Susy States in N=4 Yang Mills from Grey Galaxies and Dressed Dual Black Holes

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References

- ArXiv 2501.17217
- "Supersymmetric Grey Galaxies, Dual Dressed black holes and the Superconformal Index"
- S. Choi, D. Jain, S. Kim, V. Krishna, G. Kwon, E. Lee, S.M., C. Patel

Introduction

- $\mathcal{N} = 4$ Yang Mills is a maximally supersymmetric [1]. theory has been intensively studied over the last 30 years, especiallly in the large N limit.
- Nonetheless, the spectrum of supersymmetric states in this theory is still poorly understood, even in the large N limit.
- In this talk I will review the situation and then present a few new conjectures about this susy spectrum.

[1] It enjoys invariance under $16 + 16$ supercharges. As susy multiplets more than 16 supercharges always have fields with spin larger than one, this is likely the largest number of susys an interacting QFT can have.

The special supercharge

- Complete information about states annihilated by any one leftmoving charge completely determines the spectrum. Choose
- $$\mathcal{Q} \sim (Q_1, Q_2, Q_3, J_1, J_2) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \qquad (1)$$
- $$2\{\mathcal{Q}, \mathcal{S}\} = E - (J_1 + J_2 + Q_1 + Q_2 + Q_3) \equiv \Delta \qquad (2)$$
- State susy same as $\Delta = 0$.

Table of Adjoint Letters in $\mathcal{N} = 4$ SYM

Letters	E	(J_L, J_R)	(J_1, J_2)	(Q_1, Q_2, Q_3)
X, Y, Z	1	$(0, 0)$	$(0, 0)$	$(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$
$\bar{\psi}_{0,\pm}$	$\frac{3}{2}$	$(0, \pm\frac{1}{2})$	$(\pm\frac{1}{2}, \mp\frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
$\psi_{+,0}$	$\frac{3}{2}$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, -$
F_{++}	2	$(1, 0)$	$(1, 1)$	$(0, 0, 0)$
$\mathcal{Q}^{+++}_{-\frac{1}{2},0}$	$\frac{1}{2}$	$(-\frac{1}{2}, 0)$	$(-\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
$D_{+\pm}$	1	$(\frac{1}{2}, \pm\frac{1}{2})$	$(1, 0), (0, 1)$	$(0, 0, 0)$

Table: The BPS letters in $\mathcal{N} = 4$ Yang-Mills and their charges. Here E is the energy, $J_{L/R}$ or $J_{1/2}$ label the angular momenta and are related by $J_{L/R} = \frac{J_1 \pm J_2}{2}$.

Bosonic and Bose Fermi Cone

- Charges of Bosonic letters:

$$v_1 = (1, 0, 0, 0, 0)\,, \qquad v_2 = (0, 1, 0, 0, 0)\,, \qquad v_3 = (0, 0, 1, 0, 0)$$
$$v_4 = (0, 0, 0, 1, 0)\,, \qquad v_5 = (0, 0, 0, 0, 1) \tag{3}$$

- Charges from multiparticling (Bosonic Cone)

$$\zeta_i \geq 0, \quad (i = 1 \dots 5) \tag{4}$$

- Charges of Fermionic Letters

$$v_6 = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), v_7 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), v_8 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$
$$v_9 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), v_{10} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \tag{5}$$

- Multiparticling (Bose Fermi Cone). Saturation $(1/8)^{th}$ BPS.

$$\zeta_i + \zeta_j \geq 0, \quad (i, j = 1 \dots 5) \tag{6}$$

Table of gas modes

Word	J_L	J_R	R_1	R_2	R_3	Numerator(n_r)
$\mathrm{Tr}(X^m)$	0	0	0	m	0	$1-d_r$
$\mathrm{Tr}(\psi X^m)$	$\frac{1}{2}$	0	0	m	1	$3e^{-\frac{\mu}{2}-\frac{\omega_L}{2}}-e^{-\frac{3\mu}{2}-\frac{\omega_L}{2}}$
$\mathrm{Tr}(\bar\psi X^m)$	0	$\frac{1}{2}$	1	m	0	$e^{-\frac{3\mu}{2}}\left(e^{-\frac{\omega_R}{2}}+e^{\frac{\omega_R}{2}}\right)$
$\mathrm{Tr}(FX^m)$	1	0	0	m	0	$e^{-\omega_L}$
$\mathrm{Tr}(\bar\psi\bar\psi X^m)$	0	0	2	m	0	$e^{-3\mu}$
$\mathrm{Tr}(\psi\bar\psi X^m)$	$\frac{1}{2}$	$\frac{1}{2}$	1	m	1	$(3e^{\mu}-1)(e^{\omega_R}+1)\,e^{-3\mu-\frac{\omega_L}{2}-}$
$\mathrm{Tr}(F\bar\psi X^m)$	1	$\frac{1}{2}$	1	m	0	$\left(e^{-\frac{\omega_R}{2}}+e^{\frac{\omega_R}{2}}\right)e^{-\frac{3\mu}{2}-\omega_L}$
$\mathrm{Tr}(\psi\bar\psi\bar\psi X^m)$	$\frac{1}{2}$	0	2	m	1	$3e^{-\frac{7\mu}{2}-\frac{\omega_L}{2}}-e^{-\frac{9\mu}{2}-\frac{\omega_L}{2}}$
$\mathrm{Tr}(F\bar\psi\bar\psi X^m)$	1	0	2	m	0	$e^{-3\mu-\omega_L}$
$\mathrm{Tr}(D_{+\dot\alpha}\bar\psi^{\dot\alpha})$	$\frac{1}{2}$	0	0	0	0	$(1-e^{-\mu})^3\left(-e^{-\frac{3\mu}{2}-\frac{\omega_L}{2}}\right)$

- No more than 3 fermionic letters per trace. So gas charges (at large charge) lie in the Bosonic Cone.

Black Holes

- Susy black holes live on the manifold

$$\frac{j_1 j_2}{2} + q_1 q_2 q_3 = \left(q_1 + q_2 + q_3 + \frac{1}{2}\right)\left(q_1 q_2 + q_1 q_3 + q_2 q_3 - \frac{1}{2}(j_1 + j_2)\right) \tag{7}$$

subject to the inequalities

$$\begin{aligned} & q_1 + q_2 + q_3 + \frac{1}{2} > 0, \\ & \frac{j_1 j_2}{2} + q_1 q_2 q_3 > 0, \\ & q_1 q_2 + q_1 q_3 + q_2 q_3 - \frac{1}{2}(j_1 + j_2) > 0. \end{aligned} \tag{8}$$

- Their entropy is given by

$$S \equiv N^2 S_{BH}(\zeta_i) = 2\pi N^2 \sqrt{q_1 q_2 + q_2 q_3 + q_3 q_1 - (j_1 + j_2)/2} \tag{9}$$

In bose fermi cone. But almost every boundary point outside bosonic cone

The black hole ‘apple’

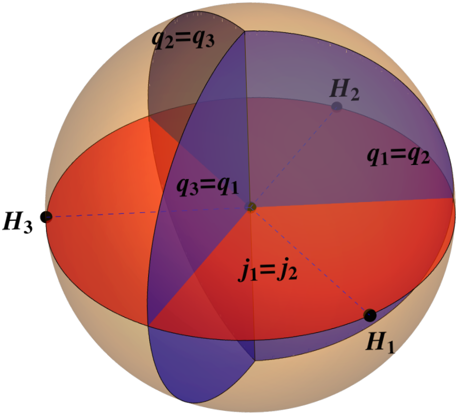


Figure: Schematic representation of the base of the black hole sheet

Partition function and reduced chemical potentials

- $$Z_{\rm gen} = {\rm Tr} e^{-\beta(E-\Omega_1 J_1-\Omega_2 J_2-\Delta_1 Q_1-\Delta_2 Q_2-\Delta_3 Q_3)} \qquad (10)$$

  Set

  $$\Omega_i = 1 - \frac{\omega_i}{\beta}, \quad (i = 1 \ldots 2)$$
  $$\Delta_j = 1 - \frac{\mu_j}{\beta}, \quad (j = 1 \ldots 3) \qquad (11)$$

- Plugging (11) into (10) we obtain

  $$Z_{\rm gen} = {\rm Tr} e^{-\beta(E-J_1-J_2-Q_1-Q_2-Q_3)} \times e^{-\omega_1 J_1-\omega_2 J_2-\mu_1 Q_1-\mu_2 Q_2-\mu_3 Q_3} \qquad (12)$$

- In the limit $\beta \to \infty$ reduces to

  $$Z_{BPS} = {\rm Tr}_{\rm BPS}\, e^{-\omega_1 J_1-\omega_2 J_2-\mu_1 Q_1-\mu_2 Q_2-\mu_3 Q_3} \qquad (13)$$

  $(\mu_1, \mu_2, \mu_3, \omega_1, \omega_2) \equiv (\nu_i)$ renormalized chemical potentials. Part fn only convergent when ${\rm Re}(\nu_i) \geq 0 \; \forall i$

Chemical potentials of susy black holes

- Define by taking zero temperature limit of nonsusy black holes. First find $\Omega_i = \Delta_m = 1$. So susy black holes compute trace on BPS manifold.
- Next compute renormalized chemical potentials. Obtain all five ν_i as function of where you are on black hole manifold. Find

$$\nu . t_I = 0, \qquad 2t_I = (1, 1, 1, -1, -1)$$

- Next check whether $\nu_i \geq 0$. Find not always the case. Define surfaces $S^{\omega_i=0}$ and $S^{\mu_i=0}$ on black hole sheet. 'Stable' and 'unstable' black holes

Intersection of surfaces of vanishing ren chem pot with boundary of black hole

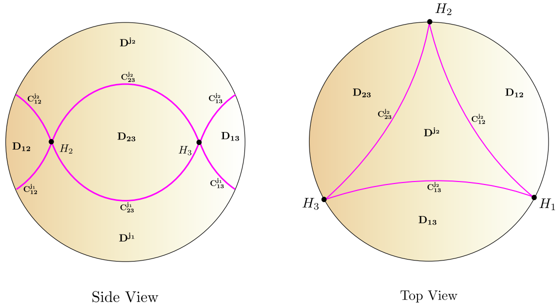
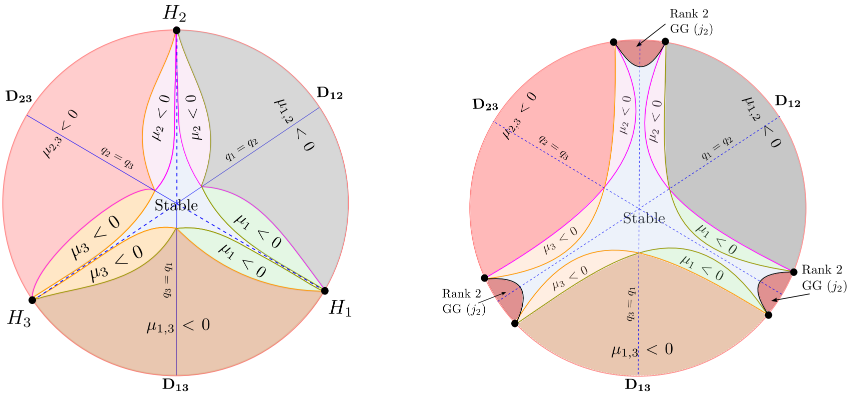


Figure: All pink boundary curves above are $(1/8)^{th}$ BPS.

Chemical potentials in black hole sheet: small charge

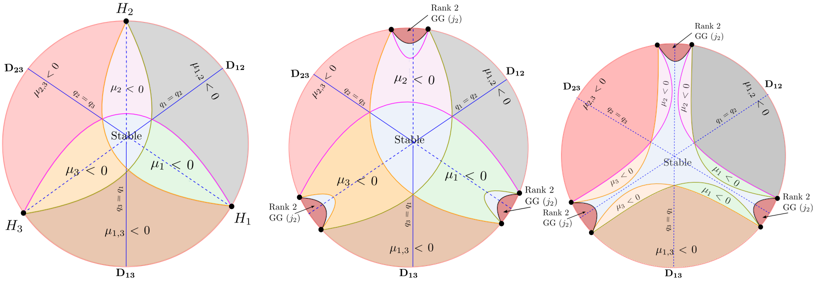


(a) For $j_R = 0$

(b) For $j_R < 0$

Figure: Cross section of the $\mathbb{B}^3$ (apple) at various values of j_R and $\frac{q_1+q_2+q_3}{3} + j_L < \frac{1}{6}$.

Chemical potentials in black hole sheet: large charge



(a) For $j_R = 0$ (b) For small negative j_R (c) For large negative j_R

Figure: Cross sections of the $\mathbb{B}^3$ (apple) at various values of j_R and $\frac{q_1+q_2+q_3}{3} + j_L > \frac{1}{6}$.

Grey Galaxy, RBH and DDBH solutions

- These new solutions can approximately be thought of as a non interacting mix of (vacuum) black holes and 'gravitons'. The 'gravitons' carry angular momentum in grey galaxy solutions, but carry $SO(6)$ charge in DDBH solutions.
- Grey galaxies/ DDBHs appear in distinct families labeled by the rank of the $SO(4)$ / $SO(6)$ angular momentum of their gravitons. Rank also equals twice the number of ω_i that are parametrically close to unity (GGs or RBHs) and twice the number of μ_i that are close to unity (for DDBHs). Grey galaxies and RBHs are either of rank 2 or 4, while DDBHs are of rank 2, 4 or 6.
- Leading large N entropy of these solutions equals the entropy of the vacuum black hole at centre.
- These new solutions (rather than vacuum black holes) dominate the phase diagram of large N $\mathcal{N} = 4$ Yang-Mills in a band of energies around the BPS plane.

Susy GGs, RBHs and DDBHS

- BPS limit. Approx non int mix of susy vacuum black holes susy 'gravitons'. Many reasons to believe exactly susy. RBHs - supersymmetric descendents of susy black hole states, so exactly susy. GGs: recent construction of large classes of fortuitous cohomologies as product of high angular momentum gravitons with (what appears to be) core black holes, Direct field theory evidence for some susy GG states.
- Susy DDBHs: special dual giants around Gutowski Real black are supersymmetric, Also of the construction similar cohomologies on the the field theory side
- While the SUSY black holes exist only on the black hole sheet, supersymmetric gravitons carry arbitrary charges $\{Z_i\}$, subject only to the restriction $Z_i \geq 0 \; \forall i$.

Dressed Concentration Conjecture

- At leading order in the large N limit, the supersymmetric entropy of $\mathcal{N} = 4$ Yang-Mills at any given values of the charges $\{Z_i\}$, is given by

$$\max_{Z_i'} S_{BH}(Z_i'), \quad (Z_i' \leq Z_i \quad \forall i = 1 \ldots 5) \tag{14}$$

  where the charges Z_i' lie on the supersymmetric black hole sheet (7).
- Reduces computation of large N susy cohomology to a maximization problem. Easily solved. Phase diagram with 10 different (3 GG and 7 DDBH) phases. Transitions from GGs to DDBHs through vacuum black holes.

Superconformal Index

- $$\mathcal{I}_W = \mathrm{Tr}\exp\left(-\sum_{i=1}^{5}\nu_i Z_i\right) \quad \text{where} \tag{15}$$
$$\nu_1+\nu_2+\nu_3-\nu_4-\nu_5 = 2n\pi i, \qquad (n \text{ is an odd integer}). \tag{16}$$

- $$(Z_1, Z_2, Z_3, Z_4, Z_5) \quad \text{and} \quad \left(Z_1+\frac{n}{2}, Z_2+\frac{n}{2}, Z_3+\frac{n}{2}, Z_4-\frac{n}{2}, Z_5-\frac{n}{2}\right), \tag{17}$$
Same contribution to index upto $(-1)^n$

- $$\mathcal{I}_W = \sum_{Z'} n_{\rm I}(Z')(-1)^{2(Z'_1+Z'_2+Z'_3-Z'_4-Z'_5)} e^{-\nu_i Z'_i} \tag{18}$$
$$n_{\rm I}(Z_i) = (-1)^{2(Z_1+Z_2+Z_3-Z_4-Z_5)} \sum_m (-1)^m \, n\left(Z_1+\frac{m}{2}, Z_2+\frac{m}{2}, Z_3+\frac{m}{2}, Z_4-\frac{m}{2}, Z_5-\frac{m}{2}\right). \tag{19}$$

Unobstructed Saddle Conjecture

- **Unobstructed Saddle Conjecture:** At leading order in the large N limit, $n_I(Z_i)$ equals the maximum of $n(Z_i)$ along the corresponding index line.
- Together with results of the cohomology, described above, allows one to compute the superconformal index as a function of 4 indicial charges. Not too hard to implement. Find a phase diagram with 9 phases (one vacuum black hole phase, 2 GG phases and 6 DDBH phases). Definite prediction for indicial entropy in each phase.
- Indices parameterized by charges ζ_i modulo shifts by t_I. Can, for instance, use the 6 charges $q_m + j_i$ with two relations. These charges convenient because positive (Bose Fermi Cone). Space of nontrivial indices a cone over R^+. Base a diamond.

The Indicial Diamond

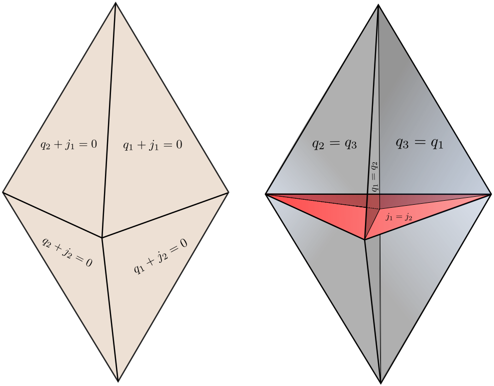


Figure: Base of the indicial cone. Every point in the indicial cone is an index line that passes through some part of the EER, and so yields and index of order N^2. Some index lines intersect black hole sheet, some dont. No line intersects more than once.

Black hole sheet on boundary of indicial diamond

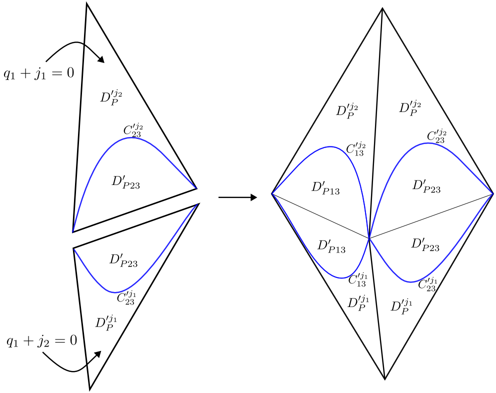


Figure: On the left figure we have two $1/8$ BPS planes $q_1 + j_{1,2} = 0$ which touch at $j_1 = j_2 = 0$. The blue curves represent the points where indicial charges match the black hole charges on this boundary. On the right, we have depicted the intersection of four such boundaries: $q_2 + j_i = 0$ and $q_1 + j_i = 0$. These surfaces meet at points where both q_1 and q_2 are zero.

Top view of boundary of indicial diamond

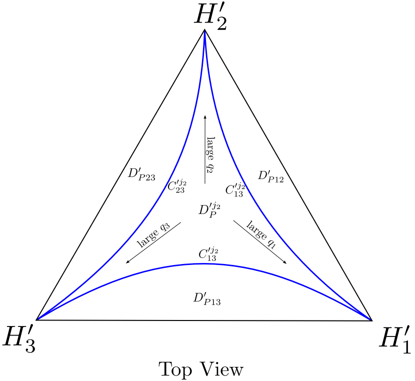
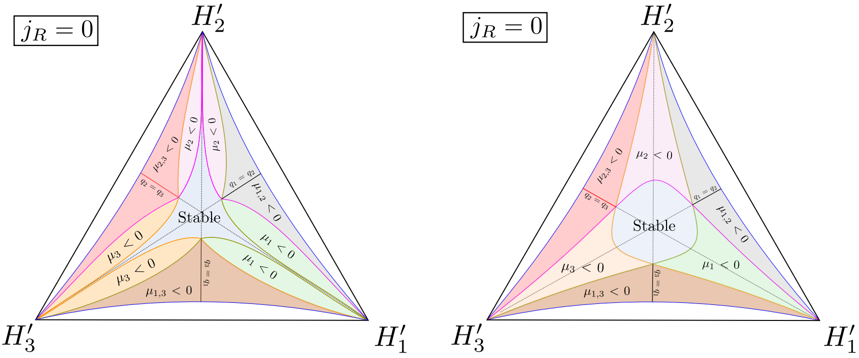


Figure: Top view of the boundary of the indicial diamond (the base of the indicial cone) shown in Figure 6

Black Hole sheet in 'equatorial' horizontal cuts

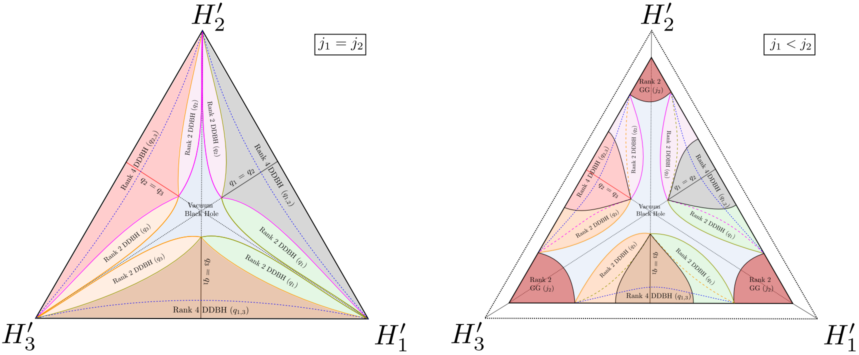


(a) For $\frac{q_1+q_2+q_3}{3} + \frac{j_1+j_2}{2} < \frac{1}{6}$

(b) For $\frac{q_1+q_2+q_3}{3} + \frac{j_1+j_2}{2} > \frac{1}{6}$

Figure: Black hole sheet inside the index diamond. The colored part of the above figures denotes the region of indicial charges that intersects the black hole sheet and the uncolored region is the space of indicial charges that do not intersect the black hole sheet.

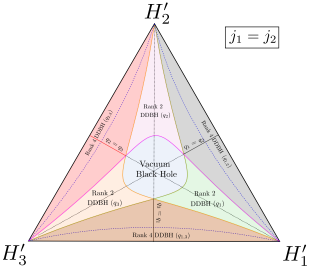
Indicial Phase diagram: Small charges



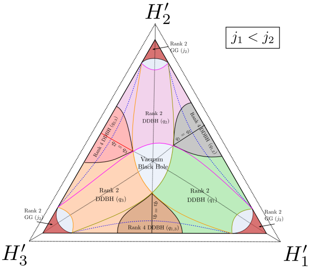
(a) For $j_1 = j_2$ (b) For $j_1 < j_2$

Figure: Horizontal cross sections of the indicial cone for $\frac{q_1+q_2+q_3}{3} + \frac{j_1+j_2}{2} < \frac{1}{6}$.

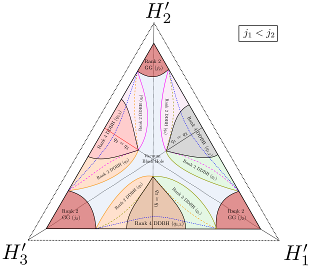
Indicial Phase diagram: Large Charges



(a) For $j_1 = j_2$

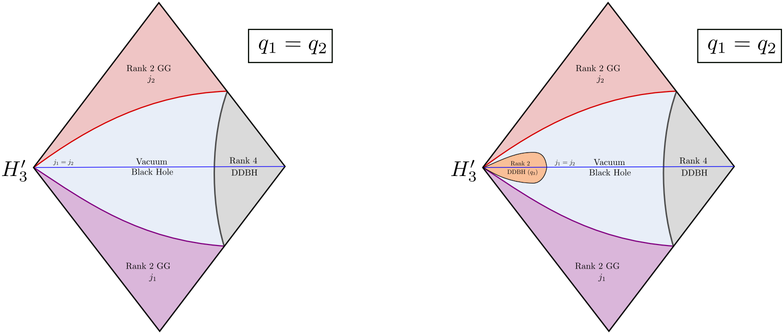


(b) For $j_R = \frac{j_1 - j_2}{2} < 0$



(c) For $|j'_R| > |j_R|$

Indicial Phase diagram: Vertical Cut



(a) For $\frac{q_1+q_2+q_3}{3} + \frac{j_1+j_2}{2} < \frac{1}{6}$

(b) For $\frac{q_1+q_2+q_3}{3} + \frac{j_1+j_2}{2} > \frac{1}{6}$

Figure: Vertical cross sections of the indicial diamond indicating the special (blue) line of indicial charges considered for the equal charges and unequal angular momentum case.

Check for $q_1 = q_2 = q_3 = q$,

- In this special case, indicial charges can be chosen as $q + j_L$ and j_R. $q + j_L$ can be thought of as 'where on R^+'. j_R as 'where on base'. See previous Fig.
- Pass through 3 indicial phases, black hole and two GGs. j_R to j_R symmetry.
- At $N = 10$, we Taylor expanded the integrand (in unitary matrix integral for index) in powers of e^{μ} and e^{ω_R}, and then evaluated integral term by term. Used that to pull out microcanonical index (though only at $N = 10$). Next slide, comparison between numerics and our prediction at $q + j_L = 0.9$.

Data vs prediction

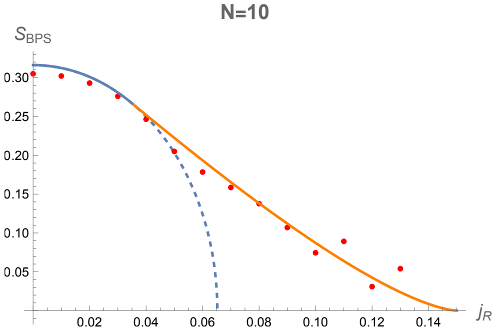
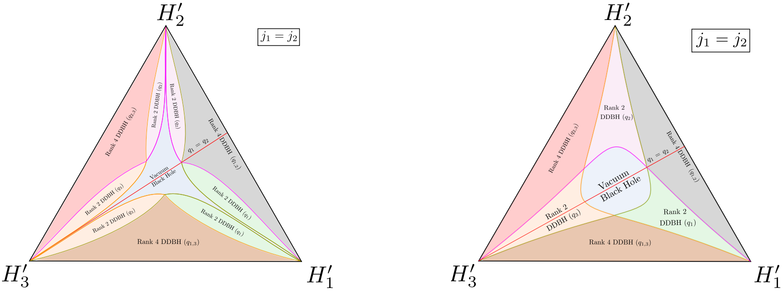


Figure: The red dots represent the numerical values of the indicial entropy, expressed as $\frac{1}{N^2}\log(|n_I(Z_i)|)$, for $N = 10$, $2(Z_1 + Z_2 + Z_3) + 3(Z_4 + Z_5) = 90$, and $j_R = \frac{Z_4 - Z_5}{N^2}$. The blue solid and dashed lines depict the black hole entropy, determined at the intersection of the black hole sheet and an index line. The solid line, combining the blue and orange segments, represents the indicial entropy S_{BPS} as computed using the Unobstructed Saddle Conjecture.

Check for $q_1 = q_2 = q$, $q_3 = q'$, $j_1 = j_2 = j$

- In this special case, indicial charges can be chosen as $\alpha = \frac{2q+q'}{3} + j_L$ and $q - q'$. First can be thought of as 'where on R^+'. j_R as 'where on base'. See fig on next slide
- Pass through rank 4 DDBH phase and black hole phase for $\alpha < \frac{1}{6}$, but through rank 4 DDBH phase, black hole phase and rank 2 DDBH phase for $\alpha > \frac{1}{6}$
- At $N = 10$, we once again compared the numerical $N = 10$ data with our predictions.



(a) For $\frac{q_1+q_2+q_3}{3} + \frac{j_1+j_2}{2} < \frac{1}{6}$

(b) For $\frac{q_1+q_2+q_3}{3} + \frac{j_1+j_2}{2} > \frac{1}{6}$

Figure: Horizontal cross sections of the indicial diamond depicting the set of indicial charges (Red line) considered in the two equal charges and equal angular momentum case.

Comparison with data

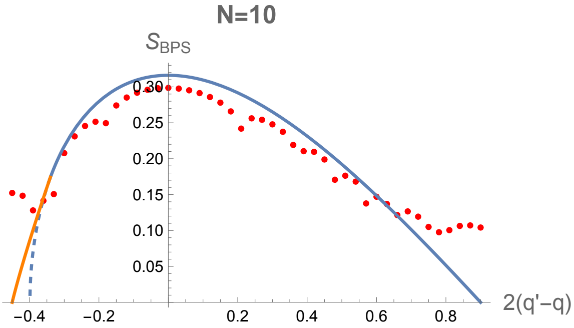


Figure: The figure compares the numerically computed indicial entropy (red dots) with the predicted formula (blue and orange solid lines) at $N = 10$. The indicial entropy S_{BPS} is calculated as $\frac{1}{N^2}\ln(|n(\mathrm{A}_1, 2Q_3 - Q_1 - Q_2)|)$, with $\mathrm{A}_1 = \frac{Q_1+Q_2+Q_3}{3} + J_L = 15$ fixed, while varying $2Q_3 - Q_1 - Q_2 = 2N^2(q' - q)$. The blue line (solid and dashed) represents the black hole entropy, determined at the intersection of the black hole sheet and an index line. The orange line depicts the entropy of DDBH solutions with $\mu_1 = \mu_2 = 0$.

Reason for tail discrepancy

- Why do tails deviate from both predictions? Reaon. Gravitons. Extreme right end half BPS. Can analytically count graviton states. Exact match with last data point. Deviation from black hole at other points evidence for DDBHS like states.
- Left end $1/4$ BPS. Again can count states. Again get exact agreement (for last point) with prediction.
- This data- while in broad qualitative agreement with our general predictions- is not as good a check as the previous data, for two reasons. First our prediction does not deviate much from the black hole prediction at these accessibble values of charges. Second, graviton entropy is significant, and must be accounted for at $N = 10$.

Conclusions

- Input: two conjectures. Reasonable evidence for veracity. Output: sharp predictions for cohomological entropy and indicial entropy. Some checks of correctness.
- Nontrivial prediction for susy states as a function of all 5 charges. Prediction of many new indicial phases
- Urgent to do: Analytically find these new phases in the unitary matrix integral for the index.