

Rapid adiabatic hydrodynamization and a generalized picture of attractors in kinetic theory

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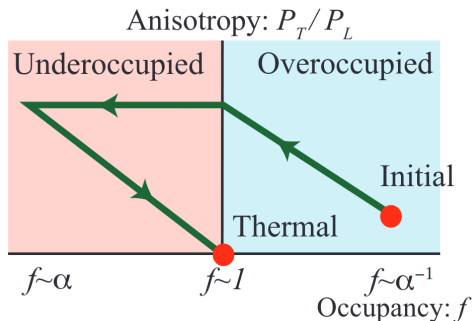
primarily based on **arXiv:2507.21232**
with Bruno Scheihing-Hitschfeld and Krishna Rajagopal

September 9, 2025

Bottom-Up Thermalization

In kinetic theory, hydrodynamization occurs in stages:

- 1 Anisotropy increases
- 2 Occupancy decreases
- 3 System becomes isotropic and thermal



Scaling and Pre-Scaling

- Many prior calculations show that distribution functions quickly take “scaling” form:

$$f(\mathbf{p}, \tau) = \tau^\alpha w(\tau^\beta p_\perp, \tau^\gamma p_z)$$

Berges, Boguslavski, Schlichting,
Venugopalan, arXiv:1303.5650

- “Prescaling” regime in QCD EKT

$$f(\mathbf{p}, \tau) = \tau^{\alpha(\tau)} w(\tau^{\beta(\tau)} p_\perp, \tau^{\gamma(\tau)} p_z)$$

Mazeliauskas, Berges, arXiv:1810.10554

- Rescaled distribution function w is a universal attractor

BMSS scaling

$$\alpha = -\frac{2}{3},$$

$$\beta = 0,$$

$$\gamma = \frac{1}{3}$$

Baier, Mueller, Schiff, Son
arXiv:hep-ph/0009237

Prescaling and Adiabatic Hydrodynamization

Early-Time Dynamics

$$\tau^{\alpha(\tau)} w(\tau^{\beta(\tau)} p_{\perp}, \tau^{\gamma(\tau)} p_z, \tau)$$



Prescaling "Ground State"

$$\tau^{\alpha_S(\tau)} w_0(\tau^{\beta_S(\tau)} p_{\perp}, \tau^{\gamma_S(\tau)} p_z)$$

Mikheev, Mazeliauskas, Berges,
arXiv:2203.02299

- Adiabatic Hydrodynamization: pre-hydro attractors are the time-dependent ground state of an effective Hamiltonian
- AH provides motivation for observed scaling/prescaling attractor behavior
- However, AH does not require the attractor to be scaling

Brewer, Yan, Yin, arXiv:1910.00021;

Brewer, Scheiing-Hitschfeld, Yin,

arXiv:2203.02427;

Prescaling and Adiabatic Hydrodynamization

Early-Time Dynamics

$$\tau^{\alpha(\tau)} w(\tau^{\beta(\tau)} p_{\perp}, \tau^{\gamma(\tau)} p_z, \tau)$$



Non-Scaling “Ground State”

$$\tau^{\alpha(\tau)} w_0(\tau^{\beta(\tau)} p_{\perp}, \tau^{\gamma(\tau)} p_z, \tau)$$

Mikheev, Mazeliauskas, Berges,
arXiv:2203.02299

- Adiabatic Hydrodynamization: pre-hydro attractors are the time-dependent ground state of an effective Hamiltonian
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arXiv:2203.02427;



Rescaled Distribution Function

- If we choose a time-dependent coordinate rescaling, e.g.

$$f(p_{\perp}, p_z, \tau) = A(\tau)w\left(\frac{p_{\perp}}{B(\tau)}, \frac{p_z}{C(\tau)}, \tau\right) = A(\tau)w(\zeta, \xi, \tau),$$

we can rewrite a Boltzmann equation

$$\frac{\partial f}{\partial \tau} - \frac{p_z}{\tau} \frac{\partial f}{\partial p_z} = -C[f],$$

as an effective Hamiltonian evolution for w :

$$Hw \equiv -\partial_y w, \quad \text{where } y \equiv \log\left(\frac{\tau}{\tau_0}\right)$$

↑
Note: -1 instead of i

- H is nonlinear, non-Hermitian, and dependent on the choice of scaling: $H = H(w, A, B, C, \tau)$.
- This is **not** a Schrödinger equation, but we can still draw analogies to Quantum Mechanics.

$$Hw \equiv -\partial_y w$$

- **Analogy to Adiabatic Theorem in QM:** if H evolves “slowly” compared to the gap, the coefficient of each eigenstate n evolves independently as

$$c_n(y) \sim e^{-\int^y \varepsilon_n(y') dy'} .$$

This means that the ground state acts as an attractor!

- The key is to choose A, B, C such that H evolves “sufficiently slowly”.

Example: Diffusive Small-Angle Elastic Scatterings

- Consider a simple longitudinally-expanding boost-invariant theory:

$$\frac{\partial f}{\partial \tau} - \frac{p_z}{\tau} \frac{\partial f}{\partial p_z} = q[f] \nabla_p^2 f$$

where $q[f] = \frac{g_s^4 N_c^2}{4\pi} \log\left(\frac{p_{UV}}{p_{IR}}\right) \int_p f(1+f)$

- Writing in terms of w , the effective Hamiltonian is

$$H = \frac{\partial_y A}{A} - \left(\frac{\partial_y C}{C} + 1\right) \xi \partial_\xi - \frac{q}{C^2} \partial_\xi^2 - \frac{\partial_y B}{B} \zeta \partial_\zeta - \frac{q}{B^2} (\partial_\zeta^2 + \frac{1}{\zeta} \partial_\zeta).$$

Solving Analytically for the Eigenstates

This H has exact analytic eigenstates and eigenvalues

$$\phi_{n,m}^R = \frac{1}{\sqrt{2\pi\tilde{q}}(2\pi)!\tilde{q}_B} \text{He}_{2n} \left(\frac{\xi}{\sqrt{\tilde{q}}} \right) {}_1F_1 \left(-2m, 1, \frac{\zeta^2}{2\tilde{q}_B} \right) e^{-\frac{\xi^2}{2\tilde{q}} - \frac{\zeta^2}{2\tilde{q}_B}}$$
$$\epsilon_{n,m} = 2n \left(1 + \left(\frac{\partial_y C}{C} \right) \right) + 2m \left(\frac{\partial_y B}{B} \right)$$

where we can make the eigenstates totally time-independent by choosing $B(y)$, $C(y)$ such that $\tilde{q} \equiv \frac{q}{C^2(1+\partial_y C/C)} = 1$ and $\tilde{q}_B \equiv \frac{q}{B\partial_y B} = 1$.

Brewer, Scheihing-Hitschfeld, Yin, arXiv:2203.02427

Using the ground state attractor

- Knowing the form of the ground state, we can test it against other prehydrodynamic models in *Trajectum*.
- First, generalize to a transversely expanding medium by replacing ∂_y with $\partial_y + \tau \hat{p}_\perp \cdot \nabla_{\mathbf{x}_\perp}$.
- We can match a T_RENTo initial condition to a free-streaming f using the energy density per unit rapidity $\frac{d\mathcal{E}}{d\eta}$, and then after some time match it to our attractor solution $\phi_{0,0}^R$.

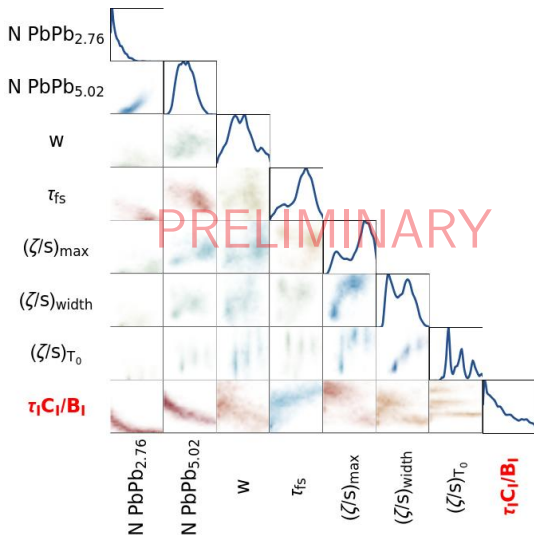
Karthein, Nijs, Scheihing-Hitschfeld, **RS** arXiv:2xxx.xxxxx

Bayesian Analysis Results

Preliminary results seem to suggest the initial anisotropy C_I/B_I is strongly correlated with other parameters.

Karthein, Nijs,
Scheiing-Hitschfeld, **RS**
arXiv:2xxx.xxxxx

However, some important physics is missing from this simplified kinetic theory.



Moving Towards Generalization...

- First, we need a theory that actually hydrodynamizes.
Small-angle elastic scattering with no Bose enhancement:

$$\frac{\partial f}{\partial \tau} + \frac{p_z}{\tau} \frac{\partial f}{\partial p_z} = \frac{g_s^4 N_c^2}{4\pi} I_{Cb}[f] (I_a[f] \nabla_p^2 f + I_b[f] \nabla_p \cdot (\hat{p}f))$$

⇒ considered in [Rajagopal, Scheihing-Hitschfeld, RS, arXiv:2405.17545](#)

- Furthermore, for consistency with the “bottom-up” picture (and for hydrodynamization on a realistic time scale) we need to find a way to include inelastic scatterings.

⇒ considered in [Rajagopal, Scheihing-Hitschfeld, RS, arXiv:2507.21232](#)

If we assume radiation of soft gluons only, it is possible to show that the leading order 1-to-2 part of the collision kernel without Bose enhancement is approximately

$$C^{1\leftrightarrow 2}[f] = -2\lambda_0 \sqrt{\frac{l_a[f] \ell_{Cb}[f]}{2\pi^3 p}} (1 - f(p=0)) \left(\frac{7}{2} + p \partial_p \right) f.$$

This expression neglects the hard splittings of relatively soft gluons, but its inclusion should still speed up hydrodynamization.

Rescaling and Basis Expansion

- For these more general kernels, it is more convenient to use the coordinates

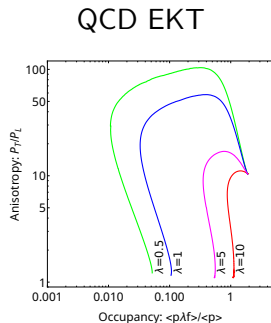
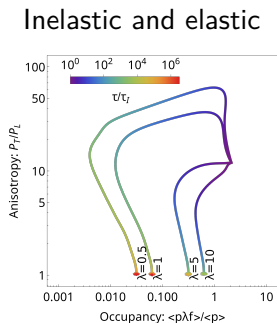
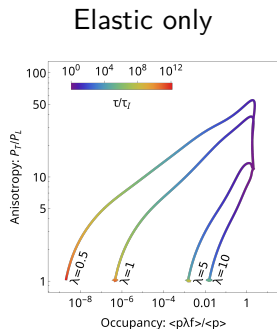
$$f(p, p_z, \tau) = A(\tau)w\left(\frac{p}{D(\tau)}, \frac{p_z}{p}, r(\tau), \tau\right) = A(\tau)w(\chi, u, r, \tau),$$

where instead of B and C we now need to choose D and r .

- Let $D(y)$ be the effective temperature I_a/I_b
- Let $r(y)$ optimize similarity of the basis to f .
- We choose a basis which can interpolate between the presumed early- and late-time ground states:

$$\psi_{ij}^{(R)} = p_i(\chi)q_j(u)e^{-(u^2/r^2+\chi)}, \quad \psi_{ij}^{(L)} = p_i(\chi)q_j(u)$$

Pressure Anisotropy vs Energy-Weighted Occupancy



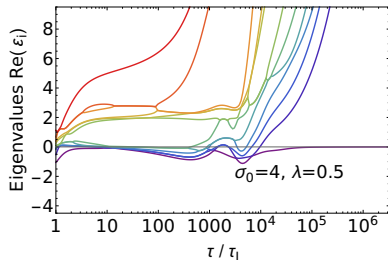
Boguslavski et al, arXiv:2303.12595

Kurkela, Zhu, arXiv:1506.06647

Rajagopal, Scheuing-Hitschfeld, **RS**, arXiv:2507.21232

As expected, things look much more physical, and hydrodynamization can happen on a much shorter timescale!

Effective Energy Levels



Rajagopal, Scheihing-Hitschfeld, **RS**,
arXiv:2507.21232

Effective energy levels suggest a pre-thermal attractor surface with “dilute” scaling exponents

$$(\alpha, \beta, \gamma) = (1, 0, 0)$$

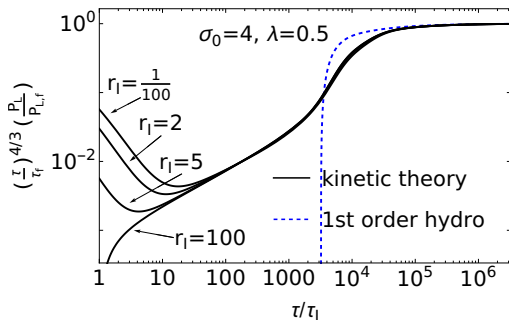
based on energy levels computed in Brewer, Scheihing-Hitschfeld, Yin, arXiv:2203.02427:

$$\epsilon_n = 2n(1 - \gamma) - 2m\beta$$

(and indeed this is the case for very weak couplings

$$g_s \lesssim 10^{-2})$$

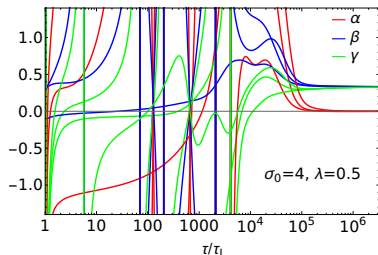
Attractor Behavior



Indeed,
far-from-equilibrium
attractor behavior is
observed.

Rajagopal, Scheiing-Hitschfeld, **RS**, arXiv:2507.21232

Scaling Is Not Observed!



Rajagopal, Scheihing-Hitschfeld, **RS**,
arXiv:2507.21232

However, solving systems of moments for α, β, γ as in [Mazeliauskas, Berges, arXiv:1810.10554](#) reveals the system is not scaling at all prior to hydrodynamization!

This is a case in which AH describes attractor behavior, but scaling does not.

Summary and Outlook

- Adiabatic hydrodynamization provides an intuitive way of describing non-thermal and hydrodynamizing attractor behavior **even in the absence of scaling**.
- By including inelastic scatterings in the AH framework, we are able to describe hydrodynamization in a **realistic time frame!**
- Open question: can this be translated into a more successful pre-hydrodynamic model in e.g. Trajectum?
- Still missing important physics: missing Bose enhancement terms, spatial gradients

Backup Slides

- For solutions in the D, r basis, we use

$$f(\mathbf{p}, \tau = \tau_I) = \frac{\sigma_0}{g_s^2} e^{-\sqrt{2}p/Q_s} e^{-r_I^2 u^2/2} Q_0^{(R)}(u; r_I)$$

with $r_I = \sqrt{3}$ and $\tau_I Q_s = 1$ unless otherwise specified.

- For solutions in the B, C basis, we use

$$f(p_\perp, p_z; \tau_I) = \frac{\sigma_0}{g_s^2} \exp\left(-\frac{p_\perp^2 + \xi_0^2 p_z^2}{Q_s^2}\right) \quad (1)$$

with $\xi_0 = 2$, $\tau_I Q_s = 70$, $B_I = Q_s/\sqrt{2}$, and $C_I = Q_s/(\xi_0\sqrt{2})$ unless otherwise specified.

Derivation of Inelastic Scattering Kernel

In the LPM regime, the leading 1-to-2 kernel in the collinear limit is

$$\begin{aligned} C^{1\leftrightarrow 2}[f] = & - \int_0^1 dx \frac{d^2 I}{dx dt} \left\{ \frac{1}{x^{5/2}} \left[f\left(\frac{p}{x}\right) (1 + f(p)) \left(1 + f\left(\frac{p(1-x)}{x}\right)\right) \right. \right. \\ & \left. \left. - f(p) f\left(\frac{p(1-x)}{x}\right) \left(1 + f\left(\frac{p}{x}\right)\right) \right] \right. \\ & \left. - \frac{1}{2} [f(p)(1 + f(px))(1 + f(p(1-x))) \right. \\ & \left. - f(px)f(p(1-x))(1 + f(p))] \right\}, \end{aligned}$$

where

$$\frac{d^2 I}{dx dt} = \lambda_0 \sqrt{\frac{I_a[f] \ell_{Cb}[f] (1-x+x^2)^{5/2}}{2\pi^3 p (x-x^2)^{3/2}}}$$

is the rate of nearly collinear splitting for a hard gluon.

Derivation of Inelastic Scattering Kernel

Expanding in $1 - x$, we can evaluate the integral to get

$$C^{1\leftrightarrow 2}[f] = -2\lambda_0 \sqrt{\frac{I_a[f] \ell_{Cb}[f]}{2\pi^3 \rho}} \left[\frac{g}{\rho} (\mathbf{p} \cdot \nabla_{\mathbf{p}})^2 + (1 + 2f) (\mathbf{p} \cdot \nabla_{\mathbf{p}}) + \frac{7}{2} \left(1 + f + \frac{g}{\rho} (\mathbf{p} \cdot \nabla_{\mathbf{p}}) \right) \right] f$$

where

$$g = \lim_{\rho \rightarrow 0} \rho f(\mathbf{p})$$

Derivation of Inelastic Scattering Kernel

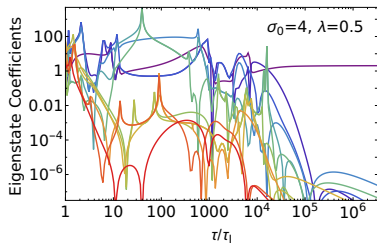
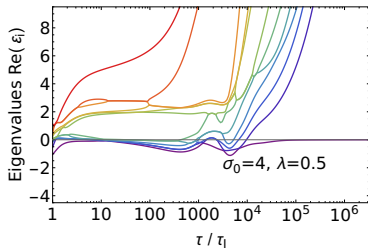
Without Bose corrections,

$$C^{1\leftrightarrow 2}[f] = - \int_0^1 dx \frac{d^2 I}{dx dt} \left\{ \frac{1}{x^{5/2}} \left[f\left(\frac{\mathbf{p}}{x}\right) - f(\mathbf{p})f\left(\frac{\mathbf{p}(1-x)}{x}\right) \right] - \frac{1}{2} [f(\mathbf{p}) - f(\mathbf{p}x)f(\mathbf{p}(1-x))] \right\}.$$

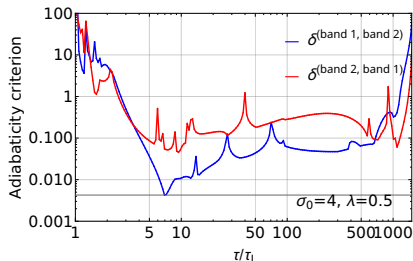
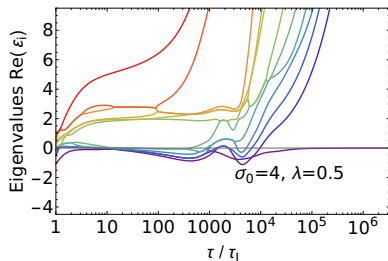
Now when we expand in $1-x$, we get

$$C^{1\leftrightarrow 2}[f] = -2\lambda_0 \sqrt{\frac{l_a[f] \ell_{Cb}[f]}{2\pi^3 \rho}} (1 - f(\mathbf{p} = 0)) \left(\frac{7}{2} + \mathbf{p} \cdot \nabla_{\mathbf{p}} \right) f.$$

Incremental Loss of Memory

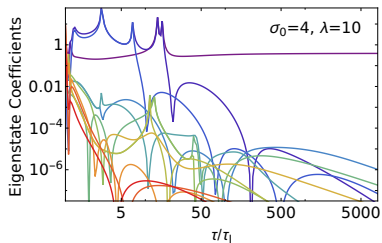
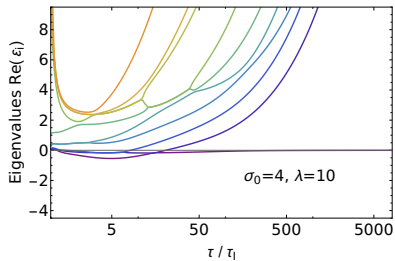


Adiabaticity



$$\delta_{\max}^{(\text{band } i, \text{band } j)} \equiv \max_{\substack{n \in \text{band } i \\ m \in \text{band } j}} \left| \frac{\langle n | L \partial_y | m \rangle_R}{\langle m | L | m \rangle_R (\epsilon_n - \epsilon_m)} \right|$$

Larger Coupling



Very Weak Coupling With Inelastic Scatterings

