ vs $\tilde{X}^r_6(M_\rho)$ (Q4)

# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

## Model-independent dispersive approach

Approximation: up to $2\pi$ as hadronic intermediate state

Under control: $\mathcal{O}(e^2p^2)$ corrections




$$f_+(s) \to \frac{1}{\pi}\int_{s_{th}}^{\infty} \mathrm{d}s' \frac{\mathrm{Im} f_+(s')}{s'-s}, \qquad f_+(0) = \frac{1}{\pi}\int_{s_{th}}^{\infty} \mathrm{d}s' \frac{\mathrm{Im} f_+(s')}{s'} = 1$$

$\to$ UV-finite but IR-divergent

# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

## $\pi$VFF

$$F_\pi^V(s) = G_{in}^N(s)\Omega(s)$$

where $\Omega(s)$ is the Omnès function [Schneider et al, '12]

$$\Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta(s')}{s'\,(s'-s)}\right\}$$

and $G_{in}^N(s)$ is a conformal polynomial taking into account inelastic channels [Colangelo et al, '19]

$$G_{\text{in}}^N(s) = 1 + \sum_{k=1}^{N} c_k \left(z^k(s) - z^k(0)\right)$$

# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

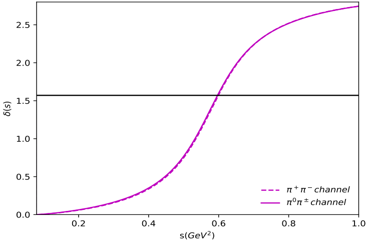
## Omnès function $\Omega(s)$

- $e^+e^- \to \pi^+\pi^-$:
  - $\rho^0$ resonance dominance
  - $\rho - \omega$ mixing

$\rightarrow$P-wave phase in the $\pi^+\pi^-$ channel

- $\tau^\pm \to \pi^\pm\pi^0\nu_\tau$:
  - $\rho^\pm$ resonance dominance
  - $\rho', \rho''$ contribution

$\rightarrow$P-wave phase in the $\pi^0\pi^\pm$ channel




$\rightarrow$ see Ruiz de Elvira's talk at KEK (Q2)

$\Rightarrow \tilde{M}_{\rho^\pm} - \tilde{M}_{\rho^0} = -1.4\,\mathrm{MeV}$ due to $\delta_\pi = M_{\pi^\pm} - M_{\pi^0}$ from $\mathrm{Re}\left[t_\ell^I(s)\right] = 0$

[Cirigliano et al, '01 & '02]: $M_{\rho^\pm} - M_{\rho^0} = (0 \pm 1)\,\mathrm{MeV}$

# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

## Numerical treatment of IR-divergent $D_0(m_\tau^2, M_\pi^2, M_\pi^2, 0, t, s, m_\tau^2, 0, M_\pi^2, s'')$

Singularity at $s'' = s$:

$$D_0(s,t) = \frac{1}{s''-s}\left[2d_0(t)\log\frac{s''}{s''-s} + D_0^{\text{rest}}(t,s'')\right]$$

$$f_+(s,t) \supset \int_{4M_\pi^2}^{\infty} \mathrm{d}s''\ \mathrm{Im}\, f_+(s'')\left(\frac{p_1(s,t)+p_2(s,t)s''}{s''-s}\right)\times \left[2d_0(t)\log\frac{s''}{s''-s} + D_0^{\text{rest}}(t,s'')\right]\ ,$$

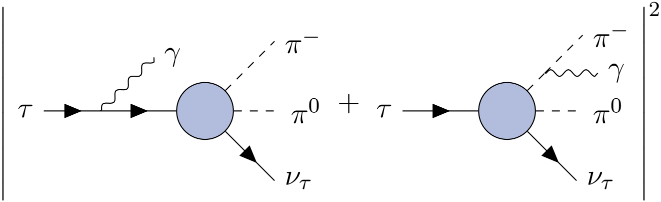
$$I_{\ell 1}(s,\Lambda^2) = 2\int_{4M_\pi^2}^{\Lambda^2}\mathrm{d}s''\log\frac{s''}{s''-s},\quad I_{\ell 2}(s) = 2\int_{4M_\pi^2}^{\infty}\mathrm{d}s''\frac{1}{s''-s-i\epsilon}\log\frac{s''}{s''-s-i\epsilon}$$

- IR-divergences in dim-reg
- application: $e^+e^- \to \pi^+\pi^-$ asymmetry
  - simplified imaginary part in [Colangelo et al, '22], no scheme ambiguity
  - see new results in [Budassi et al, '24]

# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

## Real emissions: full photon-energy spectrum

$|\mathcal{M}_{\mathrm{ReEm}}|^2 =$




From [Cirigliano et al, '01 & '02]:

$$\mathcal{M}_{\mathrm{ReEm}} = eG_F V_{ud}^* \epsilon^\mu(k)^* \Big[ F_\nu \bar{u}(l_2)\gamma_\nu (1-\gamma_5)\left(m_\tau + \not{l}_1 - \not{k}\right)\gamma_\mu u(l_1) + (V_{\mu\nu} - A_{\mu\nu})\,\bar{u}(l_2)\gamma^\nu (1-\gamma_5)\, u(l_1)\Big]$$

- bremsstrahlung off initial $\tau$: $F_\nu = \frac{(q_2-q_1)_\nu f_+(s)}{2l_1\cdot k}$
- vector and axial–vector components of $W^-(l_1-l_2) \to \pi^-(q_1)\pi^0(q_2)\gamma(k)$

# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

## Real emissions: full photon-energy spectrum

- Leading Low approximation, for $(s,t)$ in the non-radiative Daliz plot

$$\frac{d\Gamma_{\pi\pi\gamma}}{ds\ dt} = \frac{G_F^2 S_{EW}\left|V_{ud}\right|^2}{64\pi^3 m_\tau^3}|f_+(s)|^2 D(s,t) g_{\text{brems}}(s,t)$$

where $g_{\text{brems}}(s,t) = \frac{\alpha}{\pi}\left[J_{11}(s,t) + J_{20}(s,t) + J_{02}(s,t)\right]$

$$J_{mn}(s,t) = \frac{c_{mn}}{2\pi}\int_{M_\gamma^2}^{x_+(s,t)} dx \int \frac{d^3 l_2}{2l_2^0}\frac{d^3k}{2k^0}\frac{\delta^{(4)}(l_1 - l_2 - q_1 - q_2 - k)}{\left(l_1\cdot k - \frac{M_\gamma^2}{2}\right)^m\left(q_1\cdot k + \frac{M_\gamma^2}{2}\right)^n}$$

In dim-reg: $\log M_\gamma^2 \to -\frac{2}{\epsilon_{\text{IR}}} + \gamma_E - \log 4\pi + \log\mu^2$ $\to$ IR-finite

- Infrared finite remainder of the rate $\to g_{\text{rest}}(s,t)$: numerically calculated

# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

## Matching with $\chi$PT

* $\chi$PT $\leftrightarrow$ low-energy theorems
* Triangle with form factors: sensitive to high-energy behaviour of $f_+(s)$

$\to$ matching procedure: expansion of $f_+(s,t)$ around $s, t = 0$

$$f_+^{\text{match}}(s,t) = f_+^{\text{VFF}}(s,t) - f_+^{\text{VFF}}(0,0) + f_+^{\chi\text{PT}}(0,0)$$

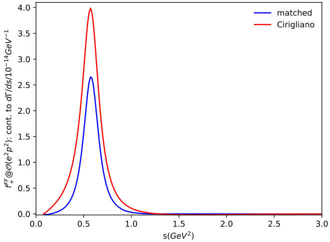
- UV divergences: $f_+^{\text{VFF}}(s,t)$ and $(f_+^{\chi\text{PT}}(s,t) + \text{LECs}) \to$ UV-finite
- IR divergences:
  - $f_+^{\text{VFF}}(s,t) + \frac{\alpha}{\pi} J_{11}(s,t) \to$ IR-finite
  - $-f_+^{\text{VFF}}(0,0) + f_+^{\chi\text{PT}}(0,0) + \frac{\alpha}{\pi}\left[J_{20}(s,t) + J_{02}(s,t)\right] \to$ IR-finite
- Correct chiral-log restored from $f_+^{\chi\text{PT}}(0,0)$
- Narrow resonance limit: $f_+^{\text{VFF}}(0,0) \to f_+^{\chi\text{PT}}(0,0)$

# Isospin-breaking corrections to $\tau^- \to \pi^- \pi^0 \nu_\tau$

## Matching with $\chi$PT

Interference with the TL:

$$\frac{d\Gamma}{ds} = \int dt \, |\mathcal{M}_{\mathsf{TL}}(s,t)|^2 \, f_+(s,t)$$



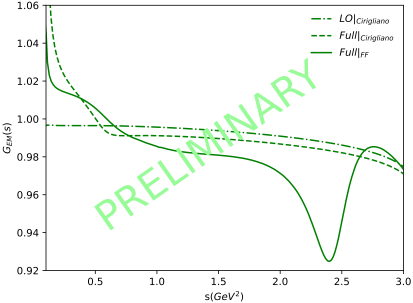

$$f_+(s,t) = f_+^{\mathrm{match}}(s,t) + \frac{e^2}{2\pi^2}\left(\frac{1}{\epsilon_{\mathsf{IR}}} - \log\mu^2\right) \times \left[-1 + \frac{m_\tau^2 + M_\pi^2 - t}{2\sqrt{\lambda(t, m_\tau^2, M_\pi^2)}} \log\left(\frac{m_\tau^2 + M_\pi^2 - t + \sqrt{\lambda(t, m_\tau^2, M_\pi^2)}}{2 m_\tau M_\pi}\right)\right]$$

$$|\mathcal{M}_{\mathsf{TL}}(s,t)|^2 = \frac{2G_F^2 |V_{ud}|^2}{(2\pi)^3 32 m_\tau^3} |f_+(s)|^2 \left[m_\tau^2 \left(t + u - 2M_\pi^2\right) + 4M_\pi^2 - 4tu\right]$$

# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

## $G_{\text{EM}}(s)$

$$G_{\text{EM}}(s) = \frac{\int_{t_{min}(s)}^{t_{max}(s)} dt D(s,t)\Delta(s,t)}{\int_{t_{min}(s)}^{t_{max}(s)} dt D(s,t)}$$




where

$$D(s,t) = \frac{m_\tau^2}{2}\left(m_\tau^2 - s\right) + 2M_\pi^2 - 2t\left(m_\tau^2 - s + 2M_\pi^2\right) + 2t^2$$

$$\Delta(s,t) = 1 + 2f_+^{e^2p^2}(s,t) + g_{\text{brems}}(s,t) + g_{\text{rest}}(s,t)$$

# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

## Results

From [Davier et al, '10]:

$$\Delta^{\mathrm{IB}} a_\mu^{\mathrm{LO,had}}[\pi\pi,\tau] = \frac{\alpha^2 m_\tau^2}{6|V_{ud}|^2\pi^2} \frac{\mathcal{B}_{\pi\pi^0}}{\mathcal{B}_e} \int_{4M_\pi^2}^{m_\tau^2} ds \frac{K(s)}{s} \frac{dN_X}{N_X ds} \times \left(1-\frac{s}{m_\tau^2}\right)^{-2} \left(1+\frac{2s}{m_\tau^2}\right)^{-1} \left[\frac{R_{\mathrm{IB}}(s)}{S_{\mathrm{EW}}} - 1\right]$$

Only $G_{\mathrm{EM}}(s)$ contribution $\Rightarrow \frac{R_{\mathrm{IB}}(s)}{S_{\mathrm{EW}}} = \frac{1}{G_{\mathrm{EM}}(s)}$:

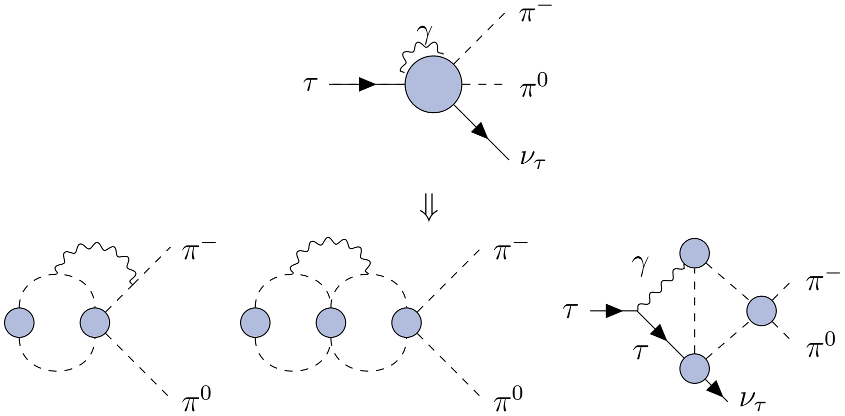
$$\Delta^{\mathrm{IB}} a_\mu^{\mathrm{LO,had}}[\pi\pi,\tau]\Big|_{G_{\mathrm{EM}}(s)} = \text{t.b.d. for} \begin{cases} X_\ell = 14\times 10^{-3} \\ X_\ell = 11\times 10^{-3} \end{cases}$$

- [Davier et al, '10]: $\Delta^{\mathrm{IB}} a_\mu^{\mathrm{LO,had}}[\pi\pi,\tau]\Big|_{G_{\mathrm{EM}}(s)} = (-1.92 \pm 0.90)\times 10^{-10}$
- [Castro et al, '24]: $\Delta^{\mathrm{IB}} a_\mu^{\mathrm{LO,had}}[\pi\pi,\tau]\Big|_{G_{\mathrm{EM}}(s)} = \left(-1.71^{+0.61}_{-1.48}\right)\times 10^{-10}$

# Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

Work in progress

## Higher order corrections




$\rightarrow$ see Monnard PhD thesis and Ruiz de Elvira's talk at KEK

# Conclusions & Outlooks

- model–independent approach for the $\mathcal{O}(e^2p^2)$ isospin-breaking corrections to the $\tau$–decay
- pion vector form factors: $F_\pi^V(s)$ vs $f_+(s)$
  - included: $\rho'$ and $\rho''$
- matching with $\chi$PT: correct low energy behavior
  - decrease w.r.t. [Cirigliano et al., '01 & '02]
- $G_{\text{EM}}(s)$ contribution to $a_\mu$: full photon-energy spectrum
- Outlooks:
  - fully consistent matching between $G_{\mathrm{EM}}(s)$, $S_{\mathrm{EW}}$, $\square_{\gamma W}|_\pi$ including RG corrections (Q4) $\rightarrow$ see [Cirigliano et al, '23] for neutron decay
  - $\rho$ parameters and isospin breaking corrections in form factor [Colangelo et al, work in progress] (Q1, Q2)

# Numerical treatment of IR-divergent $D_0$

$$\begin{aligned}
f_+(s,t) \supset \int_{4M_\pi^2}^{\infty} \mathrm{d}s'' \, \left\{ \operatorname{Im} F_\pi^V(s'') - \operatorname{Im} F_\pi^V(s) \right\} \left( \frac{p_1(s,t) + p_2(s,t)s}{s''-s} + p_2(s,t) \right) \\
\times \left[ 2d_0(t) \log \frac{s''}{s''-s} + D_0^{\mathsf{rest}}(t,s'') \right] \\
+ \operatorname{Im} F_\pi^V(s) \Bigg\{ \big(p_1(s,t) + p_2(s,t)s\big) \int_{4M_\pi^2}^{\infty} \mathrm{d}s'' \, \frac{D_0^{\mathsf{rest}}(t,s'')}{s''-s} \\
+ p_2(s,t) \int_{4M_\pi^2}^{\infty} \mathrm{d}s'' \, D_0^{\mathsf{rest}}(t,s'') \\
+ d_0(t) \Big[ p_2(s,t) I_{\ell 1}(s,\Lambda^2) + \big(p_1(s,t) + p_2(s,t)s\big) I_{\ell 2}(s) \Big] \Bigg\},
\end{aligned}$$

# Leading Low approximation

Taking into account $(l_1 - l_2)^2 = s + 2(q_1 + q_2) \cdot k$, Low's theorem is manifestly satisfied [Cirigliano et al, '02]:

$$\begin{aligned} V_{\mu\nu} =& f_+(s) \frac{q_{1\mu}}{q_1 \cdot k} \left(q_1 - q_2\right)_\nu \\ &+ f_+(s) \left( \frac{q_{1\mu} k_\nu}{q_1 \cdot k} - g_{\mu\nu} \right) \\ &+ 2 \frac{df_+(s)}{ds} \left( \frac{q_{1\mu} q_2 \cdot k}{q_1 \cdot k} - q_{2\mu} \right) \left(q_1 - q_2\right)_\nu + \mathcal{O}(k) \end{aligned}$$

# Endpoint singularity in $e^+e^- \to \pi^+\pi^-$ asymmetry

Result for the imaginary part including
$D_0(m_e^2, m_e^2, M_\pi^2, M_\pi^2, s, t, 0, m_e^2, s'', M_\pi^2)$ in the $e^+e^- \to \pi^+\pi^-$ asymmetry

$$\begin{aligned}
\delta_\lambda(s) =& 2\Bigg\{ \log\left(\frac{1-z\beta}{1+z\beta}\right)\left[\log\frac{4\lambda^2}{s} + 2\log\left(\frac{1-\beta^2}{\beta^2}\right)\right] + \log^2(1+z\beta) \\
&+ \frac{\log\left(1-\beta^2\right)}{\left(1-z^2\right)\beta^2}\left[2z\beta\log\left(\frac{1-z^2\beta^2}{1-\beta^2}\right) + z\left(1+\beta^2\right)\log\left(\frac{1-\beta}{1+\beta}\right)\right. \\
&\left. - \left(1+z^2\beta^2\right)\log\left(\frac{1-z\beta}{1+z\beta}\right)\right] - \log^2(1-z\beta) - \mathrm{Li}_2\left(\frac{(z-1)\beta}{1-\beta}\right) \\
&- \mathrm{Li}_2\left(\frac{(1+z)\beta}{1+\beta}\right) + \mathrm{Li}_2\left(\frac{(1+z)\beta}{\beta-1}\right) + \mathrm{Li}_2\left(\frac{(1-z)\beta}{1+\beta}\right)\Bigg\}
\end{aligned}$$

# Endpoint singularities in the phase space

$$f_+^{\mathsf{box},F_\pi^V}(s,t) = f_+^{fin}(s,t) + \frac{N(s,t)}{s(t-t_{\mathsf{min}})(t-t_{\mathsf{max}})}$$

$\rightarrow$ endpoint singularity in the $t$ phase space integral BUT numerically showed that the two infinities cancel $\rightarrow$ finite result.
Analytically:

$$\begin{aligned} N(s,t) &= (t-t_{\mathsf{max}})N_+(s,t) \\ &= (t-t_{\mathsf{max}})(N_+(s,t) - N_+(s,t_{\mathsf{min}}) \\ &= (t-t_{\mathsf{max}})(t-t_{\mathsf{min}})\bar{N}(s,t)\ . \end{aligned}$$

$\rightarrow$ expand $\bar{N}(s,t)$ around $t = t_{\mathsf{max/min}}$ when the integration in $t$ is close to the boundaries $t_{\mathsf{max/min}}$.

# $\rho - \gamma$ mixing

- Vector meson dominance approach [Jegerlehner and Szafron, '11]:

$$\hat{D}^{-1} = \begin{pmatrix} q^2 + \Pi_{\gamma\gamma}(q^2) & \Pi_{\gamma\rho}(q^2) \\ \Pi_{\gamma\rho}(q^2) & q^2 - M_\rho^2 + \Pi_{\rho\rho}(q^2) \end{pmatrix} \Rightarrow D_{\gamma\gamma},\ D_{\gamma\rho},\ D_{\rho\rho}$$

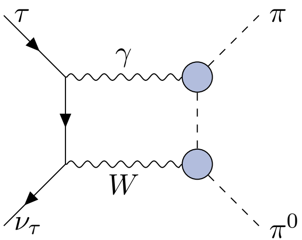
$$F_\pi(s) = \frac{e^2 D_{\gamma\gamma} + e\,(g_{\rho\pi\pi} - g_{\rho ee})\, D_{\gamma\rho} - g_{\rho ee} g_{\rho\pi\pi} D_{\rho\rho}}{e^2 D_{\gamma\gamma}}$$

- Dispersive approach:

$$\frac{F_\pi(s)}{s} = \frac{1}{s} + \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\operatorname{Im} F_\pi(s')}{s'(s'-s)}$$

$\rightarrow$ $\gamma$ and $\rho$ poles with the right masses by construction

# Seagull diagram




Analogy with $e^+e^- \to \pi^+\pi^-$: