Thermalization in Φ^4 theory via quantum computing

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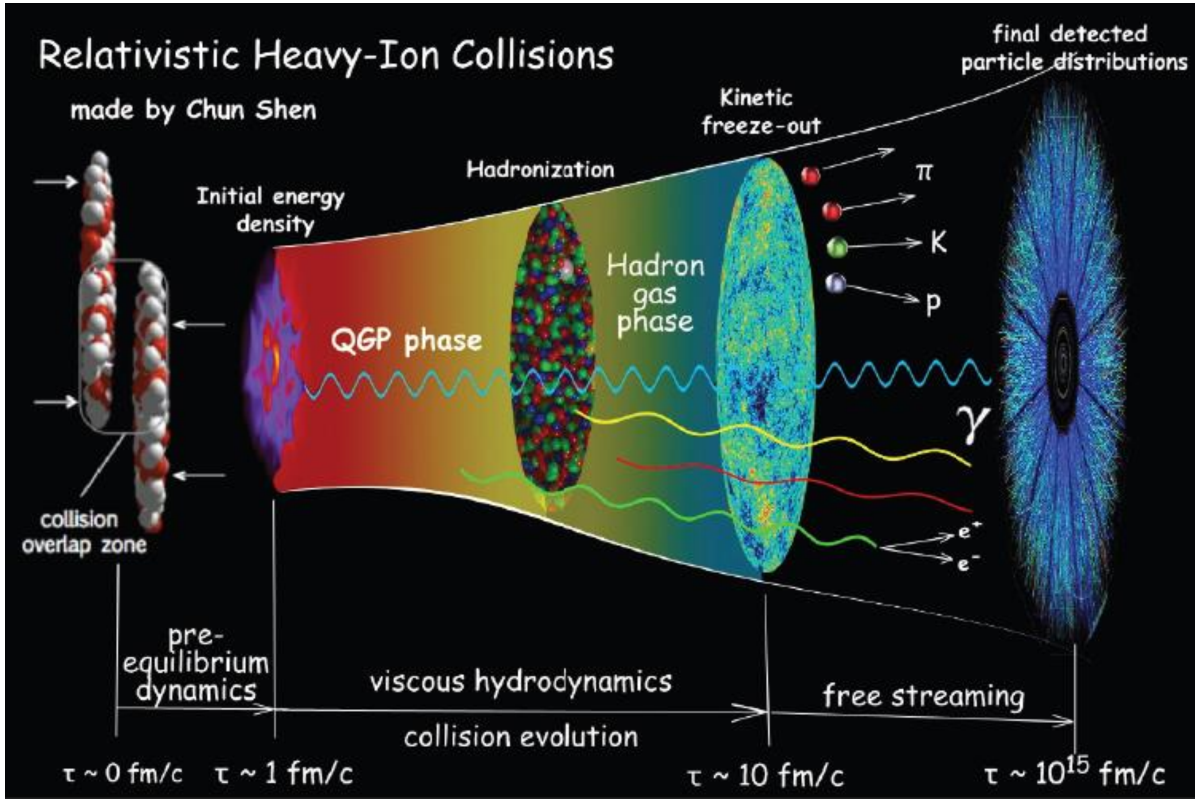
XUNTA DE GALICIA

State of the art in quantum computer technology

- Quantum computing (QC) is a rapidly-emerging technology that harnesses the laws of quantum mechanics to solve problems too complex for classical computers.
- Currently, we are in the Noisy Intermediate-Scale quantum (NISQ) era:
  - Quantum processors containing up to ~1000 qubits
  - Sensitive to their environment
  - Prone to quantum decoherence
- Quantum information science has proved useful in a broad range of physics applications
- Quantum simulation is potentially more advantageous in reducing the problem complexity from exponential to polynomial

IBM Condor employs over 1000 qubits ⇨

Thermalisation in heavy-ion collisions



- QGP constitutes one of the main research areas in QCD physics.
- Time evolution of QCD matter before thermalisation has been studied using classical approaches such as classical field simulations or kinetic theory
- It has not yet been resolved from first principles of QCD
- Lattice QCD is only applicable at low baryon densities in imaginary time where the numerical sign problem does not interfere with calculations

Motivation

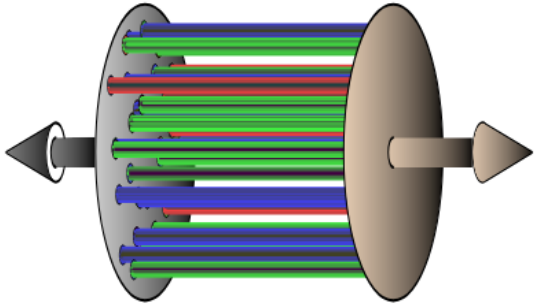
Gelis, 2012 [arXiv:1211.3327v2]

- Color Glass Condensate (CGC) can be described as a classical field at leading order with $\mathcal{A}^\mu \sim \frac{1}{g}$

Glasma: state during which the classical field stays coherent

- It remains an open question how these classical fields lose their coherence and form a plasma of quarks and gluons in local thermal equilibrium.

- Quantum computing is a potential tool for solving this issue from QCD first principles



Glasma flux tubes, along which glasma fields are coherent

Warm up: Φ^4 theory

- Before solving this issue in QCD, we start with Φ^4 theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{1}{4!}\lambda_0\phi^4$$

Jordan, Lee & Preskill, 2011 [arXiv:1112.4833]

Klco & Savage, 2018 [arXiv:1808.10378]

- In order to mimic classical field (CGC) background, we use coherent states and study the real-time evolution and thermalisation from them.

Coherent states: states that have the minimal extent in phase-space allowed by the uncertainty principle.

Thermalisation process

- Real-time evolution:

$$|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle = \sum_n c_n e^{-iE_n t}|n\rangle$$

Initial state $|\psi(t_i = 0)\rangle$

Trotterisation: $$|\psi(t)\rangle \approx \prod_{j=0}^{n_s-1} e^{-\frac{iHt}{n_s}}|\psi_0\rangle$$

- Fixed-point thermal property after thermalisation is established:

$$\langle\psi(t)|O|\psi(t)\rangle \longrightarrow \langle O\rangle_\beta = \frac{\mathrm{Tr}(e^{-\beta H}O)}{Z_\beta}$$

Partition function $Z_\beta = \mathrm{Tr}(e^{-\beta H})$

The trace can be calculated by summing the expectation values over the complete set of states

Representing fields with qubits

1. We approximate the continuum theory with a discrete theory that's similar to lattice QCD but with continuous time.

$$x \in \Omega_x \equiv \{0, a, \cdots, (N-1)a\}$$

$$\Omega_p \equiv \begin{cases} \{(\frac{-N+1}{2})\Delta p, \cdots, (\frac{N-1}{2})\Delta p\} & \text{if } N \text{ is odd} \\ \{(\frac{-N+2}{2})\Delta p, \cdots, (\frac{N}{2})\Delta p\} & \text{if } N \text{ is even} \end{cases}$$

$$\Delta p = \frac{2\pi}{L} \qquad L \equiv aN$$

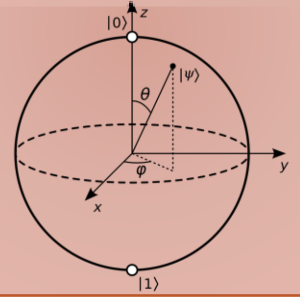
2. We map the fields into qubits

$$O^{(i)} = \mathbb{I}_0 \otimes \cdots \otimes \mathbb{I}_{i-1} \otimes O_i \otimes \mathbb{I}_{i+1} \otimes \cdots \otimes \mathbb{I}_{n_T-1}$$

$$|k\rangle = |k_0\rangle \otimes |k_1\rangle \otimes |k_2\rangle \otimes \cdots \otimes |k_{n_T-1}\rangle = \bigotimes_{i=0}^{n_T-1} |k_i\rangle$$

bit → {0,1}

qubit → $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$



SCALAR FIELD THEORY

IC, Qian & Wu, 2024 [arXiv:2411.19601]

Lagrangian density for the ϕ^4 theory in d + 1 $\mathcal{L} = \frac{1}{2}\left(\partial_\mu \Phi \partial^\mu \Phi - m^2\Phi^2\right) + \mathcal{L}_{\text{int}}\ , \quad \mathcal{L}_{\text{int}} = -\frac{\Phi^4}{4!}$

Field and conjugate-field operators $[\Phi(x), \Phi(y)] = [\Pi(x), \Pi(y)] = 0 \quad [\Phi(x), \Pi(y)] = i\delta(x-y)$

$$\mathcal{H} = \frac{1}{2}m^2\Phi^2 + \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\Phi)^2 + \lambda\frac{\Phi^4}{4!}$$

- Step 1: Discretised hamiltonian

$$H = a\sum_{x=0}^{N-1}\left[\frac{1}{2}m^2\Phi_x^2 + \frac{1}{2}\Pi_x^2 + \frac{1}{2}\frac{(\Phi_{x+1}-\Phi_x)^2}{a^2} + \lambda\frac{\Phi_x^4}{4!}\right]$$

- Step 2: Map it onto qubits

FIELD OPERATOR (FO) BASIS: Spatial Lattice Sites → Eigenstates of **Φ** fields

HARMONIC OSCILLATOR (HO) BASIS: Particle quanta → Eigenstates of occupancy

FO Basis: Field operator digitisation

Macridin et al., 2021 [arXiv:2108.10793]

- The lattice Hilbert space is a tensor product of local Hilbert space at each lattice site
- The local Hilbert space at a single lattice site is infinite dimensional because there are infinitely many bosons contributing to the local wave function

$$\varphi_\alpha = (-\infty, \infty)$$

- We truncate the number of bosons by a cutoff number N_b and then digitize the continuous field operators to discretized values

$$\varphi_\alpha = [-\varphi_{\max}, \varphi_{\max}]$$

Digitised field operators

$$\Phi_x |\varphi_\alpha\rangle_x = \varphi_\alpha |\varphi_\alpha\rangle_x, \qquad \alpha = 0, 1, \ldots, N_\varphi - 1$$

$$\varphi_\alpha = \Delta_\varphi \left(\alpha - \frac{N_\varphi - 1}{2}\right), \qquad \Delta_\varphi = \sqrt{\frac{2\pi}{N_\varphi \bar{m}}}$$

Digitised conjugate-field operators

$$\Pi_x = m \mathcal{F}_x \Phi_x \mathcal{F}_x^{-1}$$

$$\Pi_x |\kappa_\beta\rangle_x = \kappa_\beta |\kappa_\beta\rangle_x, \qquad \beta = 0, 1, \ldots, N_\varphi - 1$$

$$\kappa_\beta = \Delta_\kappa \left(\beta - \frac{N_\varphi - 1}{2}\right), \qquad \Delta_\kappa = \sqrt{\frac{2\pi \bar{m}}{N_\varphi}}$$

$$[\Phi_x, \Pi_x]|n\rangle = i|n\rangle + \mathcal{O}(\epsilon)$$

Field operator representation

- To obtain the correct eigenvalues, we represent the field operator in this way

$$\Phi = -\frac{\Delta_\varphi}{2} \sum_{k=0}^{n_Q-1} 2^{n_Q-1-k} \sigma_z^{(k)}$$

- The symmetric Quantum Fourier Transform (sQFT) can be obtained from the regular QFT

$$\mathcal{F} = e^{-i\frac{n_\varphi \delta^2}{2\pi}} \prod_{k=0}^{n_Q-1} R_z^{(k)}(-2^{n_Q-1-k}\delta) \; \mathrm{QFT} \prod_{k=0}^{n_Q-1} R_z^{(k)}(-2^{n_Q-1-k}\delta)$$

$$\mathrm{QFT} = \frac{1}{\sqrt{n_\varphi}} \sum_{\mu,\nu=0}^{n_\varphi-1} e^{i\frac{2\pi}{n_\varphi}\mu\nu} |\varphi_\mu\rangle\langle\varphi_\nu| \quad , \quad R_z^{(k)}(\theta) = \exp\left(-i\theta\frac{\sigma_z^{(k)}}{2}\right)$$

FO Basis: Scalar field theory on the qubit

- We map bosons onto qubits using eigenstates of the field operator as the computational basis

$$\langle \varphi_n | = (0 \quad \cdots \quad 0 \quad \overbrace{1}^{n} \quad 0 \quad \cdots \quad 0)$$

- We use a 1D lattice of N quantum registers to represent N lattice points.
  In each register, we use n_Q qubits to digitise the fields

$$N_\varphi = 2^{n_Q}$$
$$n_T = N n_Q$$

$$|\varphi_n\rangle = \bigotimes_{x=0}^{N-1} |\varphi_{n_x}\rangle = \bigotimes_{x=0}^{N-1} \bigotimes_{q=0}^{n_Q-1} |\varphi_{n_{x_q}}\rangle \qquad \begin{cases} n = 0, 1, \ldots, n_\varphi^N - 1 \\ n_x = 0, 1, \ldots, n_\varphi - 1 \\ n_{x_q} = 0, 1 \end{cases}$$

$$\varphi_{\max} \equiv \frac{N_\varphi - 1}{2} \sqrt{\frac{2\pi}{N_\varphi \bar{m}}}$$

HO Basis

Klco & Savage, 2019 [arXiv:1808.10378]

- Step 1: Hamiltonian in terms of creation and annihilation operators in momentum space

1 $$a = \sqrt{N}\sum_{n=0}^{n_\varphi}\sqrt{n}|n-1\rangle\langle n| \quad , \quad a^\dagger = \sqrt{N}\sum_{n=0}^{n_\varphi-1}\sqrt{n+1}|n+1\rangle\langle n| \qquad N_p = \frac{1}{N}a_p^\dagger a_p$$

2 $$\Phi_x = \frac{1}{Na}\sum_p \sqrt{\frac{1}{2\omega_p}}\left(a_p e^{ipax} + a_p^\dagger e^{-ipx}\right) \quad , \quad \Pi_x = \frac{-i}{Na}\sum_p \sqrt{\frac{\omega_p}{2}}\left(a_p e^{ipax} - a_p^\dagger e^{-ipax}\right)$$

$$\omega_p = E_p = \sqrt{m^2 + \frac{4}{a^2}\sin^2\frac{ap}{2}}$$

3 $$H = \sum_p \omega_p\left(N_p + \frac{\mathbb{1}}{2}\right) + \frac{\lambda}{4!N^3a^3}\sum_{p_i}\frac{\delta_{p_1+p_2+p_3+p_4,0}}{4\sqrt{\omega_{p_1}\omega_{p_2}\omega_{p_3}\omega_{p_4}}}\left(a_{p_1}+a_{-p_1}^\dagger\right)\left(a_{p_2}+a_{-p_2}^\dagger\right)\left(a_{p_3}+a_{-p_3}^\dagger\right)\left(a_{p_4}+a_{-p_4}^\dagger\right)$$

- Step 2: Map it onto qubits using eigenstates of $a_p^\dagger a_p$ as the computation basis

$$\langle n| = (0 \ \cdots \ 0 \ \overbrace{1}^{n} \ 0 \ \cdots \ 0) \qquad |n\rangle = \bigotimes_{p=0}^{N-1}|n_p\rangle = \bigotimes_{p=0}^{N-1}\bigotimes_{q=0}^{n_Q-1}|n_{p_q}\rangle$$

Max number of bosons → $n_\varphi = 2^{n_Q}$

Coherent states

The coherent state $|\alpha\rangle$ of a HO is defined as the eigenstate of the annihilation operator

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{\sum_k \alpha_k a_k^\dagger - \alpha_k^* a_k}|0\rangle$$

- Infinite-dimensional Hilbert space:

Glauber, 1963 [DOI:10.1103/PhysRev.131.2766]

$$|\alpha\rangle = e^{-\frac{1}{2}\sum_{k=0}^{N-1}\alpha_k^2} \sum_{n=0}^{\infty} \prod_{k=0}^{N-1} \frac{\alpha_k^{n_k}}{\sqrt{n_k!}}|n\rangle \xrightarrow{\text{small } \alpha} |\alpha\rangle \approx e^{-\frac{1}{2}\sum_{k=0}^{N-1}\alpha_k^2} \sum_{n=0}^{n_\varphi^N-1} \prod_{k=0}^{N-1} \frac{\alpha_k^{n_k}}{\sqrt{n_k!}}|n\rangle$$

- Finite-dimensional Hilbert space:

Miranowicz, Piątek and Tanaś, 1994 [DOI:10.1103/PhysRevA.50.3423]

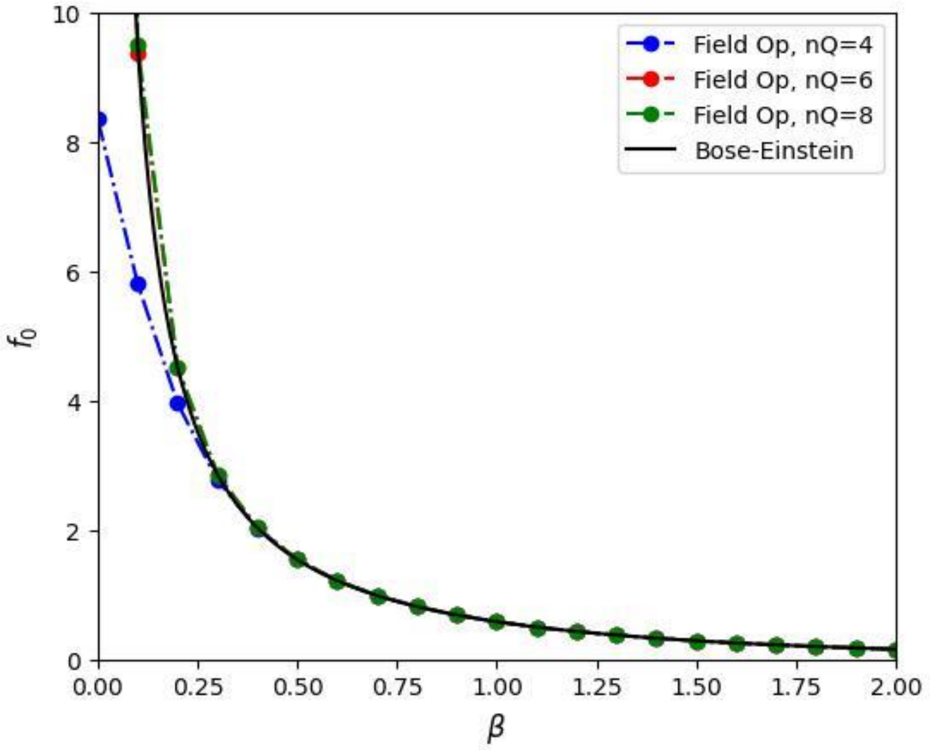
$$|\alpha\rangle = \sum_{n=0}^{n_\varphi^N-1} \prod_{k=0}^{N-1} C_{n_k}(\alpha_k)|n\rangle \qquad C_n(\alpha) = \frac{(n_\varphi - 1)!}{n_\varphi} \frac{1}{\sqrt{n}} \sum_{i=0}^{n_\varphi-1} e^{i(x_i|\alpha| - n\frac{\pi}{2})} \frac{\mathrm{He}_n(x_i)}{\mathrm{He}^2_{n_\varphi-1}(x_i)}, \qquad \mathrm{He}_{n_\varphi}(x_i) = 0$$

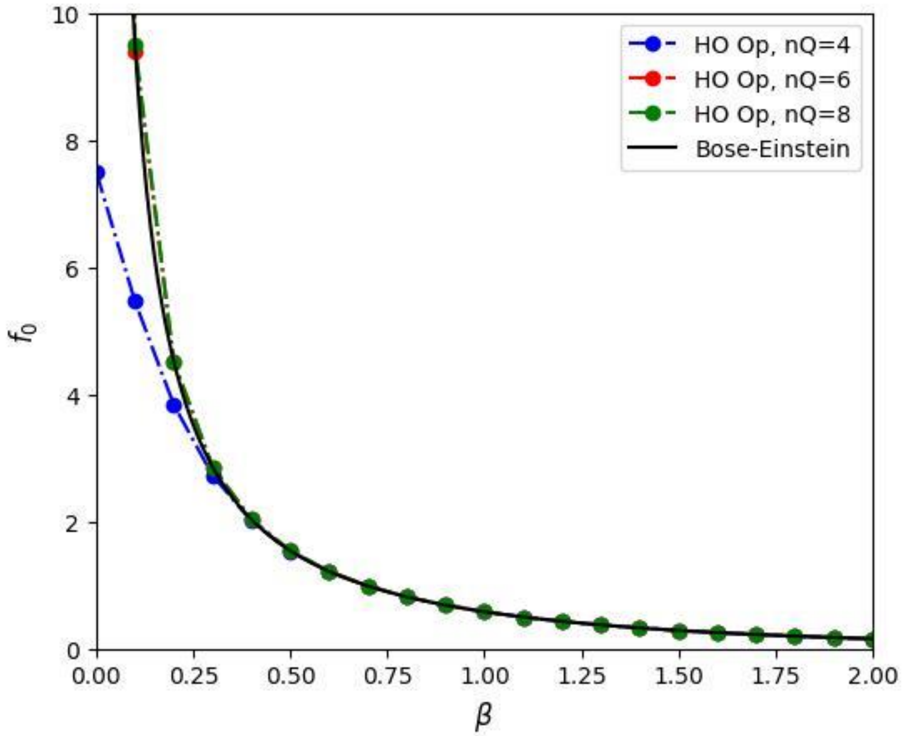
Results for Scalar Fields in Thermal Equilibrium ($N = 1$)

IC, Qian & Wu, 2024 [arXiv:2411.19601]

Analytical results:
Bose-Einstein distribution

$$f_p \equiv \langle \hat{a}_p^\dagger \hat{a}_p \rangle_\beta \quad \rightarrow \quad f_p = \frac{1}{e^{\beta E_p} - 1}$$





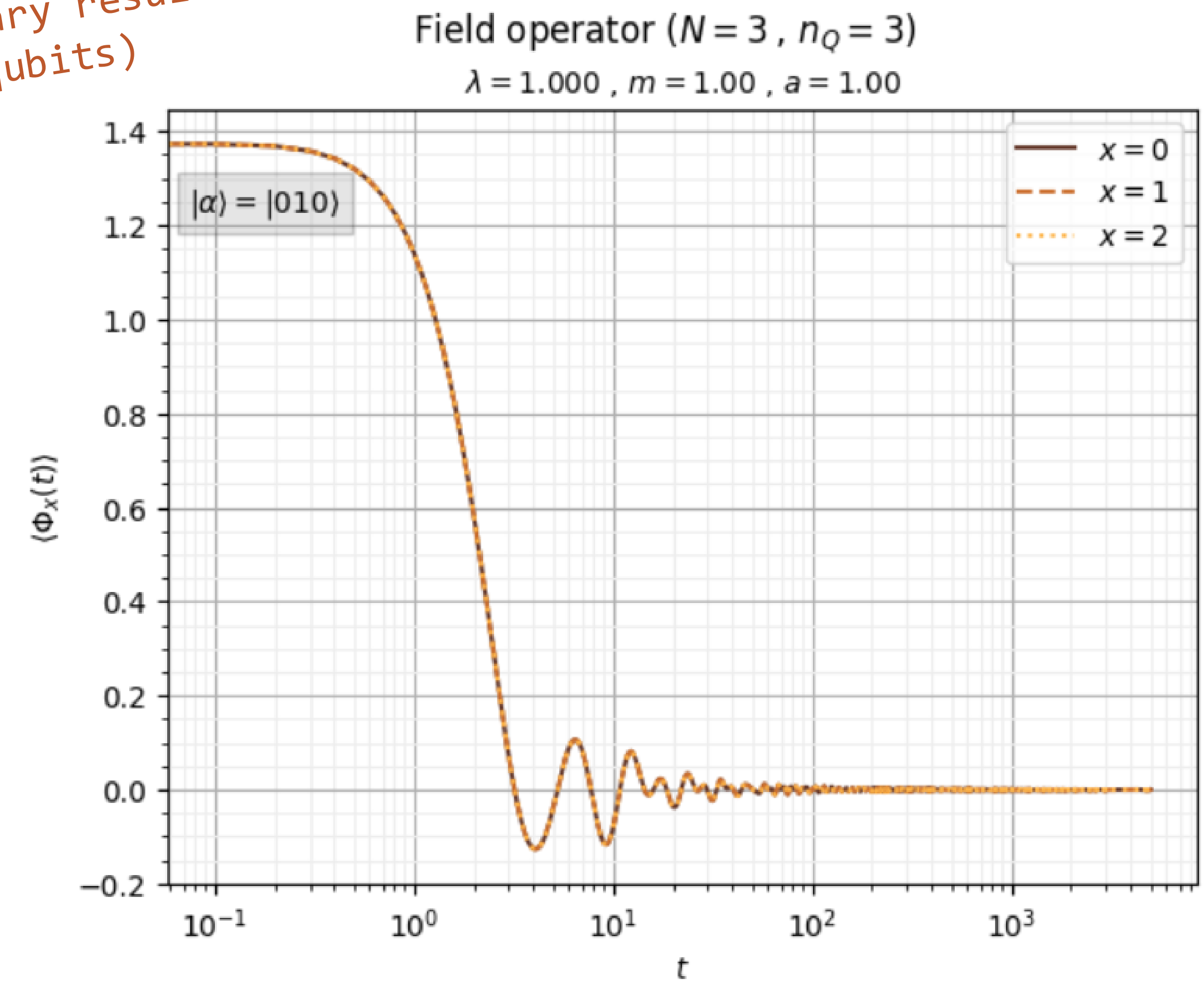
$$a_p = \sum_{x=0}^{N-1} e^{-ipax} \left[\sqrt{\frac{\omega_p}{2}} \Phi_x + i \sqrt{\frac{1}{2\omega_p}} \Pi_x \right] \qquad a_p^\dagger = \sum_{x=0}^{N-1} e^{ipax} \left[\sqrt{\frac{\omega_p}{2}} \Phi_x - i \sqrt{\frac{1}{2\omega_p}} \Pi_x \right]$$

HO Results: Real-time evolution

Initial state for homogeneous field:

$$|\alpha\rangle = |010\rangle \equiv \exp\left(\sum_{k=0}^{N-1} \alpha_k a_k^\dagger - \alpha_k^* a_k\right)|0\rangle \text{ with } \alpha_k = \{0,1,0\}$$

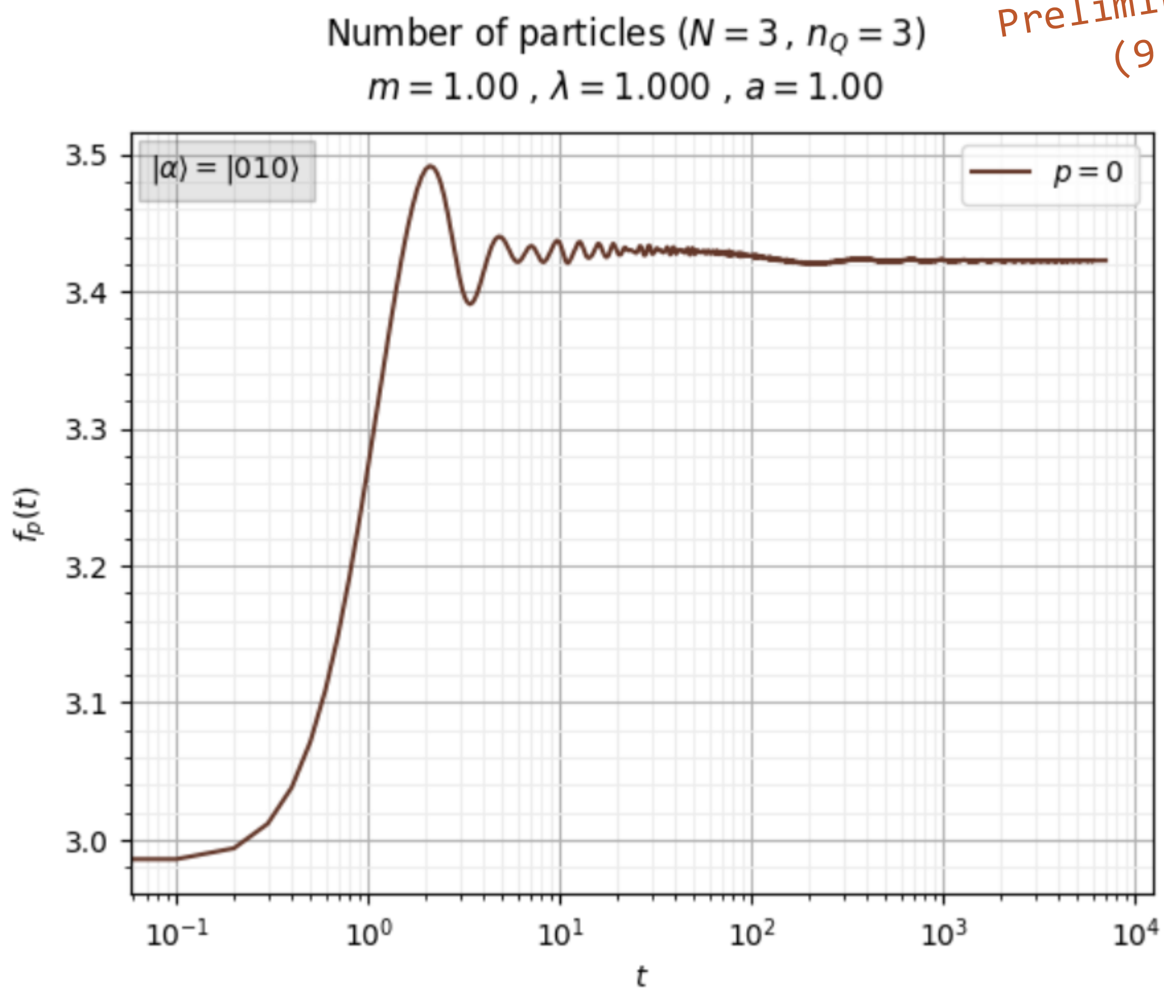
Preliminary results
(9 qubits)

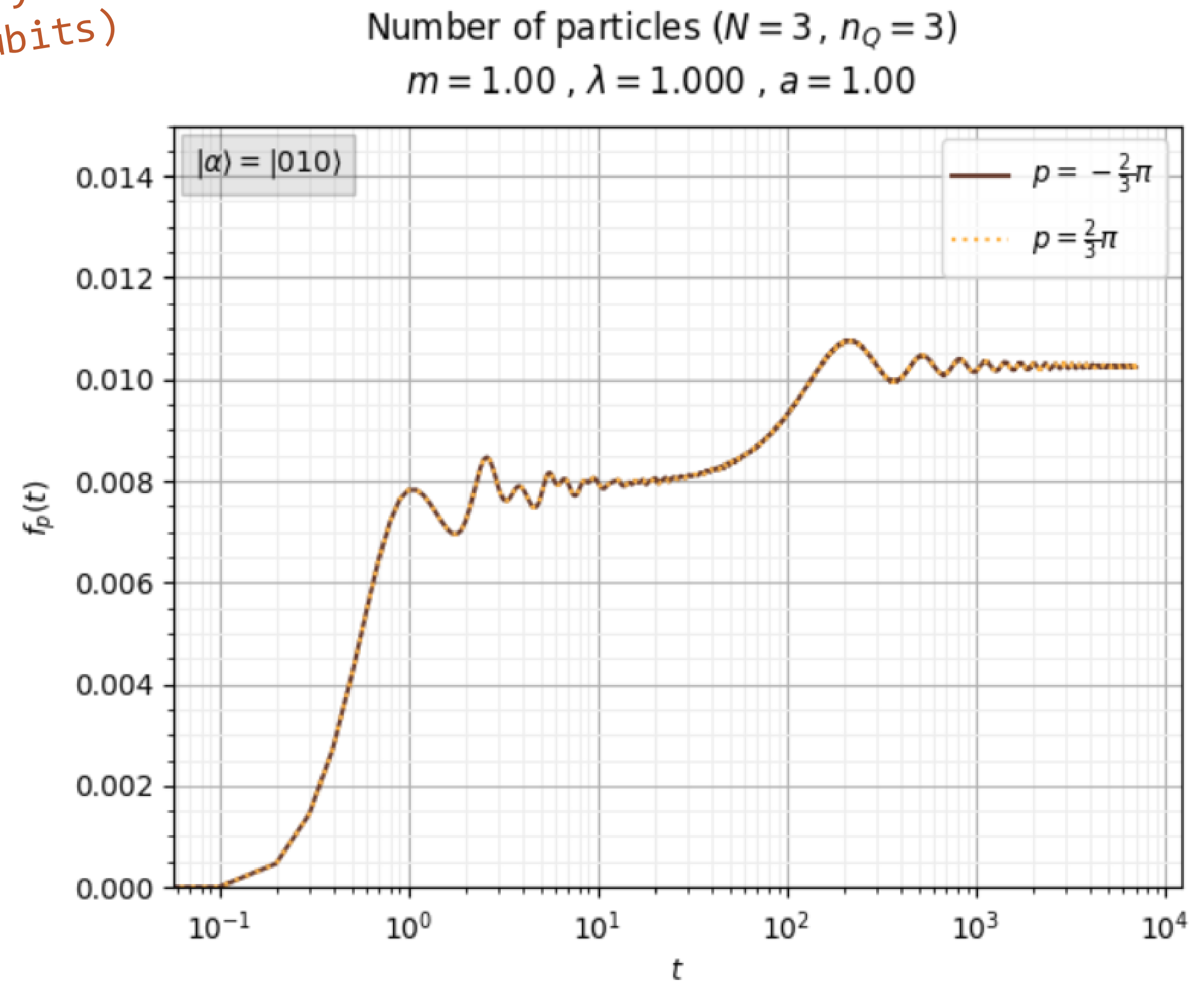


HO Results: Real-time evolution

IC, Qian & Wu [work in progress]

Preliminary results
(9 qubits)

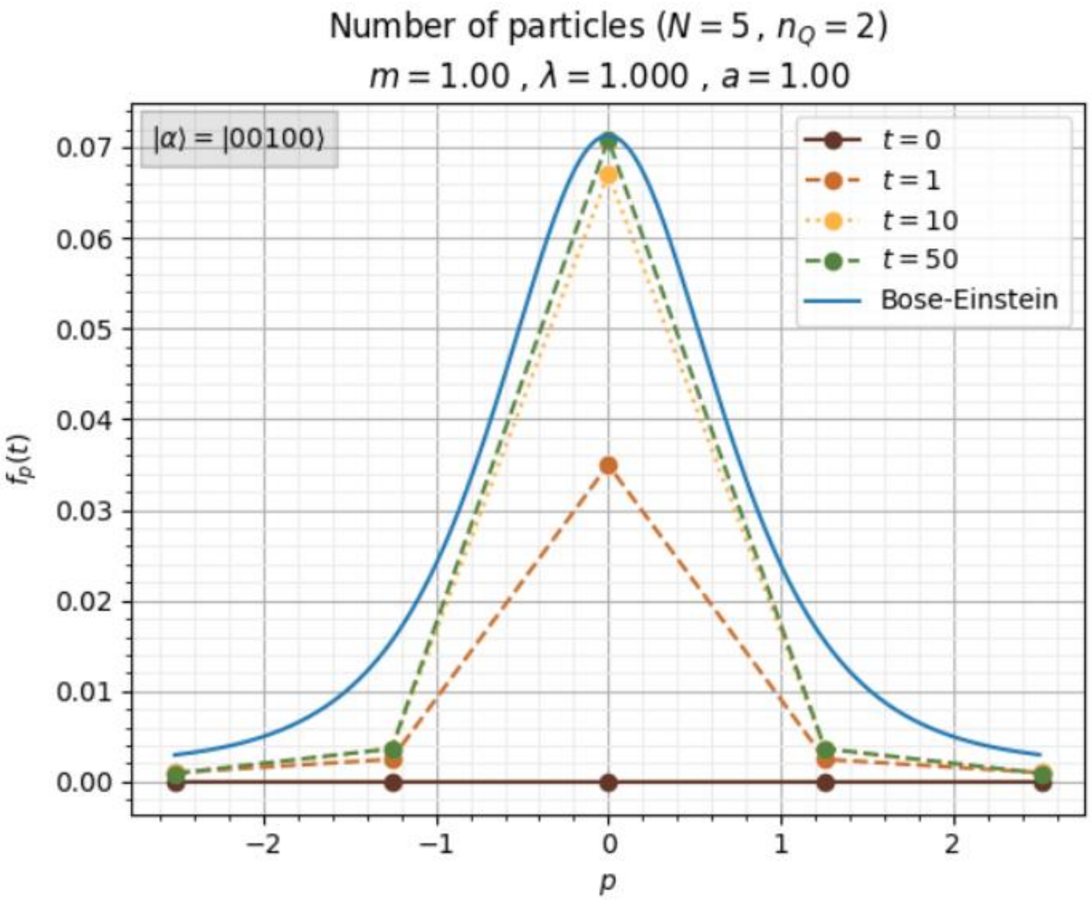




HO Results: Thermalisation

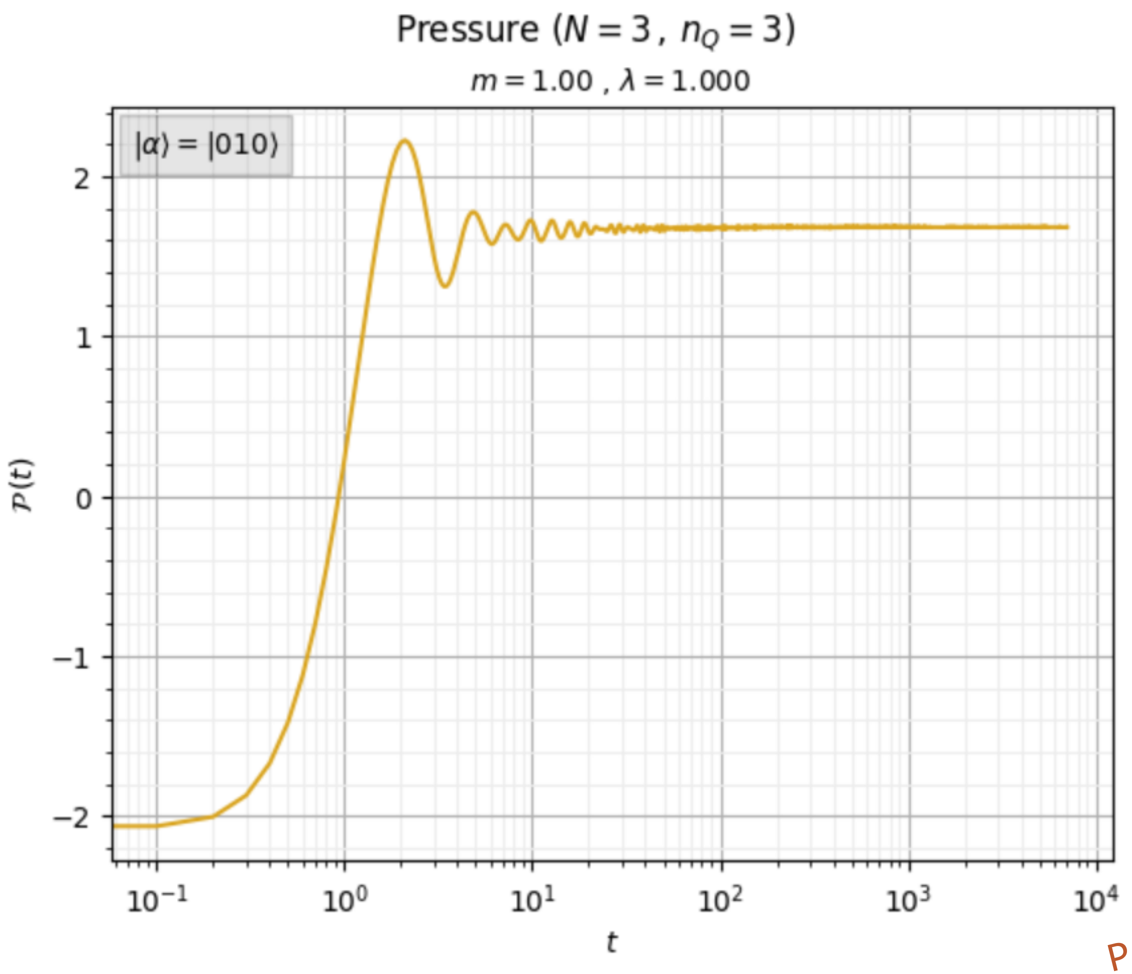
IC, Qian & Wu [work in progress]

Preliminary results
(10 qubits)



HO Results: Real-time evolution

IC, Qian & Wu [work in progress]



$$T_{\mu\nu} = \partial_\mu \Phi \partial_\mu \Phi - \mathcal{L} g_{\mu\nu} \Rightarrow T^{xx} \equiv P$$

$$P = a \sum_{x=0}^{N-1} \left[-\frac{1}{2} m^2 \Phi_x^2 + \frac{1}{2} \Pi_x^2 + \frac{1}{2} \frac{(\Phi_{x+1} - \Phi_x)^2}{a^2} - \lambda \frac{\Phi_x^4}{4!} \right]$$

Preliminary results
(9 qubits)

Summary and outlook

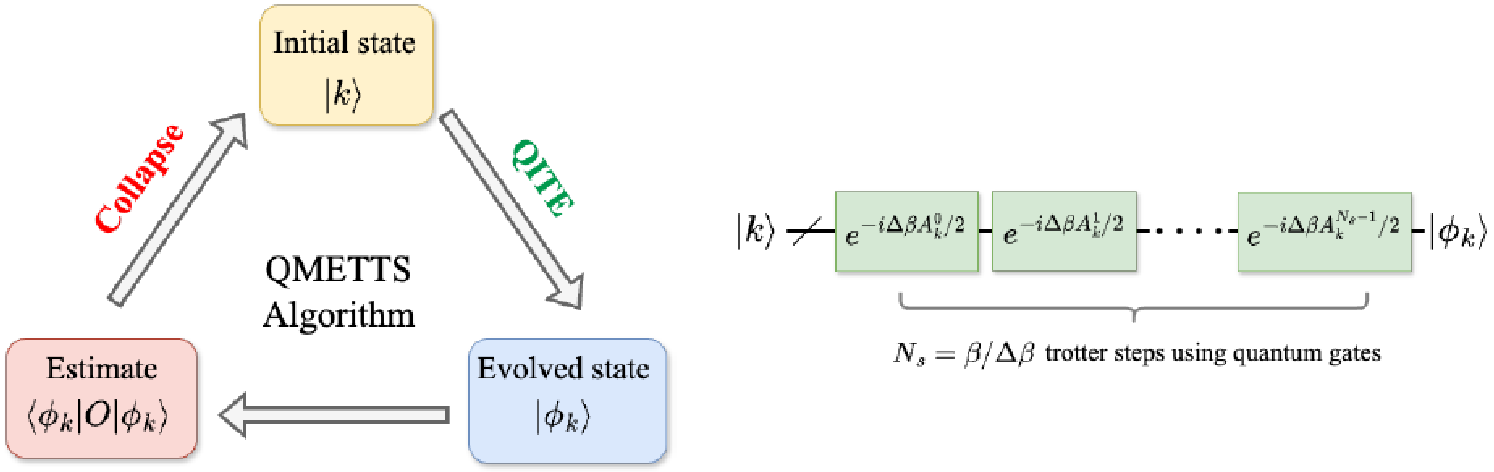
- Quantum computing technology is available today and developing fast for HEP
- We formulated the quantum field theory for scalar fields in 1+1 dimensions on the qubits and studied its various thermal properties at finite temperature as well as thermalisation using quantum simulation algorithms
- Quantum simulation helps to understand real-time evolution of non-equilibrium dynamics relevant to heavy-ion collisions.
- Larger scale simulation under development. It remains interesting to explore other options such as tensor networks

THANK YOU FOR YOUR ATTENTION!

Quantum imaginary time evolution

Motta, 2019 [arXiv:1901.07653]

Trotter formula: $|\psi_k(\beta/2)\rangle = e^{-\beta\hat{H}/2}|k\rangle = \prod_{i=1}^{N_s} e^{-\Delta\beta\hat{H}/2}|k\rangle$



(a) QMETTS algorithm

(b) QITE algorithm

$$|\psi_k^{i+1}\rangle = e^{-\Delta\beta\hat{H}/2}|\psi_k^i\rangle = \sqrt{c_k^i(\Delta\beta)}\,e^{-i\Delta\beta\hat{A}_k^i/2}|\psi_k^i\rangle + \mathcal{O}(\Delta\beta^2)$$

Real-time Hamiltonian operator

$$\hat{A}_k^i = \sum_I a_I\sigma_I$$

determined by solving the linear matrix equation

$$(\boldsymbol{S} + \boldsymbol{S}^T)\boldsymbol{a}_i = \boldsymbol{b}$$

whose matrix elements are evaluated as expectation values on the quantum circuit

$$b_I = -i\,\langle\psi_k^i|(\hat{H}\sigma_I - \sigma_I^\dagger\hat{H})|\psi_k^i\rangle \,/\, \sqrt{c_k^i(\Delta\beta)}$$

$$S_{IJ} = \langle\psi_k^i|\sigma_I^\dagger\sigma_J|\psi_k^i\rangle$$

Thermal expectation on the circuit

White, 2009 [arXiv:0902.4475]

Minimally entangled typical thermal state method (METTS)

Redefined quantum state $$|\phi_k\rangle = P_k^{-1/2} e^{-\beta \hat{H}/2} |k\rangle\,, \quad P_k = \langle k| e^{-\beta \hat{H}} |k\rangle$$

in QITE: $$|\phi_k\rangle \equiv \prod_{i=1}^{N_s} e^{-i\Delta\beta \hat{A}_k^i/2} |k\rangle\,, \qquad P_k \equiv \prod_{i=1}^{N_s} c_k^i(\Delta\beta)$$

Thermal observables $$\langle \hat{O} \rangle_\beta = \sum_{k\in\mathcal{S}} \frac{P_k}{Z} \langle \hat{O}_k \rangle_\beta\,, \qquad \langle \hat{O}_k \rangle_\beta = \langle \phi_k | \hat{O} | \phi_k \rangle\,, \qquad Z = \sum_{k\in\mathcal{S}} P_k$$

Equivalent to sampling $|\phi_k\rangle$ with probability P_k/Z and summing its expectations $\langle O_k \rangle_\beta$