Bootstrapping Black Holes

Henry Lin, Stanford University

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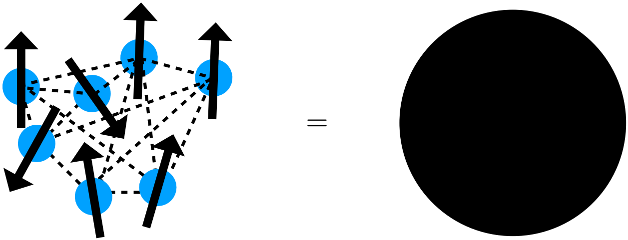

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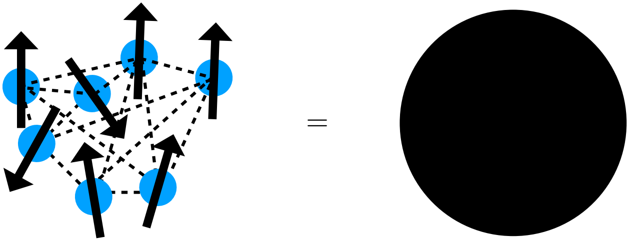

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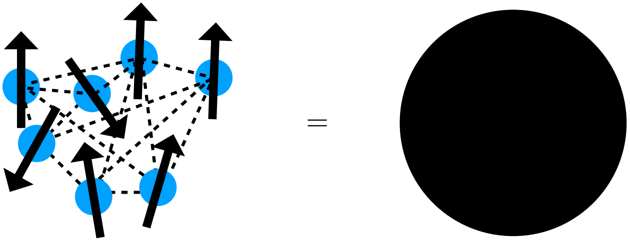

It would be nice to solve the quantum system. This means, e.g., computing correlators like $\frac{1}{Z}\,\mathrm{tr}\big(e^{-\beta H}\mathcal{O}\big)$ on the LHS.

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So why haven't we solved these black holes yet?

Why bootstrap?

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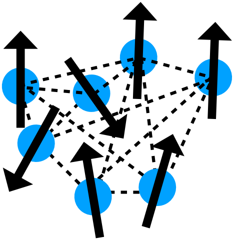
Review gauge/gravity duality for maximally supersymmetric Yang Mills (SYM) theory in $p+1$ dimensions, for $p<3$.
[Itzhaki, Maldacena, Sonneschein, Yankielowicz '00]

$\mathrm{SYM}_{p+1} \quad = $



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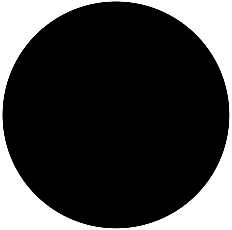

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In slightly different regimes, terminology:

- $p = 0$: BFSS
- $p = -1$: IKKT
- $p = 1$: matrix string theory

- Review of non-conformal holography/black holes
- Matrix bootstrap for simple models
- Matrix bootstrap for BFSS, future directions

Consider the effective field theory of Dp-branes, for $p \leq 3 \Rightarrow$ SU(N) SYM in $p+1$ dimensions.

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't Hooft coupling $\lambda = g^2_{\mathrm{YM}} N \Rightarrow \tilde{\lambda} = g^2_{\mathrm{YM}} N / \beta^{p-3}$.

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Decoupling limit: $E^2(\alpha') \to 0, \quad E^{p-3}\, g_{\mathrm{YM}}^2 = \text{fixed}$

Note that for $p < 3$ this implies that $g_s \to 0$.

Black brane solution

In the 't Hooft limit, with $\tilde{\lambda} \gg 1$, Type II supergravity.

$$I = \frac{1}{(2\pi)^7(\alpha')^4} \int \mathrm{d}^{10}x\sqrt{g}\left[e^{-2\phi}(R + 4(\nabla\phi)^2) - \frac{1}{2(p+2)!}F_{p+2}^2\right]$$

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This has the famous black brane solution:

$$\begin{aligned}
\mathrm{d}s^2 &= f^{-1/2}[h(r)\mathrm{d}t^2 + \mathrm{d}x_p^2] + f^{1/2}\left[h^{-1}(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{8-p}^2\right], \\
e^{-2\phi} &= g_{\mathrm{s}}^{-2} f^{(p-3)/2}, \quad A_{0\cdots p} = f^{-1}, \\
f &= 1 + \frac{d_0 g_{\mathrm{YM}}^2 N}{(\alpha')^2 (r/\alpha')^{7-p}}, \quad d_p = 2^{7-2p}\pi^{\frac{9-3p}{2}}\Gamma\left(\tfrac{1}{2}(7-p)\right), \\
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In the decoupling limit, delete 1.

This solution is somewhat similar to the AdS black brane $\times S_{8-p}$

$$\frac{\mathrm{d}s^2}{\alpha'} = \left(\frac{z}{R_{\mathrm{AdS}}}\right)^{\frac{3-p}{5-p}} \left[R_{\mathrm{AdS}}^2 \left(\frac{h(z)\,\mathrm{d}\tau^2 + h^{-1}(z)\mathrm{d}z^2 + \mathrm{d}x_p^2}{z^2}\right) + \mathrm{d}\Omega_{8-p}^2\right],$$

$$h = 1 - \frac{z^d}{z_0^d}, \quad d = 1 + \frac{9-p}{5-p}, \quad R_{\mathrm{AdS}} = \frac{2}{5-p},$$

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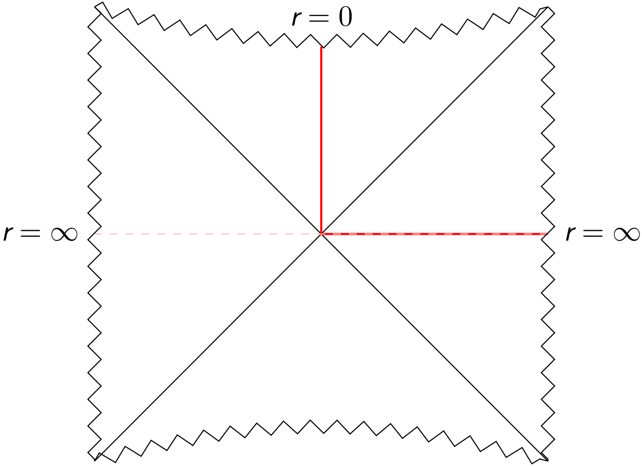
Exercise: work out the change of coordinates and check that finite τ, z is consistent with the decoupling limit. Work out z_0 as a function of temperature.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$r = \infty$
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Some features for $p = 0$:

$$\frac{\mathrm{d}s^2}{\alpha'} = \left(\frac{z}{R_{\mathrm{AdS}}}\right)^{\frac{3}{5}} \left[R_{\mathrm{AdS}}^2 \left(\frac{h(z)\,\mathrm{d}\tau^2 + h^{-1}(z)\mathrm{d}z^2}{z^2}\right) + \mathrm{d}\Omega_8^2\right],$$

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- Exercise: compute the proper distance from the boundary to the horizon. Use this to estimate the thermal 2-pt function of a massive stringy mode $e^{-m\ell}$ and its β dependence.

Extrapolation to strong coupling

- $p = 0$ view D0s as gravitons in 11d $\Rightarrow$ boosted Schwarzschild black hole (homogeneous in the 11th dimension) $\Rightarrow$ BFSS conjecture
- $p = 2$, view the D2 branes as M2 branes, $\mathrm{AdS}_4 \times S_7$ ABJM
- $p = 1$, S-duality relates D1 solution to F1s $\Rightarrow$ matrix string ($(R^8)^N/S_N$ CFT)
- $p = -1$ ...?

Relation to AdS

Fluctuations of the dilaton $\phi = \phi_{\mathsf{sol}} + \chi$

$$I \propto \int \mathrm{d}^{10}x\sqrt{g}e^{-2\phi_{\mathsf{sol}}}(\nabla\chi)^2$$
$$= \int \mathrm{d}^{8-p}\Omega\,\mathrm{d}^{d-1}\vec{x}\,\mathrm{d}z\,\mathrm{d}\tau\,\sqrt{g_{\mathsf{AdS}}}\left[(\nabla_{\mathrm{AdS}}\chi)^2 + m_k^2\chi^2\right]$$

Using $m_k = k(k+7-p) \Rightarrow$ fields in AdS_{d+1}:

$$\langle\mathcal{O}_\phi(x)\mathcal{O}_\phi(0)\rangle \sim \frac{1}{|x|^{2(\Delta-d-p-1)}}, \quad \Delta = R_{\mathrm{AdS}}(k+2)+2.$$

This applies to SUGRA modes.

The GKP dictionary

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$$I_{DBI} \sim \int \mathrm{d}^{p+1}x\, e^{-\phi(x,X)} F^2(x) + \cdots$$

Boundary operators schematically of the form

$$S_{\mathsf{SYM}} \to S_{\mathsf{SYM}} + \mathcal{N} \sum_j \frac{1}{k!} \int \mathrm{d}^{p+1}x\, \partial_{I_1} \cdots \partial_{I_k} \phi \, \mathrm{Tr}\Big(F^2_{\mu\nu} X^{(I_1} \cdots X^{I_k)} \Big).$$

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This is a super-descendant of the $1/2$ BPS operator:

$$\mathrm{Tr}\Big(F^2_{\mu\nu} X^{(I_1} \cdots X^{I_k)} \Big) \sim QQQQ\, \mathrm{Tr}\, X^{(I_1} \cdots X^{I_{k+2})}$$

We learn that the dimensions of the $\frac{1}{2}$ BPS operator:

$$\Delta = R_{\mathrm{AdS}}(k+2) + 2 \Rightarrow \Delta_{\frac{1}{2}\mathsf{BPS}} = R_{\mathrm{AdS}} k$$

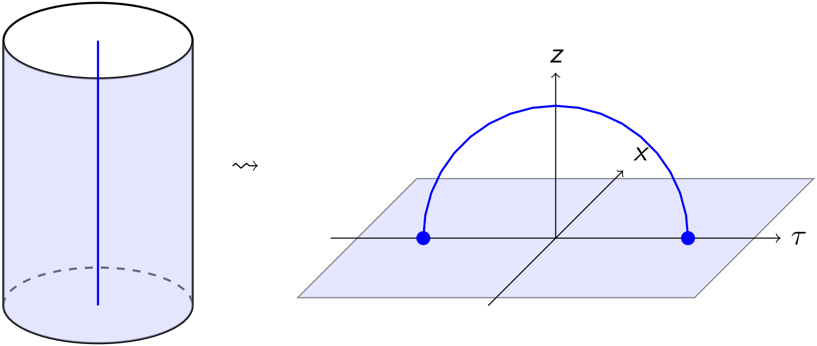
Giant gravitons

As a fun aside [WIP w/ Gauri Batra], we can also reproduce the relation

$$\Delta_{\frac{1}{2}\mathsf{BPS}} = R_{\mathrm{AdS}} k$$

For values of k that are very large $k \sim N$ by considering a classical solution where a giant graviton $D(6-p)$ brane that couples to magnetically-dual RR flux.

Giant gravitons



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$$S = N^2 \left(c_0 (T/\lambda)^{9/5} + c_1 (T/\lambda)^{18/5} + \cdots \right) + N^0 b_0 T^{-3/5}$$

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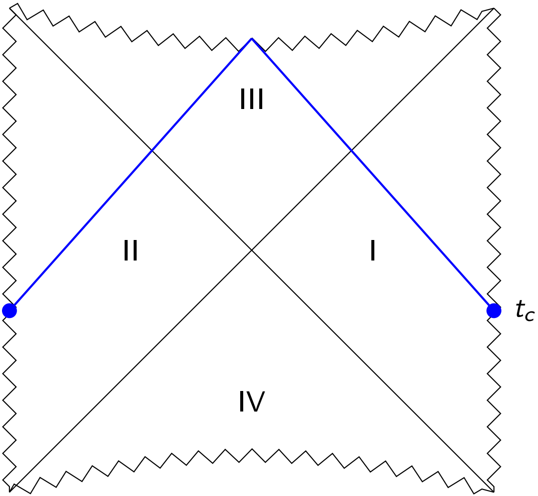
5. exercise: debug the following wrong argument. In the decoupling limit $E^2 \alpha' \to 0$, so we should be able to neglect all α' corrections to SUGRA.


III
II
I
t_c
IV

End of the gravity review
⇒ Boundary methods for analyzing low dimensional SYM

Refs for bootstrap:
[Anderson & Kruczenski, 1612.08140],
[HL, 2002.08387], [Kazakov & Zheng, 2108.04830]
[Han, Harnoll, Kruthoff, 2004.10212]
[Fawzi, Fawzi, Scalet, 2311.18706] [Cho, Gabai, Sandor, Yin, 2410.04262]
[HL, 2302.04416] [HL & Zheng, 2410.14647]

Numerically, heroic Monte Carlo simulations have been performed [Kabat *et al.*, Anagnostopoulos *et al.*, Hanada *et al.*, ..., Berkowitz *et al.*, Pateloudis *et al.*].

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- metastability: some problems ill-defined at finite N

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  Number of correlators of fixed length grows $\sim D^L$.

$$\langle \mathrm{Tr}\, ABABBBA \rangle \,, \langle \mathrm{Tr}\, AAABBBB \rangle \,, \cdots$$

1-Matrix model

Probability distribution over $N \times N$ Hermitian matrix M_{ij}:

$$p(M) = \frac{1}{\mathcal{Z}} e^{-N^2 \,\mathrm{tr}\, V(M)}, \quad V(M) = \frac{1}{2}M^2 + \frac{g}{4}M^4$$

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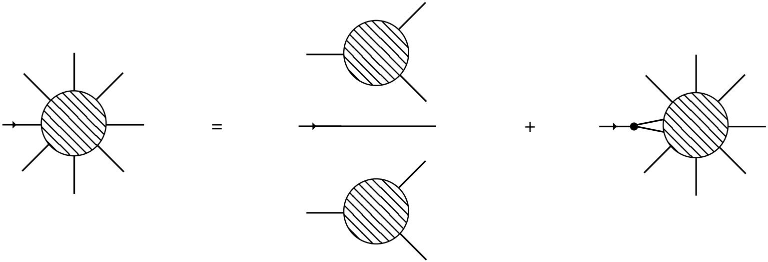
Goal: compute moments $\langle \operatorname{tr} M^k \rangle$ as a function of g.

$$\langle \operatorname{tr} M^2 \rangle = \lim_{N \to \infty} \frac{1}{\mathcal{Z}} \int \mathrm{d}M \, e^{-N^2 \operatorname{tr} V(M)} \operatorname{tr} M^2$$

Bootstrapping matrices

1. Guess the value of some simple correlator, e.g. $\langle \operatorname{tr} M^2 \rangle$
2. Feed it through the loop eqns to generate more correlators
3. Demand that $\langle \operatorname{tr} \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0$.

Loop (Schwinger-Dyson) equations



- relates lower-pt correlators to higher-pt correlators
- uses large N factorization ('t Hooft)

$$\langle \mathrm{tr}\, M^k \rangle = \sum_{\ell=0}^{k-1} \langle \mathrm{tr}\, M^\ell \rangle \langle \mathrm{tr}\, M^{k-\ell-2} \rangle + g \langle \mathrm{tr}\, M^{k+2} \rangle$$

Positivity

Naive algorithm: starting with some guess for $\langle \operatorname{tr} M^2 \rangle$, generate moments $\langle \operatorname{tr} M^4 \rangle, \langle \operatorname{tr} M^6 \rangle, \langle \operatorname{tr} M^8 \rangle, \cdots$.

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Positivity

More systematically, we can consider a general polynomial in the matrix M:

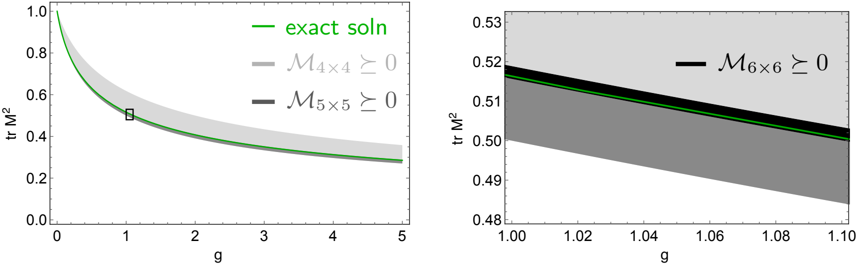
$$\mathcal{O} = \sum \alpha_k M^k \Rightarrow \operatorname{tr} \mathcal{O}^\dagger \mathcal{O} \geq 0.$$

This implies that $\alpha_i^* \mathcal{M}_{ij} \alpha_j \geq 0$ for all coefficients α, where we have assembled all the correlators into a big matrix $\mathcal{M}_{ij} = \left\langle \operatorname{tr} M^{i+j} \right\rangle$:

$$\mathcal{M} = \begin{pmatrix} 1 & \langle \operatorname{tr} M \rangle & \left\langle \operatorname{tr} M^2 \right\rangle \\ \langle \operatorname{tr} M \rangle & \left\langle \operatorname{tr} M^2 \right\rangle & \left\langle \operatorname{tr} M^3 \right\rangle \\ \left\langle \operatorname{tr} M^2 \right\rangle & \left\langle \operatorname{tr} M^3 \right\rangle & \left\langle \operatorname{tr} M^4 \right\rangle \end{pmatrix} \succeq 0$$

Here $\mathcal{M}_{ij} = \left\langle \operatorname{tr} M^{i+j} \right\rangle$.

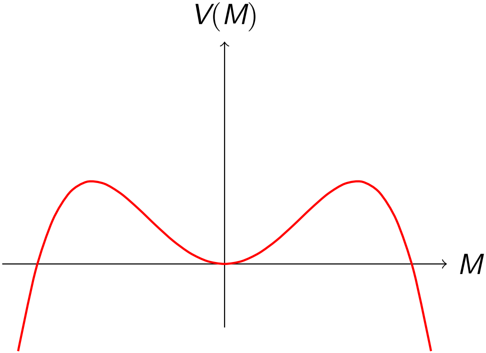
Review of the matrix bootstrap



As the size of $\mathcal{M}$ increases, rapid convergence to the exact solution.

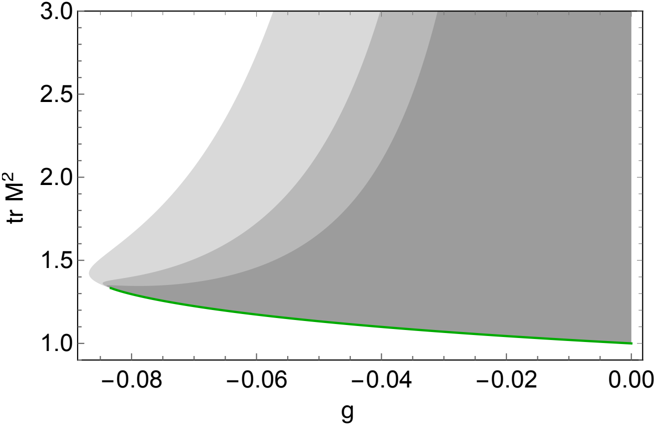
Metastability

To address the issue of metastability, consider $g < 0$. The potential is unbounded from below:



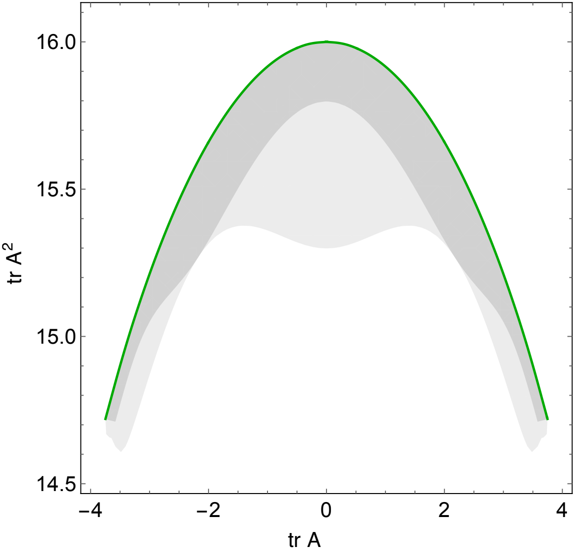
In the large N limit, tunneling is suppressed.

Metastability



For $-g_* < g < 0$ the model still makes sense at $N = \infty$

Metastability $V = -\frac{1}{2}A^2 + \frac{1}{4}gA^4$



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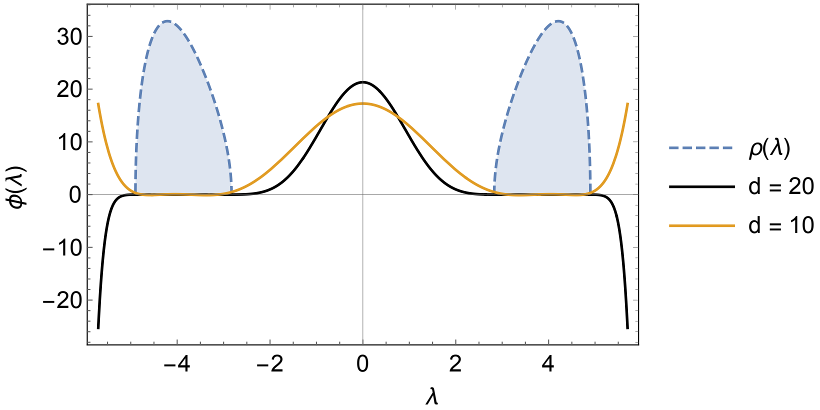

	$d = 0$ (stat mech)	$d = 1$
solvable	1-matrix integral	1-matrix quantum mech $c = 1$ matrix model
unsolvable	**multi-matrix integral**	D0-brane quantum mech BFSS matrix theory

Multi-matrix integrals

Main challenge: exponentially many correlators for a given length L, e.g., for $L = 7$:

$$\langle \mathrm{Tr}\, ABABBBA \rangle, \langle \mathrm{Tr}\, BBBABAB \rangle, \cdots$$

Also more loop equations and more positivity constraints:

$$\mathcal{M} = \begin{pmatrix} 1 & \mathrm{Tr}\, A & \mathrm{Tr}\, B & \cdots \\ \mathrm{Tr}\, A & \mathrm{Tr}\, A^2 & \mathrm{Tr}\, AB & \\ \mathrm{Tr}\, B & \mathrm{Tr}\, BA & \mathrm{Tr}\, B^2 & \\ \vdots & & & \ddots \end{pmatrix}$$

Despite these challenges, the bootstrap gives strong results for multi-matrix integrals [HL '20].

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$$\mathcal{Z} = \int \mathrm{d}A \, \mathrm{d}B \, e^{-N^2 \operatorname{tr} V(A,B)}$$

$$V(X, Y) = -\frac{1}{2}[A, B]^2 + v(A) + v(B),$$

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Using non-linear relaxation, one can convert it to a standard semi-definite programming problem [Kazakov & Zheng '22].

$$0.4217836 \leq \left\langle \operatorname{tr} A^2 \right\rangle \leq 0.4217847$$

$$0.3333413 \leq \left\langle \operatorname{tr} A^4 \right\rangle \leq 0.3333421$$

~ 6 decimal digits on a laptop!

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1-matrix QM

N^2 non-relativistic particles arranged in a matrix.

$$\mathrm{i}[X_{ij}, P_{kl}] = \delta_{il}\delta_{jk}.$$

Hamiltonian:

$$H = N\left(\frac{1}{2}\,\mathrm{Tr}\,P^2 + \frac{m^2}{2}\,\mathrm{Tr}\,X^2 + \frac{g}{4}\,\mathrm{Tr}\,X^4\right).$$

U(N) gauge constraint:

$$J_{ik} = \mathrm{i}(X_{ij}P_{jk} - P_{ij}X_{jk}) + N\delta_{ik} = 0$$

[for a review, see Klebanov hep-th/9108019] [Brezin, Itzykson, Parisi, Zuber, Douglas, Klebanov, Kutasov, Maldacena, Martinec, Takayangi, Toumbas, Verlinde, $\cdots$]

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known as $c = 1$ or $\hat{c} = 1$ matrix model[1].

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[1] in the double scaling limit

Review of the quantum mechanical bootstrap

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   example: $0 = \langle [\mathrm{tr}\, XP, H] \rangle = -\mathrm{tr}\, P^2 + \mathrm{tr}\, X^2 + g\, \mathrm{tr}\, X^4$

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$$\langle E|\,\mathrm{tr}\,\mathcal{O}^\dagger\mathcal{O}\,|E\rangle \geq 0 \Rightarrow \mathcal{M}_{ij} = \langle E|\,\mathrm{tr}\,\mathcal{O}_i^\dagger\mathcal{O}_j\,|E\rangle \geq 0$$
3. Optional: ground state bootstrap positivity:
$$\langle O^\dagger[H, O]\rangle_{\mathrm{gs}} = \langle O^\dagger H O\rangle_{\mathrm{gs}} - E_{\mathrm{gs}}\langle O^\dagger O\rangle_{\mathrm{gs}} \geq 0$$

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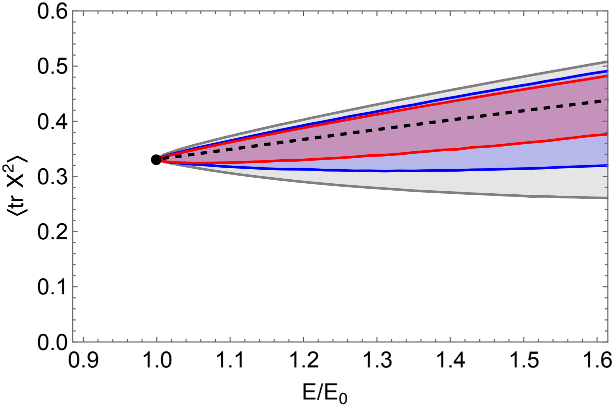
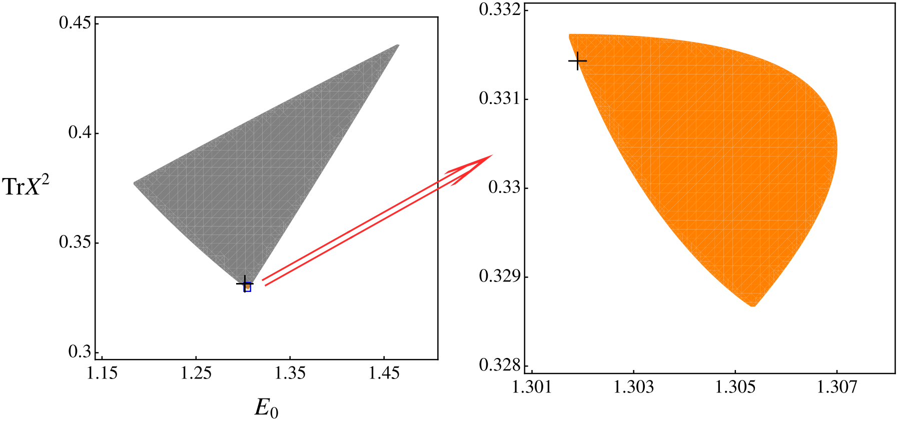
Finite energy bootstrap



Dashed line is the exact solution for $g = 1$.

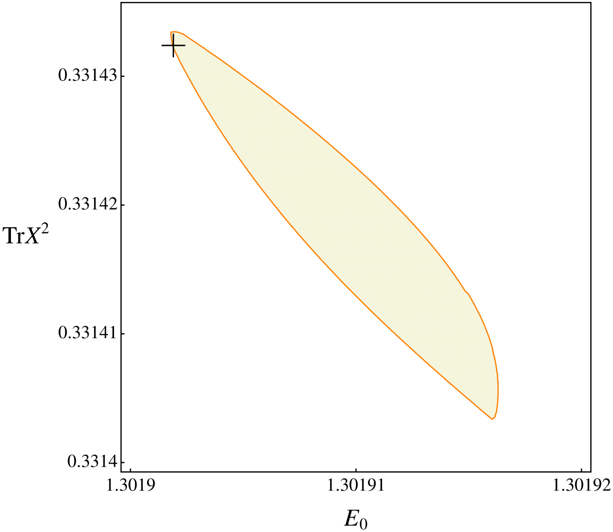
[WIP w/ Zechuan Zheng; see also Han, Hartnoll, Kruthoff '20]

Ground state bootstrap



$+$ denotes the exact solution for $g = 1$

Ground state bootstrap



$+$ denotes the exact solution for $g = 1$.
~ 6 digit precision on a laptop.

D0-brane quantum mechanics

9 bosonic matrices and 16 fermionic matrices. Transform as a vector and spinor of $SO(9)$.

$$H = \frac{1}{2}\mathrm{Tr}\left(P_I^2 - \frac{1}{2}\left[X_I, X_J\right]^2 - \psi_\alpha \gamma^I_{\alpha\beta}\left[X_I, \psi_\beta\right]\right)$$

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$\mathcal{N} = 16$ SUSY:

$$Q_\alpha = \mathrm{Tr}\, P_I \gamma^I_{\alpha\beta}\psi_\beta - \frac{\mathrm{i}}{2}\,\mathrm{Tr}\left[X^I, X^J\right]\gamma^{IJ}_{\alpha\beta}\psi_\beta,$$
$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta} H + 2\gamma^I_{\alpha\beta}\,\mathrm{Tr}\, X^I C$$

$C_{ij} =$ generator of SU(N).

Review of known finite N BFSS properties

Believed to have 1 normalizable energy eigenstate $|\Omega\rangle$ with $E = 0$.

The bound state has power law tails $\psi(r) \sim 1/r^9$. This implies that at finite N, $\langle\Omega| \operatorname{Tr} X^L |\Omega\rangle \sim \int r^8 \mathrm{d}r |\psi|^2 \to \infty$ if $L \leq 9$.

All other states are believed to be scattering states $E > 0$.

Some limited information about the S-matrix is known. [Douglas, Kabat, Pouliot, Shenker, Paban, Sethi, Stern, Becker, Becker, Polchinski][Maldacena Herderschee]

[Polchinski '99] gave a lower bound on $\langle\Omega| \operatorname{Tr} X^4 |\Omega\rangle$.

Bosonic constraints: round 1

Commutator constraints

$$: \left\langle [H, \mathrm{Tr}\, X^2] \right\rangle = 0 \Rightarrow \left\langle \mathrm{Tr}\, X^I P_I + P^I X_I \right\rangle = 0.$$

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Positivity:

$$\mathcal{M} = \begin{pmatrix} \mathrm{Tr}\, X^2 & \mathrm{Tr}\, XP \\ \mathrm{Tr}\, PX & \mathrm{Tr}\, P^2 \end{pmatrix} \succeq 0$$

$$\Rightarrow \quad \sum_I \left\langle \mathrm{Tr}\, X^2 \right\rangle \left\langle \mathrm{Tr} \left(P^I P_I \right) \right\rangle \geq \frac{9}{4} N^4.$$

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Next: replace $\mathrm{Tr}\, P^2$ (kinetic energy) with potential energy.

Bosonic constraints: round 2

$$\boxed{\left\langle N \operatorname{Tr} X^4 \right\rangle^{1/2} \left(\frac{144}{g^2} \left\langle \operatorname{Tr} X^4 \right\rangle + \frac{2E}{3} \right) \geq \frac{9}{4} g^2 N^4}$$

Comments:

- Setting $E = 0$ recovers Polchinski point. Assuming parametric saturation of the bd implies that ``typical eigenvalue'' $r \sim \lambda^{1/3}$, which is the size of the gravity region.
- Scale at which the bd varies is $E/N^2 \sim \lambda^{1/3}$, regime of validity of gravity.
- No good bound on $\left\langle \operatorname{Tr} X^2 \right\rangle$.

Fermionic constraints

Had two eqns:

$$-2\langle K\rangle + 4\langle V\rangle + \langle F\rangle = 0, \quad \langle K\rangle + \langle V\rangle + \langle F\rangle = E$$

In addition to solving for V, can solve for F:

$$\langle F\rangle = 2\left(\frac{1}{3}\langle E\rangle - \langle V\rangle\right)$$

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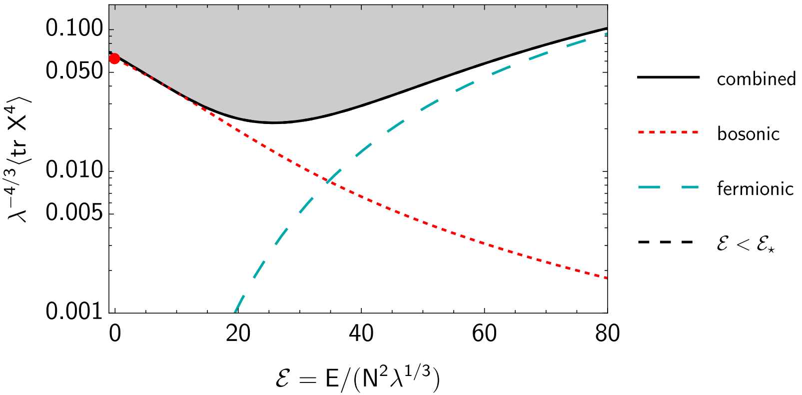
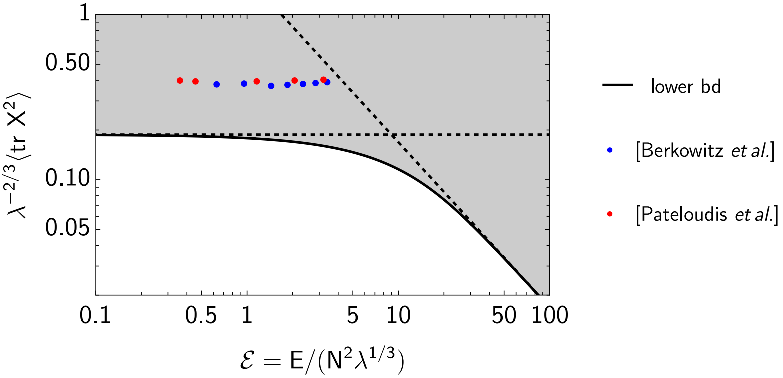
Therefore, as F gets large, X cannot be too small $\Rightarrow \operatorname{tr} X^2$ has a lower bound.

Lower bounds on $\langle \mathrm{tr}\, X^4 \rangle$



Constraints on $\langle \mathrm{tr}\, X^2 \rangle$



Large N extrapolation of Monte Carlo simulations [Pateloudis *et al.*] are $\sim 1/2$ from the lower bound.

Now let's set up the bootstrap more systematically and explain the numerics.

- **Dynamics**:

$$\langle\Omega|\left\{Q_\alpha, O_\alpha\right\}|\Omega\rangle = 0.$$

O_α is any single trace, SO(9) spinor.

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example:

$O_\alpha \propto \gamma^{IJ}_{\alpha\beta}\,\mathrm{tr}\,\psi_\beta X^I P^J \Rightarrow -2\mathrm{i}\left\langle \mathrm{tr}\left[X^I, X^J\right] X^I P^J \right\rangle = \left\langle \mathrm{tr}\,\psi_\alpha \psi_\alpha X^I X^I \right\rangle.$

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- **Hierarchy:**

$\langle \mathrm{tr}\, W\rangle$ where W a word made of X, ψ, P.
level: $\ell(X) = 1, \ell(\psi) = \frac{3}{2}, \ell(P) = 2.$

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  - SO(9) invariance, SU(N) invariance $\langle \mathrm{tr}(OC)\rangle = 0$.
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  - example:

$$\left\langle \mathrm{tr}\, X^{I_1} X^{I_2} X^{I_3} P^{I_4} X^{I_5} X^{I_6} \right\rangle = \left\langle \mathrm{tr}\, X^{I_2} X^{I_3} P^{I_4} X^{I_5} X^{I_6} X^{I_1} \right\rangle + \mathrm{i} \left\langle \mathrm{tr}\, X^{I_2} X^{I_3} \right\rangle \left\langle \mathrm{tr}\, X^{I_5} X^{I_6} \right\rangle \delta^{I_1 I_4}$$

SO(9) group theory

Only SO(9)-invariant operators have non-zero vevs.

However, positivity requires that we consider non-singlet operators in intermediate steps.

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However, positivity requires that we consider non-singlet operators in intermediate steps.

Trivial example: $\langle \mathrm{tr}(X^J X^J) \rangle \geq 0$. Derived by observing that it is the sum of squares of X^J (a non-invariant operator).

Kinematic constraints

Less trivial example:

$$\mathcal{M}_{\alpha\beta\gamma\eta} = \left\langle \mathrm{tr}\left(\psi^{\alpha}\psi^{\beta}\psi^{\delta}\psi^{\eta}\right)\right\rangle$$

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$$\mathcal{M}_{\alpha\beta\gamma\eta} = \left\langle \mathrm{tr}\left(\psi^\alpha \psi^\beta \psi^\delta \psi^\eta\right)\right\rangle$$

Viewed as a matrix in the $\{\alpha, \beta\}$ and $\{\gamma, \eta\}$ indices, positivity requires $\mathcal{M} \succeq 0$.

Kinematic constraints

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$$\mathcal{M}_{\alpha\beta\gamma\eta} = \left\langle \mathrm{tr}\Big(\psi^{\alpha}\psi^{\beta}\psi^{\delta}\psi^{\eta}\Big)\right\rangle$$

Using the "addition of SO(9) angular momentum" rules:

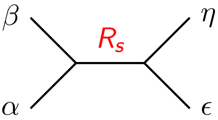
$$\begin{aligned}(\mathbf{16})^4 &= (\mathbf{16}\times\mathbf{16})^2 = (\mathbf{1}+\mathbf{9}+\mathbf{36}+\mathbf{84}+\mathbf{128})^2 \\ &= 5(\mathbf{1}) + \text{non-singlets}\end{aligned}$$

Thus group theory determines this $16^4 = 65536$ to just 5 unknowns.

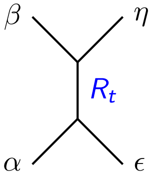
$$\begin{aligned}\mathcal{M}_{\alpha\beta\gamma\eta} = {}& \delta_{\alpha\beta}\delta_{\eta\epsilon} a_{\mathbf{1}} + \gamma^{I}_{\alpha\beta}\gamma^{I}_{\eta\epsilon} a_{\mathbf{9}} + \gamma^{IJ}_{\alpha\beta}\gamma^{IJ}_{\eta\epsilon} a_{\mathbf{36}} + \gamma^{IJK}_{\alpha\beta}\gamma^{IJK}_{\eta\epsilon} a_{\mathbf{84}} \\ & + \gamma^{IJKL}_{\alpha\beta}\gamma^{IJKL}_{\eta\epsilon} a_{\mathbf{128}}\end{aligned}$$

Cyclicity and the fermion anti-commutation relations cuts this further to just 2 unknowns.

Expand *s*-channel block in terms of *t*-channel blocks:



$$= \sum_{R_t} \mathbb{F}_{R_s,R_t}\begin{bmatrix} \mathbf{16} & \mathbf{16} \\ \mathbf{16} & \mathbf{16} \end{bmatrix}$$



$\Rightarrow$ 6j symbol. At higher levels, need higher-pt crossing kernels.

Kinematics determined $\mathcal{M}_{\alpha\beta\gamma\eta}$ in terms of 2 unknowns. We still need to impose positivity of a large matrix $\mathcal{M}_{\alpha\beta,\gamma\eta}$. By decomposing $\psi_\alpha\psi_\beta$ into irreps, one can easily diagonalize $\mathcal{M}$.

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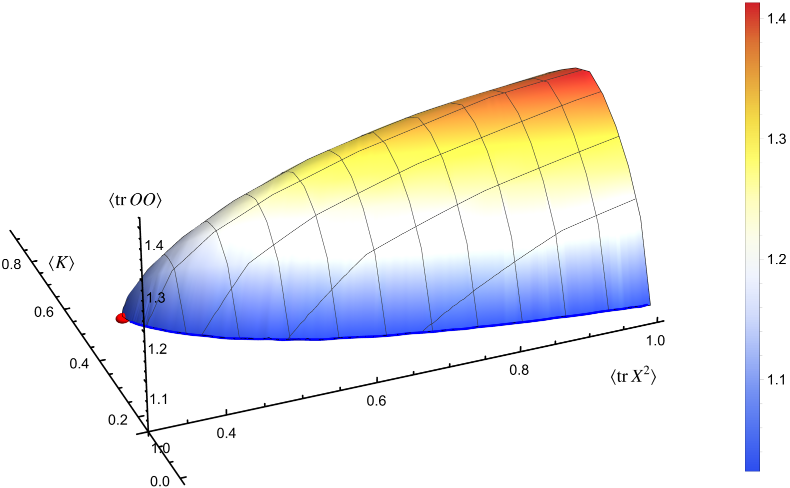
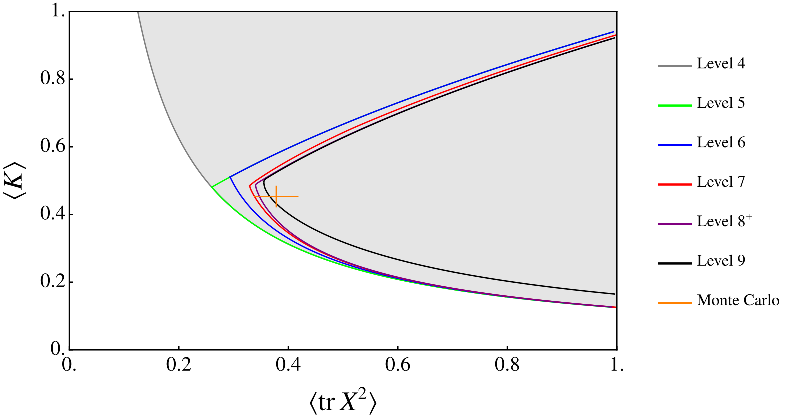
The upshot is that by leveraging the symmetries of the model, the D0-brane bootstrap is practical. ☺

level	total variables	free variables
4	11	3
5	38	4
6	140	11
7	569	18
8	2528	59
9	12077	149

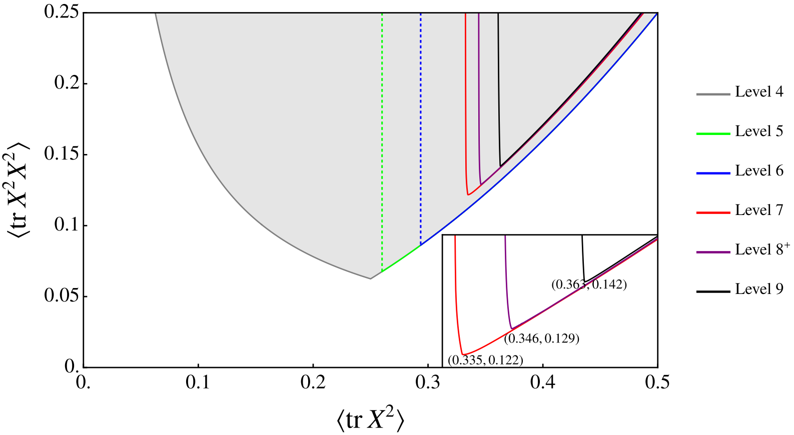
# of single trace SO(9) singlets, before and after modding out by the EOM/kinematic constraints.



⟨tr OO⟩
1.4
1.3
1.2
1.1
1.0
⟨K⟩
0.8
0.6
0.4
0.2
0.0
⟨tr X^2⟩
0.4
0.6
0.8
1.0
1.4
1.3
1.2
1.1



Cross + is the Monte Carlo result* of [Berkowitz *et al.*'16].



The lower bound on $\langle \mathrm{tr}\, X^2 X^2 \rangle$ was derived (up to some factors) in [Polchinski '99]. It can also be improved to finite energy [HL '23].

	method	$\langle \mathrm{tr}\, X^2 \rangle$
	Monte Carlo [Pateloudis *et al.*'22]	$\approx 0.37 \pm 0.05$
	primitive bootstrap [HL '23]	≥ 0.1875
	bootstrap level 6	≥ 0.294
	bootstrap level 7	≥ 0.331
	bootstrap level 8^+	≥ 0.3401
	bootstrap level 9	≥ 0.3451

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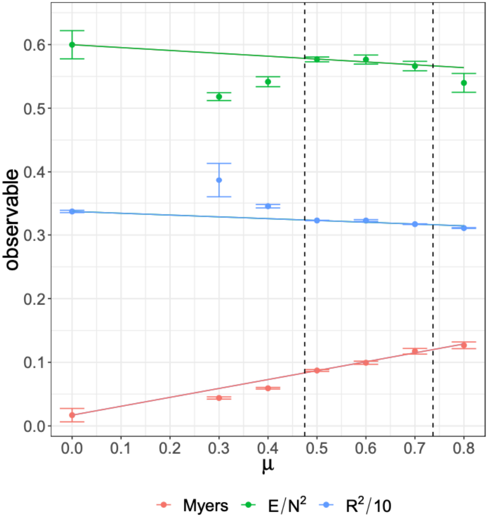
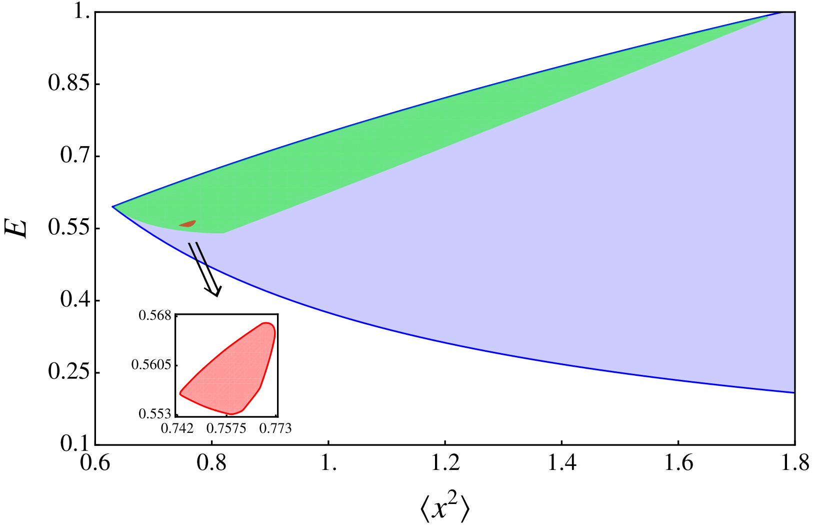
$\sim 90\%$ of the MC value with just level 7:
19 variable SDP, ~ 170 EoMs, matrices of size $\lesssim 20 \times 20$.

Metastability in Monte Carlo



Monte Carlo results [Pateloudis *et al.* '22]

Toy supermembrane problem



In a simpler toy problem, we see a similar-looking peninsula at low levels, but an island at higher levels.

Finite temperature generalization

[Araki Sewell '77] Somewhat surprising reformulation of KMS condition:

$$\left\langle \mathcal{O}^\dagger \mathcal{O} \right\rangle \log \frac{\langle \mathcal{O}^\dagger \mathcal{O} \rangle}{\langle \mathcal{O} \mathcal{O}^\dagger \rangle} \leq \beta \left\langle \mathcal{O}^\dagger [H, \mathcal{O}] \right\rangle$$

KMS $\iff$ inequality holds for all operators $\mathcal{O}$.

Can deal with the log using non-linear relaxation. [Fawzi, Fawzi, Scalet '24] [Cho,Gabai,Sandor,Yin '24]

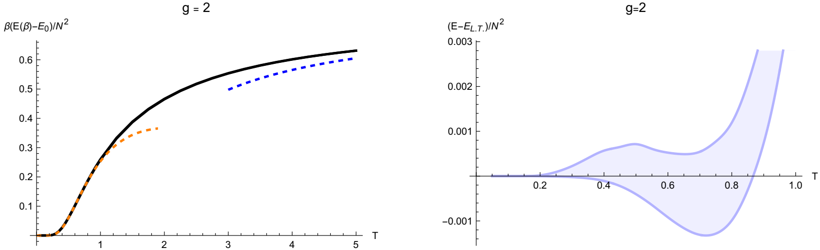
Finite temperature ungauged 1-matrix QM

Back to the 1-matrix model [Cho,Gabai,Sandor,Yin '24]

$$H = N\left(\frac{1}{2}\,\mathrm{Tr}\,P^2 + \frac{m^2}{2}\,\mathrm{Tr}\,X^2 + \frac{g}{4}\,\mathrm{Tr}\,X^4\right).$$

Choose not to impose the gauge constraint, e.g., sum over all SU(N) charge sectors.

Finite temperature ungauged 1-matrix QM



Level 10 results, $\delta E/E \sim 10^{-3}$. [Cho,Gabai,Sandor,Yin '24]

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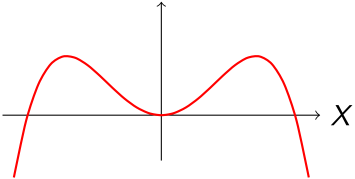
$V(X) \quad =$ 

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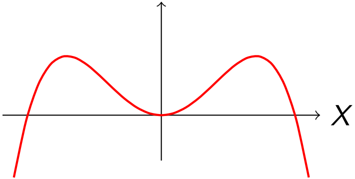

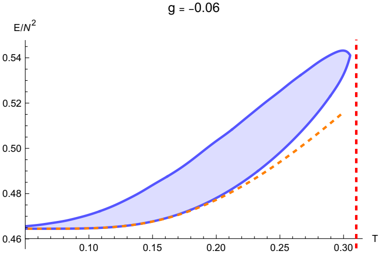

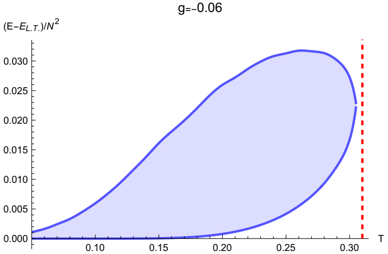

Finite temperature bootstrap

Ungauged 2-matrix quantum mechanics [Cho,Gabai,Sandor,Yin '24]:

$$H = \frac{1}{2}\,\mathrm{tr}\big(P_X^2 + P_Y^2 + (X^2 + Y^2) - 2g[X, Y]^2\big)$$

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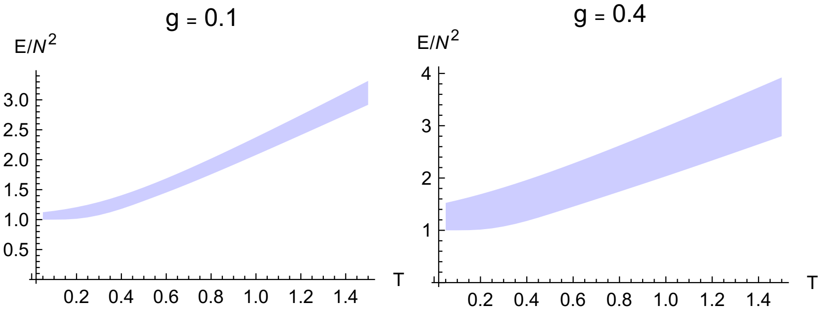

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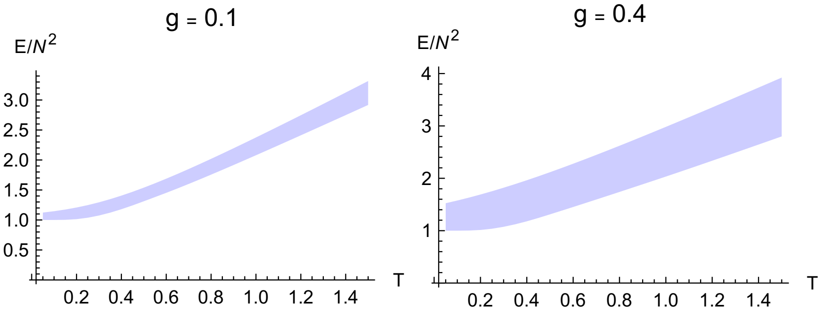

In the future, BFSS at finite temp?!

Future directions

- Islands?
- Constraints on the bound state?
- Finite energy/temperature BFSS
- Large N lattice systems, especially those with sign problems?
  [Anderson & Kruczenski, Kazakov & Zheng, ...]

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Thanks!

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$$\frac{\mathrm{d}s^2}{\alpha'} = -f(r) r_c^2 \,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r) r_c^2} + \left(\frac{r}{r_c}\right)^{-3/2} \mathrm{d}\Omega_8^2$$

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S_8 *shrinks* with r. At $r \sim \lambda^{1/3} \Rightarrow$ string scale curvature.

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The semiclassical BH geometry and its stringy corrections

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The semiclassical BH geometry and its stringy corrections

In principle, this includes properties that are currently inaccessible by worldsheet methods.

[See Hanada *et al.*, Berkowitz *et al.*, Pateloudis, *et al.*for similar discussions involving the BH thermodynamics. In principle using the Fawzi, Fawzi, & Scalet one can bootstrap the thermodynamics.]

In principle, a theory of quantum gravity should predict the higher-derivative corrections to Einstein gravity, e.g.,

$$\mathcal{L} \sim R + \#\alpha'^3 R^4 + \#\alpha'^3 R^3 F^2 + \cdots .$$

For charged black holes (with Ramond-Ramond gauge fields), the *leading* correction is unknown.

A precision measurement of certain correlators will give us information about these corrections. Similar program in the CFT bootstrap; e.g., [Binder, Chester, Pufu, Wang '19]

A clear target is the only SO(9) singlet field in this background χ.

χ has scaling dimension $\Delta = 28/5$ [Sekino & Yoneya '00, Biggs & Maldacena '23]. The leading α'^3 correction breaks the scaling symmetry and gives rise to a non-trivial 1-pt function:

$$S_{\text{eff}} \supset \frac{(\alpha')^3}{G_N} \int \sqrt{g} e^{-2\phi} \chi \left(\#_1 R^4 + \#_2 e^{2\phi} R^3 F^2 + \cdots + \right)$$

$$\langle \mathcal{O}_\chi \rangle \propto T^{\Delta+\delta} = T^{28/5}$$

On the matrix side, the operator $\mathcal{O}_\chi$ is a known level 8 operator [Van Raamsdonk and Taylor '98] :

$$\mathcal{O}_\chi \sim \mathrm{Tr}\, P^I P^I P^J P^J + \mathrm{Tr}[X_I, X_J][X_J, X_K] P^K P^I + \cdots + \text{fermions}$$

χ is also expected to contribute to a generic SO(9) singlet due to operator mixing, e.g.,

$$\left\langle \mathrm{tr}\, X^2 \right\rangle \sim \#_1 + \#_H T^{14/5} + \#_{H'} T^{23/5} + \#_\chi T^{28/5} + \cdots$$

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1. solvable matrix models can also be solved by bootstrap
2. for "unsolvable" models like BFSS, bootstrap yields non-trivial bounds. Old results from the matrix side [Polchinski '99] can be reformulated and improved as a bootstrap result.
3. In principle, we could learn about stringy black holes using the bootstrap. We are in the process of putting this into practice.

Future directions I.

Bootstrapping the thermal entropy, e.g.,

$$S = A/(4G_N) + \text{corrections.}$$

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Recent progress [Fawzi, Fawzi & Scalet '23] in inputting the KMS condition into the bootstrap (in the Hamiltonian approach). Uses a non-linear relaxation of the relative entropy.

Can be applied to large N matrix quantum mechanics [Cho, Sandor, & Yin, WIP]

Future directions II.

	$d = 0$	$d = 1$	$d \geq 2$
	1-matrix integral	1-matrix model $c = 1$ matrix model	
	multi-matrix integral	D0-brane BFSS	

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	$d = 0$	$d = 1$	$d \geq 2$
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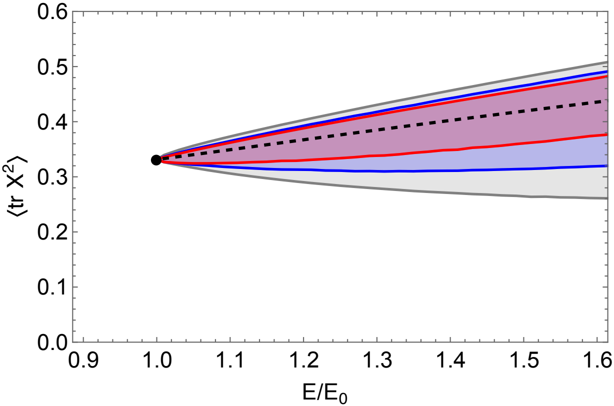
	multi-matrix integral	D0-brane BFSS	large N Yang Mills large N QCD

Already some interesting progress...

[Anderson & Kruczenski '16] [Kazakov & Zheng '22] [Kazakov & Zheng, '24]

Many other strongly-coupled lattice systems seem possible...

Finite energy bootstrap



Dashed line is the exact solution for $g = 1$.

[WIP w/ Zechuan Zheng; see also Han, Hartnoll, Kruthoff '20]

(note: we are considering high energies $E \sim N^2$ even though for the 1-matrix model there are only N eigenvalues).