

**All  
you  
need  
is  
logs**

**Uli Haisch, MPI Munich, ZPW2025, 7.1.25**



# During the years, the ZPW ...

has developed into a forum to discuss in an informal setting the latest developments on different aspects of particle physics phenomenology. In 2025 we will have a special focus on present and future prospects in precision physics, and a special session dedicated to the evolution of effective field theory methods from low to high energies, in occasion of the 75th birthday of Daniel Wyler.

This talk explores the highlighted aspects & their link to Daniel's & my research

# Large logarithms in flavor physics

$$Q_2 = (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L),$$

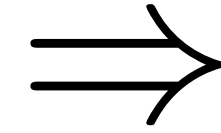
current-current  
operator

$$Q_7 = \frac{e}{16\pi^2} m_b \bar{s}_L \sigma_{\mu\nu} b_R F^{\mu\nu}$$

electromagnetic  
dipole operator

# Large logarithms in flavor physics

$$\frac{dC_7(\mu)}{d \ln \mu} = \frac{\gamma_{27}}{16\pi^2} C_2(\mu)$$



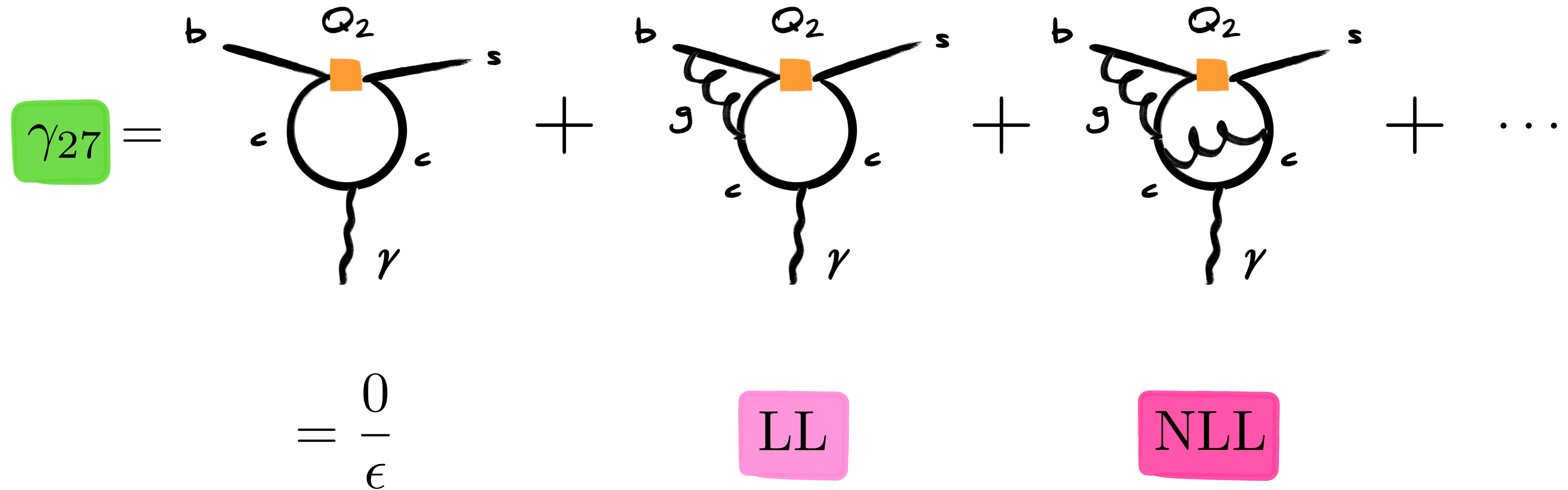
$$\frac{C_7(m_b)}{C_2(M_W)} \simeq -\frac{\gamma_{27}}{16\pi^2} \ln \frac{M_W}{m_b}$$

anomalous dimension  
encoding mixing  $Q_2 \rightarrow Q_7$

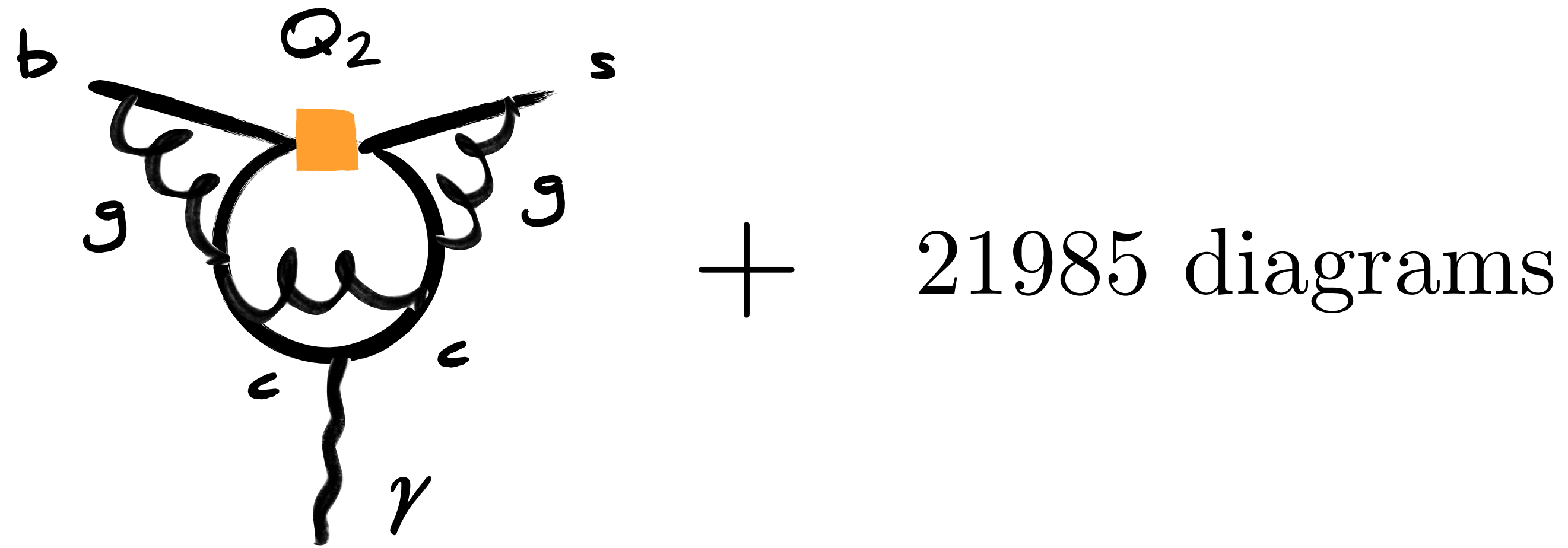
leading logarithm (LL)  
resummed by RGE



# Large logarithms in flavor physics



# NNLL endeavor in 2006

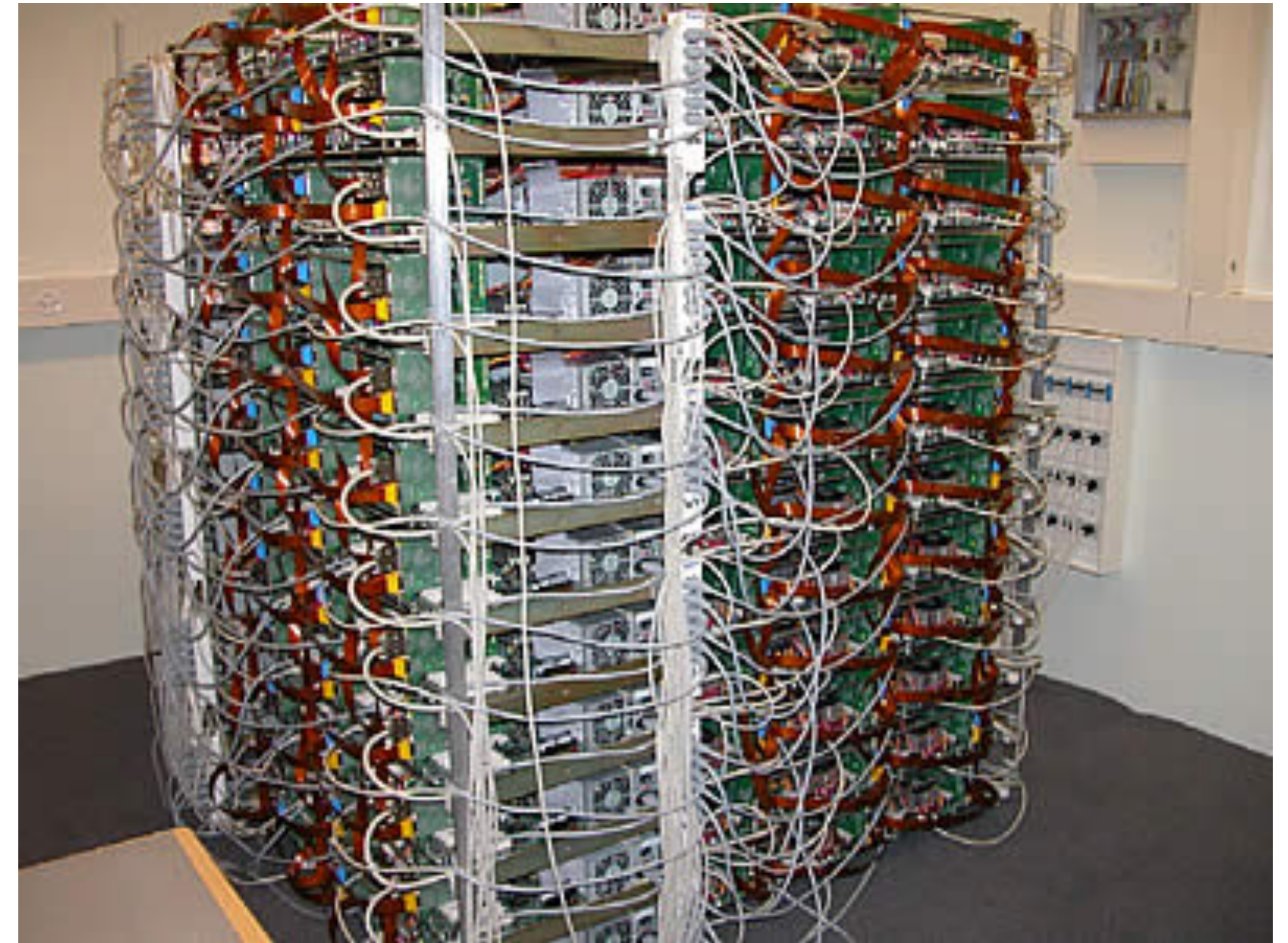


Calculation of 4-loop anomalous dimensions proved to be a challenge due to large number of diagrams, emergence of numerous “unphysical” operators as ultraviolet subdivergences caused by infrared rearrangement, ...

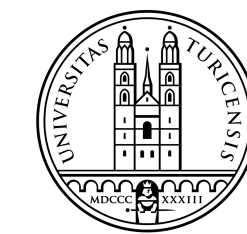


# NNLL endeavor in 2006

NNLL calculation would not have been possible without (mis)use of zBox1 supercomputer, built in 2002 for cosmological N-body simulations by astrophysics group of University of Zurich



zBox1



Universität  
Zürich<sup>UZH</sup>

[Daniel provided funding for zBox4, the great-granddaughter of zBox1]



# NNLL endeavor in 2006

$$\begin{aligned}
 Z_{27} = & \frac{\tilde{\alpha}^2}{\epsilon} \left( \frac{8}{81} + 2Q_u \right) + \frac{\tilde{\alpha}^3}{\epsilon} \left\{ \frac{1}{\epsilon} \left[ -\frac{4636}{2187} + \frac{80}{729} f - \left( \frac{206}{9} - \frac{16}{9} f \right) Q_u \right] + \frac{6698}{2187} + \frac{64}{2187} f + \left( \frac{128}{27} \right. \right. \\
 & \left. \left. - \frac{2}{27} f \right) Q_u \right\} + \frac{\tilde{\alpha}^4}{\epsilon} \left\{ \frac{1}{\epsilon^2} \left[ \frac{560390}{19683} - \frac{7400}{2187} f + \frac{208}{2187} f^2 + \left( \frac{4379}{18} - \frac{323}{9} f + \frac{4}{3} f^2 \right) Q_u + \frac{5}{18} \overline{Q} \right] \right. \\
 & \left. + \frac{1}{\epsilon} \left[ -\frac{735973}{13122} + \frac{173111}{39366} f + \frac{200}{6561} f^2 - \left( \frac{39427}{216} - \frac{613}{27} f + \frac{2}{27} f^2 \right) Q_u + \frac{245}{162} \overline{Q} \right] + \frac{3142663}{1417176} + \frac{5068}{2187} \zeta_3 \right. \\
 & \left. - \left( \frac{79754}{59049} + \frac{146}{243} \zeta_3 \right) f - \frac{140}{6561} f^2 + \left[ \frac{11699}{36} - \frac{6140}{27} \zeta_3 - \left( \frac{17153}{972} + \frac{128}{27} \zeta_3 \right) f - \frac{79}{243} f^2 \right] Q_u \right. \\
 & \left. + \left( -\frac{3695}{648} + \frac{25}{9} \zeta_3 \right) \overline{Q} \right\},
 \end{aligned}$$

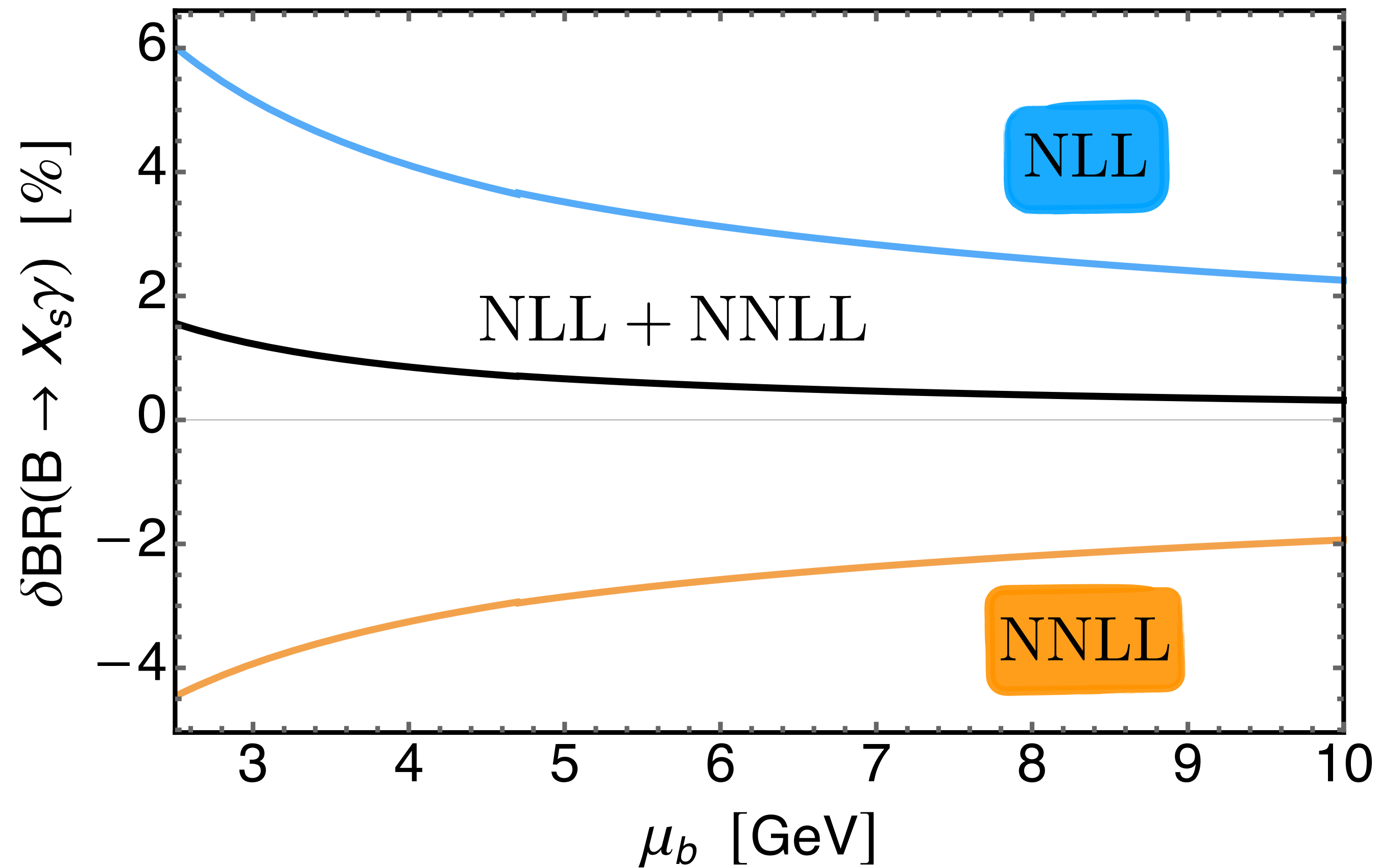
LL

NLL

NNLL



# NNLL endeavor in 2006



Large 4-loop anomalous dimensions make NNLL & NLL comparable in size. Cancellation between NLL & NNLL contributions results in total RGE effects of  $O(1\%)$

# NNLL endeavor was important part of

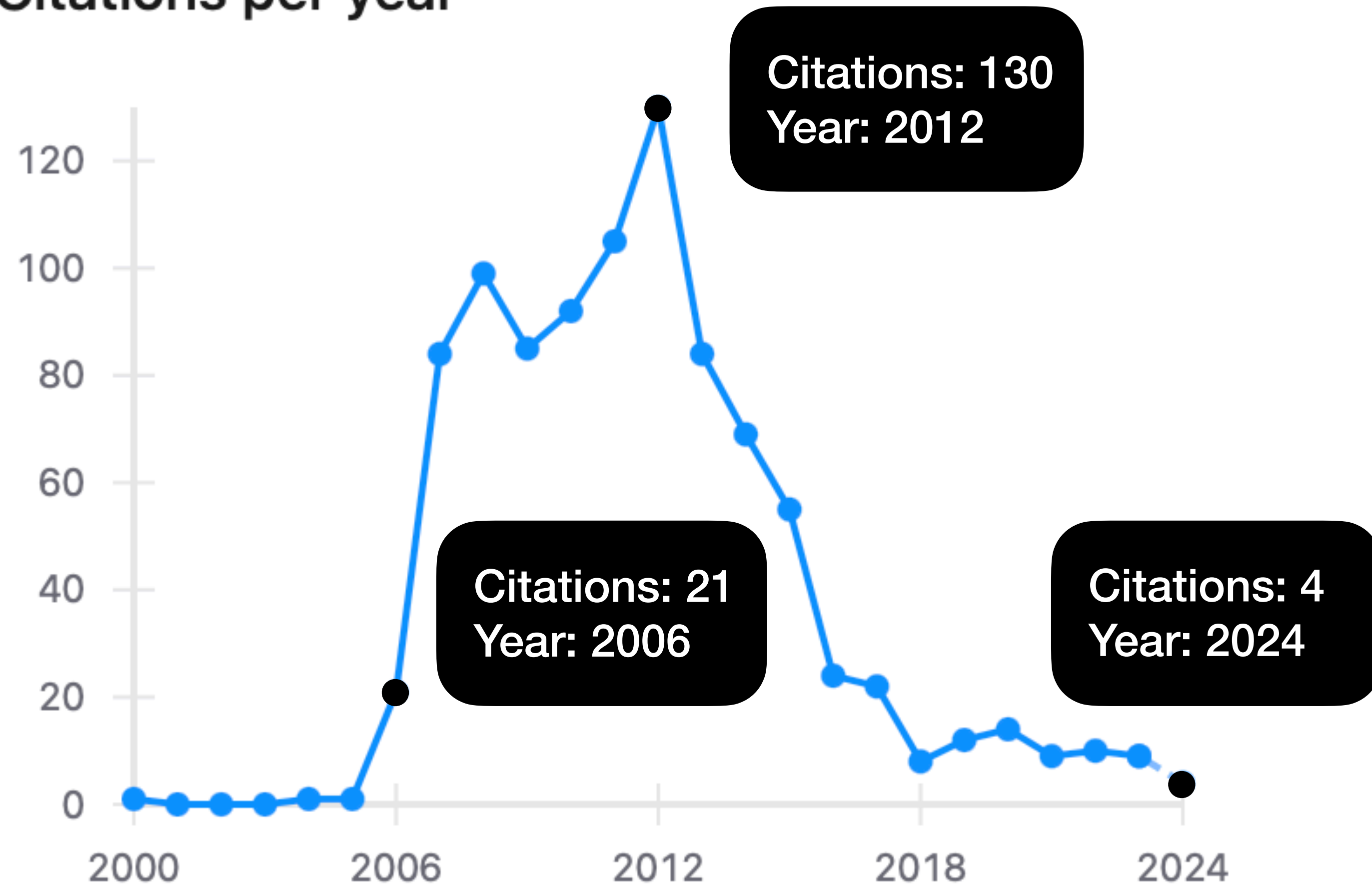
## Estimate of $\mathcal{B}(\bar{B} \rightarrow X_s \gamma)$ at $\mathcal{O}(\alpha_s^2)$

M. Misiak,<sup>1,2</sup> H. M. Asatrian,<sup>3</sup> K. Bieri,<sup>4</sup> M. Czakon,<sup>5</sup> A. Czarnecki,<sup>6</sup> T. Ewerth,<sup>4</sup>  
A. Ferroglia,<sup>7</sup> P. Gambino,<sup>8</sup> M. Gorbahn,<sup>9</sup> C. Greub,<sup>4</sup> U. Haisch,<sup>10</sup> A. Hovhannisyan,<sup>3</sup>  
T. Hurth,<sup>2,11</sup> A. Mitov,<sup>12</sup> V. Poghosyan,<sup>3</sup> M. Ślusarczyk,<sup>6</sup> and M. Steinhauser<sup>9</sup>

Combining our results for various  $\mathcal{O}(\alpha_s^2)$  corrections to the weak radiative  $B$ -meson decay, we are able to present the first estimate of the branching ratio at the next-to-next-to-leading order in QCD. We find  $\mathcal{B}(\bar{B} \rightarrow X_s \gamma) = (3.15 \pm 0.23) \times 10^{-4}$  for  $E_\gamma > 1.6$  GeV in the  $\bar{B}$ -meson rest frame. The four types of uncertainties: nonperturbative (5%), parametric (3%), higher-order (3%) and  $m_c$ -interpolation ambiguity (3%) have been added in quadrature to obtain the total error.

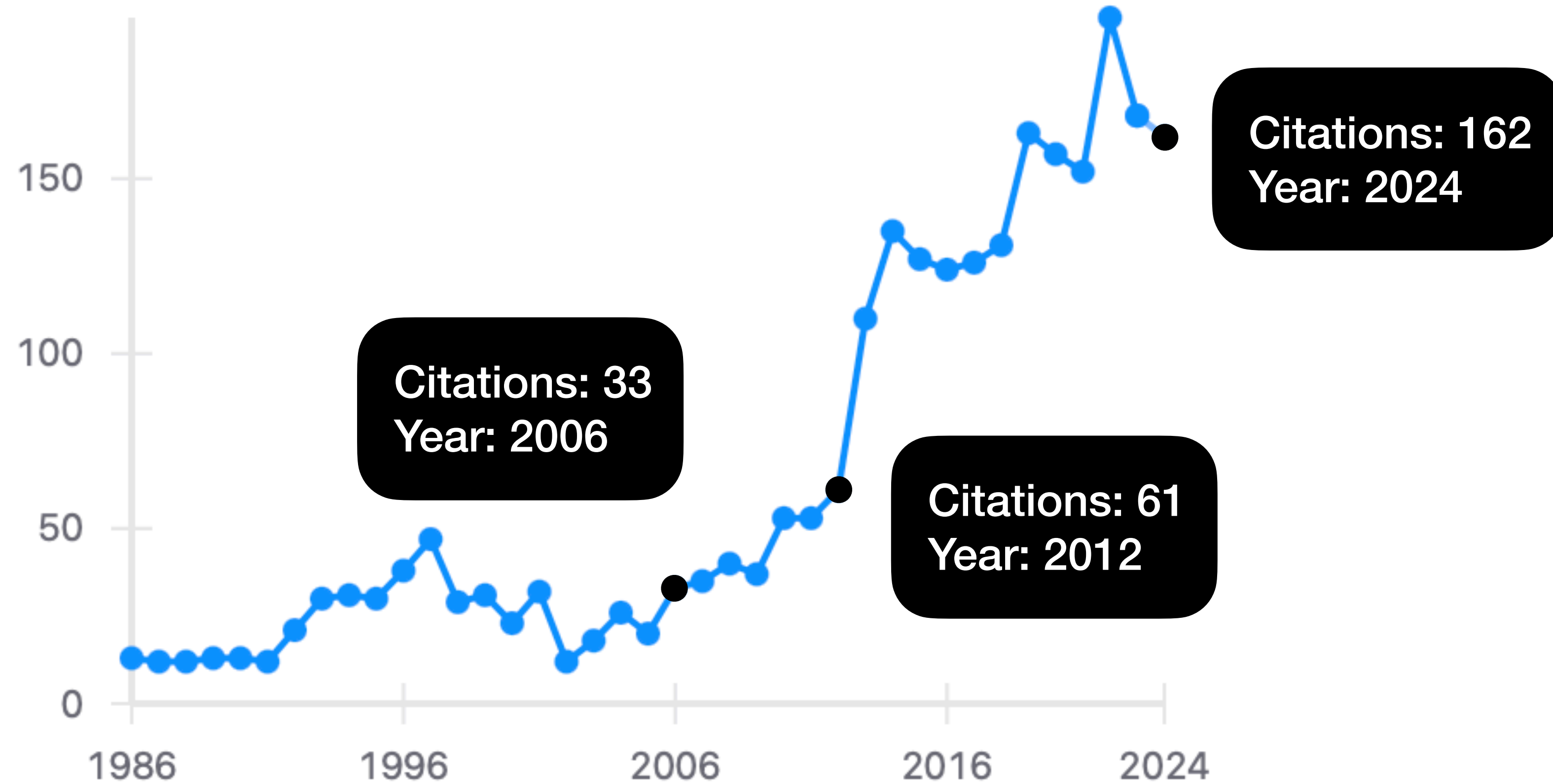
# But physics landscape has changed

Citations per year



# Today more popular to work on

Citations per year





# Who is this sleeping beauty?

## EFFECTIVE LAGRANGIAN ANALYSIS OF NEW INTERACTIONS

### AND FLAVOUR CONSERVATION

#### A B S T R A C T

New interactions with a scale  $\Lambda$  larger than the Fermi scale  $G_F^{-\frac{1}{2}}$  will manifest themselves at energies below  $\Lambda$  through small deviations from the standard model, which can be described by an effective Lagrangian containing non-renormalizable  $SU(3)\times SU(2)\times U(1)$  invariant operators. We construct the first two terms of this Lagrangian in an expansion in powers of  $1/\Lambda$  and study systematically possible effects of new interactions such as anomalous magnetic moments, deviations from universality in weak interactions and rare processes. Among the flavour conserving processes the universality of the charged current weak interactions yields the strongest bound on the new interaction scale,  $\Lambda > 5 \text{ TeV}$ .

# Now with 40, she is called

## The Standard Model effective field theory at work

The striking success of the Standard Model in explaining precision data and, at the same time, its lack of explanations for various fundamental phenomena, such as dark matter or the baryon asymmetry of the universe, suggests new physics at an energy scale much larger than the electroweak scale. In the absence of a short-range–long-range conspiracy, the Standard Model can be viewed as the leading term of an effective ‘remnant’ theory (referred to as the SMEFT) of a more fundamental structure. Over the last years, many aspects of the SMEFT have been investigated and it has become a standard tool to analyze experimental results in an integral way. In this article, after briefly presenting the salient features of the Standard Model, we review the construction of the SMEFT. We discuss the range of its applicability and bounds on its coefficients imposed by general theoretical considerations. Since new-physics models are likely to exhibit exact or approximate accidental global symmetries, especially in the flavor sector, we also discuss their implications for the SMEFT. The main focus of our review is the phenomenological analysis of experimental results. We show explicitly how to use various effective field theories to study the phenomenology of theories beyond the Standard Model. We give a detailed description of the matching procedure and the use of the renormalization group

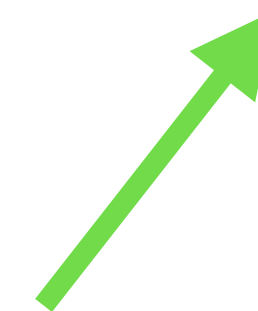
# LL in SMEFT: a simple example

$$Q_{Hq,33}^{(1)} = (H^\dagger i \overleftrightarrow{D}_\mu H) (\bar{q}_3 \gamma^\mu q_3), \quad Q_{Hq,33}^{(3)} = (H^\dagger i \overleftrightarrow{D}_\mu^i H) (\bar{q}_3 \gamma^\mu \sigma^i q_3)$$

$$\mathcal{L}_{\text{LEFT}} \supset -\frac{e}{2s_w c_w} \left( C_{Hq,33}^{(1)} + C_{Hq,33}^{(3)} \right) \sum_{i,j} V_{ti}^* V_{tj} \bar{d}_{L,i} \gamma_\mu d_{L,j} Z^\mu$$

# LL in SMEFT: a simple example

$$C_{Hq,33}^{(1)}(\Lambda) + C_{Hq,33}^{(3)}(\Lambda) = 0$$



absence of tree-level modifications of  $d_j \bar{d}_i Z$  couplings at scale  $\Lambda$



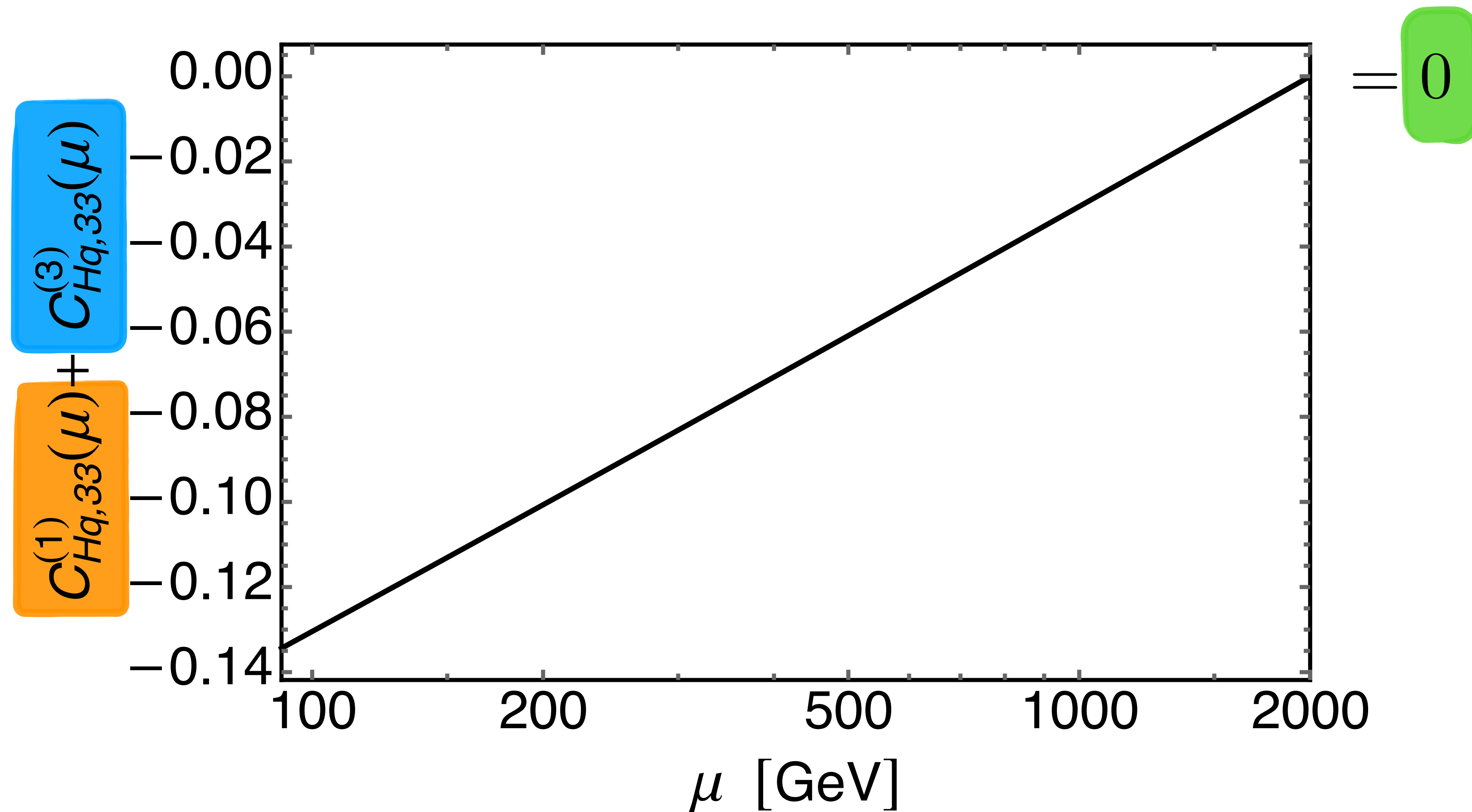
# LL in SMEFT: a simple example

$$\frac{dC_{Hq,33}^{(1)}}{d \ln \mu} = \frac{y_t^2}{16\pi^2} \left( 10C_{Hq,33}^{(1)} - 9C_{Hq,33}^{(3)} \right) + \dots$$

$$\frac{dC_{Hq,33}^{(3)}}{d \ln \mu} = \frac{y_t^2}{16\pi^2} \left( -3C_{Hq,33}^{(1)} + 8C_{Hq,33}^{(3)} \right) + \dots$$

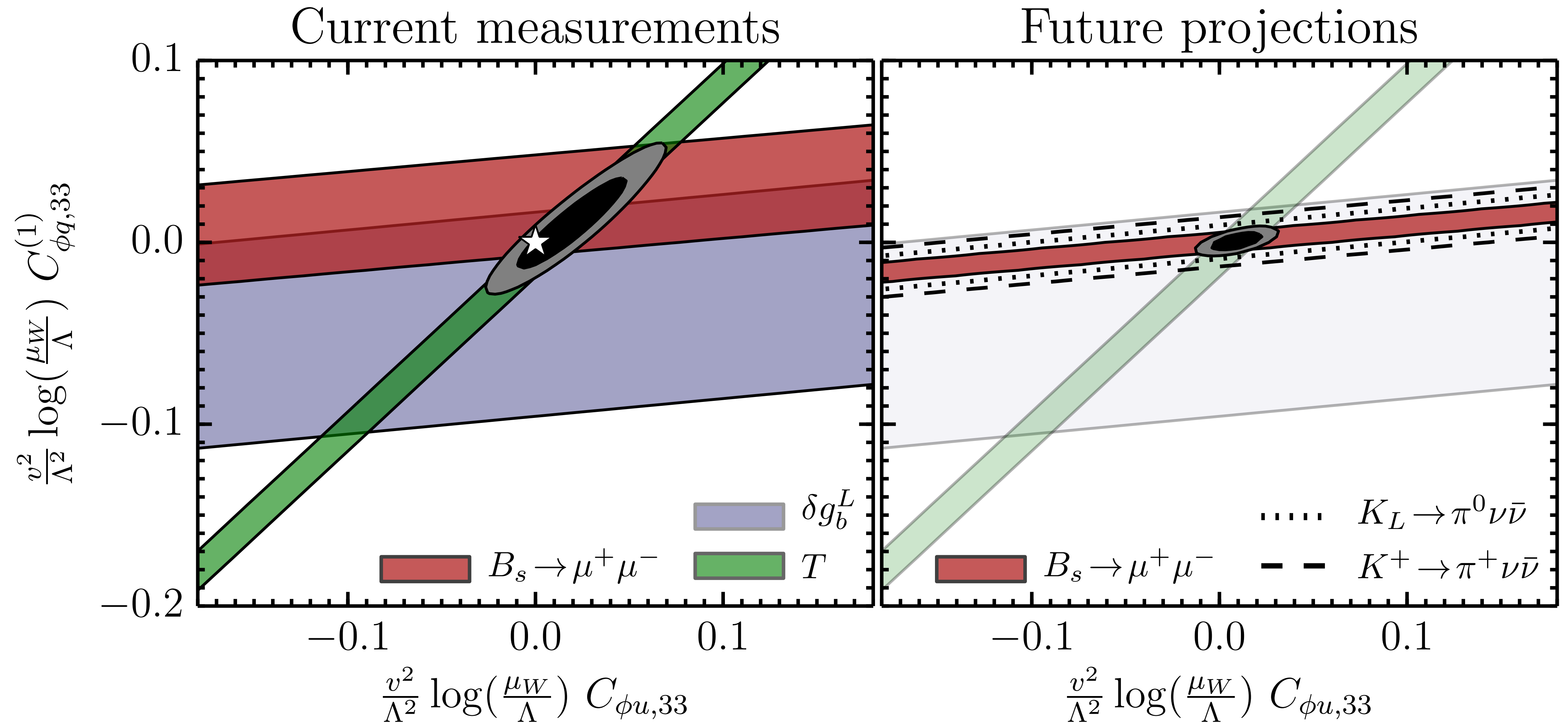
[discussion follows Brod et al., 1408.0792]

# LL in SMEFT: a simple example

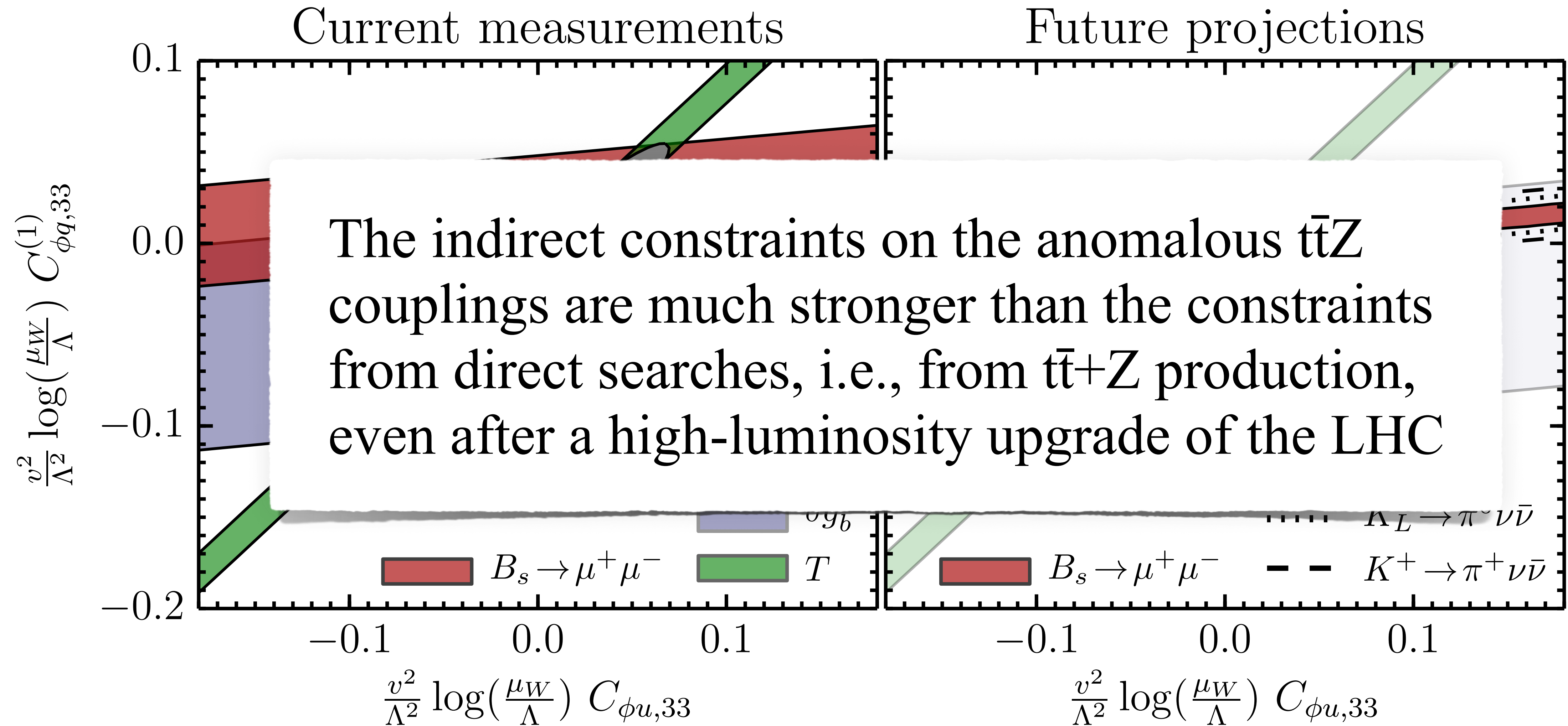


[discussion follows Brod et al., 1408.0792]

# LL in SMEFT: a simple example



# LL in SMEFT: a simple example





# System of three operators $Q_i$ , $Q_m$ & $Q_j$

$$\frac{C_j(\mu)}{C_i(\mu_0)} \simeq -\frac{\gamma_{ji}}{16\pi^2} \ln\left(\frac{\mu_0}{\mu}\right) + \frac{\gamma_{jm}\gamma_{mi}}{512\pi^4} \ln^2\left(\frac{\mu_0}{\mu}\right)$$

mixing  $Q_i \rightarrow Q_j$

mixing  $Q_i \rightarrow Q_m \rightarrow Q_j$

[discussion follows Buras & Jung, 1804.05852]

# System of three operators $Q_i$ , $Q_m$ & $Q_j$

$$\frac{C_j(\mu)}{C_i(\mu_0)} \simeq \underbrace{-\frac{\gamma_{ji}}{16\pi^2} \ln\left(\frac{\mu_0}{\mu}\right)}_{\text{1-loop LL effect}} + \underbrace{\frac{\gamma_{jm}\gamma_{mi}}{512\pi^4} \ln^2\left(\frac{\mu_0}{\mu}\right)}_{\text{2-loop LL effect}}$$

# System of three operators $Q_i$ , $Q_m$ & $Q_j$

$C(\mu)$   $\alpha_s(\mu)$   $\alpha_s(\mu)$   $\alpha_s(\mu)$

$\overline{C}$  The 2-loop LL effects can be derived from the known 1-loop anomalous dimensions. The 2-loop anomalous dimensions solely generate NLL corrections

ct

# Examples of 2-loop LL effects

$$C_{H\Box}(\mu) \propto \frac{y_t^4}{(4\pi)^4} C_{tt}^{(1)}(\mu_0) \ln^2\left(\frac{\mu_0}{\mu}\right),$$

$$C_{HGG}(\mu) \propto \frac{g_s^3 y_t^2}{(4\pi)^4} C_G(\mu_0) \ln^2\left(\frac{\mu_0}{\mu}\right),$$

$$C_{HW}(\mu) \propto \frac{g^2 y_b y_t}{(4\pi)^4} C_{qtqb}^{(1)}(\mu_0) \ln^2\left(\frac{\mu_0}{\mu}\right), \dots$$



# 2-loop LL effects in $gg \rightarrow h$

$$\begin{aligned}\delta\kappa_g &\simeq \frac{12\pi^2}{\alpha_s} \frac{\gamma_{HG,tG} \gamma_{tG,G}}{512\pi^4} \frac{v^2}{\Lambda^2} C_G(\Lambda) \ln^2\left(\frac{\Lambda}{m_h}\right) \\ &\simeq -\frac{27g_s y_t^2}{8\pi} \frac{v^2}{\Lambda^2} C_G(\Lambda) \ln^2\left(\frac{\Lambda}{m_h}\right) \simeq -0.09 C_G\end{aligned}$$

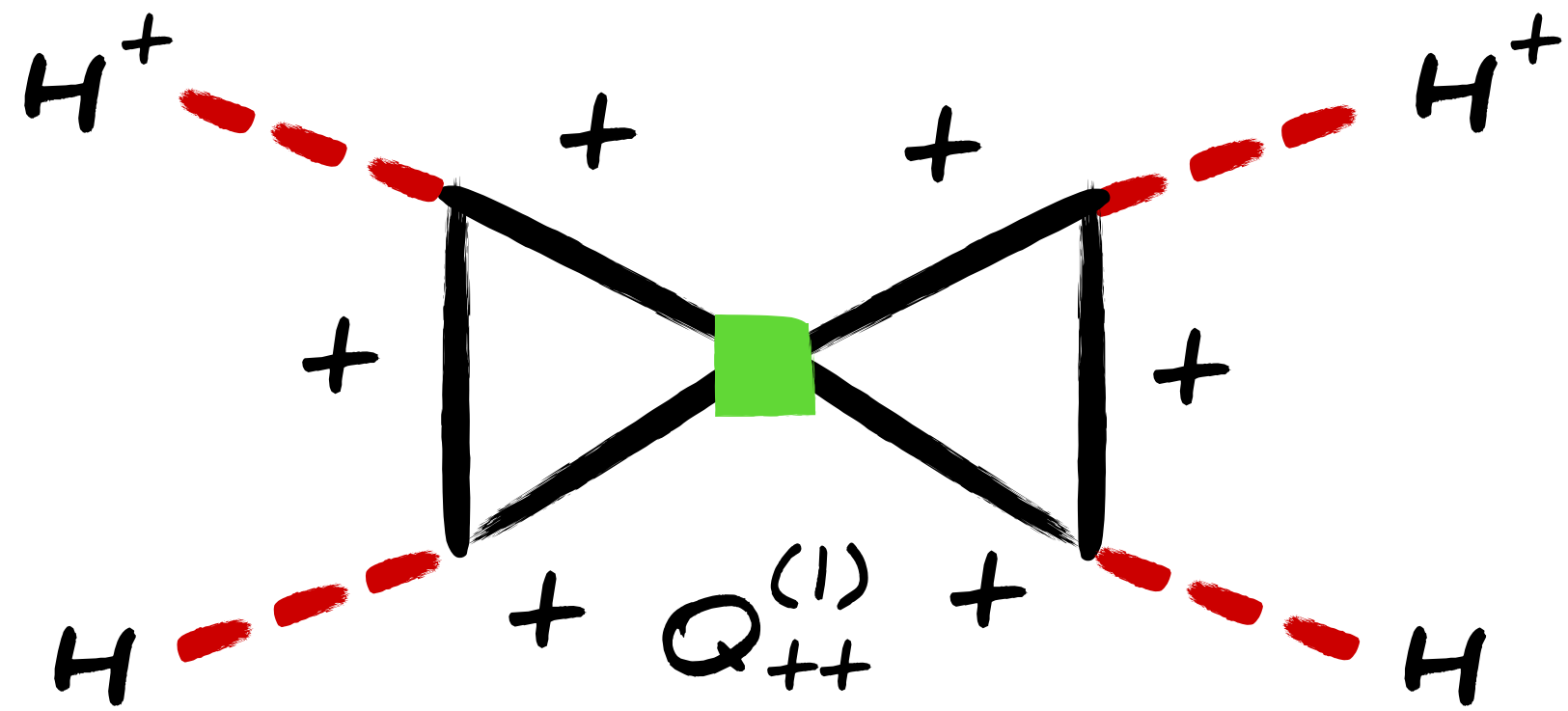
# 2-loop LL effects in $gg \rightarrow h$

$\delta\kappa_g$

A 10% measurement of the signal strength in gluon-gluon-fusion Higgs production enables setting an indirect bound on the triple gluon operator, which is as good or better than direct limits obtained from di-jet or top-pair production

$C_G$

# How big are 2-loop NLL effects?



$$\Rightarrow Q_{HD} = (H^\dagger D_\mu H)^* (H^\dagger D^\mu H)$$

$$\gamma_{HD,qq}^{(1)}(\mu) = \frac{3y_t^4}{2\pi^2}, \quad \gamma_{HD,qq}^{(3)}(\mu) = \frac{9y_t^4}{2\pi^2}, \quad \gamma_{HD,tt}^{(1)}(\mu) = \frac{3y_t^4}{\pi^2}$$

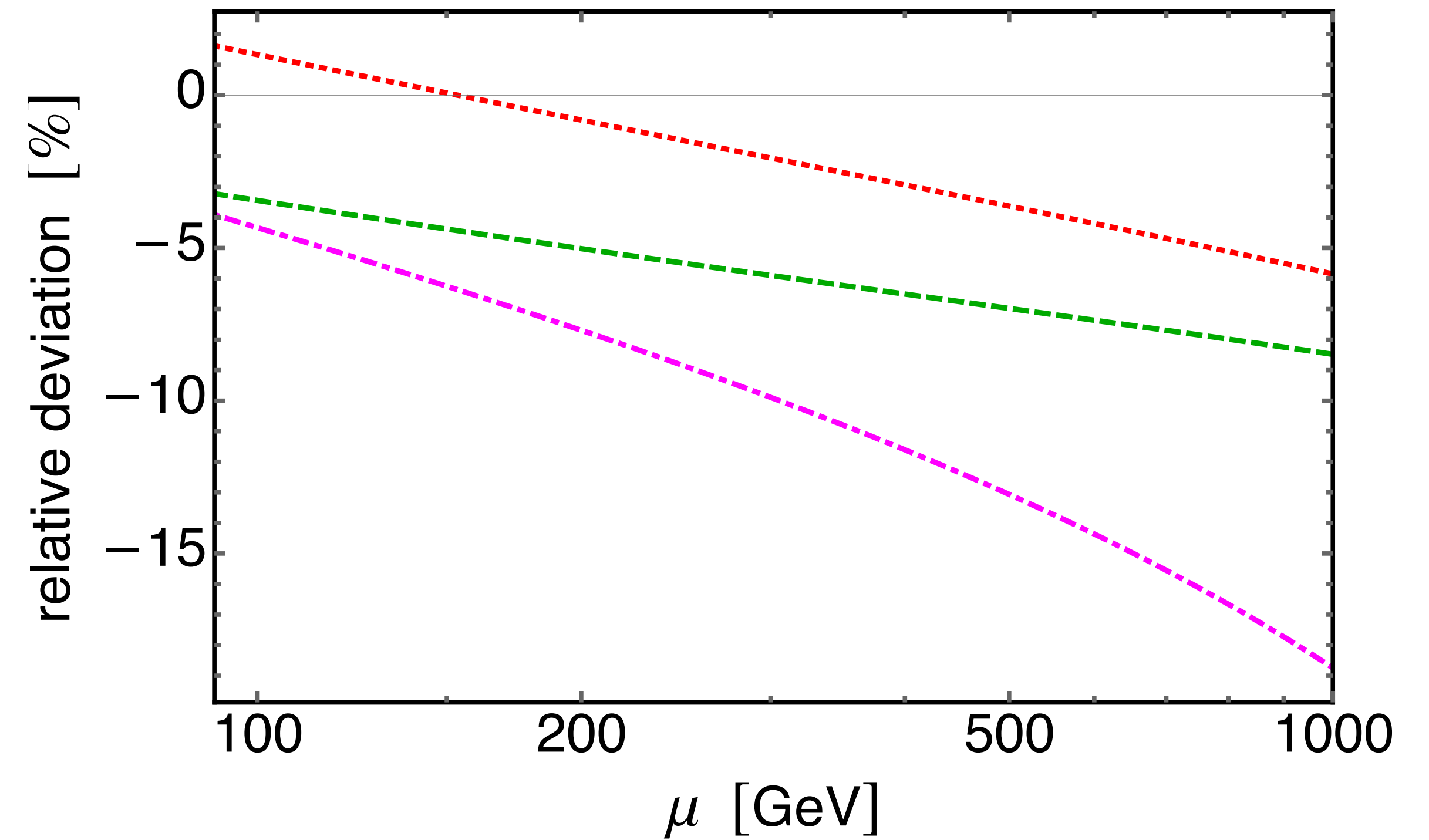
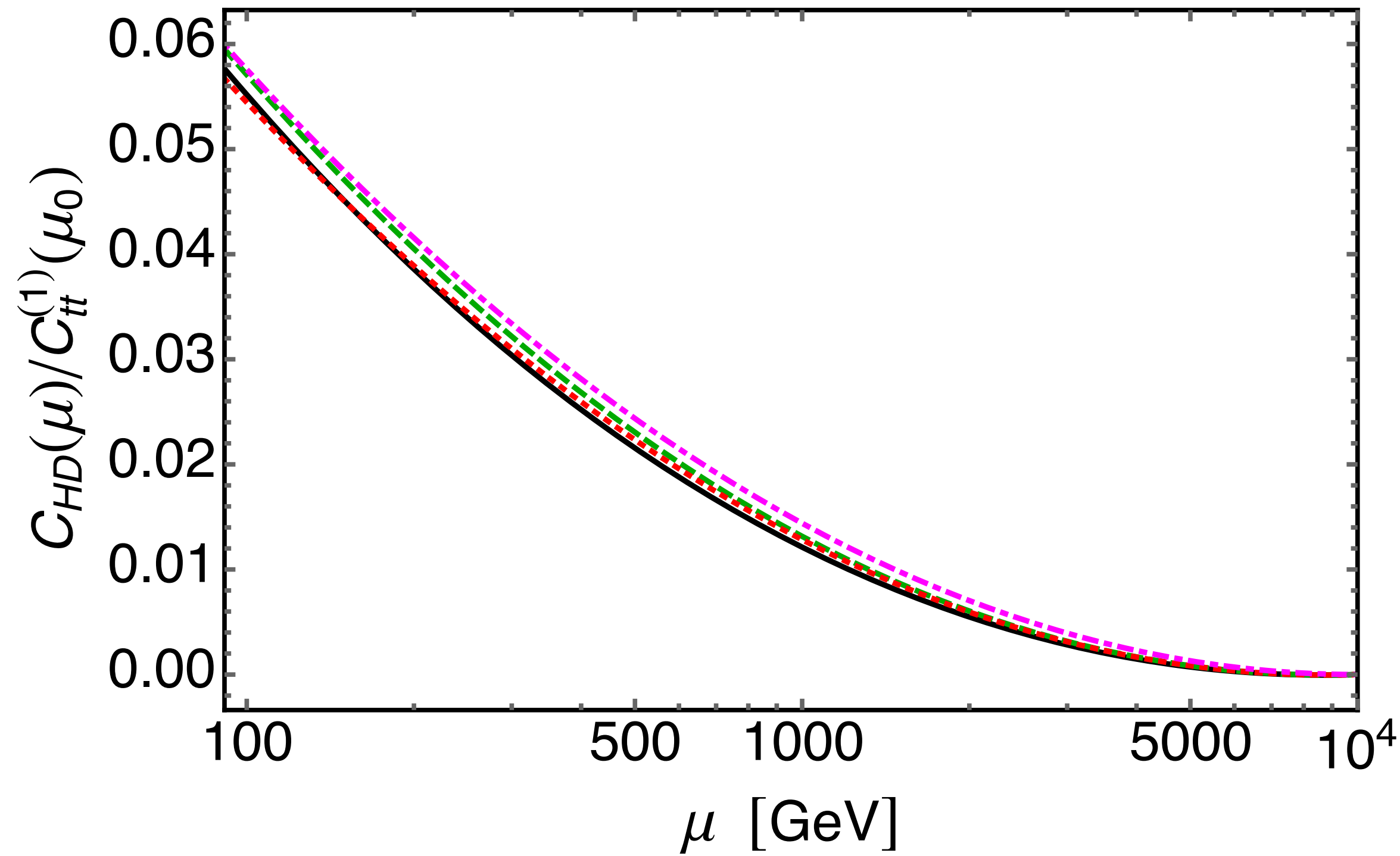
# How big are 2-loop NLL effects?

exact

resummed analytic

NLL accurate

LL accurate



[based on UH & Schnell, 2410.13304; unpublished]

# How big are 2-loop NLL effects?

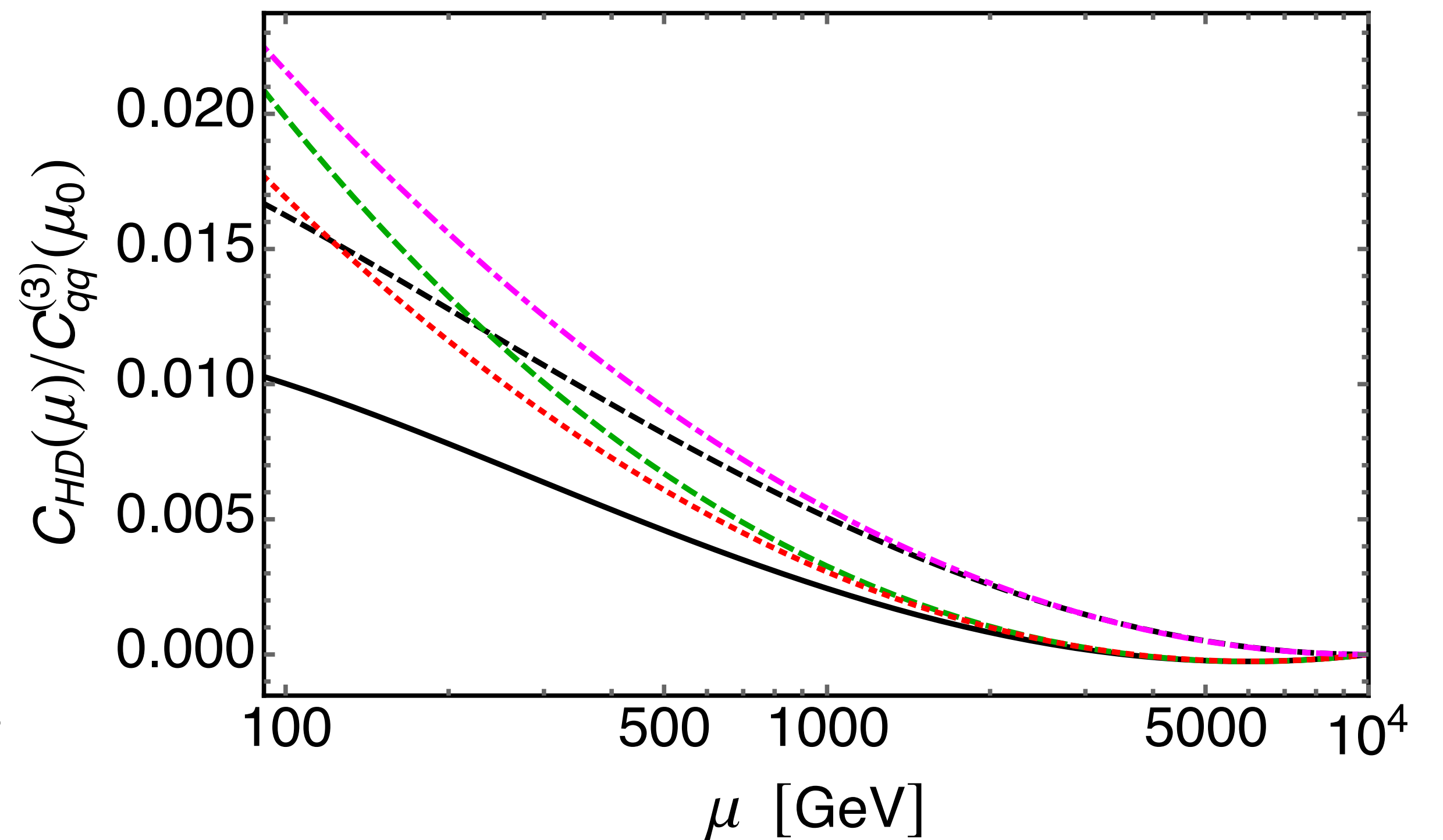
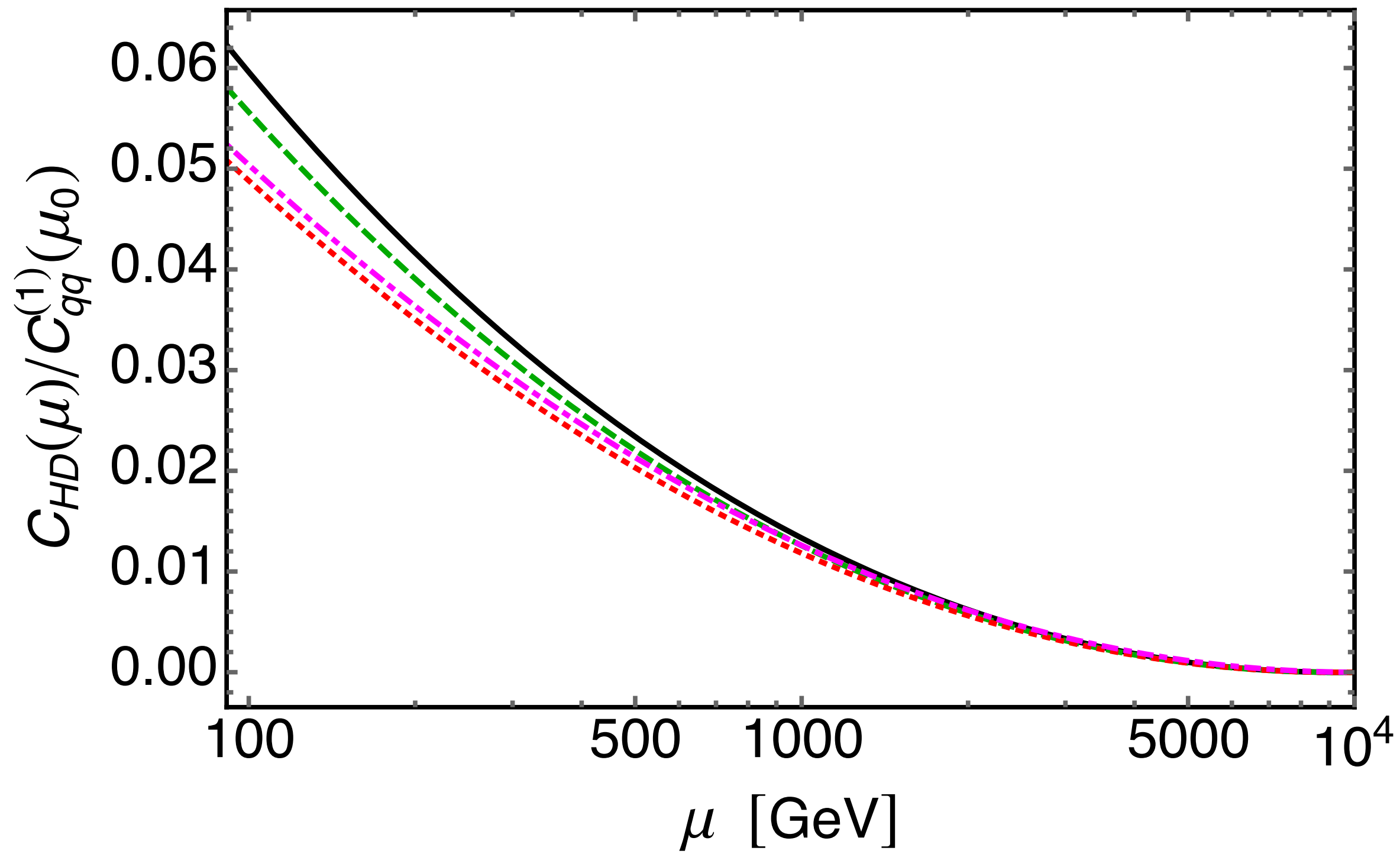
exact

exact 1-loop

resummed analytic

NLL accurate

LL accurate



[based on UH & Schnell, 2410.13304; unpublished]



# How big are 2-loop NLL effects?

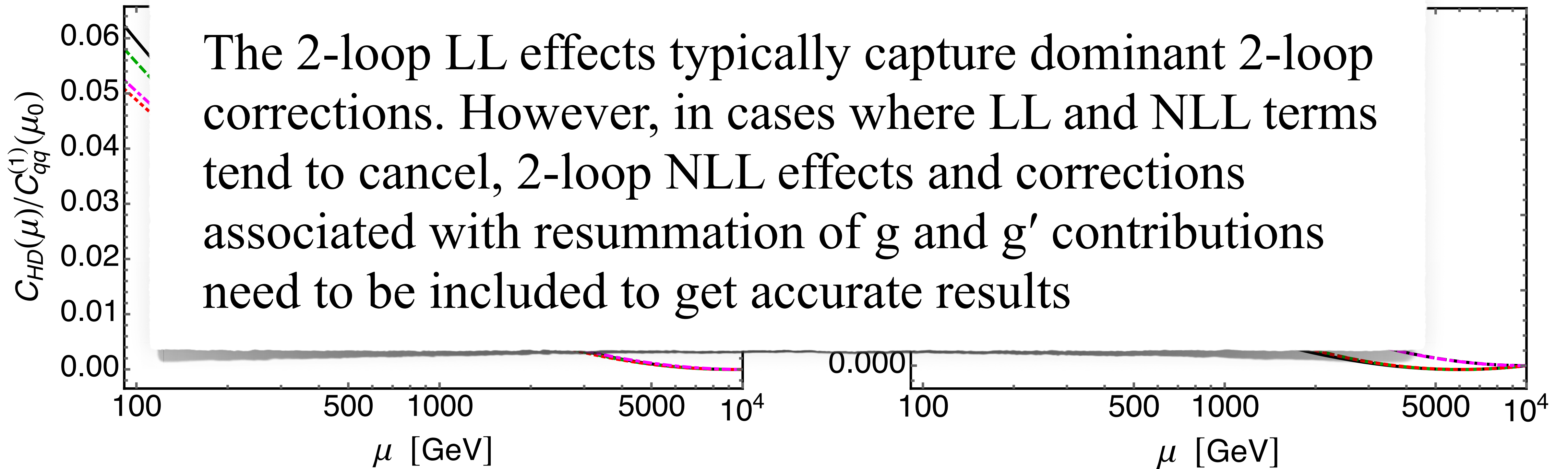
exact

exact 1-loop

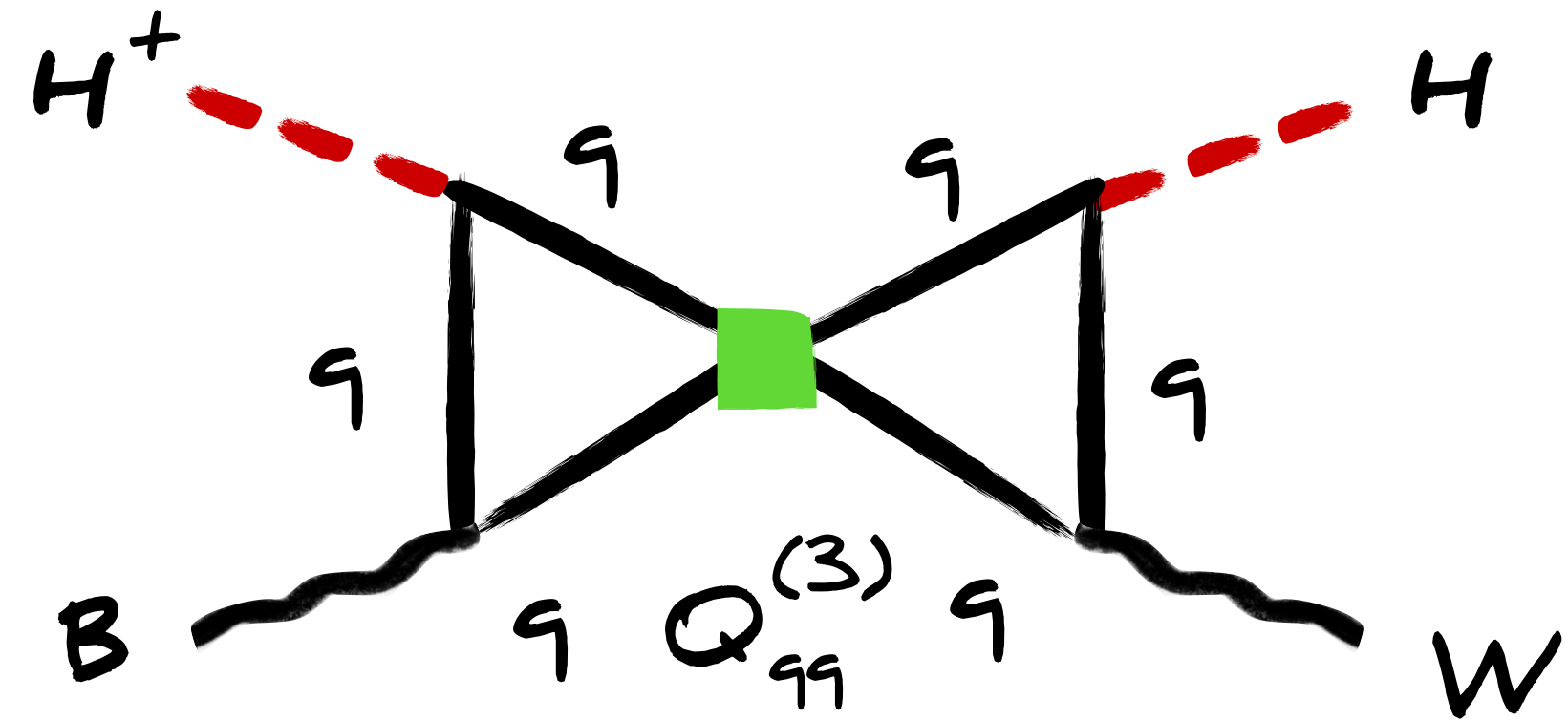
resummed analytic

NLL accurate

LL accurate



# An example with only 2-loop NLLs



$$\Rightarrow Q_{HWB} = (H^\dagger \sigma^i H) W_{\mu\nu}^i B^{\mu\nu}$$

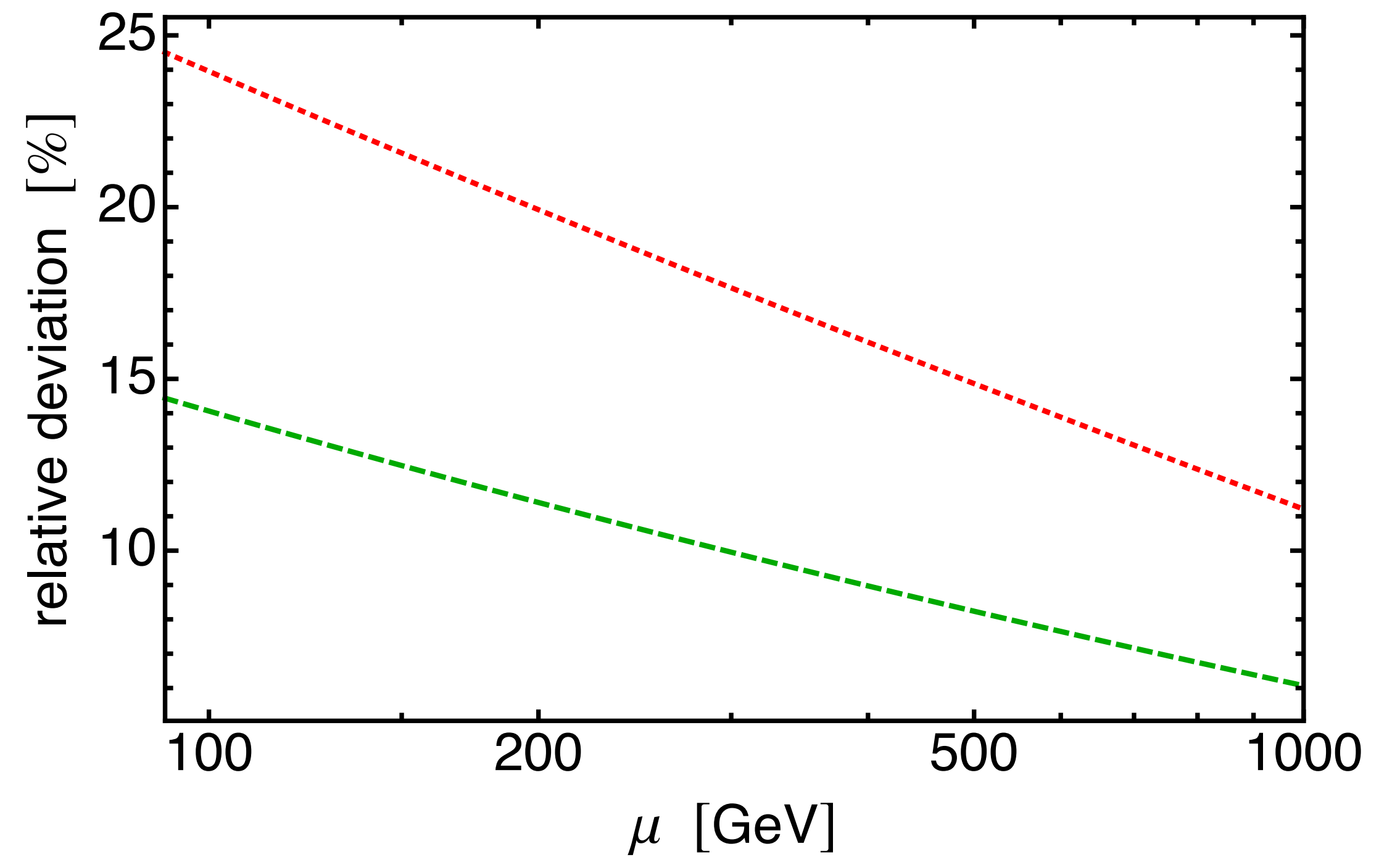
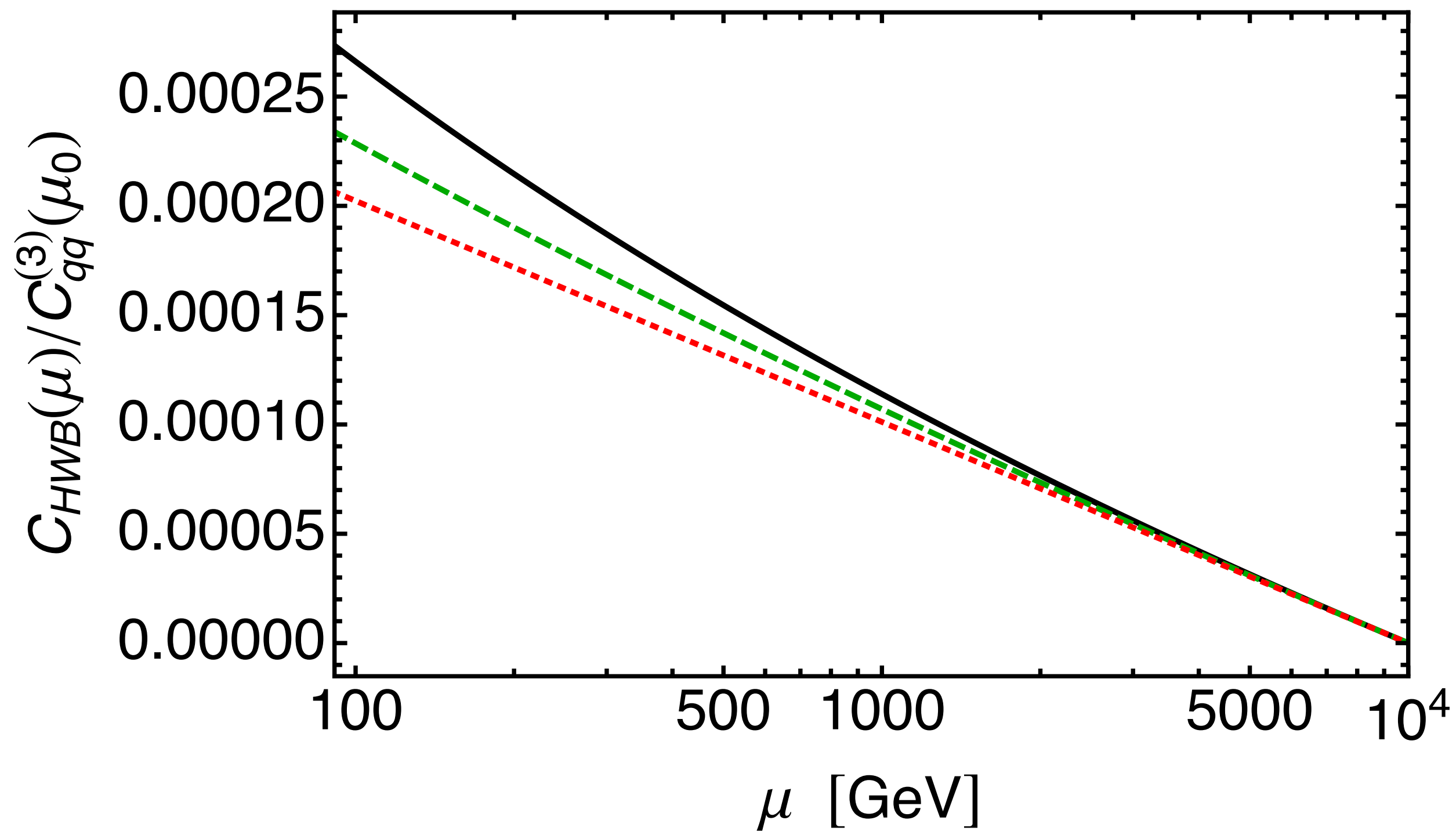
$$\gamma_{HWB,qq}^{(3)}(\mu) = -\frac{gg' y_t^2}{2\pi^2}$$

# An example with only 2-loop NLLs

exact

resummed analytic

single-logarithm accurate



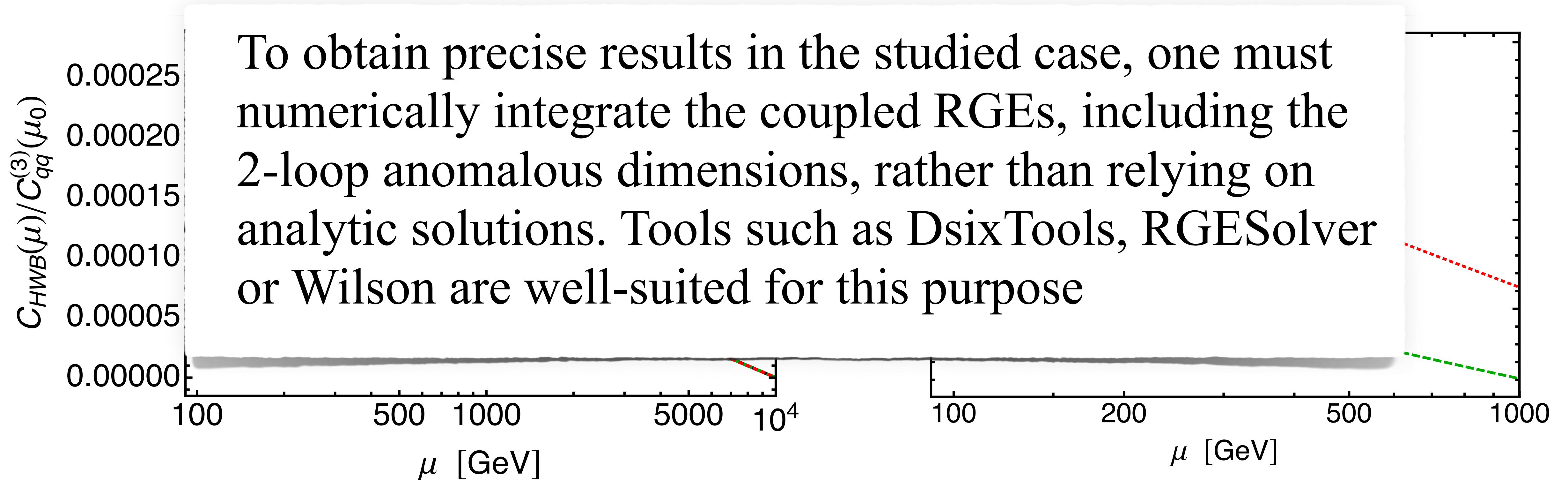
[based on UH & Schnell, 2410.13304; unpublished]

# An example with only 2-loop NLLs

exact

resummed analytic

single-logarithm accurate



# New 2-loop anomalous dimensions @ work

$$T \simeq -\frac{3y_t^4}{8\pi^4\alpha} \frac{v^2}{\Lambda^2} C_{tt}^{(1)}(\Lambda) \left[ \ln^2 \left( \frac{\Lambda}{M_Z} \right) - \frac{1}{4} \ln \left( \frac{\Lambda}{M_Z} \right) \right]$$

$$T \in [-0.23, 0.25] \Rightarrow \frac{C_{tt}^{(1)}}{\Lambda^2} \in \frac{[-2.04, 1.87]}{\text{TeV}^2}$$



# New 2-loop anomalous dimensions @ work

$$S \simeq \frac{y_t^2}{2\pi^3} \frac{v^2}{\Lambda^2} C_{qq}^{(3)}(\Lambda) \ln\left(\frac{\Lambda}{M_Z}\right)$$

$$S \in [-0.24, 0.16] \Rightarrow \frac{C_{qq}^{(3)}}{\Lambda^2} \in \frac{[-63.2, 95.5]}{\text{TeV}^2}$$

# New 2-loop anomalous dimensions @ work

Indirect 2-loop constraints on Wilson coefficients from EW precision measurements can match or even surpass the sensitivity of direct tree-level probes (such as 4-top production in the examples shown) only if the indirect probe receives a LL correction at the 2-loop level

$\Lambda^2$        $1\text{eV}^2$

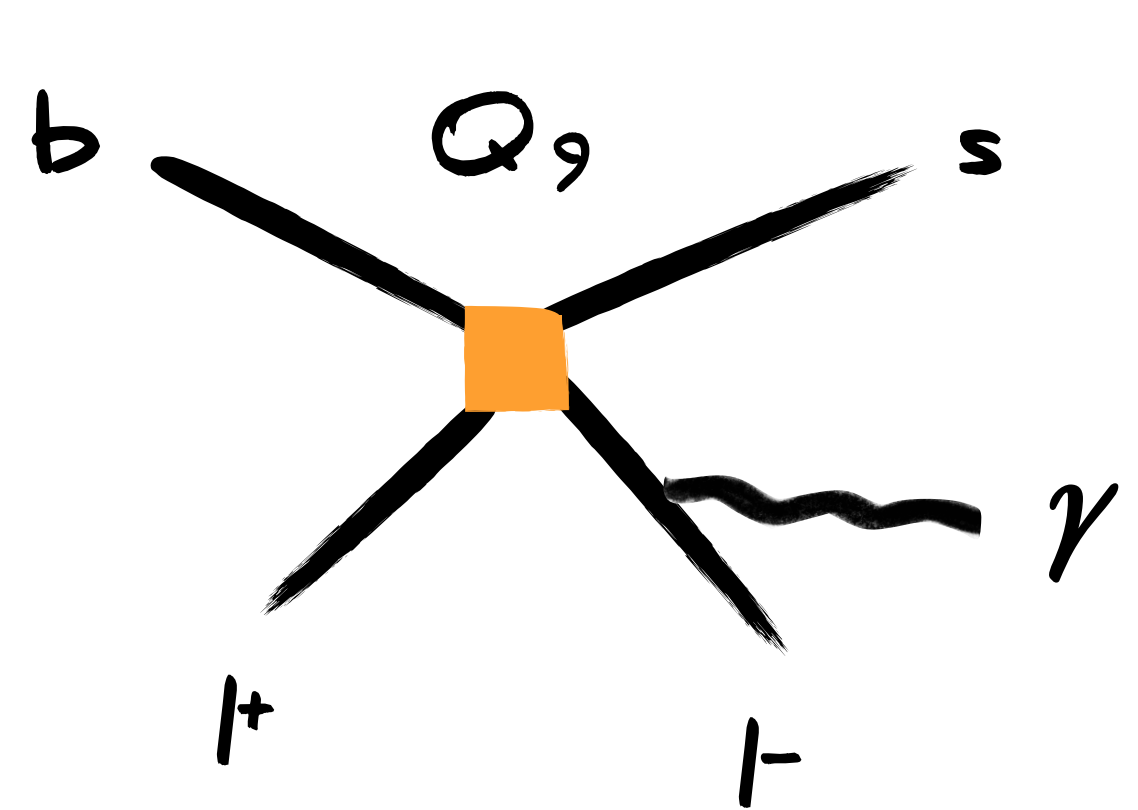
# What was Daniel doing around 2005?

## Electromagnetic Logarithms in $\bar{B} \rightarrow X_s \ell^+ \ell^-$

### Abstract

The  $\bar{B} \rightarrow X_s \ell^+ \ell^-$  decay rate is known at the next-to-next-to-leading order in QCD. It is proportional to  $\alpha_{em}(\mu)^2$  and has a  $\pm 4\%$  scale uncertainty before including the  $\mathcal{O}(\alpha_{em} \ln(M_W^2/m_b^2))$  electromagnetic corrections. We evaluate these corrections and confirm the earlier findings of Bobeth *et al.*. Furthermore, we complete the calculation of logarithmically enhanced electromagnetic effects by including also QED corrections to the matrix elements of four-fermion operators. Such corrections contain a collinear logarithm  $\ln(m_b^2/m_\ell^2)$  that survives integration over the low dilepton invariant mass region  $1 \text{ GeV}^2 < m_{\ell\ell}^2 < 6 \text{ GeV}^2$  and enhances the integrated decay rate in this domain. For the low- $m_{\ell\ell}^2$  integrated branching ratio in the muonic case, we find  $\mathcal{B}(B \rightarrow X_s \mu^+ \mu^-) = (1.59 \pm 0.11) \times 10^{-6}$ , where the error includes the parametric and perturbative uncertainties only. For  $\mathcal{B}(B \rightarrow X_s e^+ e^-)$ , in the current BaBar and Belle setups, the logarithm of the lepton mass gets replaced by angular cut parameters and the integrated branching ratio for the electrons is expected to be close to that for the muons.

# Origin of QED logarithms



The diagram shows a quark line with an orange square vertex labeled  $Q_9$ . The incoming quark is labeled  $b$  and the outgoing quark is labeled  $s$ . A photon  $\gamma$  is emitted from the vertex. The quark line continues as  $l^+$  and  $l^-$ .

$$\Rightarrow \frac{\alpha}{\pi} \ln \left( \frac{m_b^2}{m_l^2} \right) \simeq \begin{cases} 4.2\% , & l = e \\ 1.7\% , & l = \mu \\ 0.4\% , & l = \tau \end{cases}$$

Hard collinear emissions of photons result in flavor-dependent logarithms.  
Understanding of effects crucial for precision measurements of  $R_K^{(*)}$ ,  $V_{cb}$ , ...

# Lepton energy spectrum in $b \rightarrow cl\nu$

$$y = \frac{2E_l}{m_b}, \quad \rho = \frac{m_c^2}{m_b^2}, \quad P_{ll}^{(0)}(z) = \left[ \frac{1+z^2}{1-z} \right]_+$$

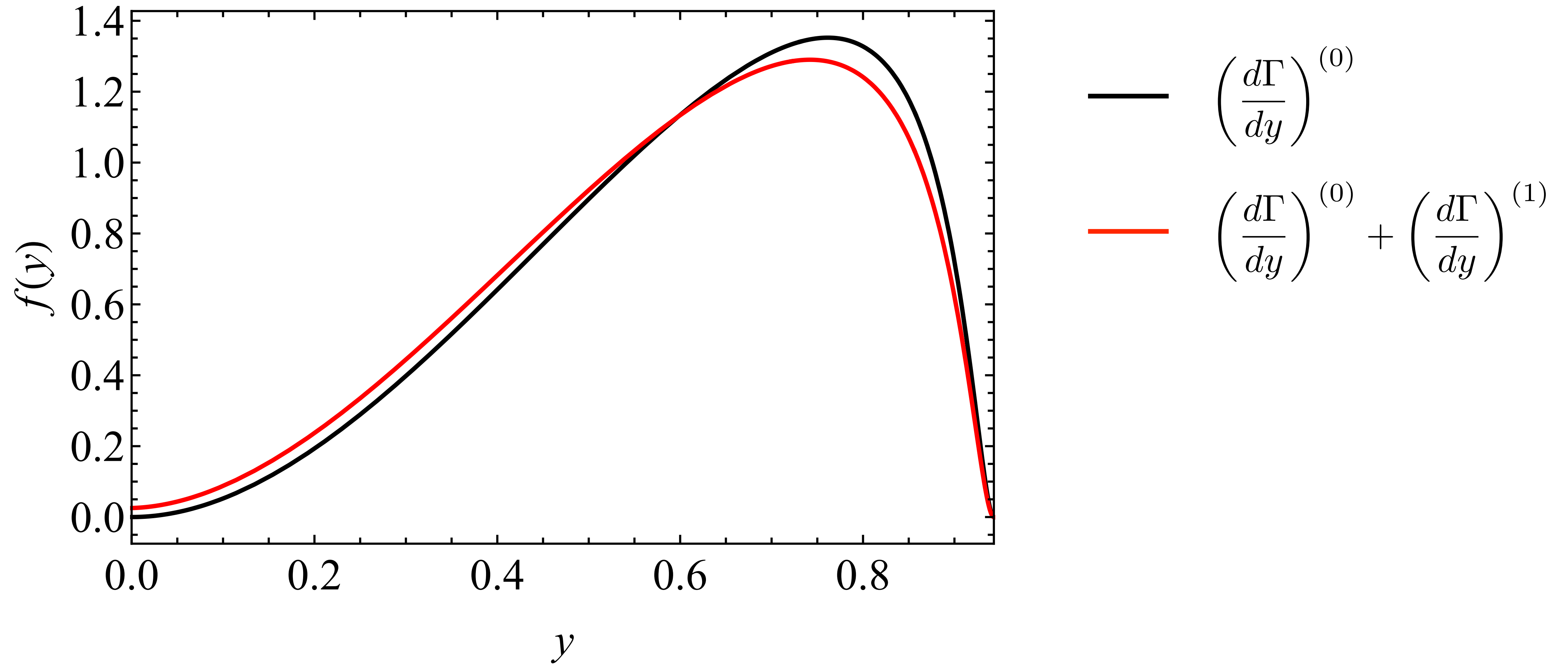
$$\left( \frac{d\Gamma}{dy} \right)^{(1)} = \frac{\alpha}{2\pi} \left[ \ln \left( \frac{m_b^2}{m_l^2} \right) - 1 \right] \int_y^{1-\rho} \frac{dx}{x} P_{ll}^{(0)} \left( \frac{y}{x} \right) \left( \frac{d\Gamma}{dx} \right)^{(0)}$$

LL corrections to spectrum obtained from tree-level result through convolution with leading-order lepton-lepton splitting function

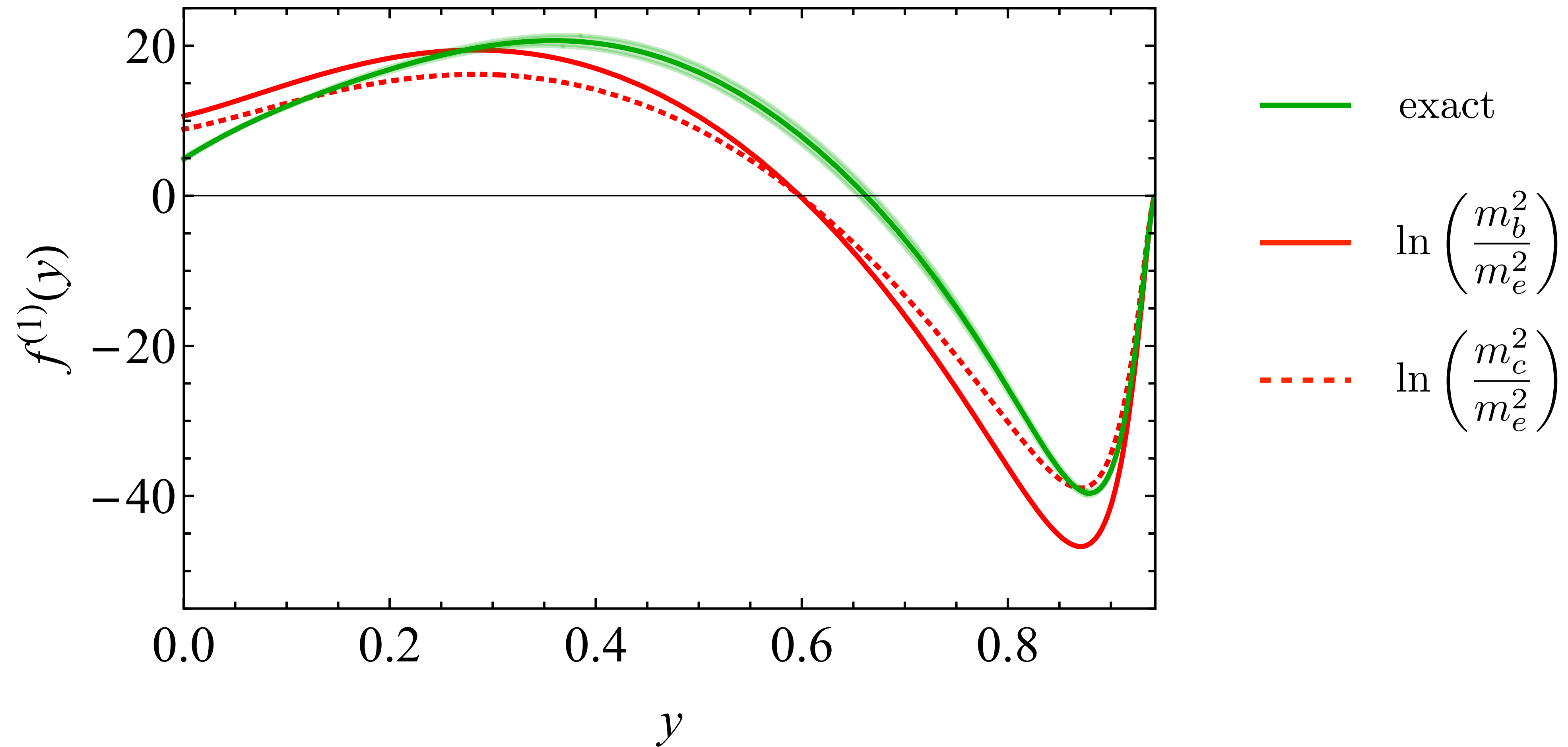
[see e.g. Arbuzov et al., hep-ph/0202102, hep-ph/0205172, hep-ph/0206036 for structure function approach]



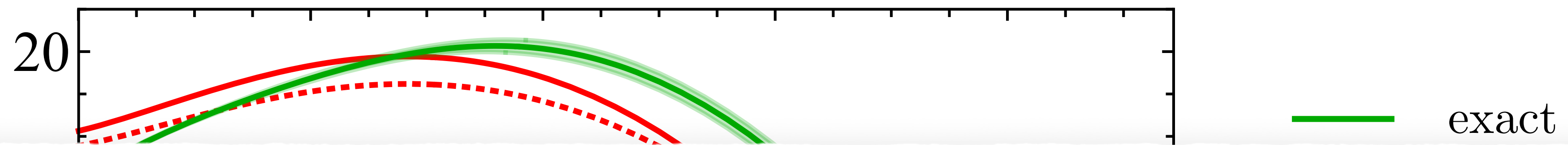
# Electron energy spectrum in $b \rightarrow ce\nu$



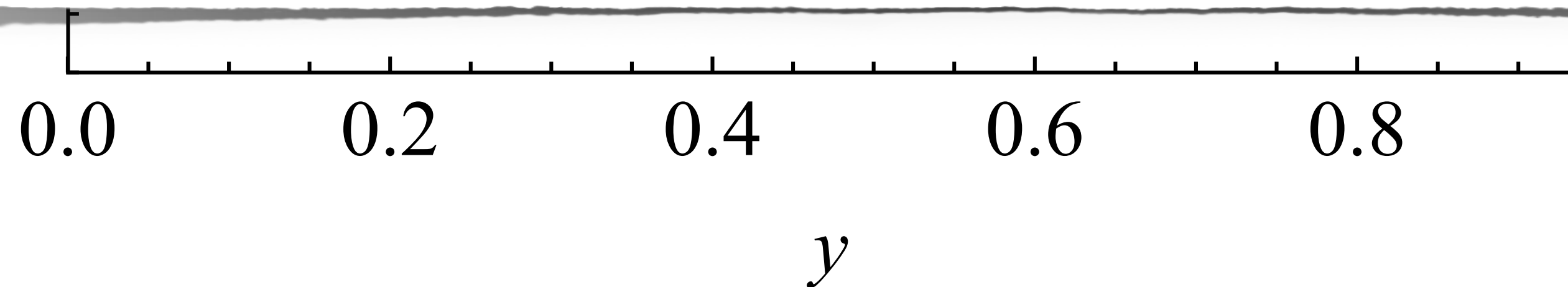
# Electron energy spectrum in $b \rightarrow ce\nu$



# Electron energy spectrum in $b \rightarrow ce\nu$



LL QED effects indeed provide dominant contributions to the electron energy spectrum in  $b \rightarrow ce\nu$ . Motivated by this observation, we have calculated the partonic NLL QED corrections as well as the LL QED contributions to the quadratic and cubic power corrections within the HQE

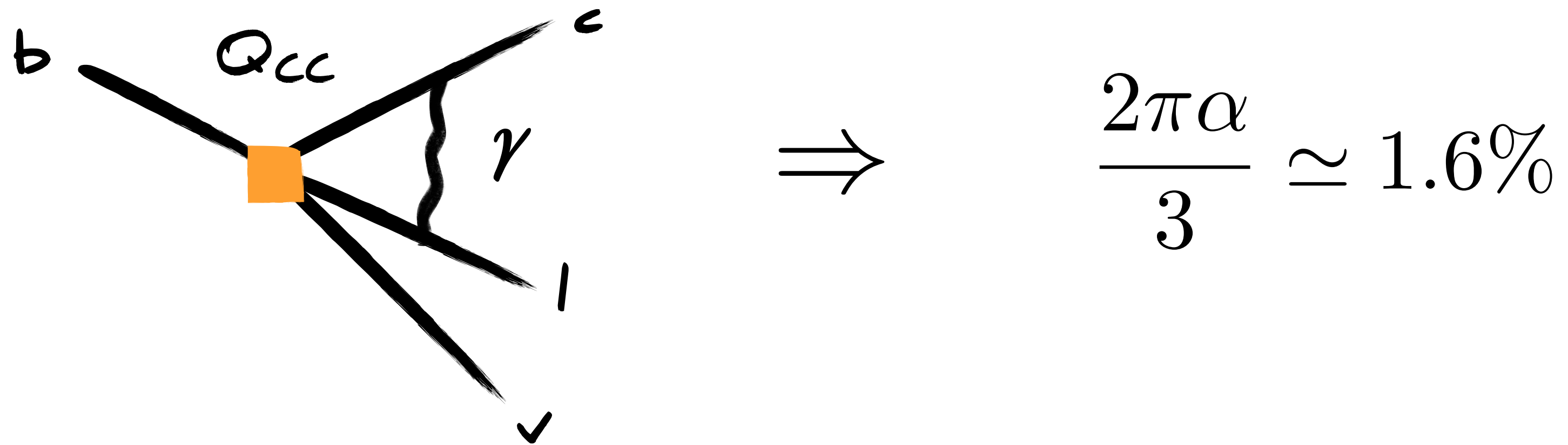


# QED effects in total decay width of $b \rightarrow cl\nu$

$$\int_0^1 dz P_{ll}^{(0)}(z) = 0 \quad \Rightarrow \quad \int_0^{1-\rho} dy \left( \frac{d\Gamma}{dy} \right)^{(1)} = 0$$

In total decay width of  $b \rightarrow cl\nu$ , LL QED corrections are not present as required by Kinoshita-Lee-Nauenberg theorem — true also for power-suppressed OPE contributions. What are dominant corrections to total decay width?

# QED effects in total decay width of $b \rightarrow cl\nu$



$\pi^2$ -enhanced QED effects, i.e. so-called Coulomb corrections, arise from soft virtual photon exchange from configurations close to charm threshold



# QED effects in total decay width of $b \rightarrow cl\nu$

$$\frac{\Gamma}{\Gamma^{(0)}} = 1 + \frac{\alpha}{\pi} \left[ \ln \left( \frac{M_Z^2}{m_b^2} \right) - \frac{11}{6} + 5.516(14) \right]$$

short-distance EW LL

pure photonic corrections

# QED effects in total decay width of $b \rightarrow cl\nu$

$$\frac{\Gamma}{\Gamma^{(0)}} = 1 + \frac{\alpha}{\pi} \left[ \ln \left( \frac{M_Z^2}{m_b^2} \right) - \frac{11}{6} + 5.516(14) \right]$$

$$= 1 + 1.43\% - 0.44\% + 1.32\% = 1 + 2.31\%$$

$\pi^2$ -enhanced terms give 1.57%

# QED effects in total decay width of $b \rightarrow cl\nu$

$$\frac{\Gamma}{\Gamma_0} = 1 + \frac{\alpha}{4\pi} \left[ \ln \left( \frac{M_Z^2}{m_b^2} \right) - \frac{11}{4} + 5.516(14) \right]$$

The QED corrections to the total decay width of  $b \rightarrow cl\nu$  amount to 2.3%. They are dominated by the short-distance EW LL contributions, derived by Sirlin in his seminal 1982 paper, and by  $\pi^2$ -enhanced threshold corrections

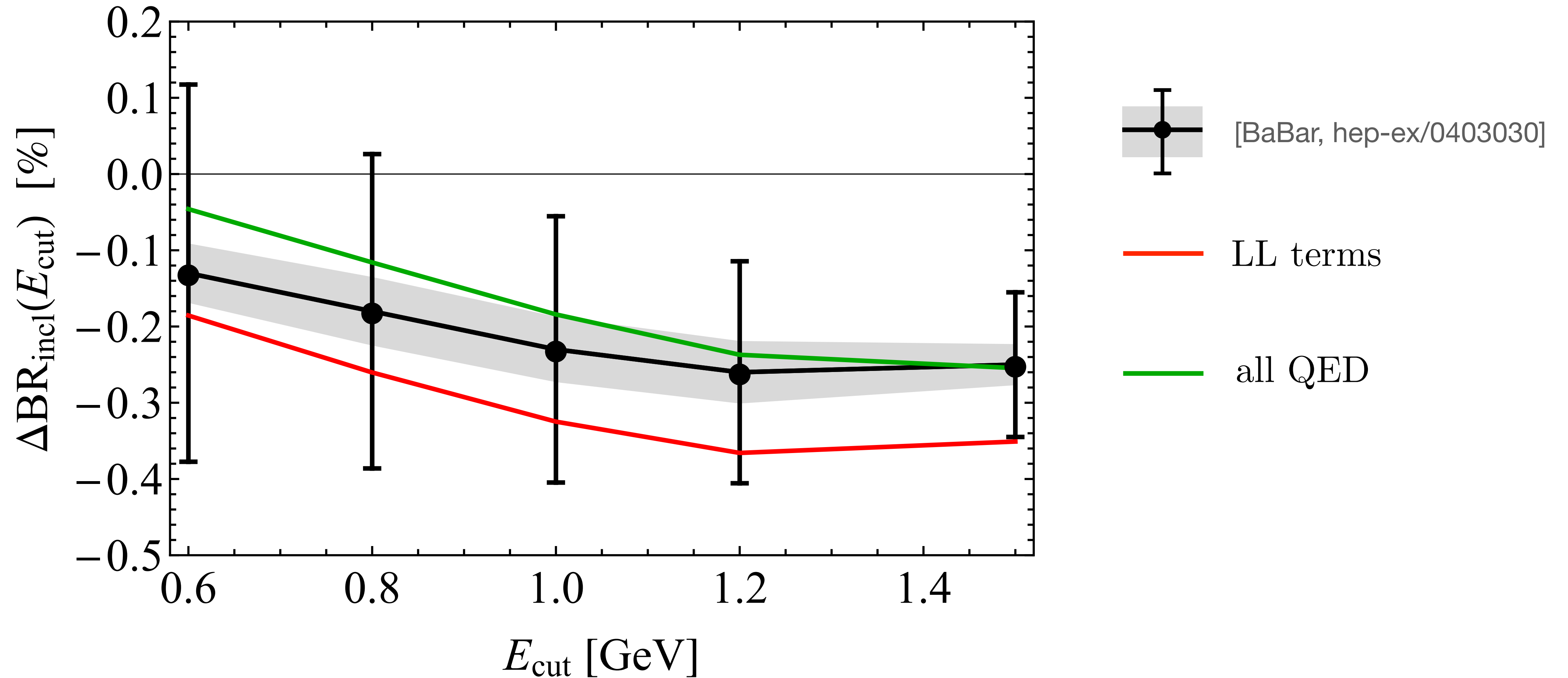
$\pi^2$ -enhanced terms give 1.57%

# Comparison to experiment

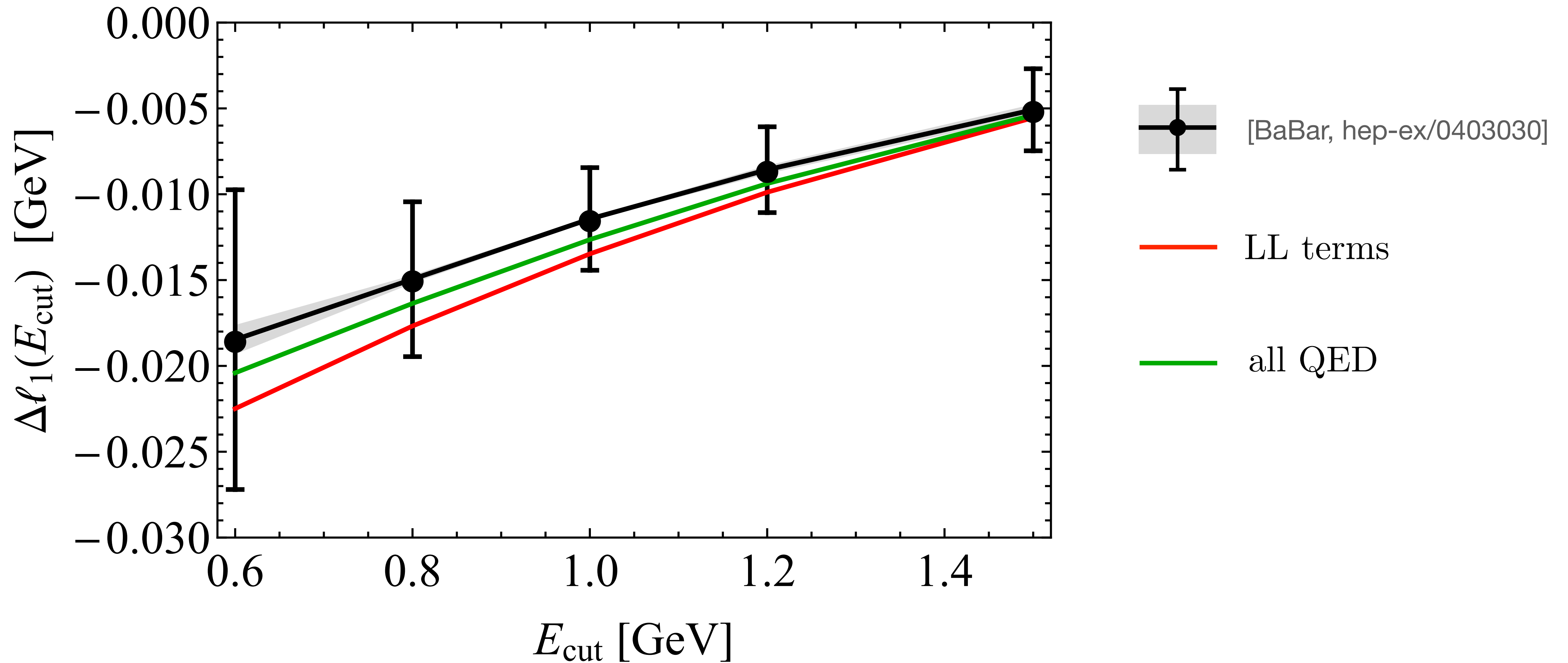
In BaBar & Belle, photon radiation is removed in analyses of  $b \rightarrow ce\nu$  spectra. Subtraction is performed using PHOTOS, which includes soft & collinear radiation from final-state charged leptons & hadrons with logarithmic accuracy. It lacks interference between initial- & final-state photons, hard & structure-dependent radiative effects as well as virtual effects, like  $\pi^2$ -enhanced terms. How good are these approximations?

[see e.g. Barberio & Was, Comput. Phys. Commun. 79 for details on PHOTOS]

# Comparison to experiment

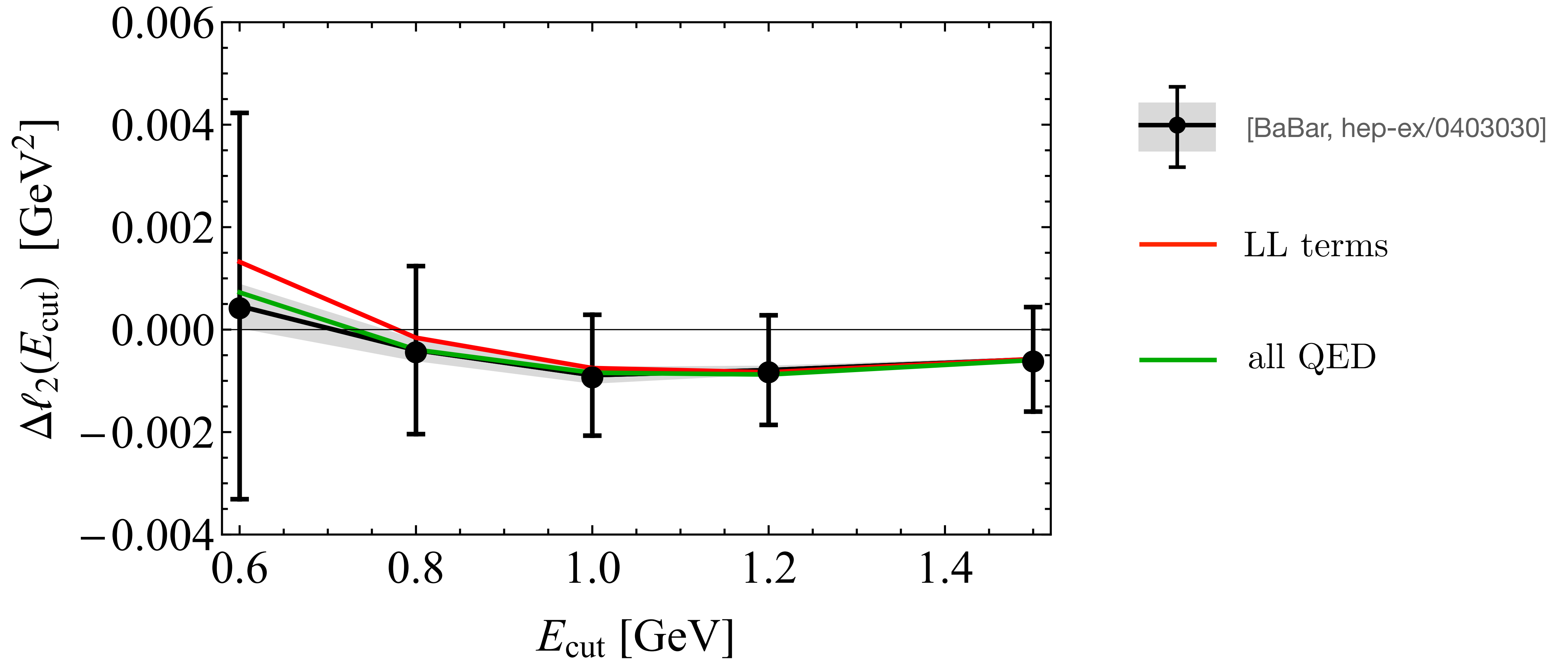


# Comparison to experiment

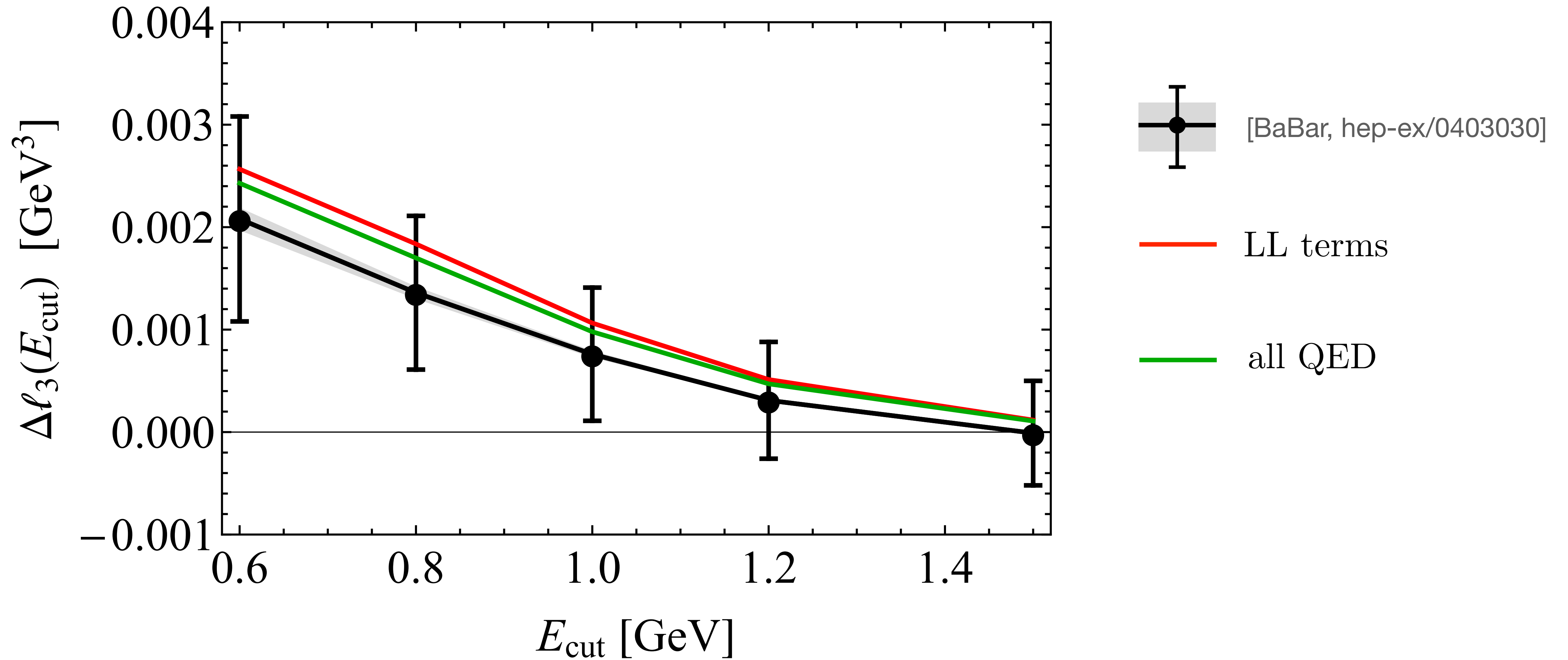




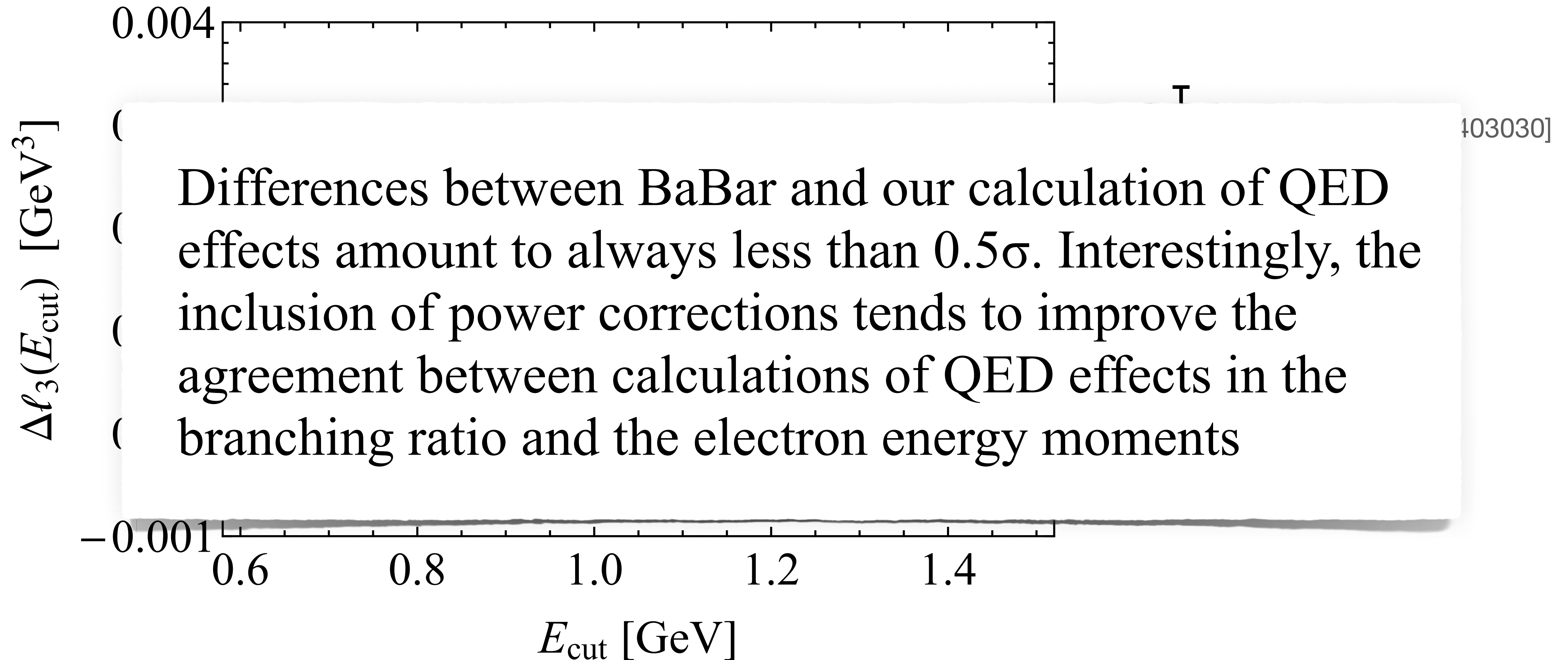
# Comparison to experiment



# Comparison to experiment



# Comparison to experiment



# Conclusions

- Daniel has envisioned and accomplished groundbreaking advancements in theoretical particle physics, far ahead of their time: SMEFT, lepto-quarks, neutrinos, flavor physics, chiral perturbation theory, ...
- While I never collaborated with him, his research has had a significant indirect impact on my own work. Yet, it still feels as though I'm the hare chasing the hedgehog — Daniel is always out in front.

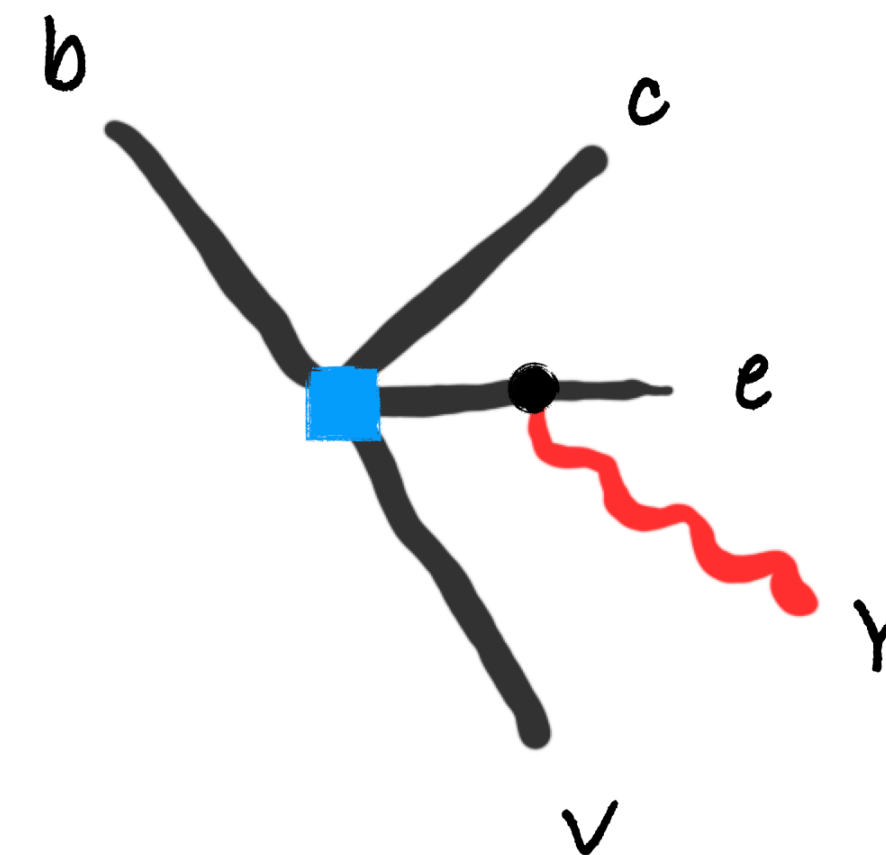
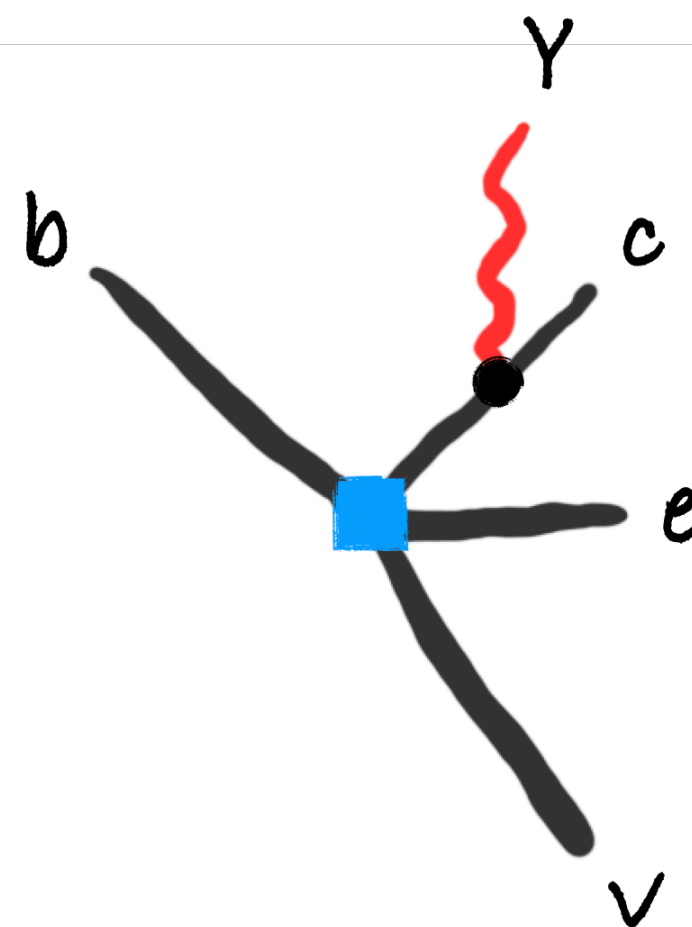
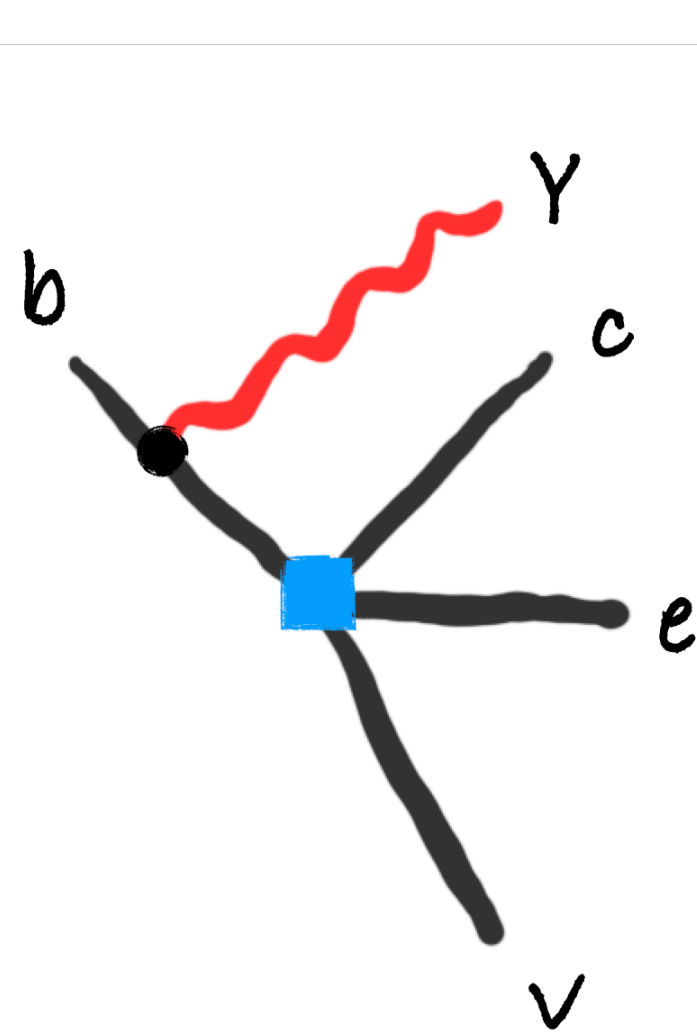
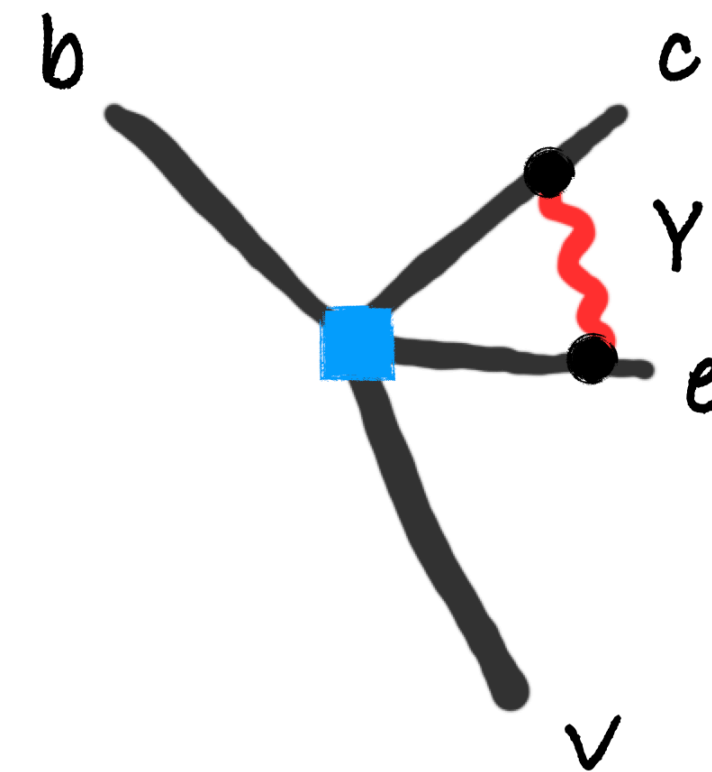
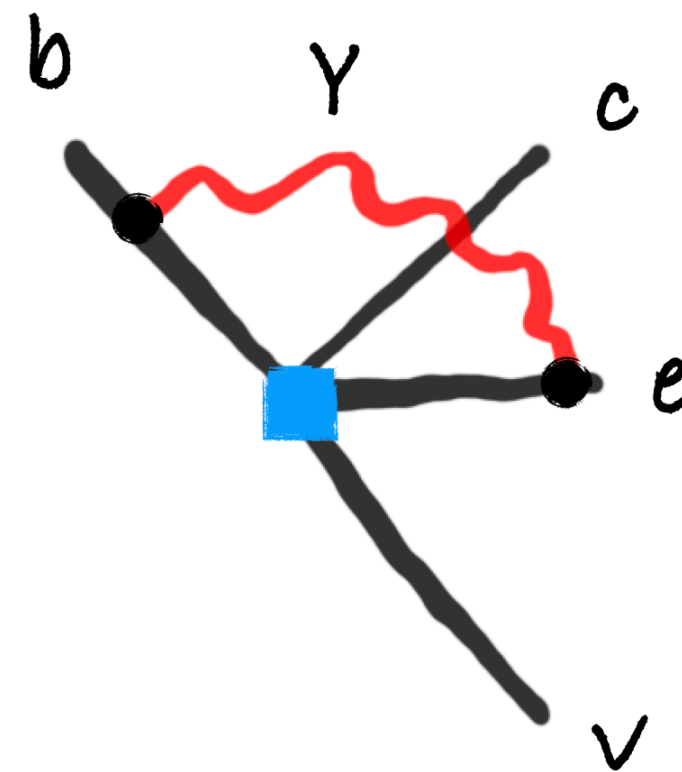
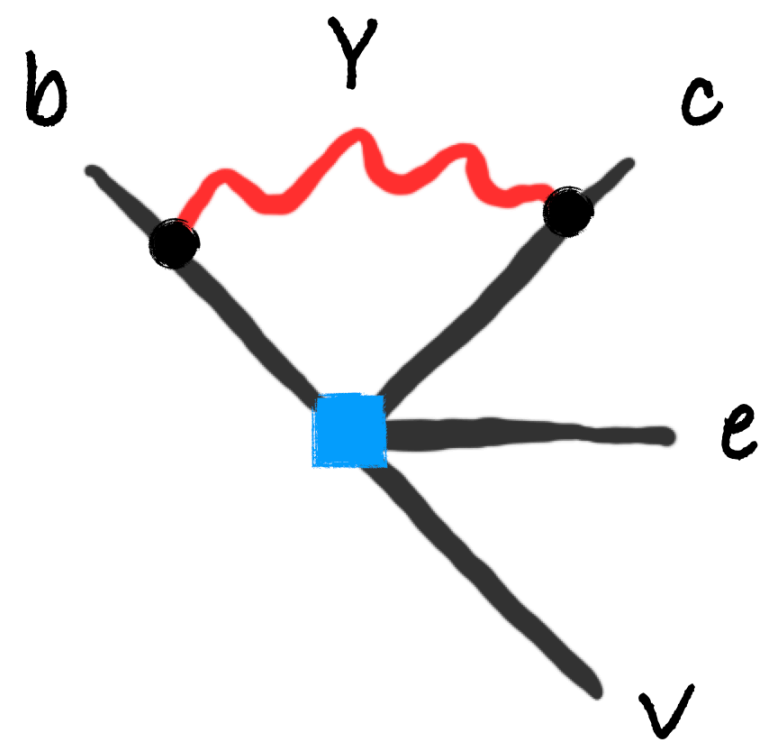


# Backup



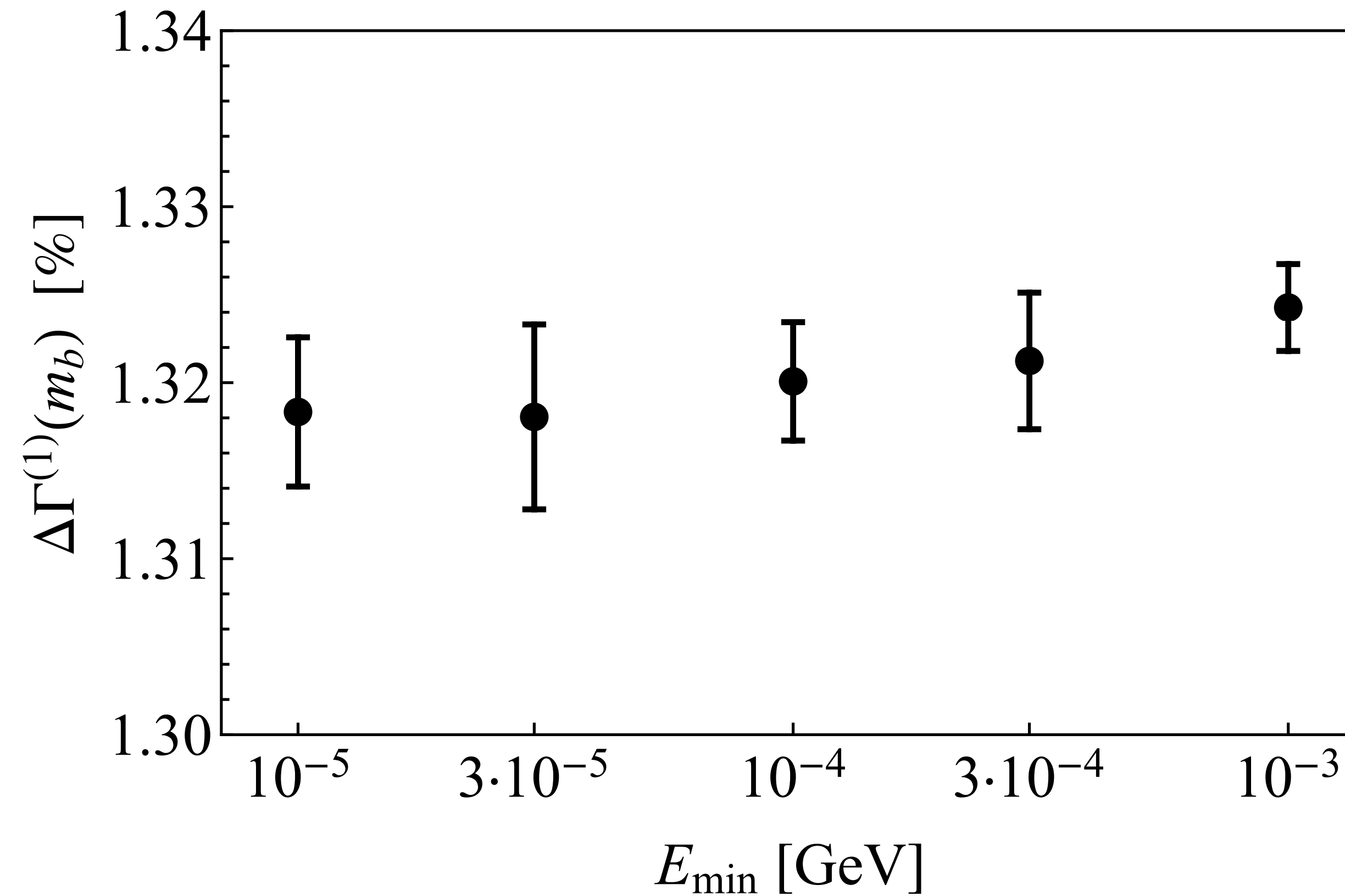


# Exact 1-loop QED effects in $b \rightarrow ce\nu$



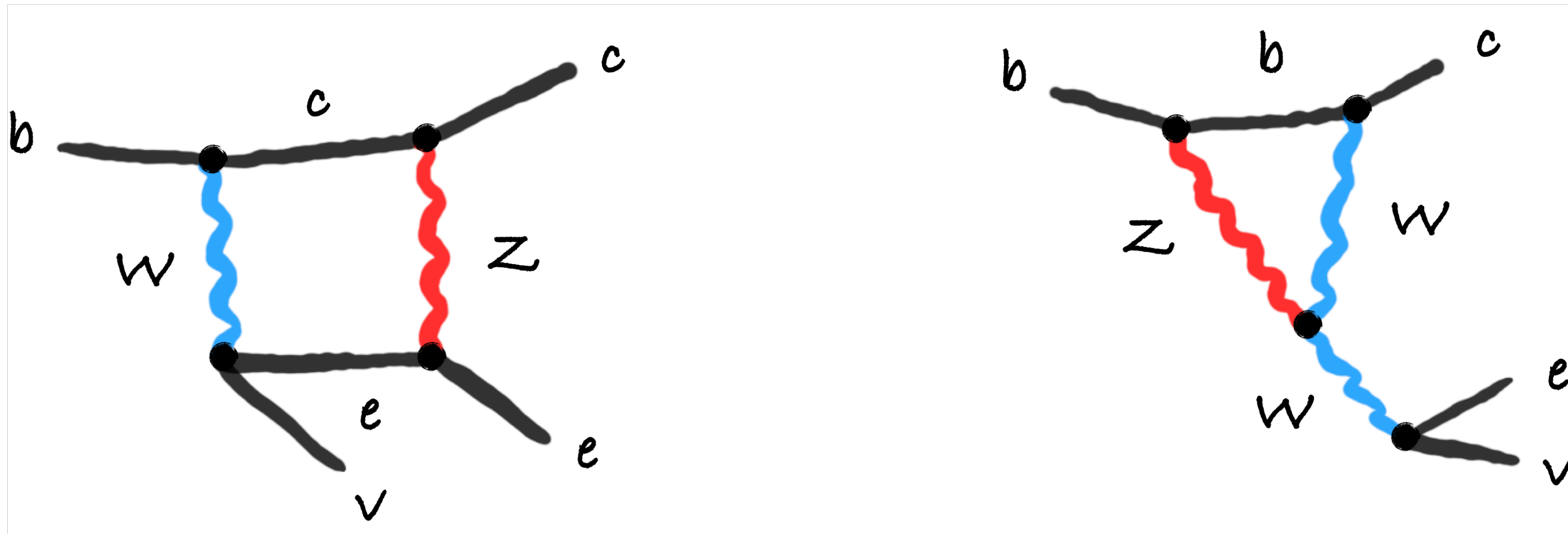


# Exact 1-loop QED effects in $b \rightarrow ce\nu$



Weak dependence of QED effects on  $E_{\min}$  parameter used for phase-space slicing

# Short-distance EW corrections



$$C_{CC}(\mu) = 1 + \frac{\alpha(\mu)}{4\pi} \begin{cases} \left[ 2 \ln \left( \frac{M_Z^2}{\mu^2} \right) - \frac{11}{3} \right], & \text{semi-leptonic} \\ \left[ \frac{4}{3} \ln \left( \frac{M_Z^2}{\mu^2} \right) - \frac{22}{9} \right], & \text{non-leptonic} \end{cases}$$

# Observables in $b \rightarrow ce\nu$

$$R^*(E_{\text{cut}}) = \frac{\int_{E_{\text{cut}}}^{E_{\text{max}}} dE_e \frac{d\Gamma}{dE_e}}{\int_{m_e}^{E_{\text{max}}} dE_e \frac{d\Gamma}{dE_e}}, \quad \langle E_e^n \rangle_{E_e > E_{\text{cut}}} = \frac{\int_{E_{\text{cut}}}^{E_{\text{max}}} dE_e E_e^n \frac{d\Gamma}{dE_e}}{\int_{E_{\text{cut}}}^{E_{\text{max}}} dE_e \frac{d\Gamma}{dE_e}}$$

$$l_1(E_{\text{cut}}) = \langle E_e \rangle_{E_e > E_{\text{cut}}}, \quad l_n(E_{\text{cut}}) = \left\langle (E - \langle E_e \rangle)^n \right\rangle_{E_e > E_{\text{cut}}}$$

# Comparison to BaBar

$E_{\text{cut}}$	$\delta\text{BR}_{\text{incl}}^{\text{BaBar}}$	$\delta\text{BR}_{\text{incl}}^{\text{LL}}$	$\delta\text{BR}_{\text{incl}}^{\text{NLL}}$	$\delta\text{BR}_{\text{incl}}^{\alpha}$	$\delta\text{BR}_{\text{incl}}^{1/m_b^2}$	$\delta\text{BR}_{\text{incl}}$	$\sigma$
0.6	-1.26%	-1.92%	-1.95%	-0.54%	-0.50%	-0.45%	+0.34
0.8	-1.87%	-2.88%	-2.91%	-1.36%	-1.29%	-1.22%	+0.30
1.0	-2.66%	-4.03%	-4.04%	-2.38%	-2.26%	-2.15%	+0.25
1.2	-3.56%	-5.43%	-5.41%	-3.65%	-3.43%	-3.27%	+0.14
1.5	-5.22%	-8.41%	-8.26%	-6.37%	-5.73%	-5.39%	-0.09

# Comparison to BaBar

$E_{\text{cut}}$	$\delta\ell_1^{\text{BaBar}}$	$\delta\ell_1^{\text{LL}}$	$\delta\ell_1^{\text{NLL}}$	$\delta\ell_1^\alpha$	$\delta\ell_1^{1/m_b^2}$	$\delta\ell_1$	$\sigma$
0.6	-1.29%	-1.60%	-1.58%	-1.48%	-1.45%	-1.42%	-0.22
0.8	-1.01%	-1.22%	-1.20%	-1.16%	-1.13%	-1.10%	-0.31
1.0	-0.74%	-0.89%	-0.88%	-0.87%	-0.84%	-0.82%	-0.40
1.2	-0.53%	-0.63%	-0.62%	-0.62%	-0.59%	-0.58%	-0.32
1.5	-0.29%	-0.33%	-0.32%	-0.34%	-0.31%	-0.30%	-0.13

# Comparison to BaBar

$E_{\text{cut}}$	$\delta l_2^{\text{BaBar}}$	$\delta l_2^{\text{LL}}$	$\delta l_2^{\text{NLL}}$	$\delta l_2^\alpha$	$\delta l_2^{1/m_b^2}$	$\delta l_2$	$\sigma$
0.6	+0.31%	+1.65%	+1.43%	+0.91%	+0.48%	+0.50%	+0.07
0.8	-0.34%	+0.50%	+0.34%	+0.04%	-0.40%	-0.33%	+0.01
1.0	-1.00%	-0.27%	-0.38%	-0.60%	-1.08%	-0.95%	+0.04
1.2	-1.27%	-0.78%	-0.85%	-1.05%	-1.60%	-1.42%	-0.08
1.5	-1.91%	-1.15%	-1.18%	-1.40%	-2.24%	-1.93%	-0.01



# Comparison to BaBar

$E_{\text{cut}}$	$\delta\ell_3^{\text{BaBar}}$	$\delta\ell_3^{\text{LL}}$	$\delta\ell_3^{\text{NLL}}$	$\delta\ell_3^\alpha$	$\delta\ell_3^{1/m_b^2}$	$\delta\ell_3$	$\sigma$
0.6	-17.3%	-22.1%	-23.1%	-22.7%	-22.6%	-22.5%	+0.35
0.8	-42.8%	-68.9%	-69.2%	-66.7%	-62.9%	-62.9%	+0.45
1.0	+63.9%	+67.3%	+66.3%	+63.9%	+56.2%	+57.6%	+0.34
1.2	+11.7%	+18.1%	+17.6%	+17.1%	+13.3%	+14.7%	+0.28
1.5	-0.47%	+5.92%	+5.69%	+5.61%	+2.10%	+4.69%	+0.23

# QED corrections in $|V_{cb}|$ extraction

Correction  $\delta BR_{\text{incl}}$  @ lowest  $E_{\text{cut}}$  value leads to largest shift of -0.4% in  $|V_{cb}|$ . QED corrections to moments of electron energy spectrum instead amount to an effect of +0.2%. Applying our new results for the QED corrections to electron energy spectrum & its moments to the BaBar data, therefore gives a total modification of around -0.2% in  $|V_{cb}|$  compared to an inclusive determination without QED effects