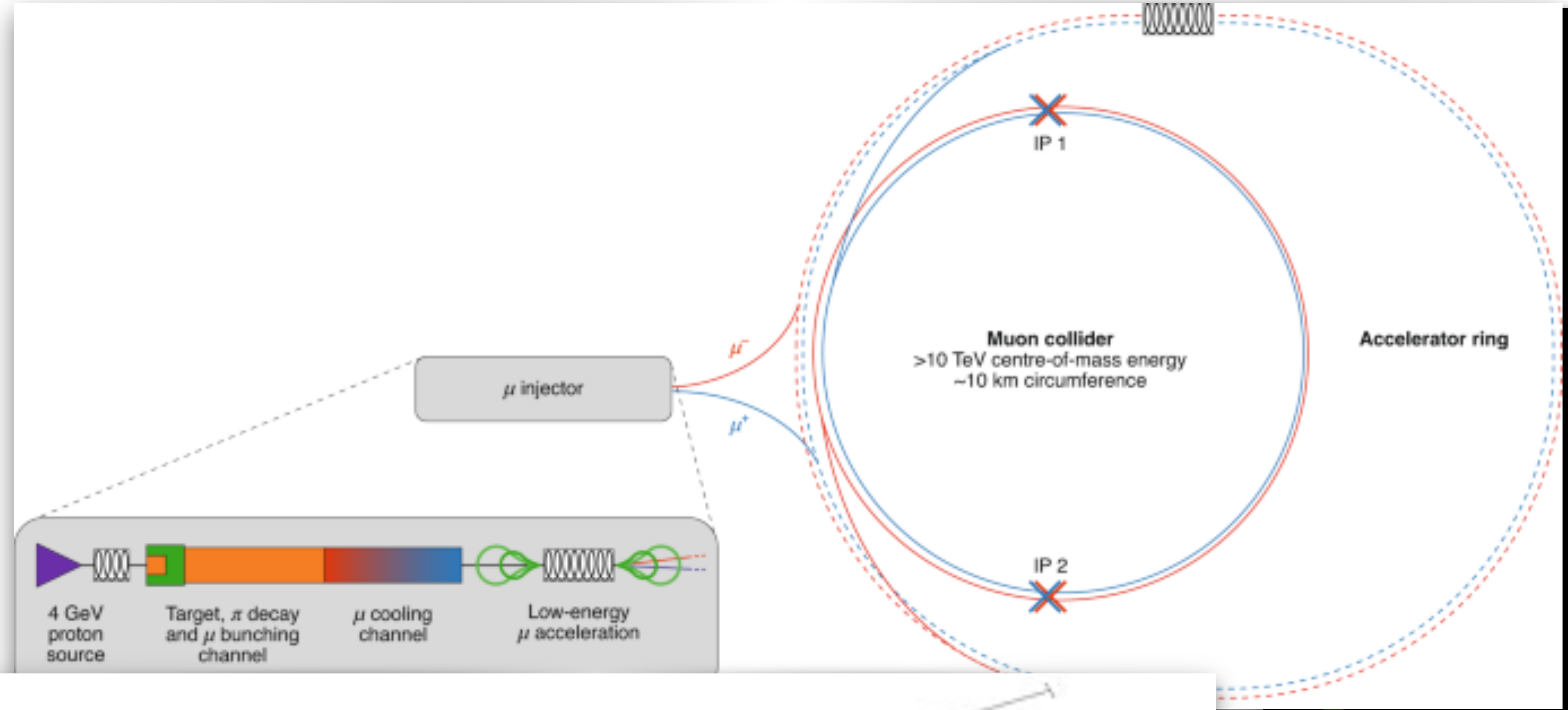
SCATTERING AMPLITUDES: FROM COLLIDER PHYSICS TO GEOMETRY

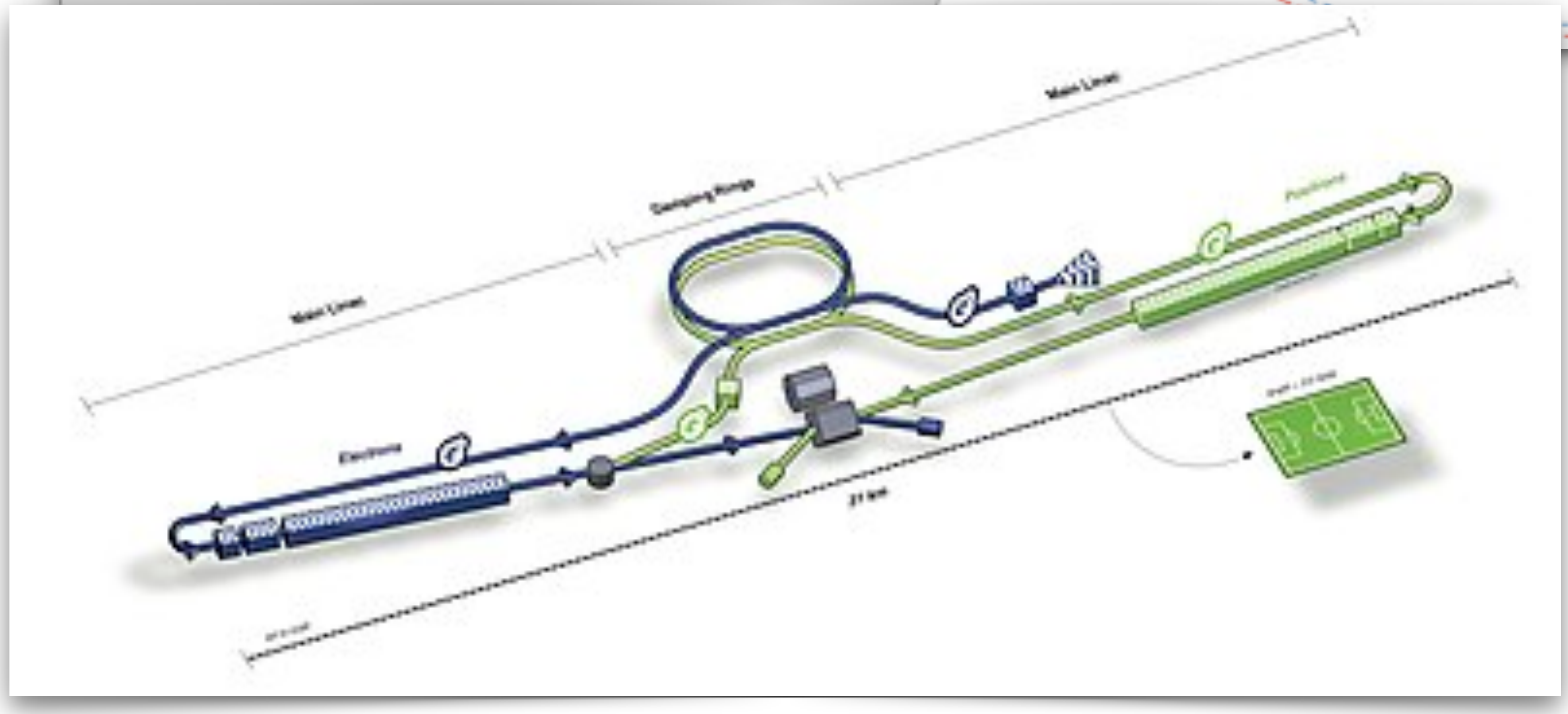
COMETA Colloquium
December 16th 2024

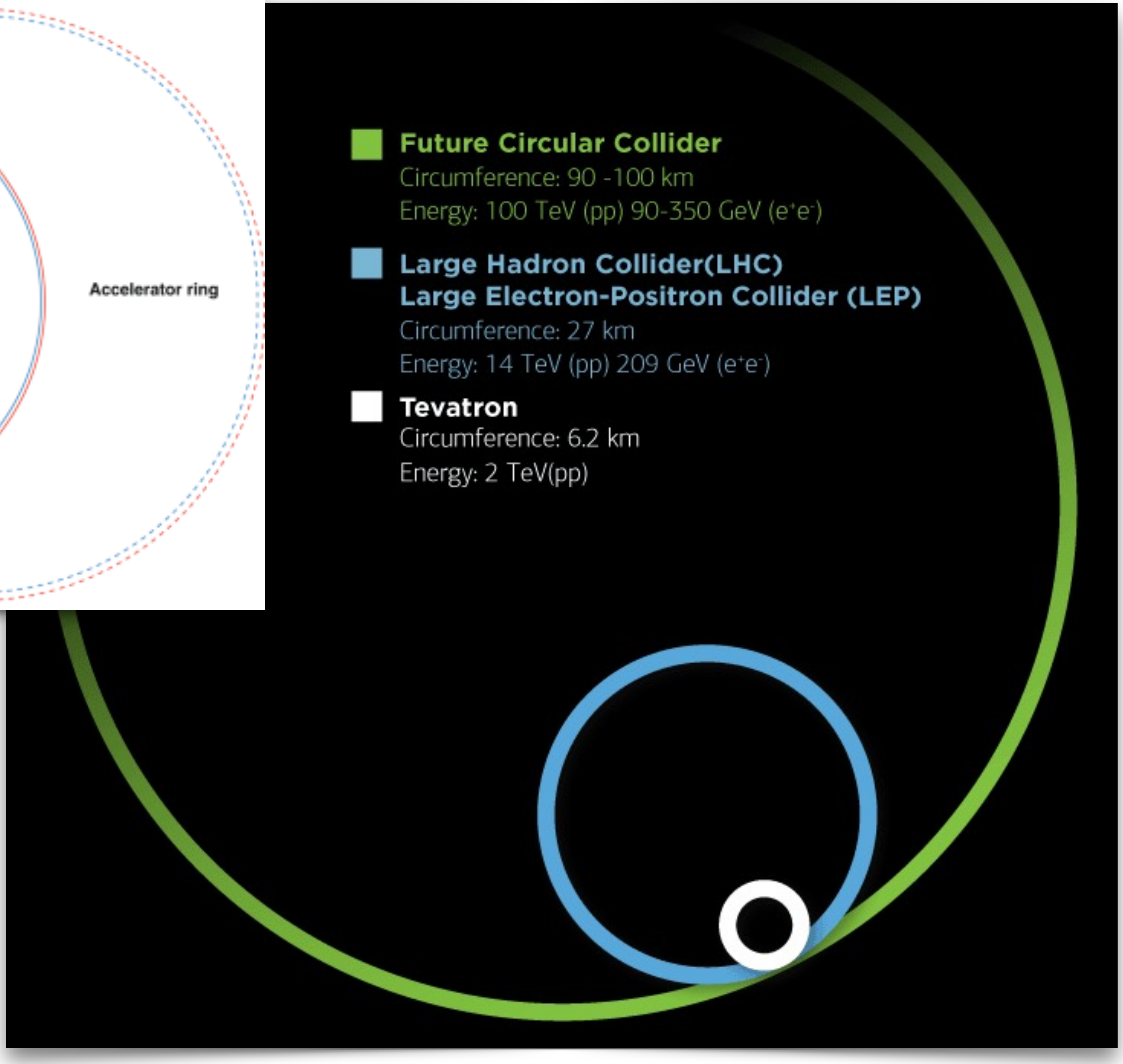
Lorenzo Tancredi – Technical University Munich

Technische Universität München

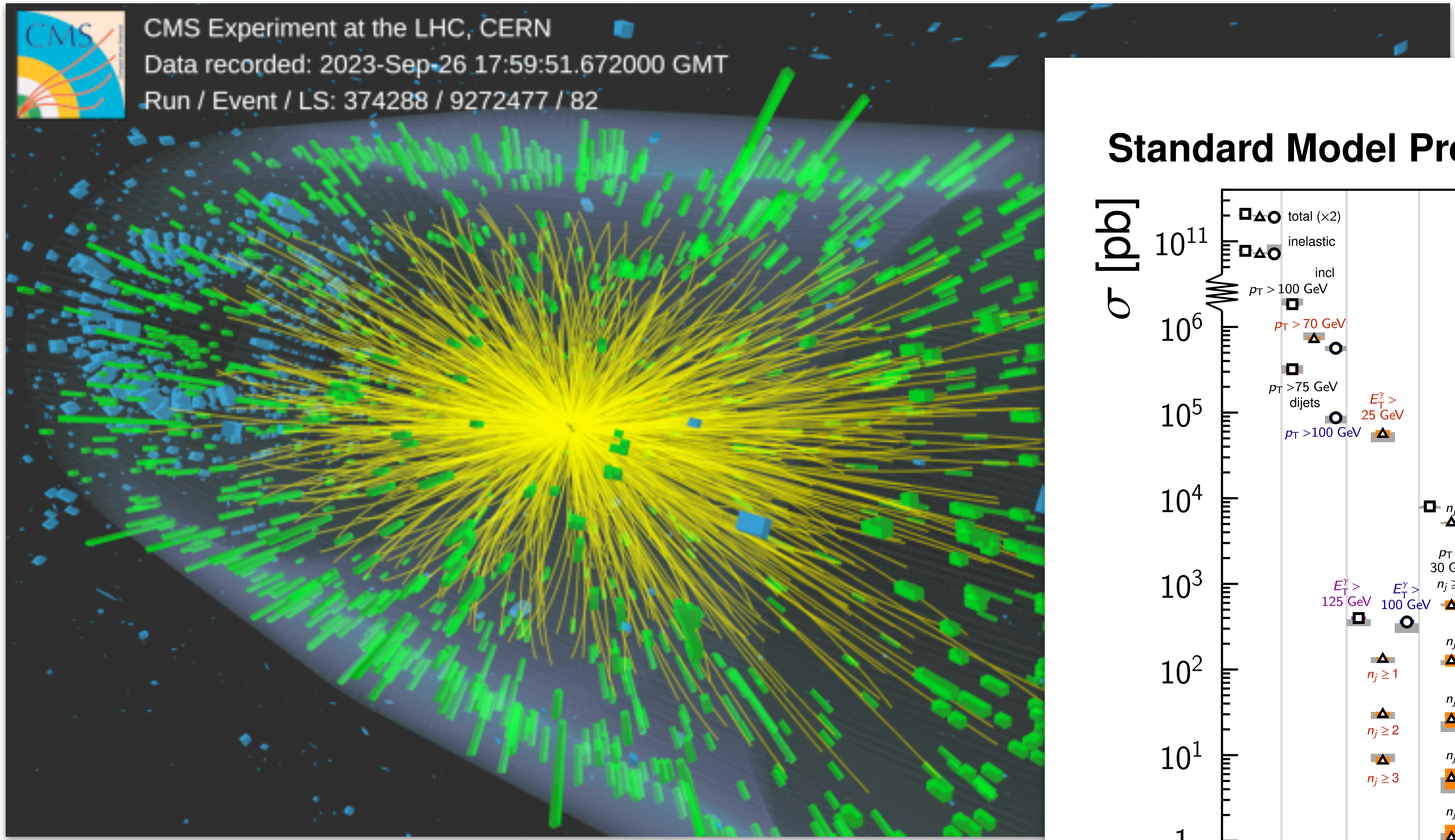
WHY (STILL) COLLIDERS? THE LHC (AND BEYOND)…

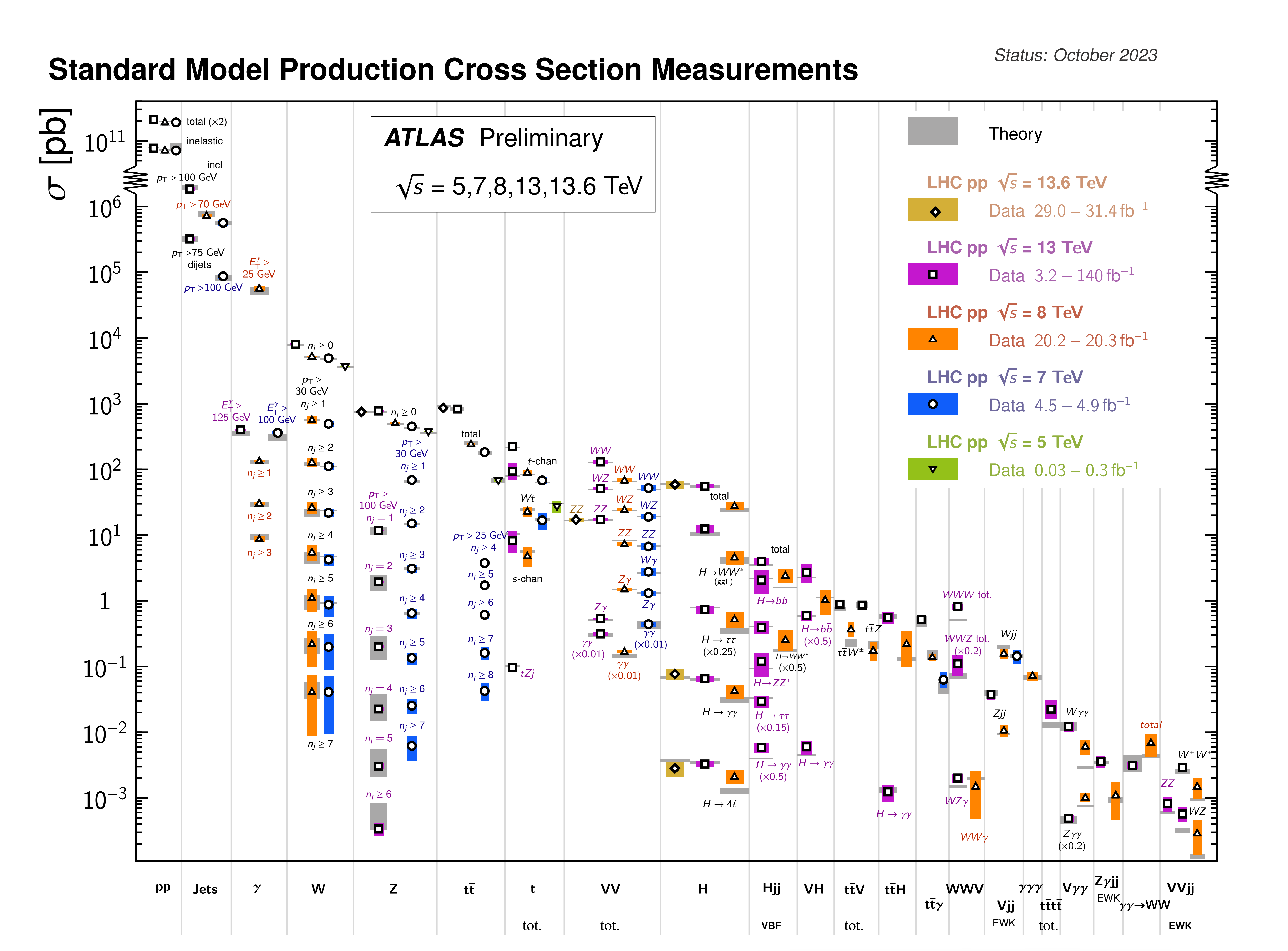






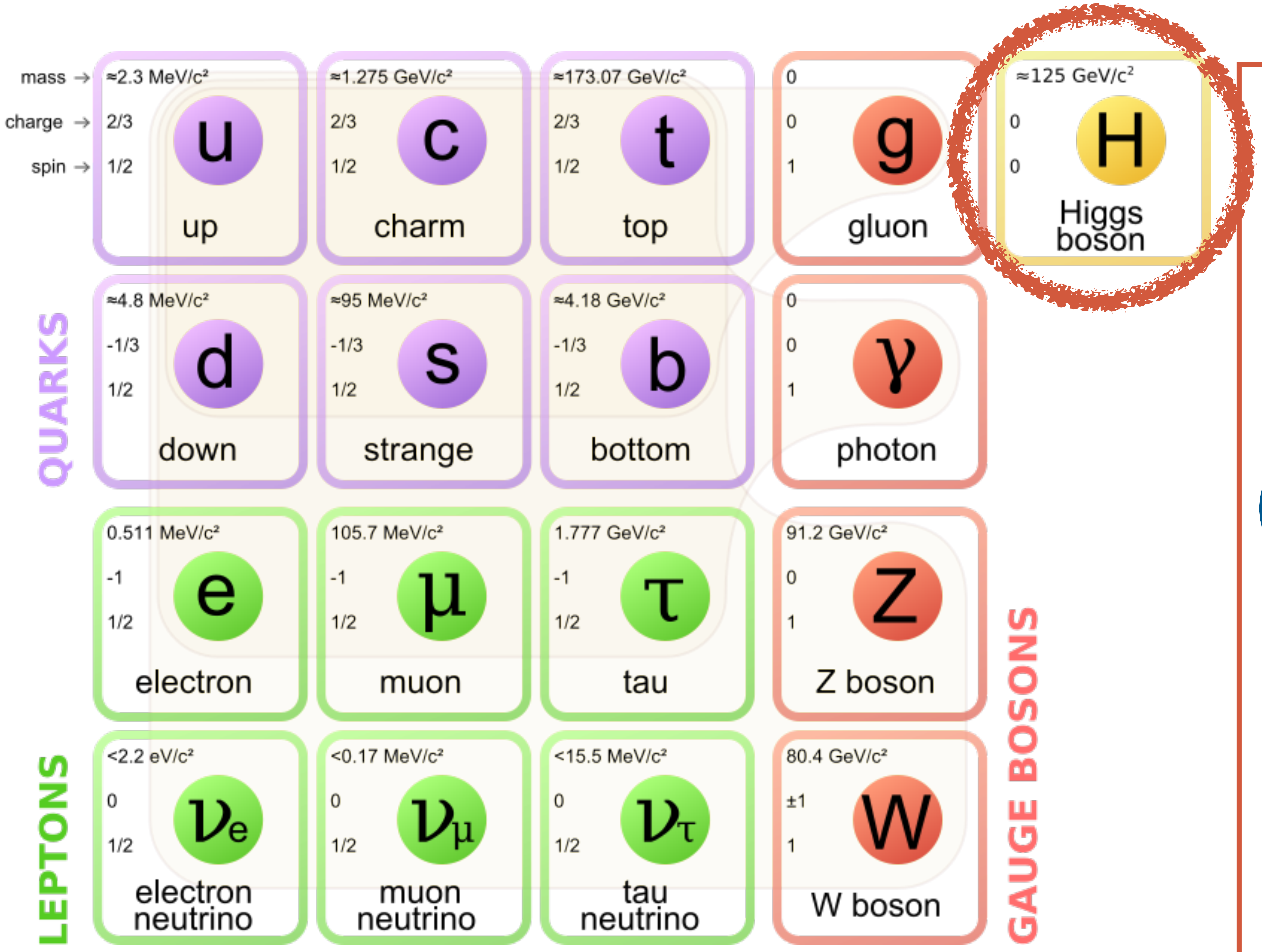
THE LHC HAS BECOME A PRECISION MACHINE

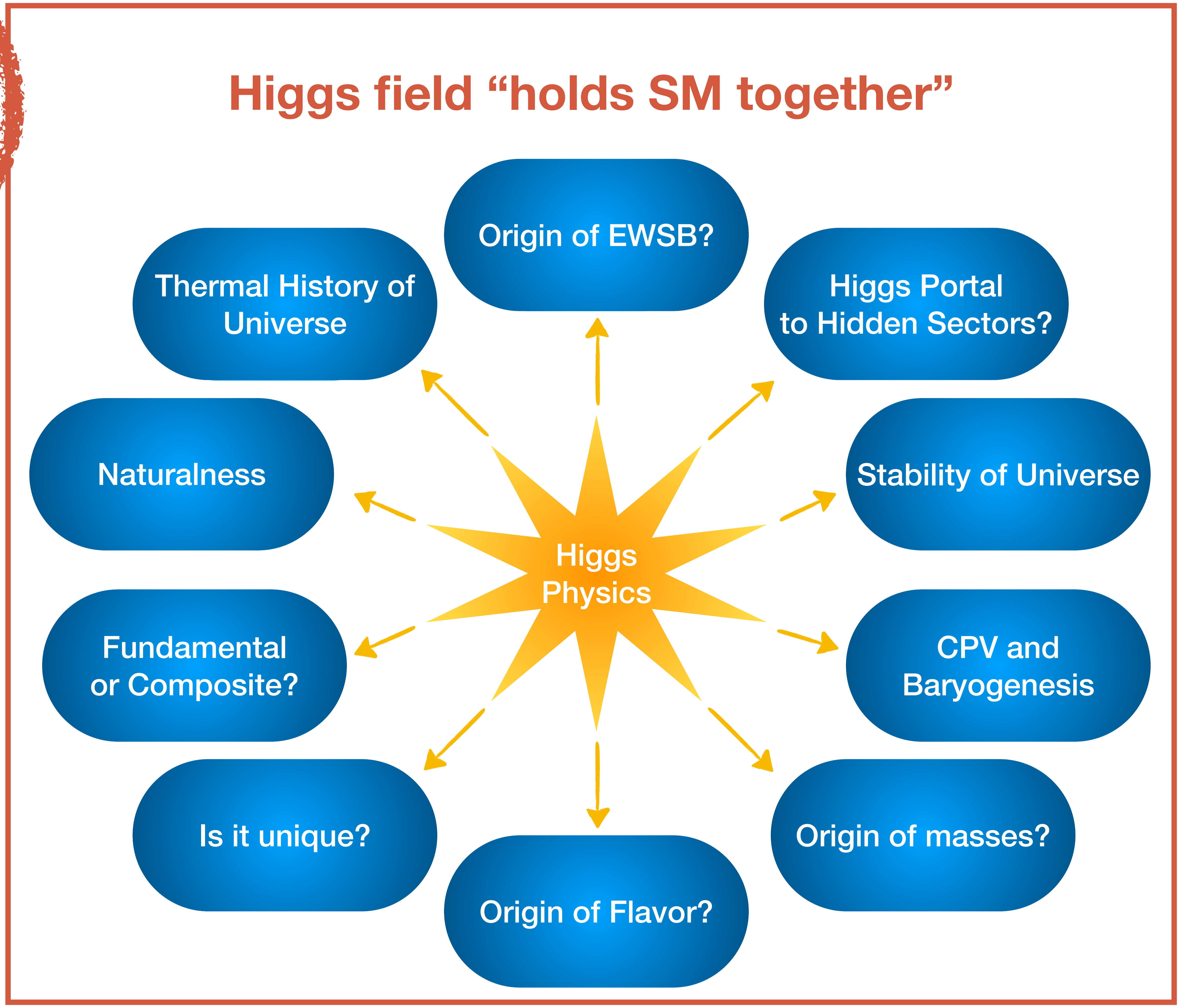




After its discovery in 2012, a lot (but not only) revolving around Higgs boson's properties

THE HIGGS BOSON: THE LAST MISSING PIECE

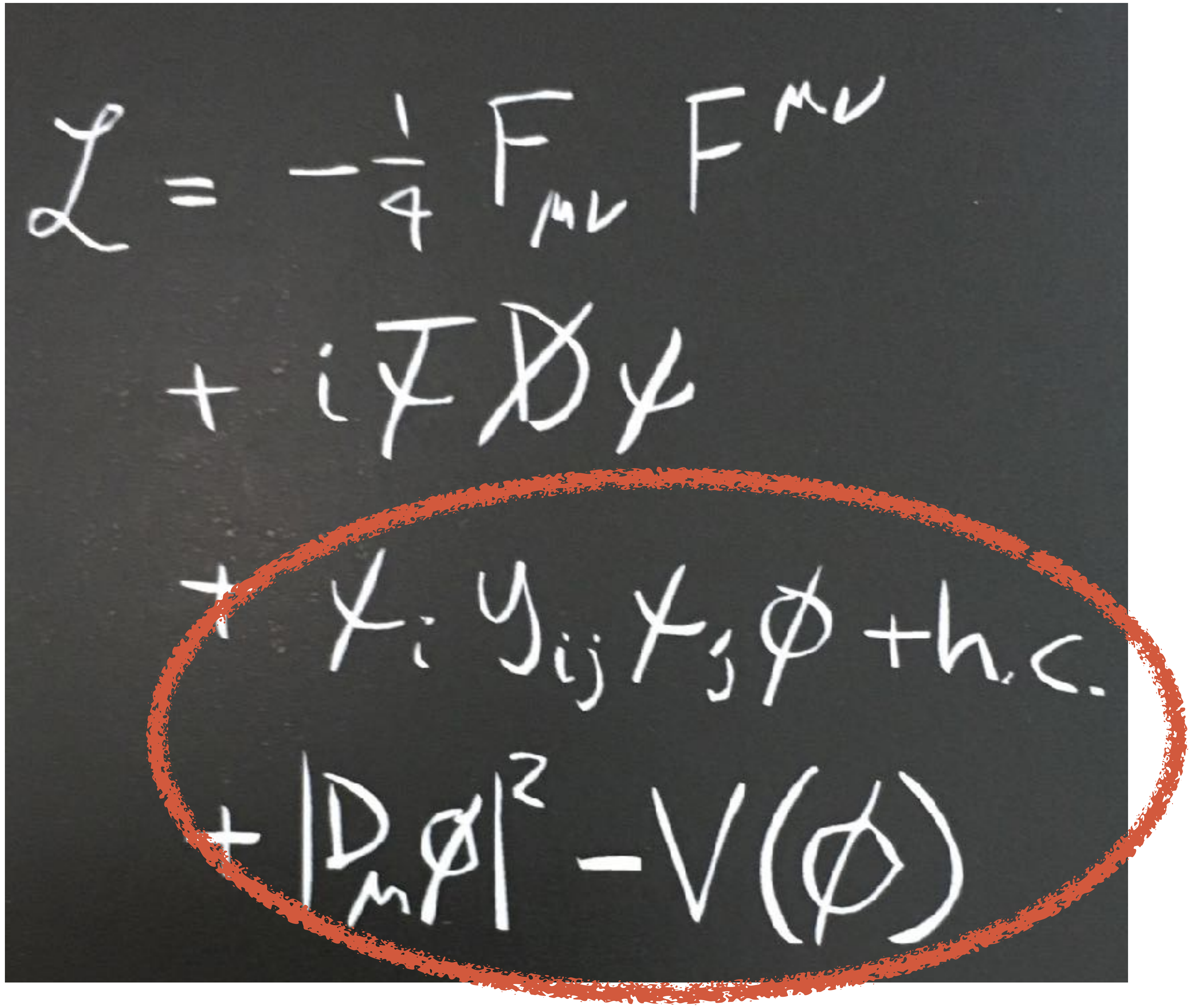




[Snowmass 2022 arXiv:2209.0751]

HIGGS INTERACTIONS AT THE LHC

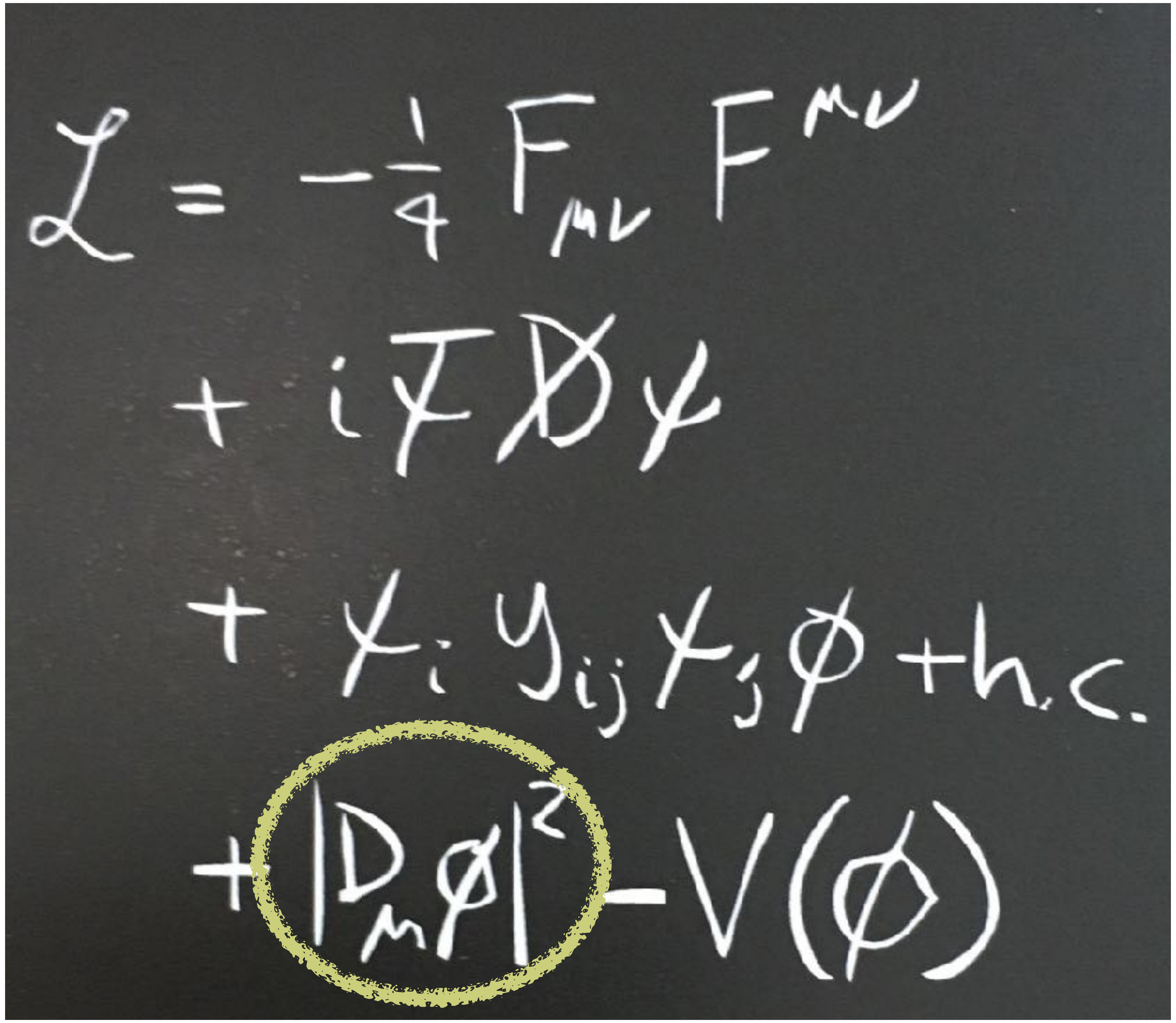
Hints to answer these questions hidden in the **details of Higgs interactions to SM particles**

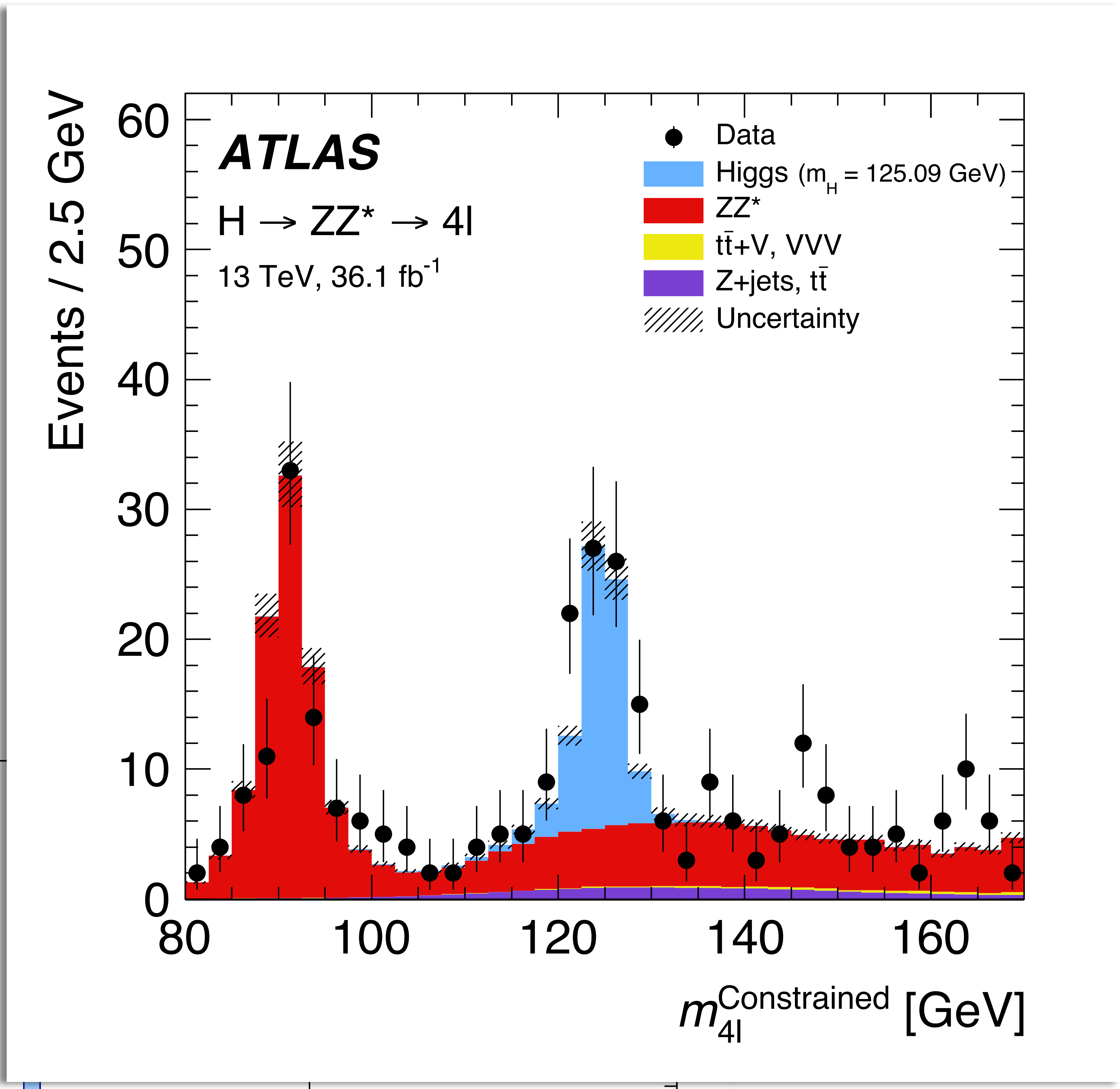


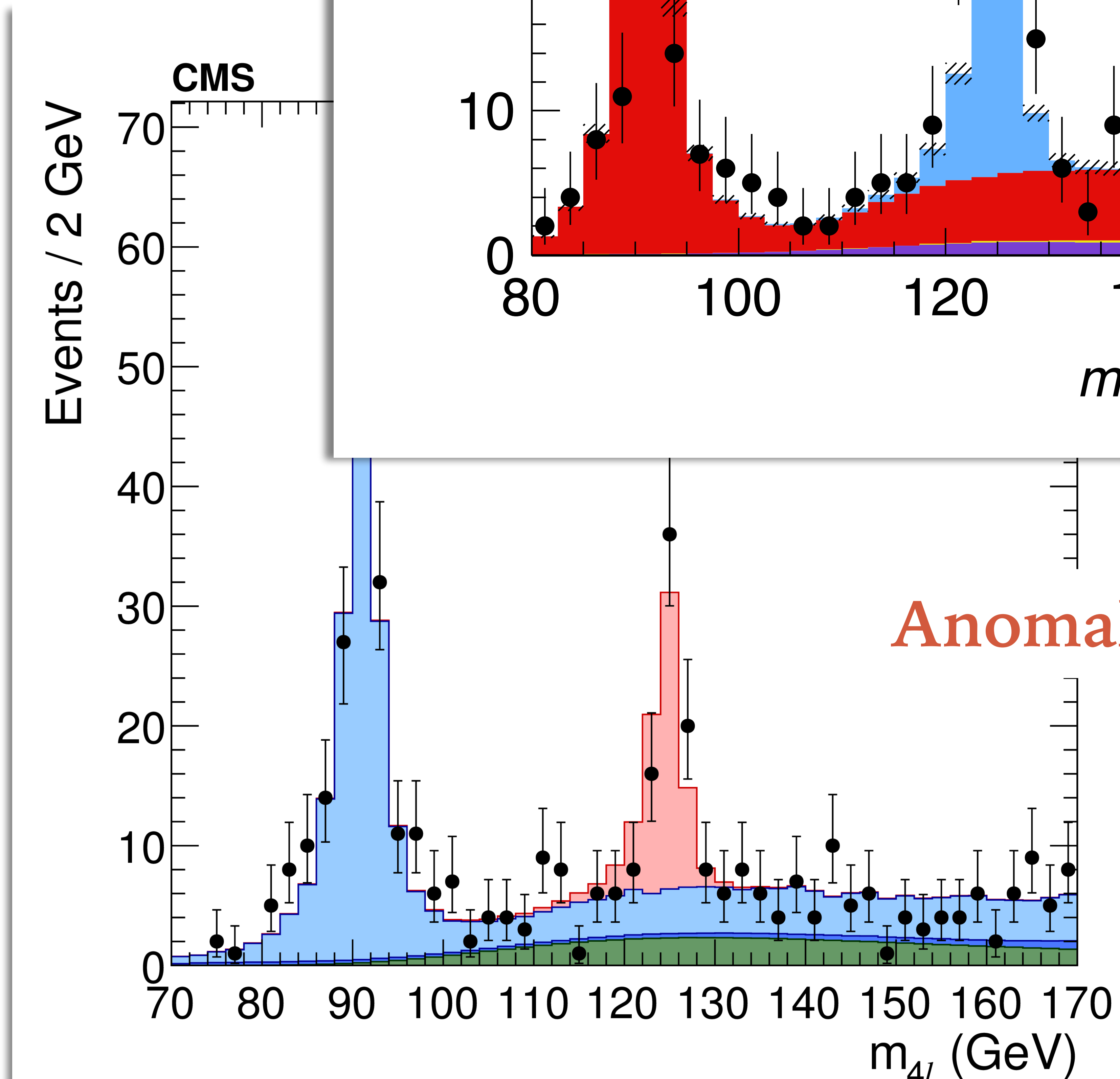
LHC has opened a window for us to peak at Higgs’ interactions

HIGGS INTERACTIONS THE GAUGE SECTOR

Higgs discovery through its **couplings to gauge sector**



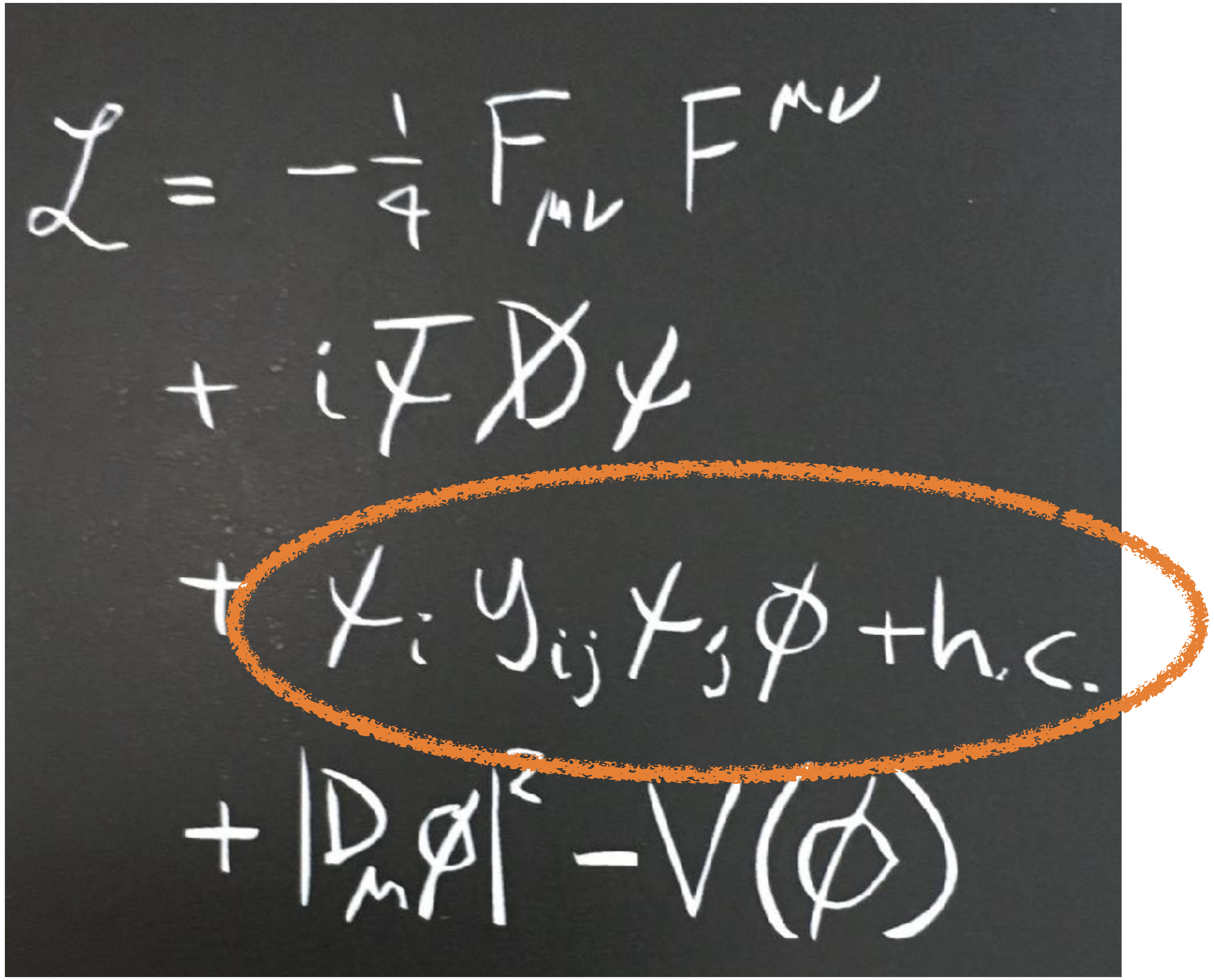


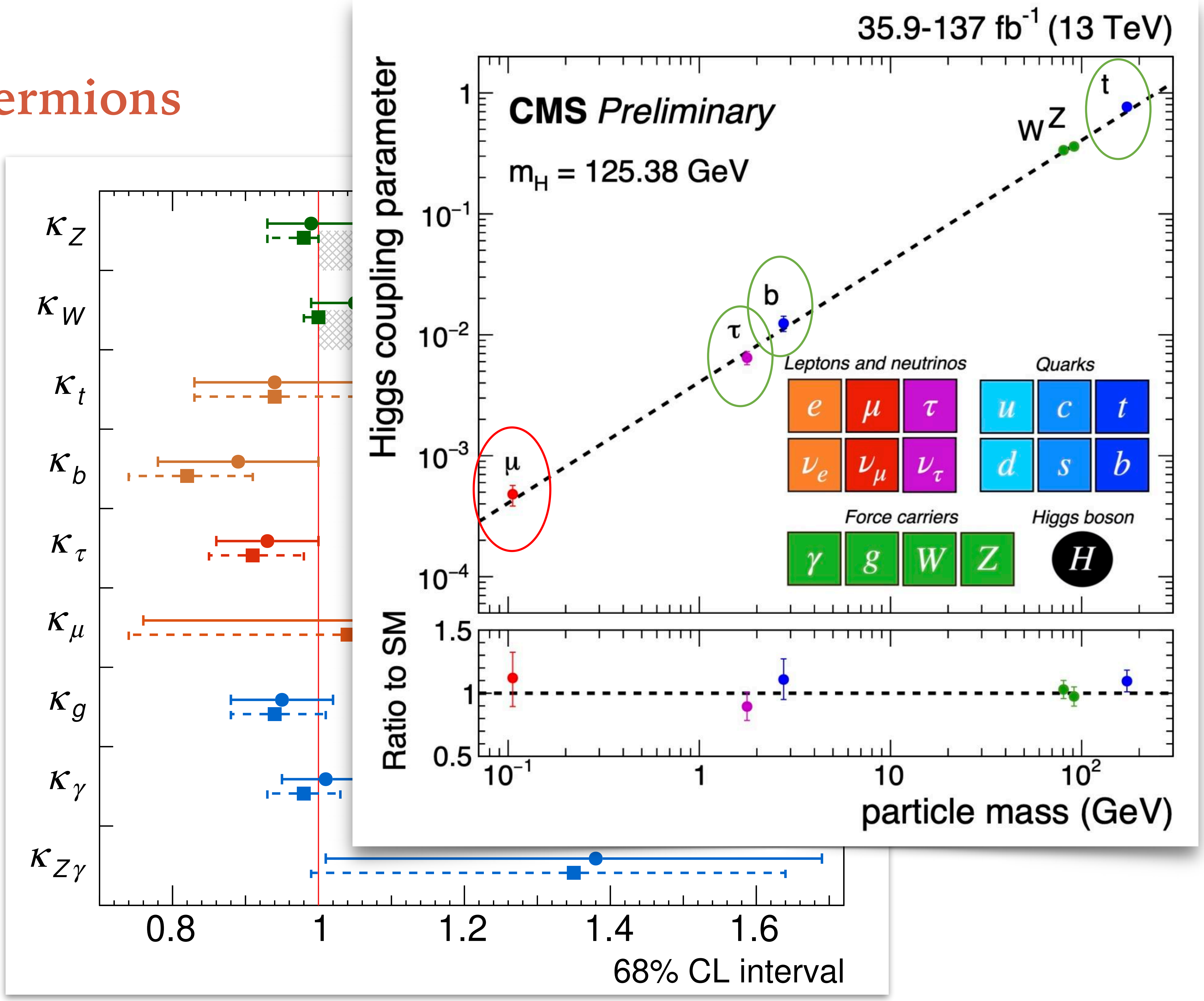


Anomalous couplings?

HIGGS INTERACTIONS THE YUKAWA SECTOR

Run 2 direct observation of H coupling to third family fermions



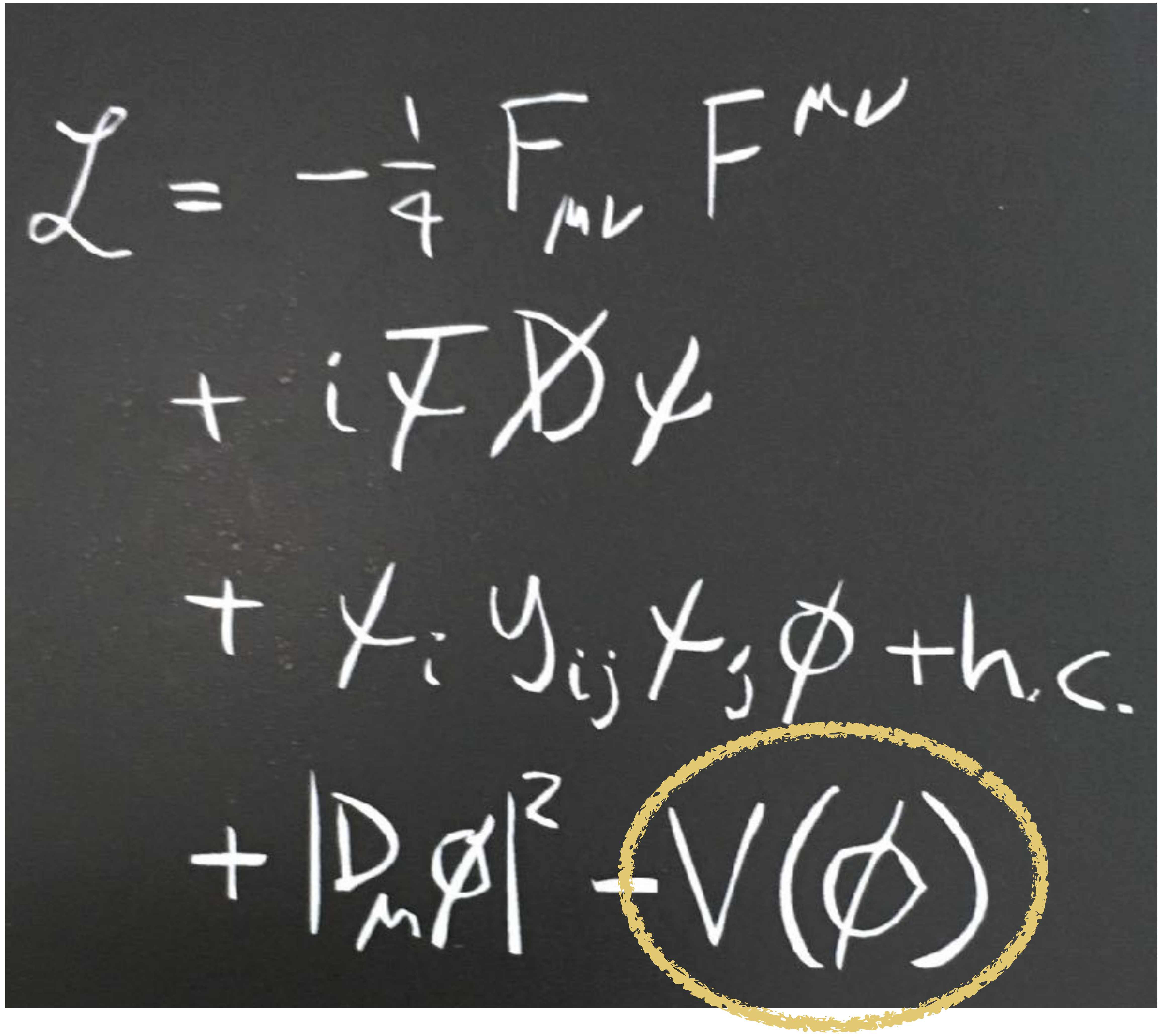


Run 3 and **HL potential:**

1. **Precision** measurements for **third family**
2. Discovery couplings to **second family** (μ & c)

HIGGS SELF INTERACTIONS THE MOST MYSTERIOUS?

HL-LHC first to see the triple-H coupling



We have seen the Higgs but

$$V(\phi) = -\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

is a “toy model”!

1. more minima?
2. more Higges?
3. microscopic model of SSB?
4. …

Higgs self coupling extremely difficult to measure.

With 2018 estimates 4σ ATLAS+CMS

HIGGS SELF INTERACTIONS THE MOST MYSTERIOUS?

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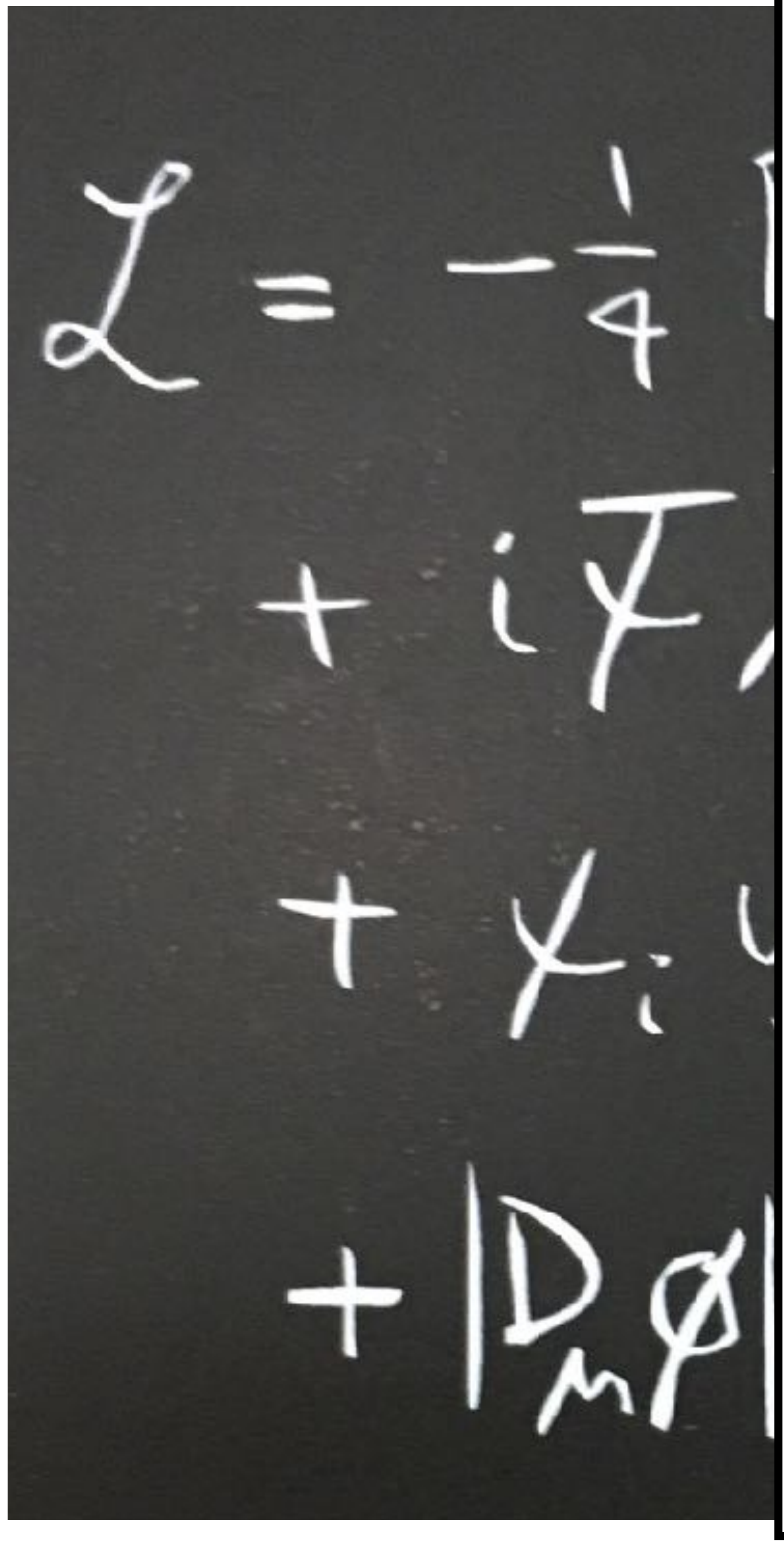

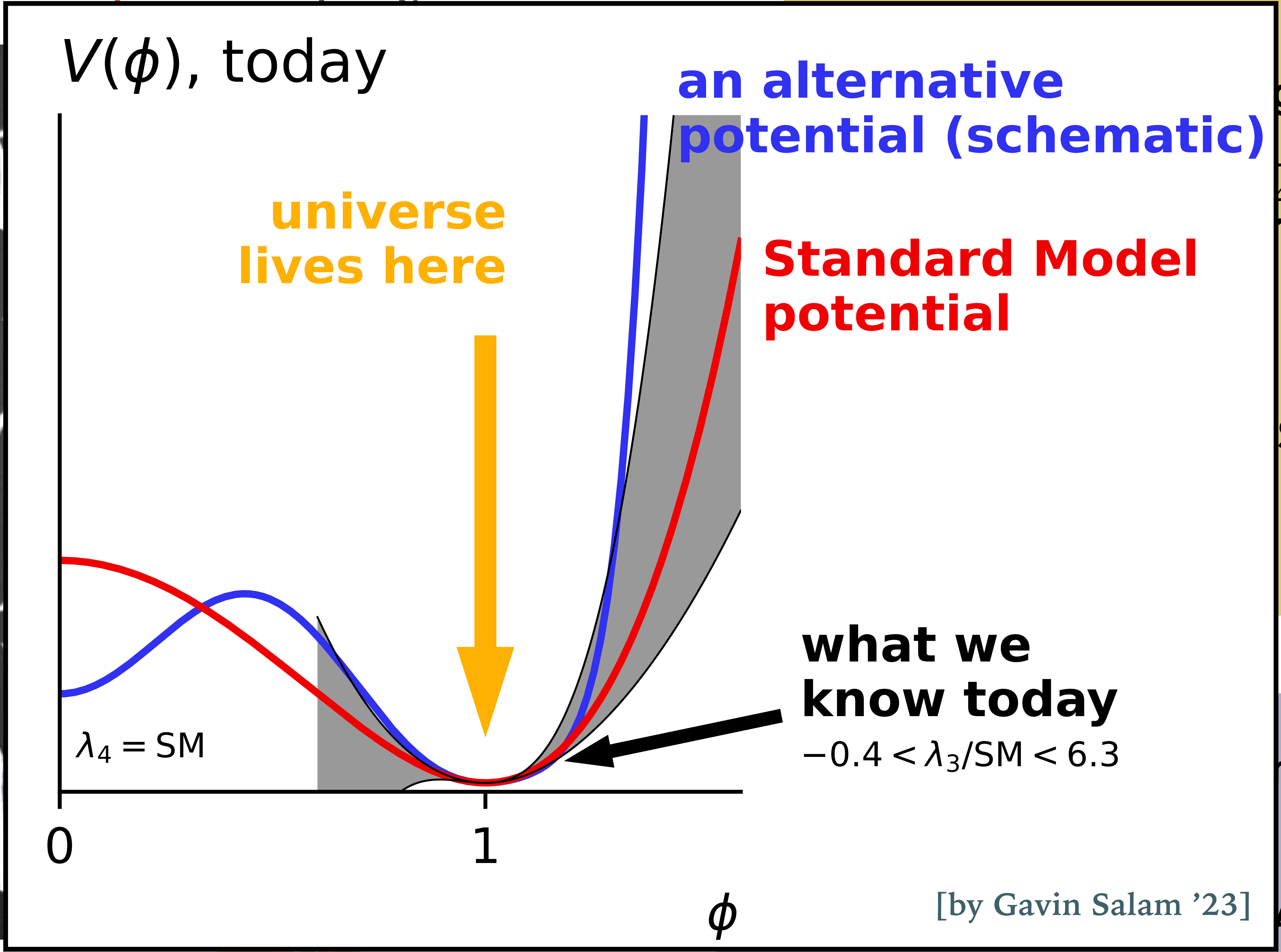

ggs but

$\frac{\lambda}{4!}\phi^4$

”!

SSB?

nely difficult to

ATLAS+CMS

HIGGS SELF INTERACTIONS THE MOST MYSTERIOUS?

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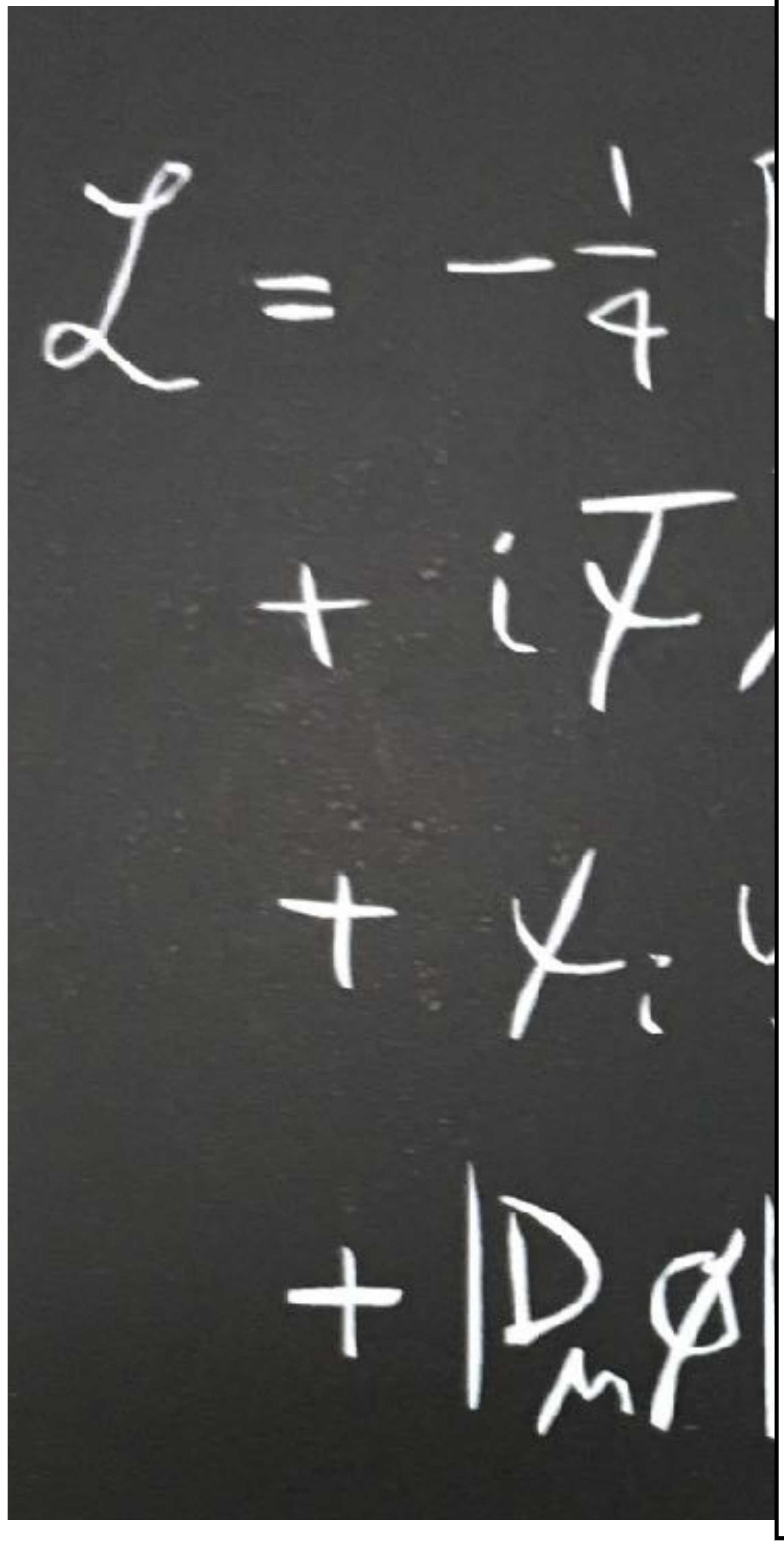

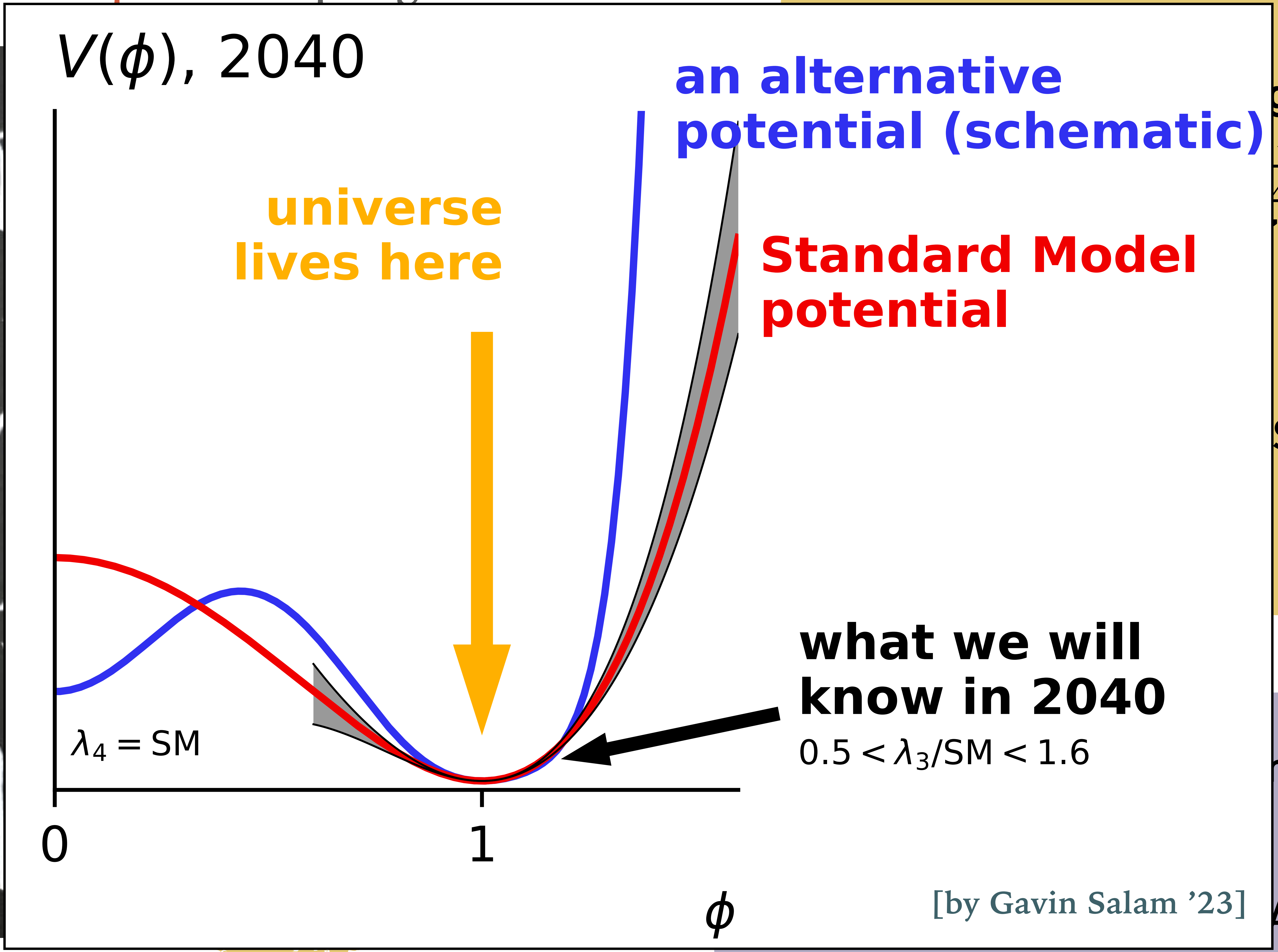

ggs but

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”!

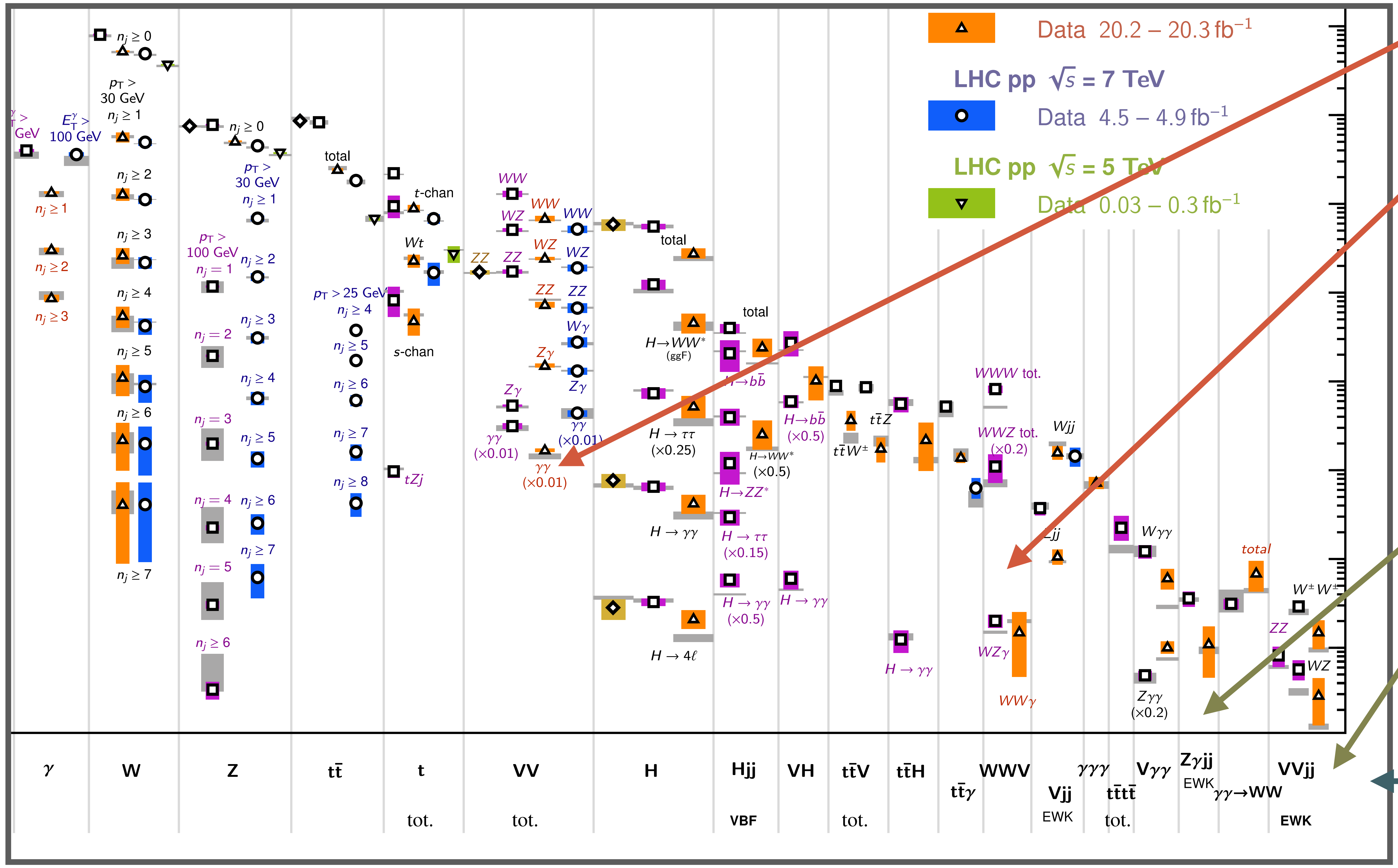
SSB?

nely difficult to

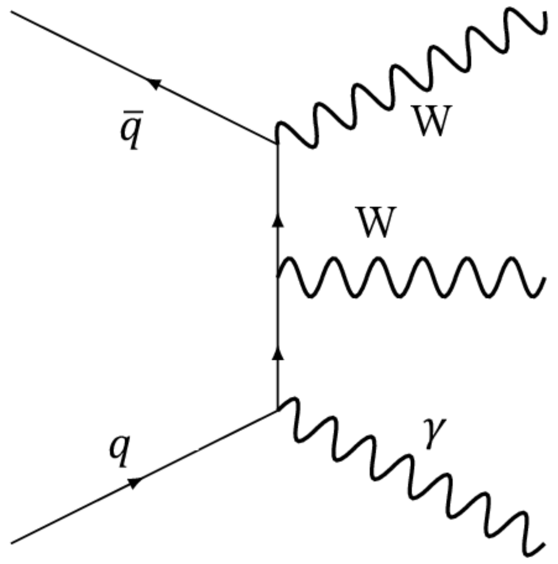
ATLAS+CMS

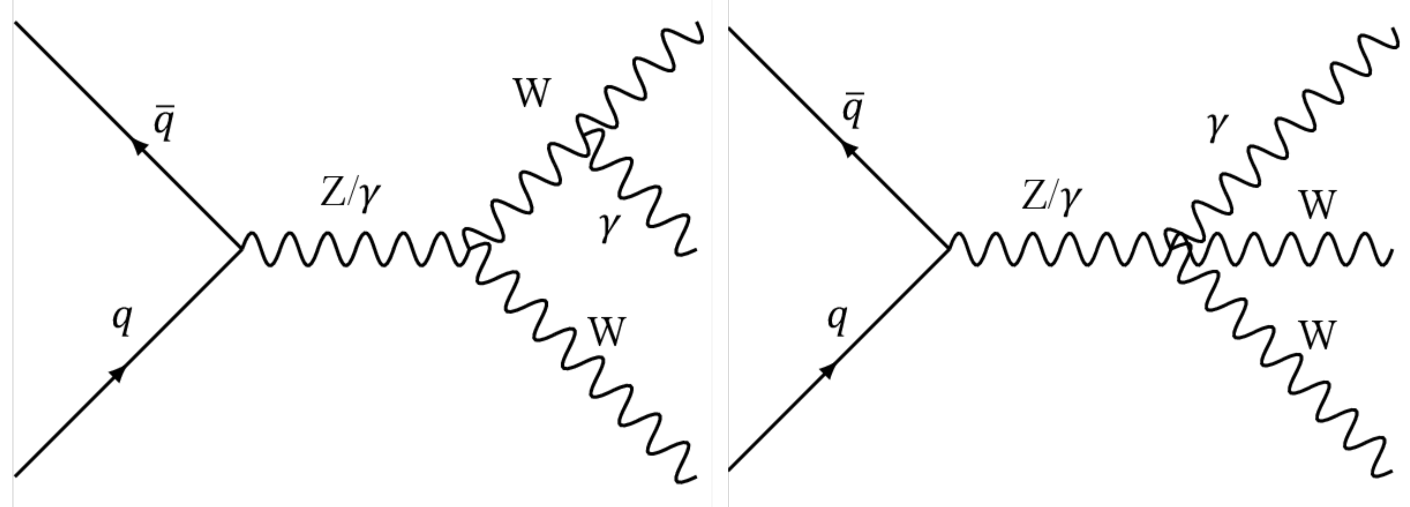
PROBING THE GAUGE SECTOR

Multiboson final states as probe of electroweak sector of SM

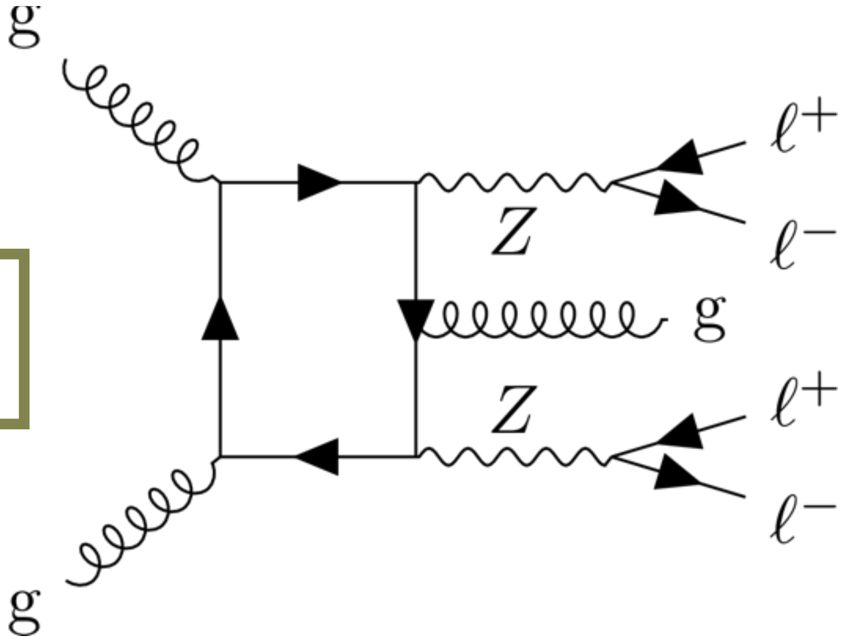


VV & VVV production

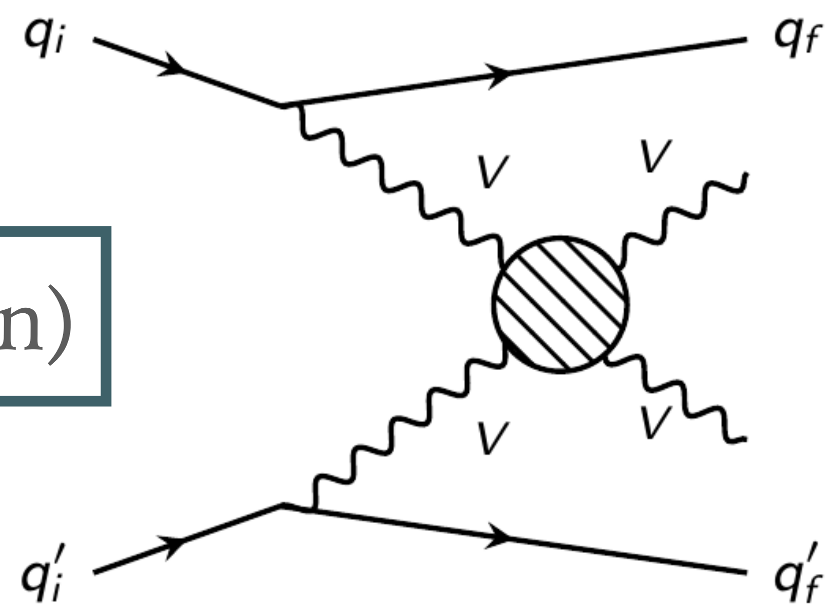




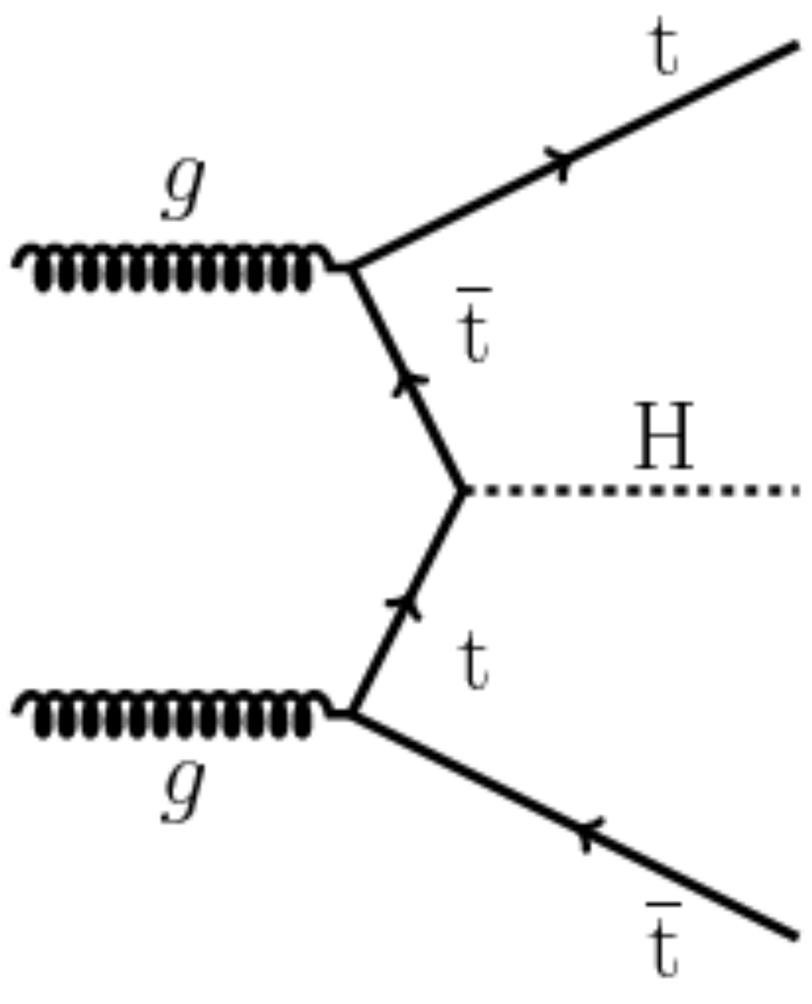
V & VV + jets production



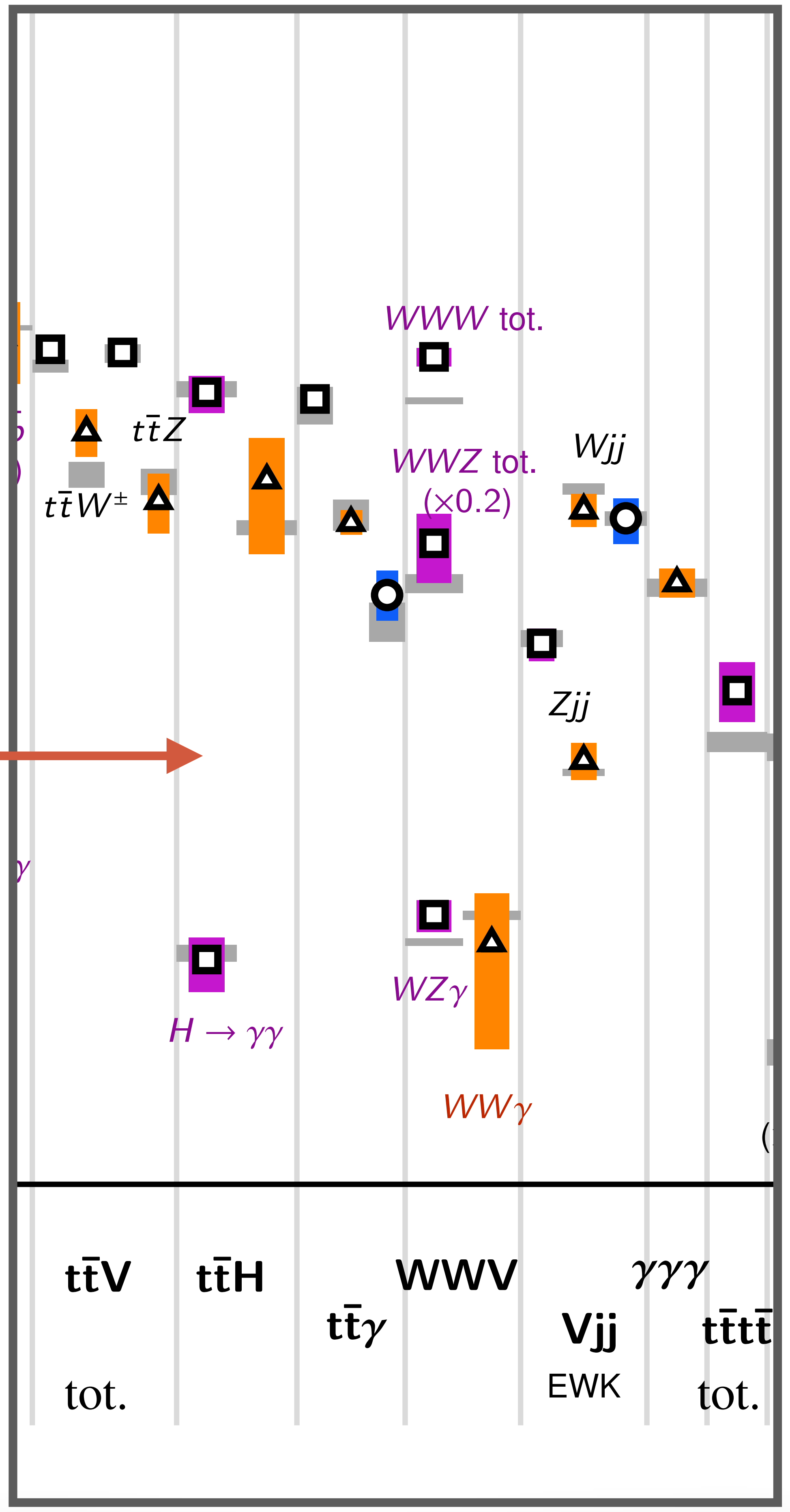
VBF (vector boson fusion)

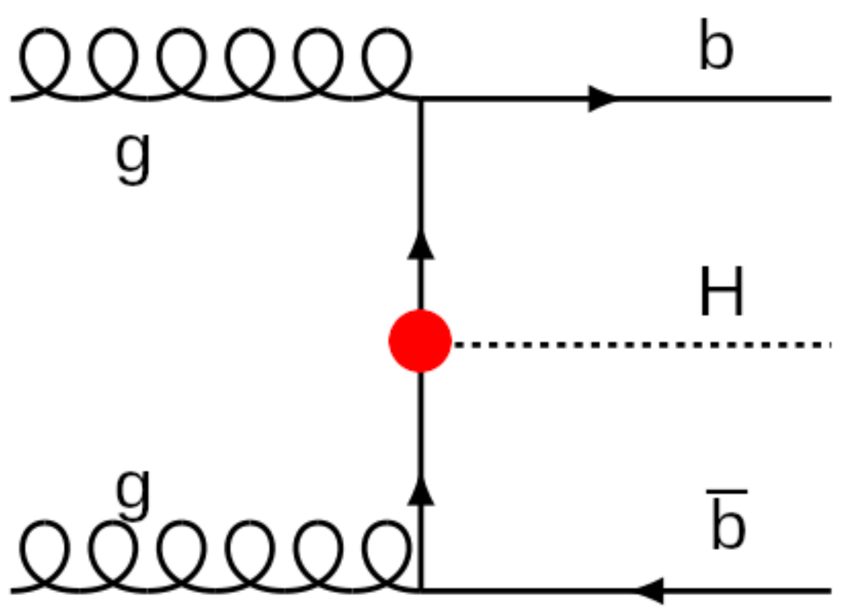


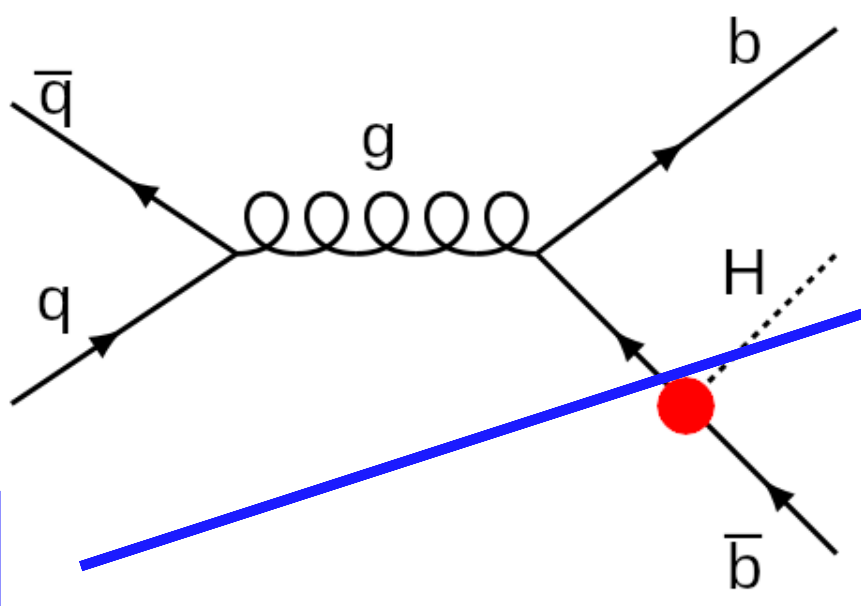
PROBING THE YUKAWA SECTOR

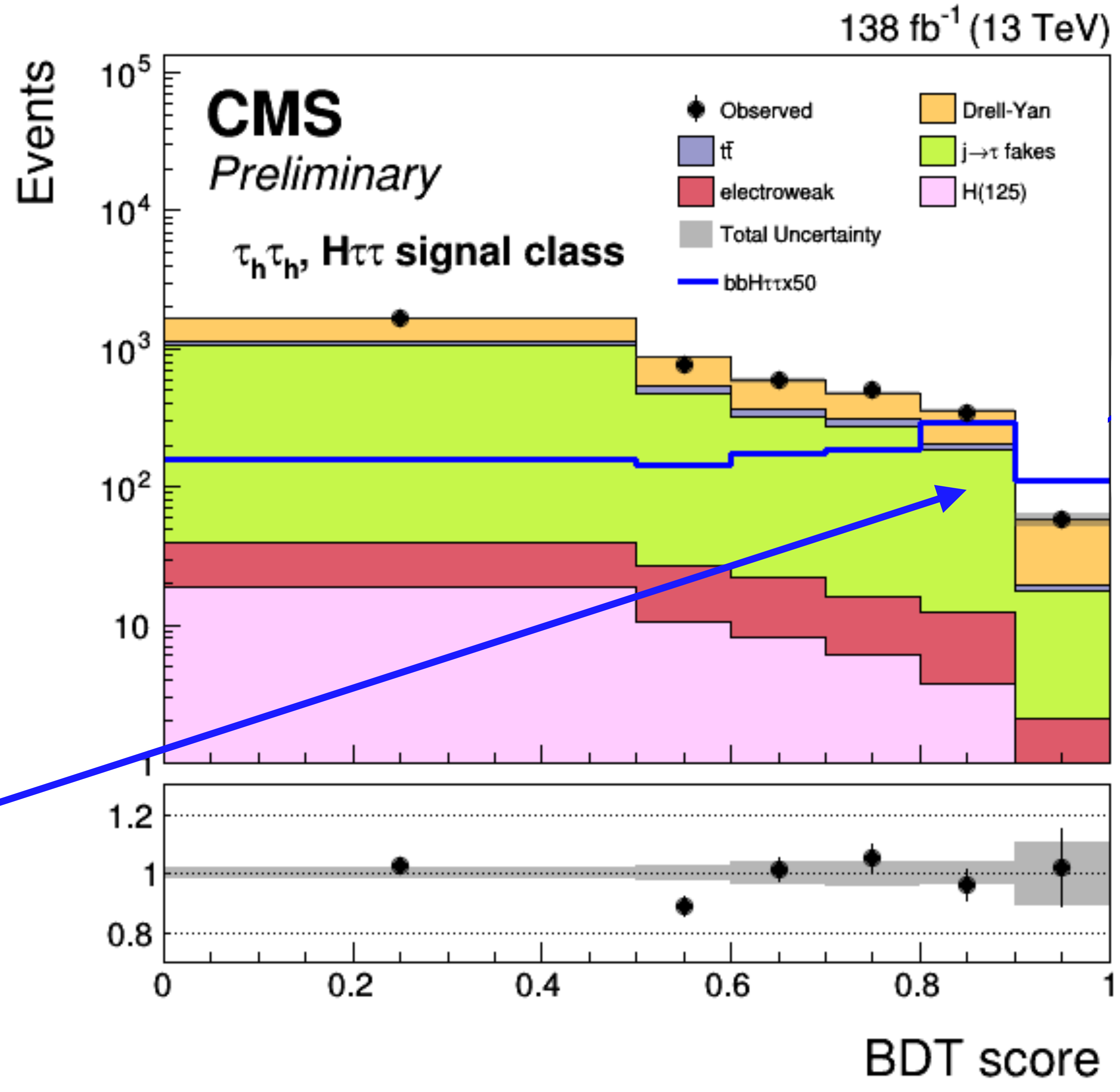


$t\bar{t}H$ production





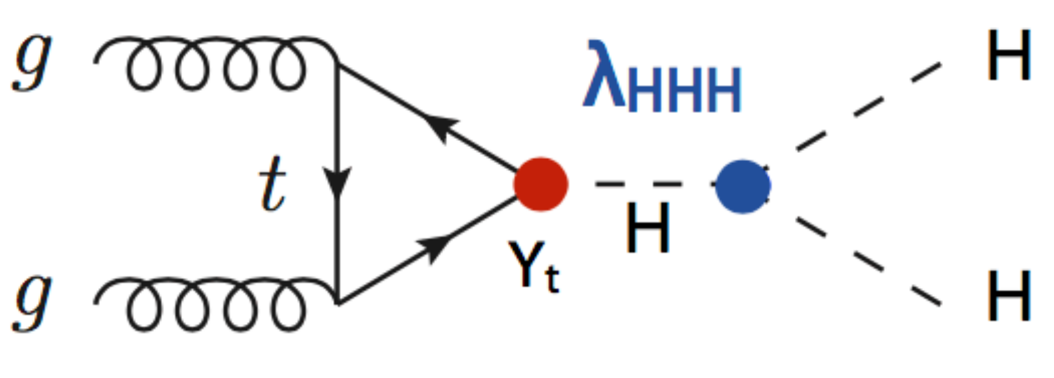


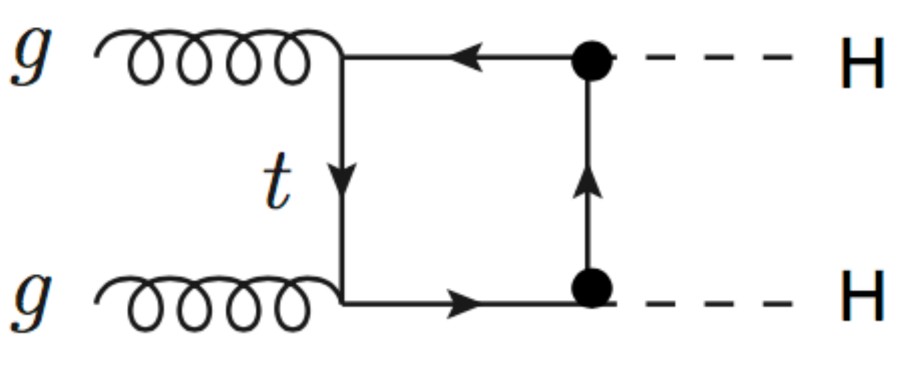


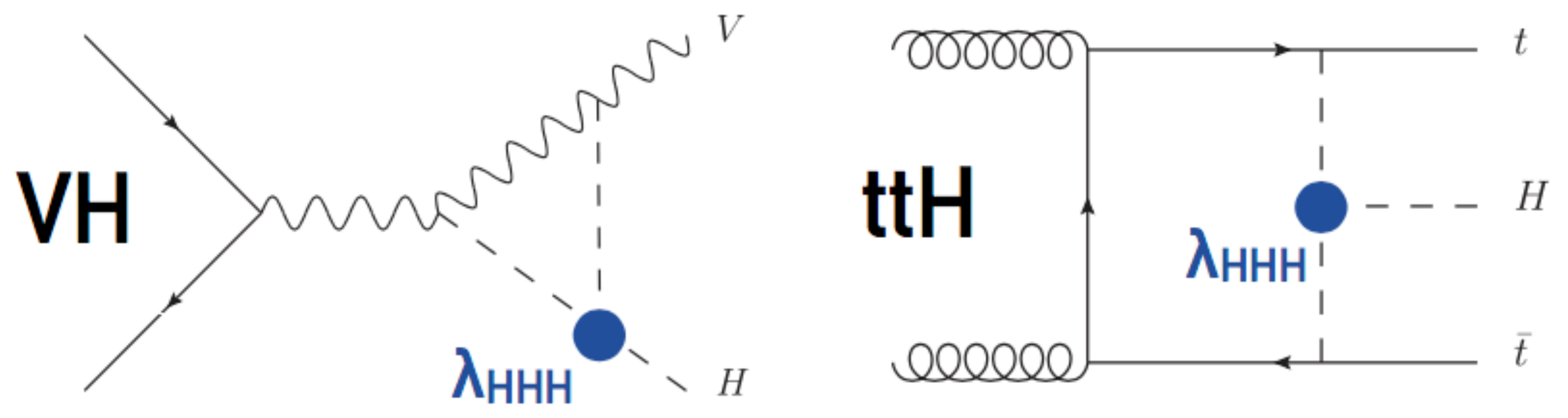
$b\bar{b}H$ production

PROBING H SELF INTERACTION THE MOST CHALLENGING?

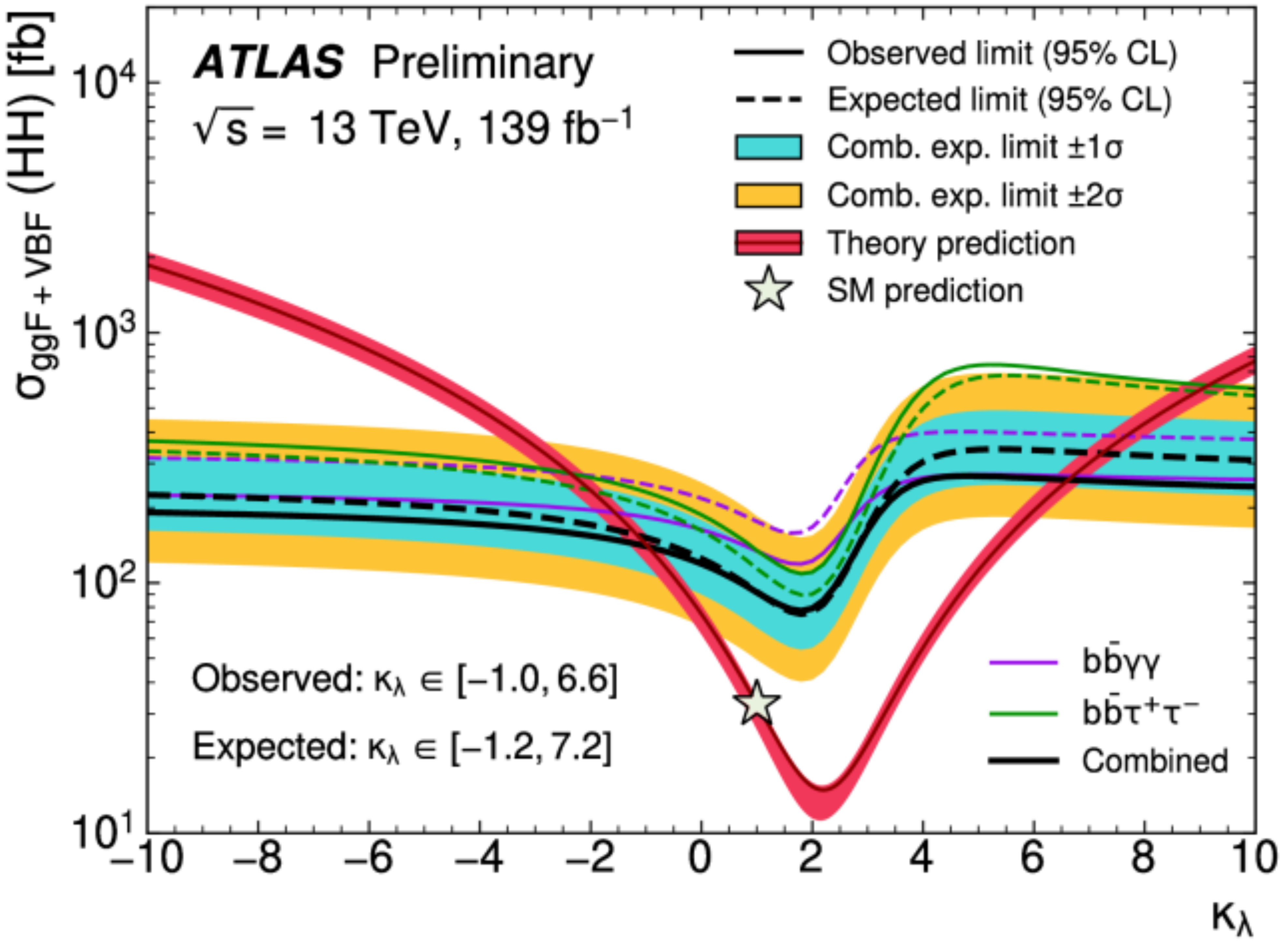
Direct sensitivity in HH production: Progress, but extremely hard to measure even at (HL-)LHC



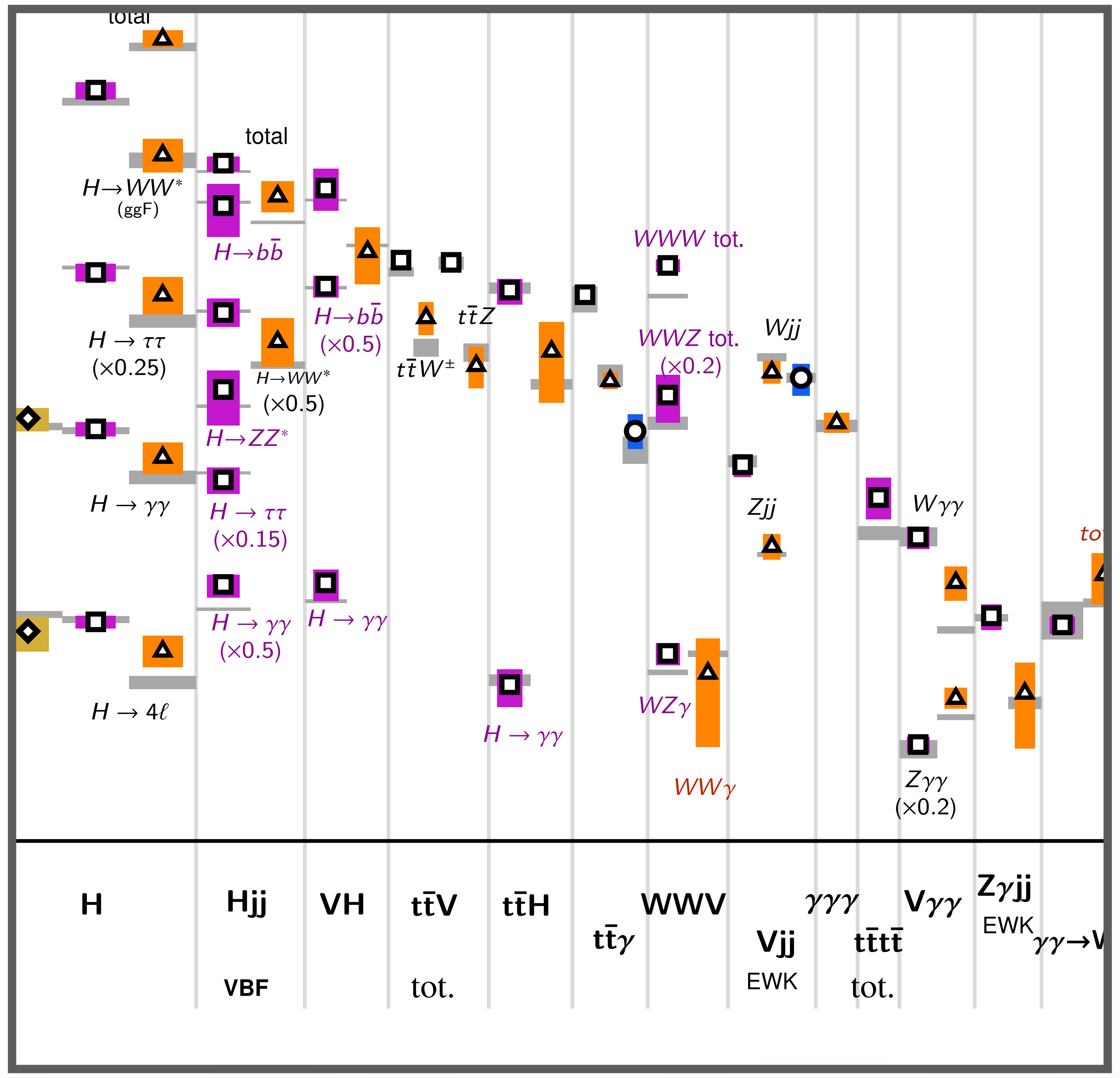




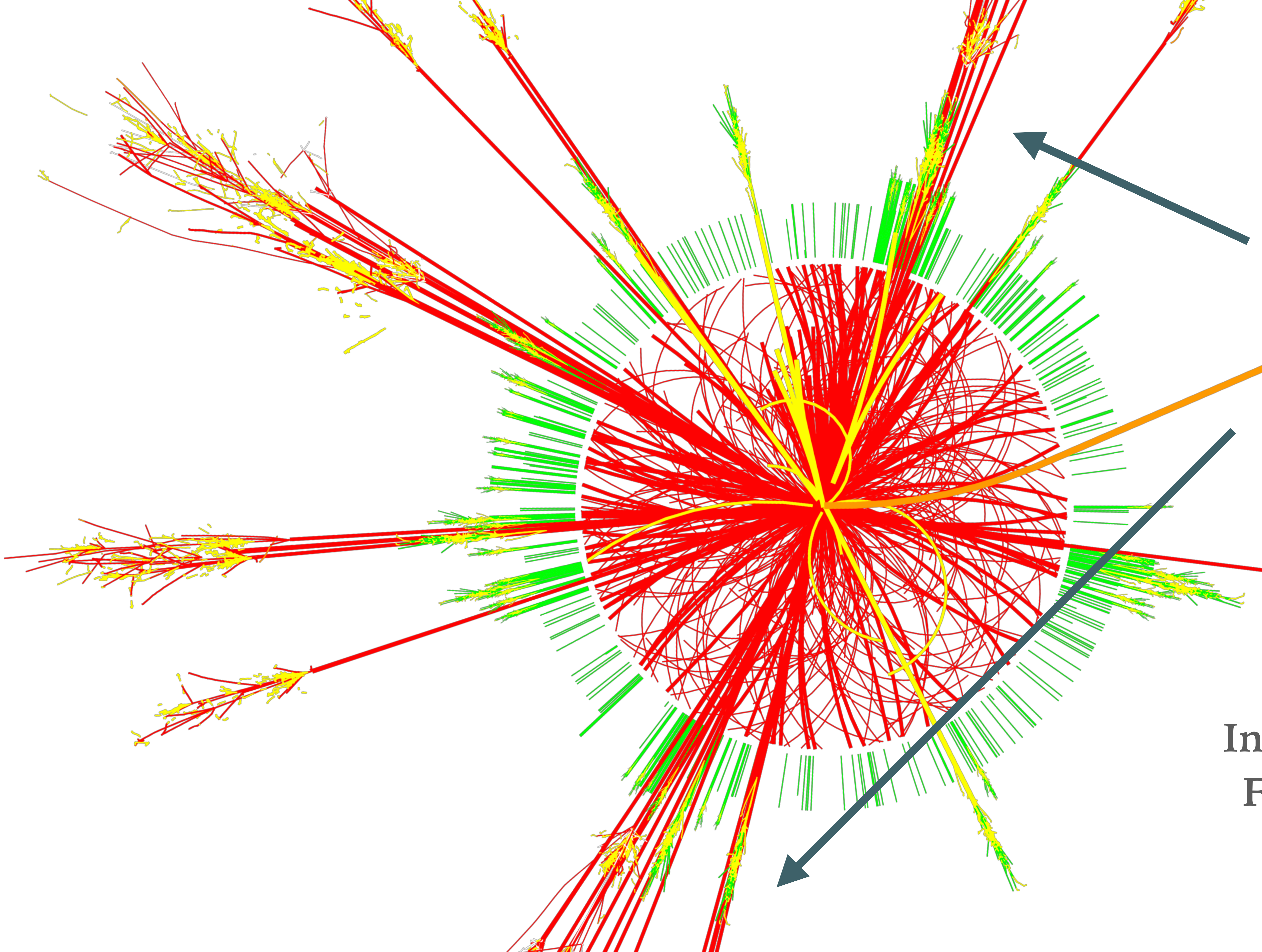
$b\bar{b}\tau\tau$ + $b\bar{b}\gamma\gamma$ + $b\bar{b}b\bar{b}$



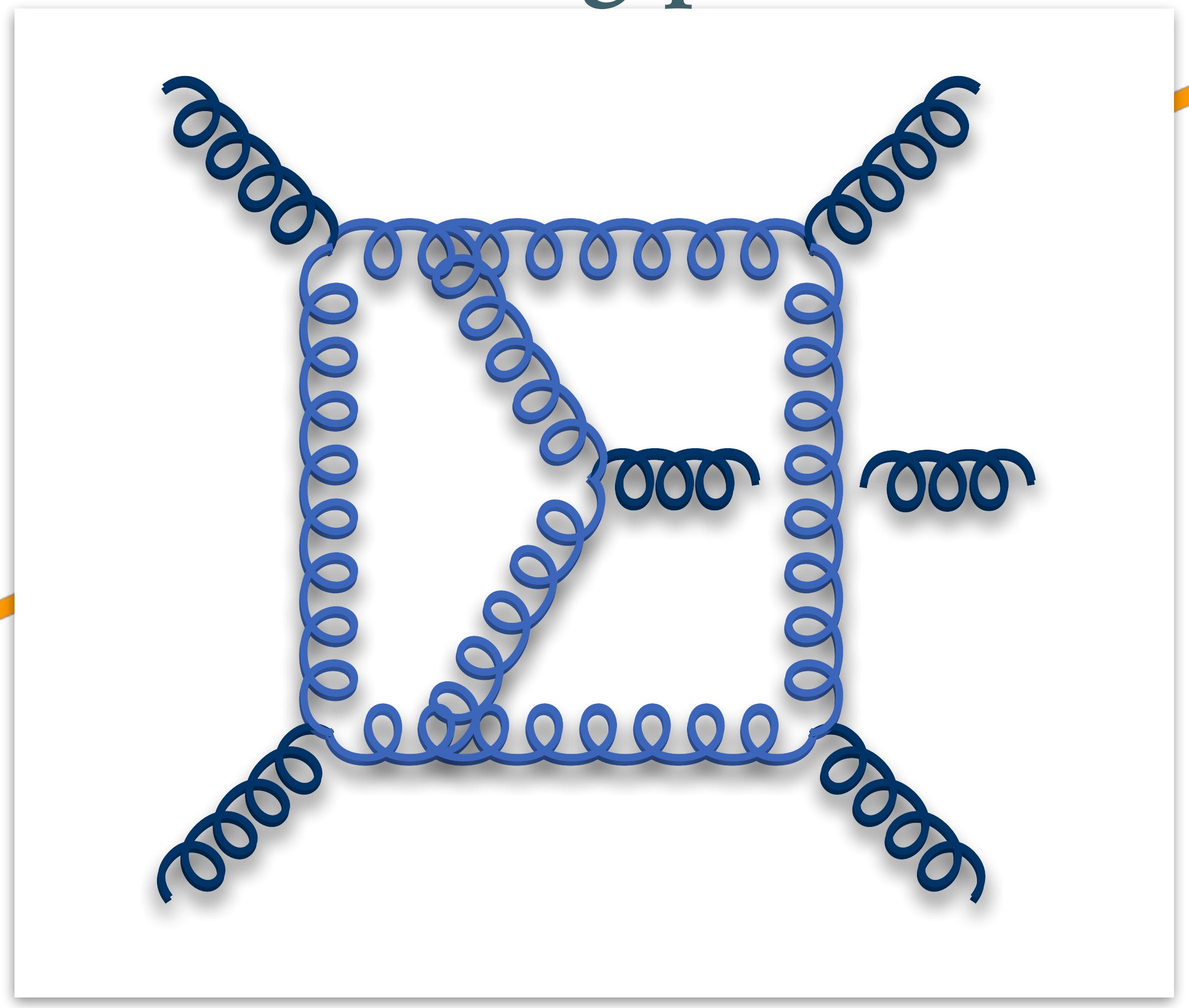
Indirect sensitivity through precision studies!



BEYOND THE HIGGS: PROBING QM AT THE HIGHEST ENERGIES

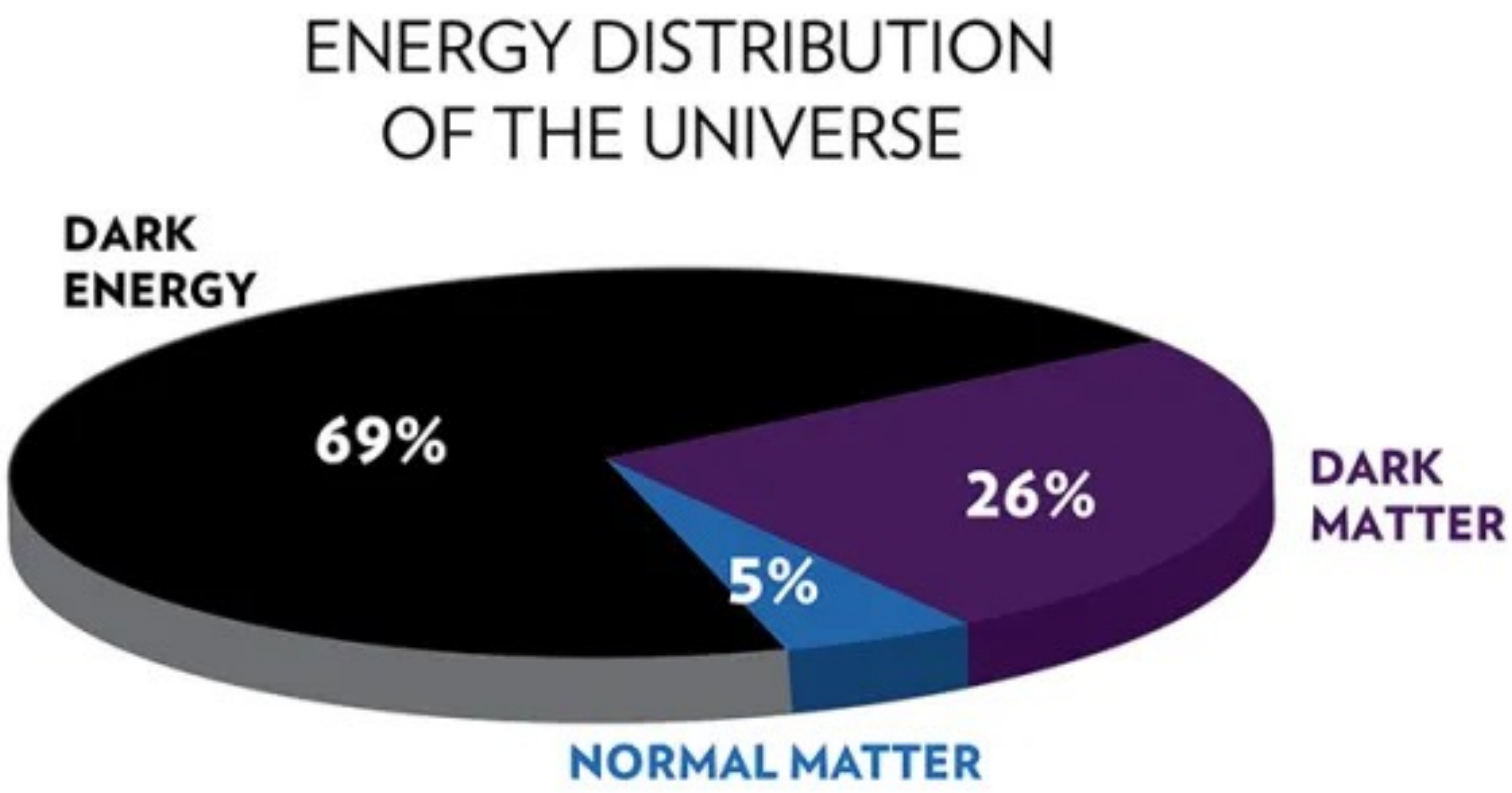


jets of strongly interacting particles



Investigate Quantum Field Theory at the highest energies!

BEYOND THE HIGGS: PROBING QM AT THE HIGHEST ENERGIES



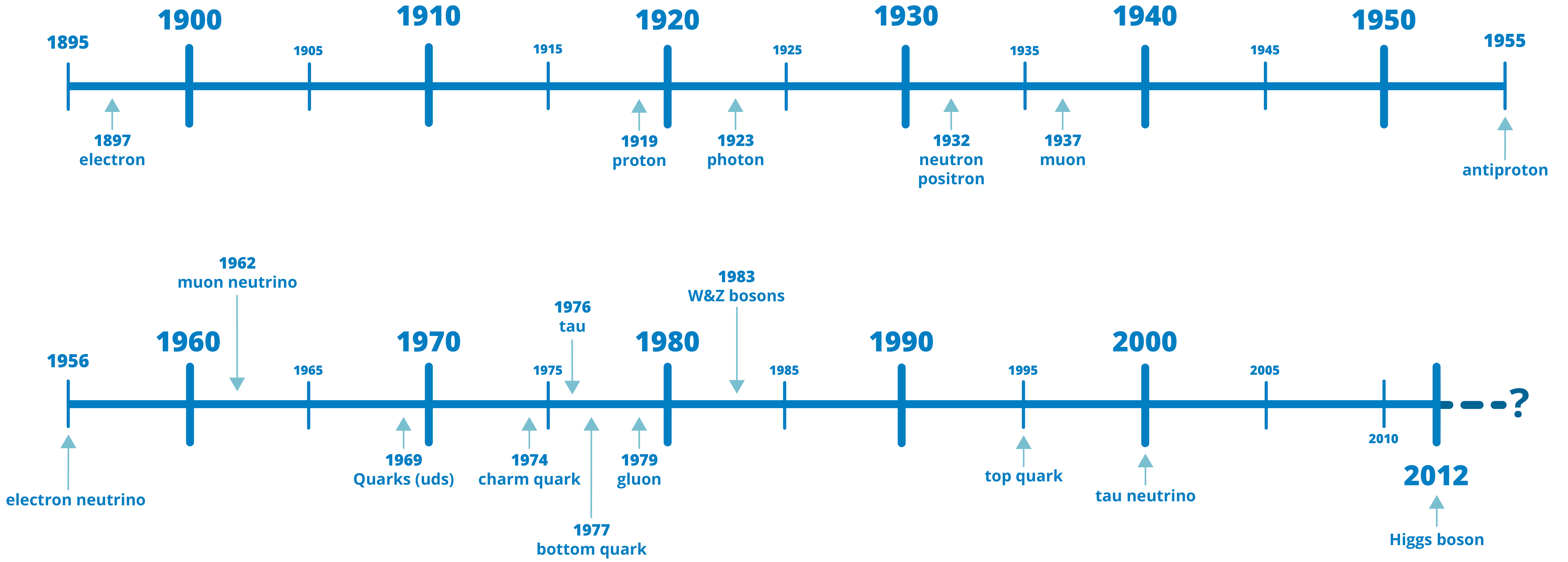
- Big Bang?
- Black Holes?
- DM?
- DE?
- ...?

Answering questions related to **Quantum Gravity** will require an entirely **point of view**.

Testing limitations of QFT ***might* suggest how to go beyond!**

PLENTY OF POTENTIAL FOR "DISCOVERY" AHEAD

ATLAS EXPERIMENT

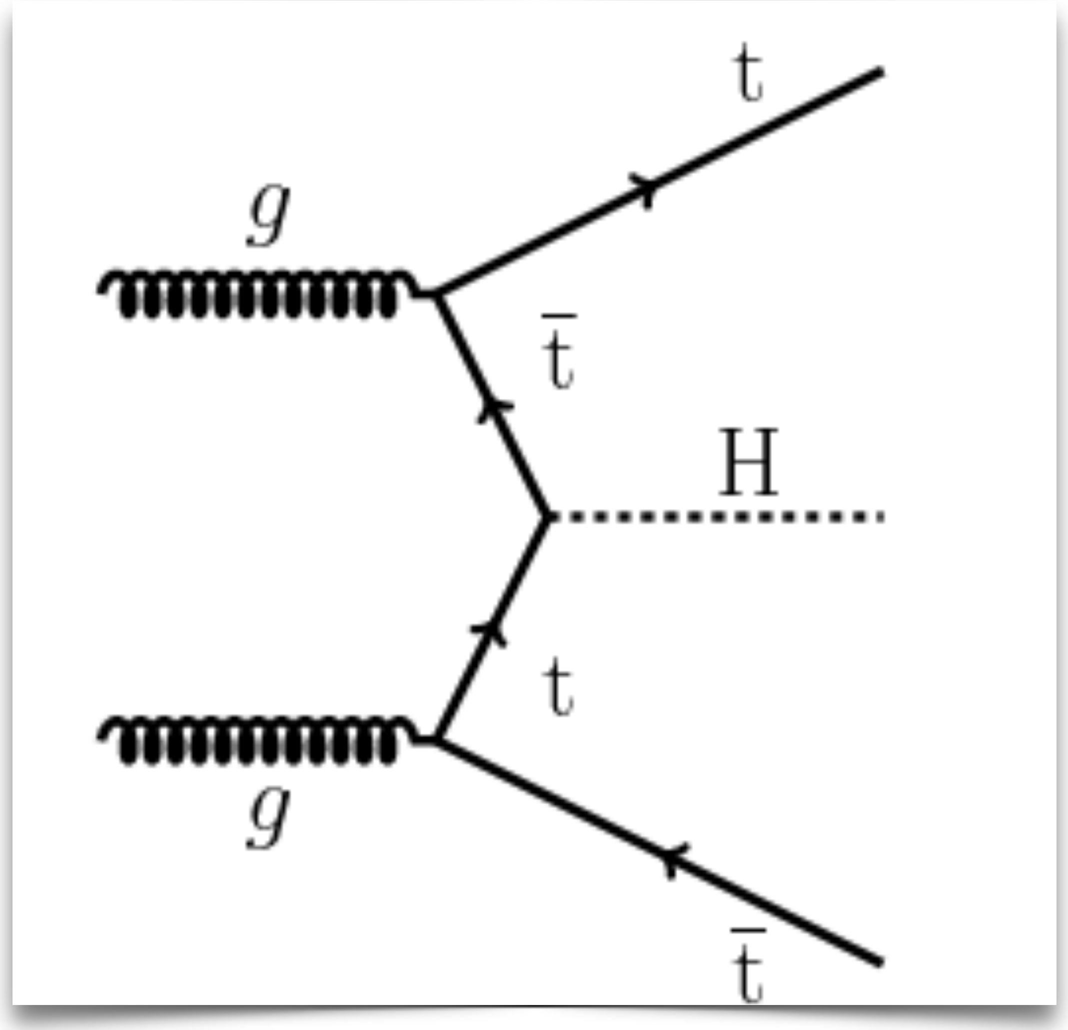


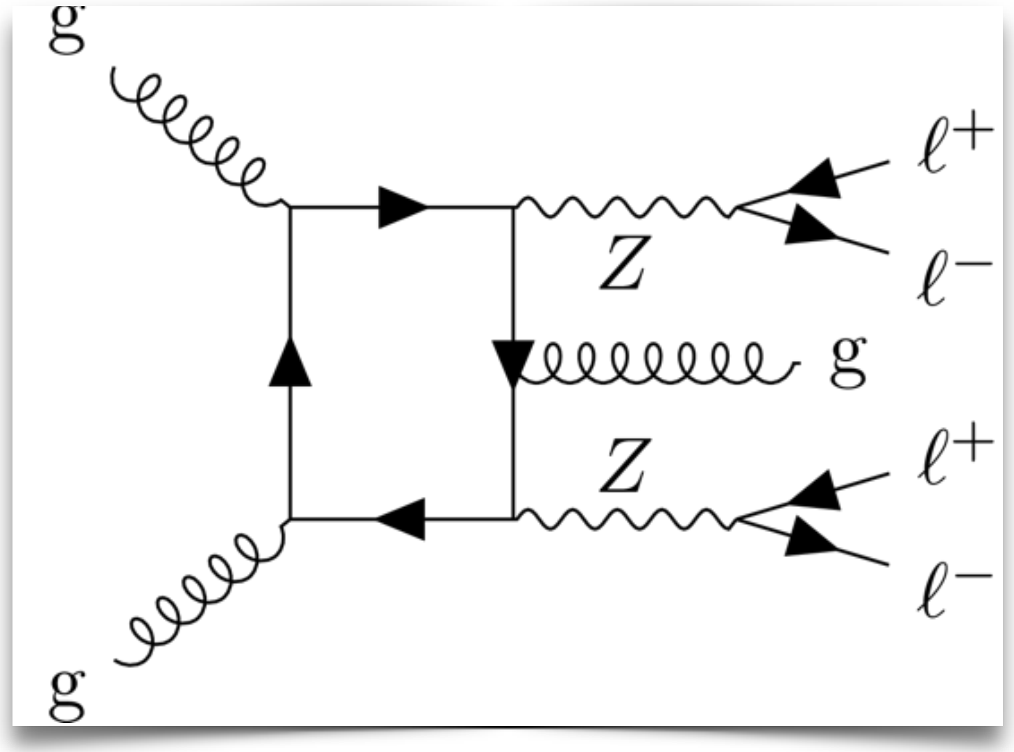
For the first time in decades, we might not expect new particles ahead…

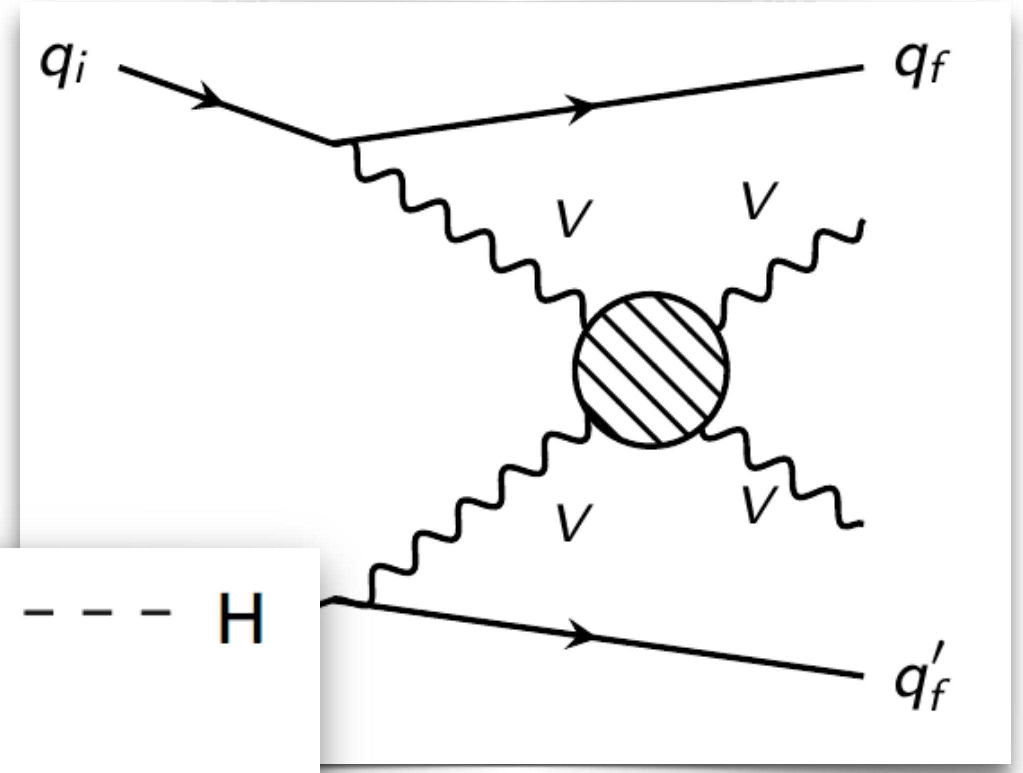
Still, thanks to the % precision physics program at colliders, we have the chance to **discover "new types of interactions"**, and **scrutinize quantum field theory to the highest precisions**

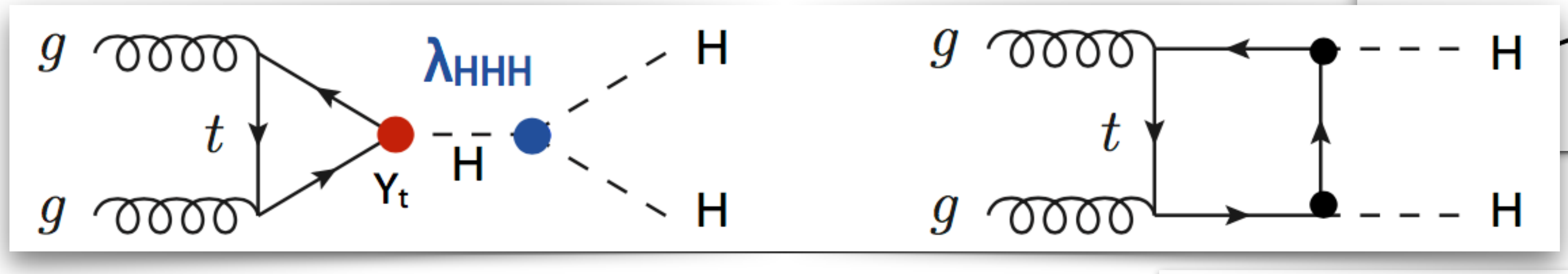
PRECISION STUDIES “OPPORTUNITIES” ALL OVER

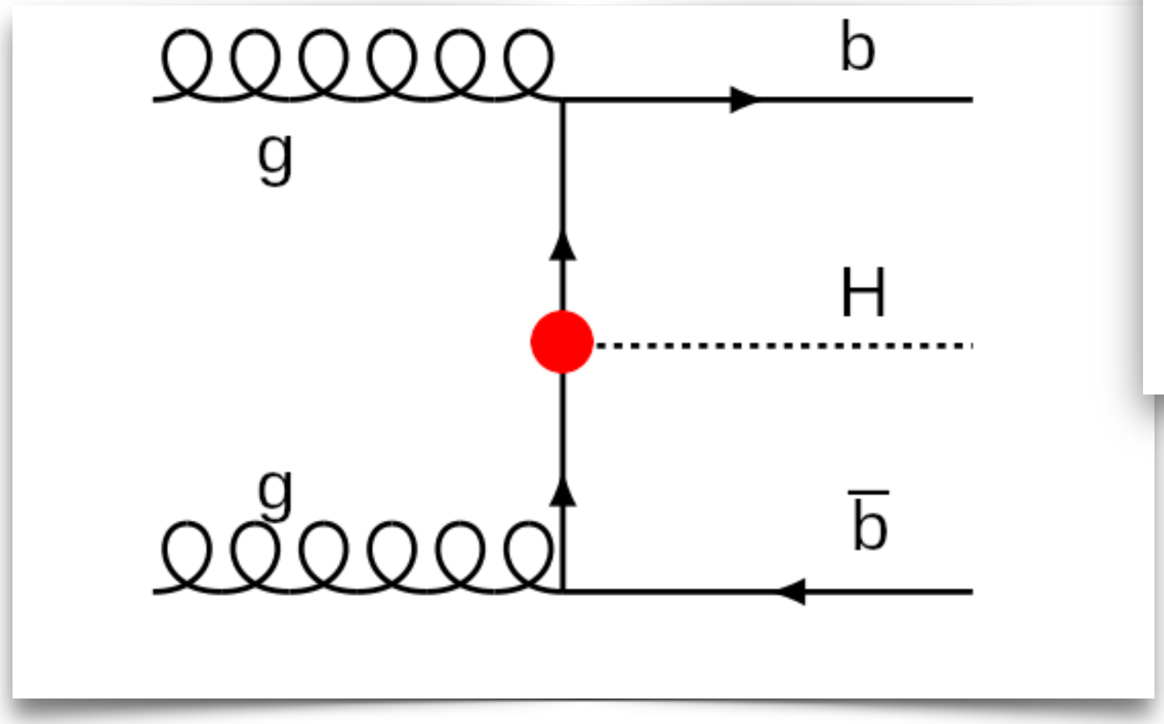
Complexity scales badly with masses and external “legs”

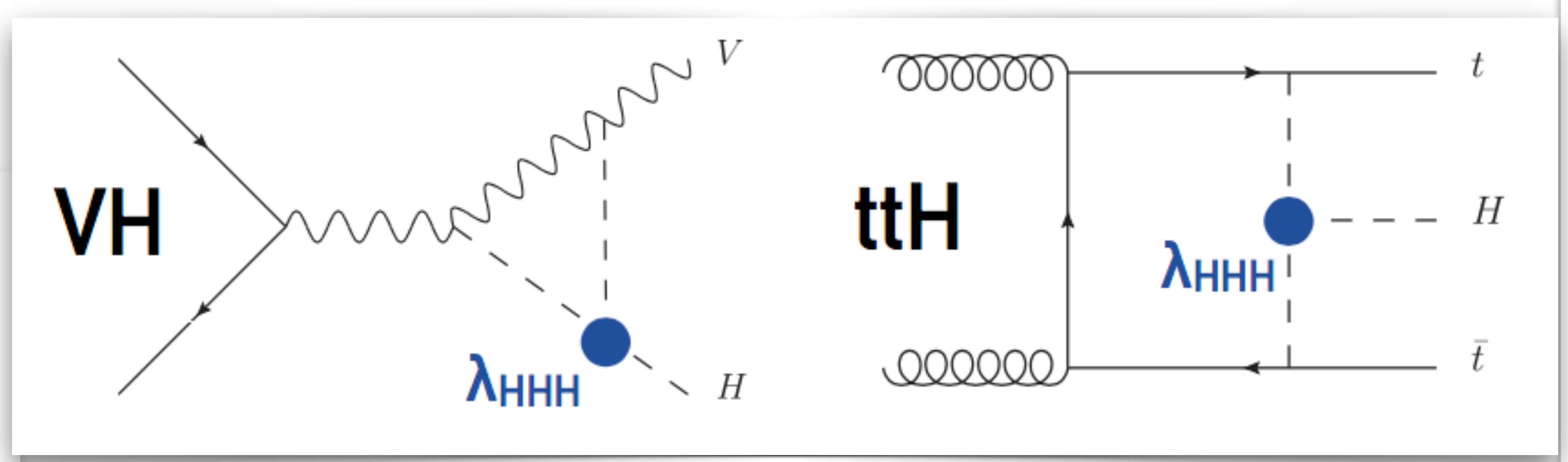


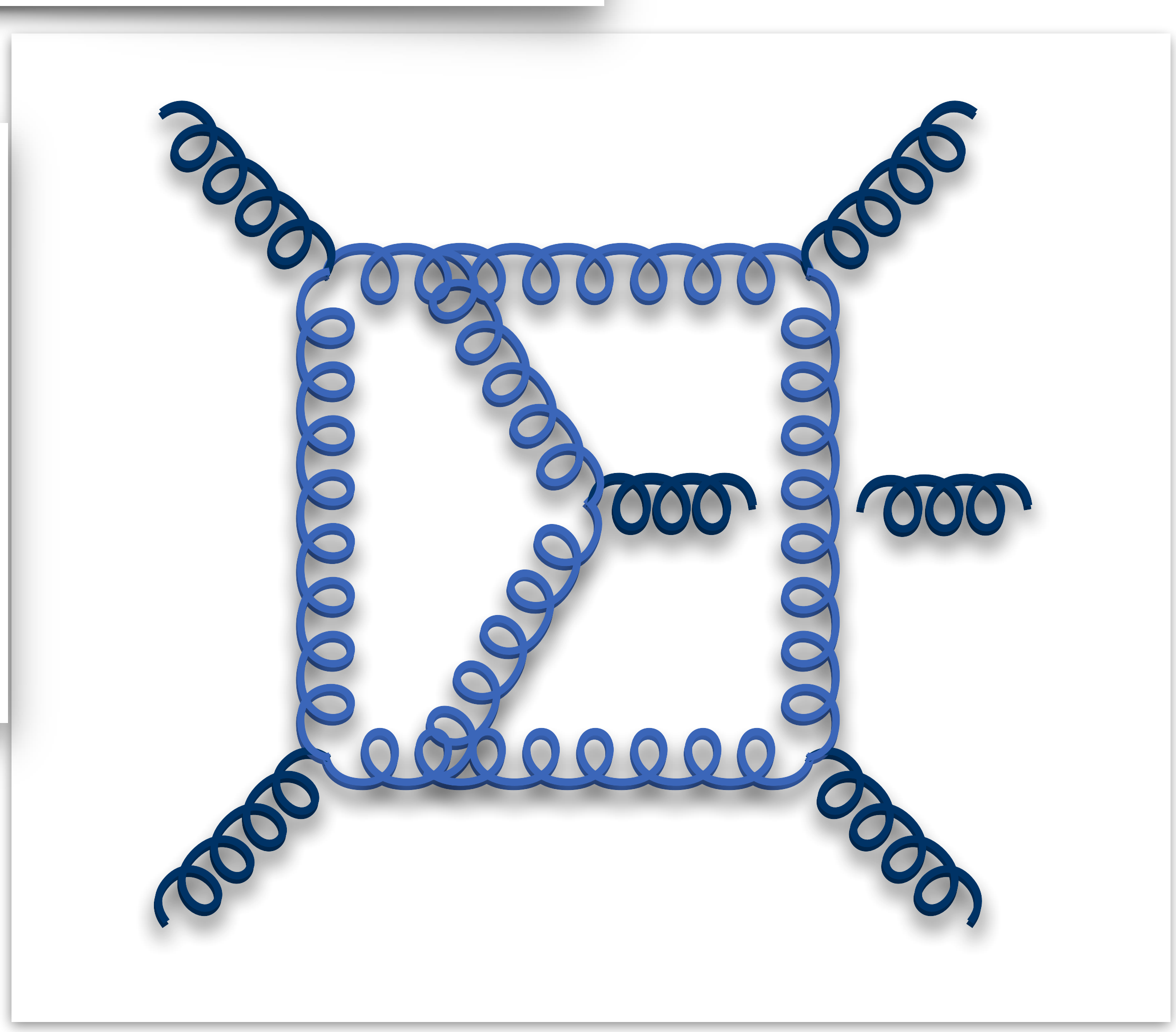




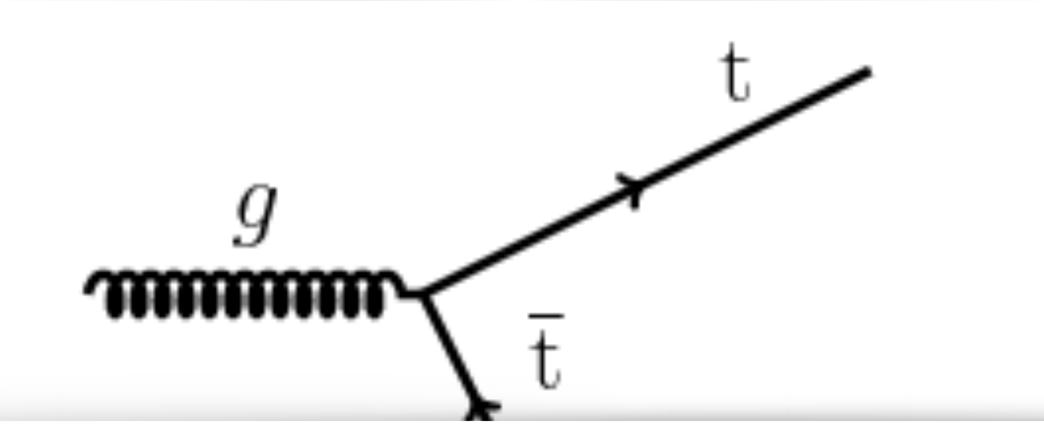


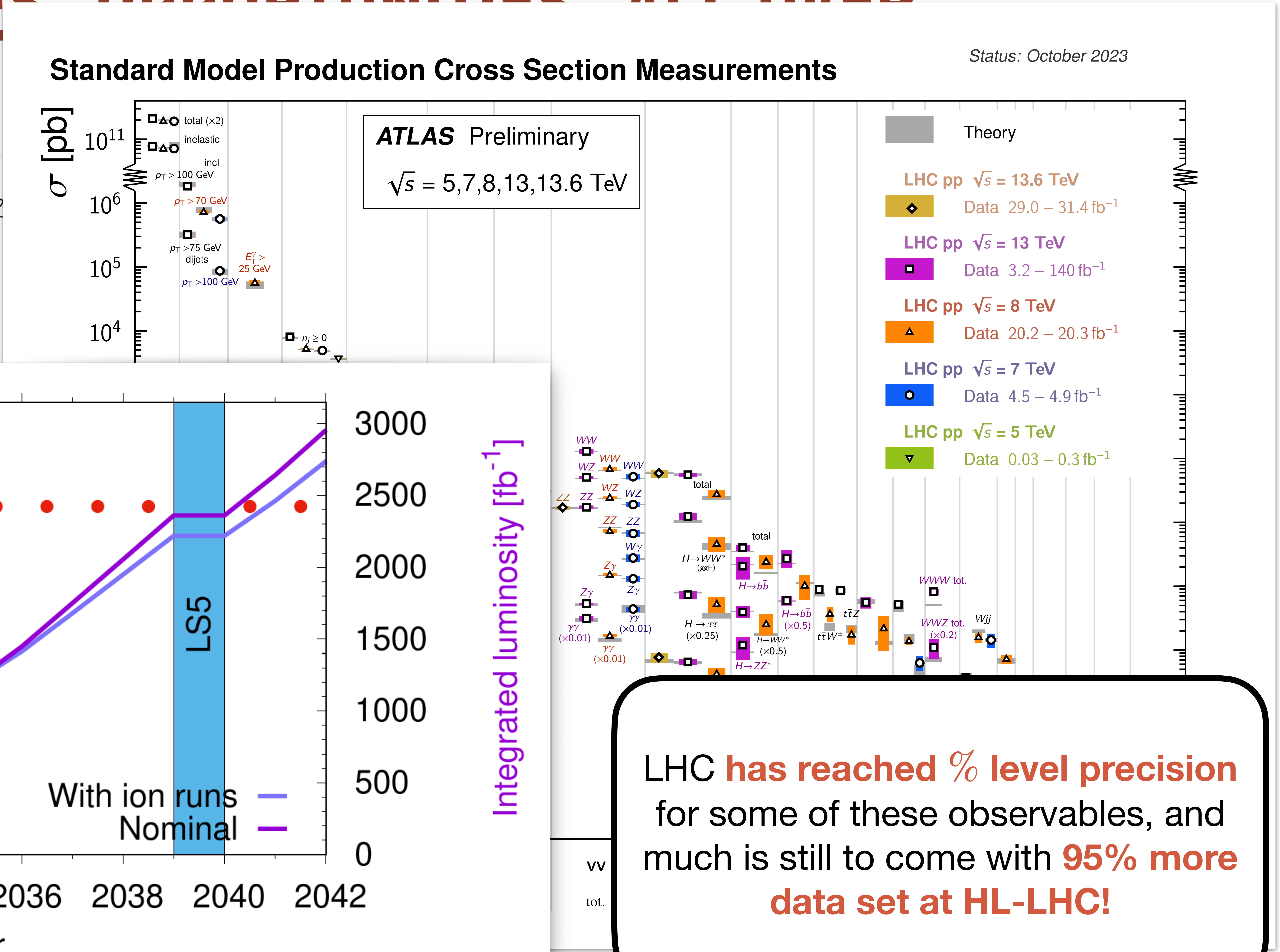


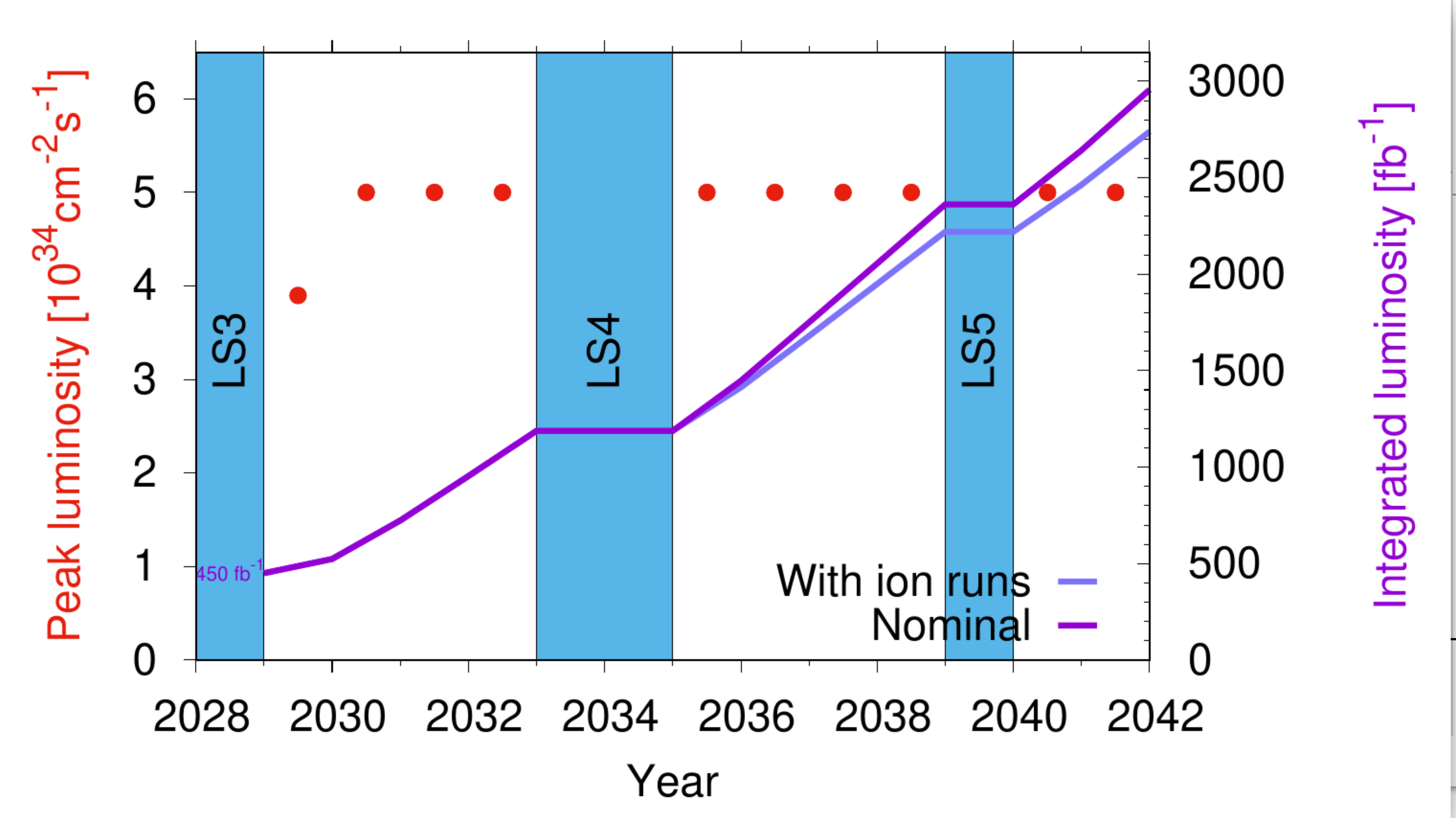




PRECISION STUDIES “OPPORTUNITIES” ALL OVER





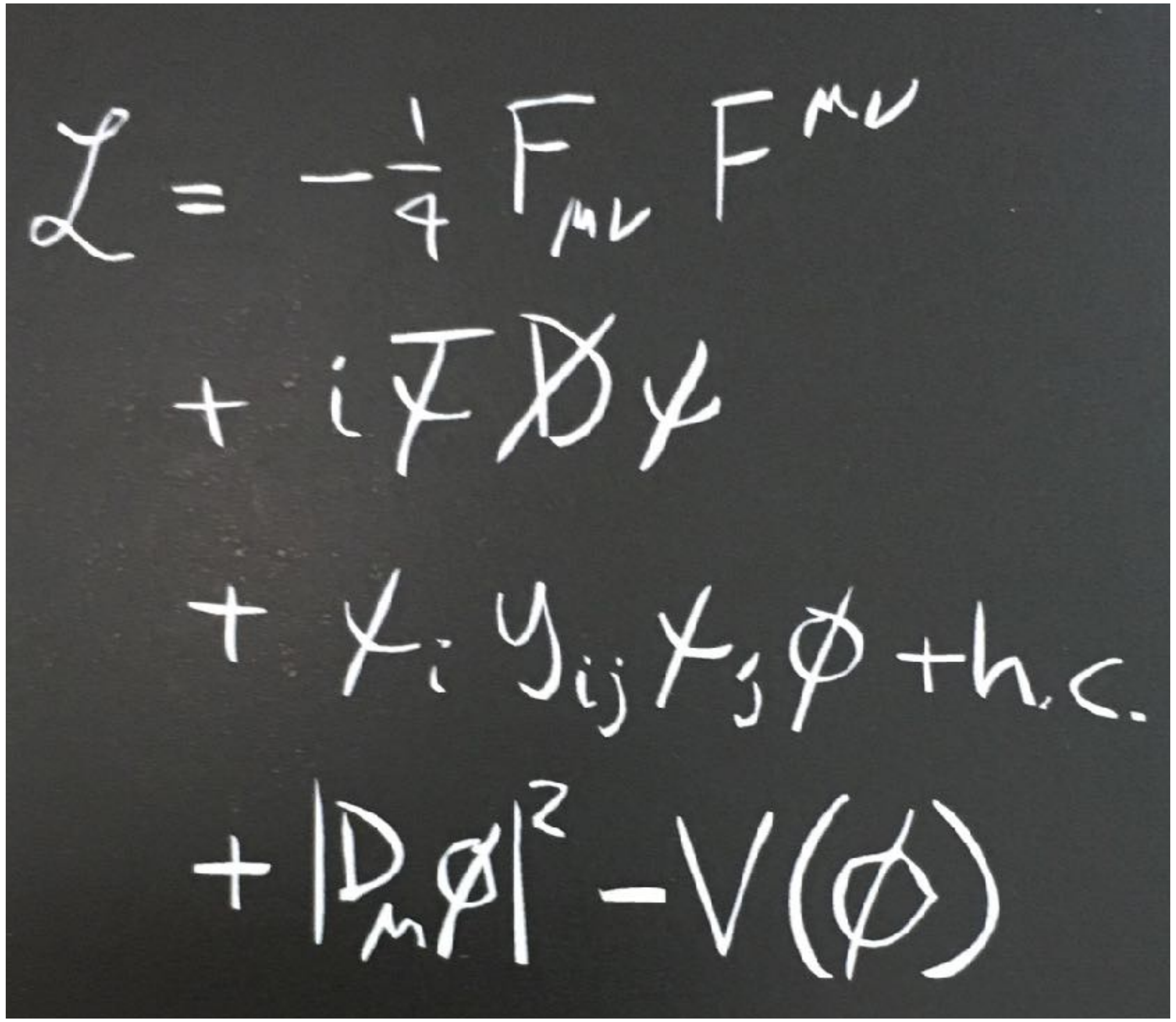


LHC **has reached % level precision** for some of these observables, and much is still to come with **95% more data set at HL-LHC!**

% PRECISION, HOW DO WE GET THERE?

(AND WHEN SHOULD WE STOP?)

FROM THEORY TO THEORY PREDICTIONS IT'S A LONG WAY!



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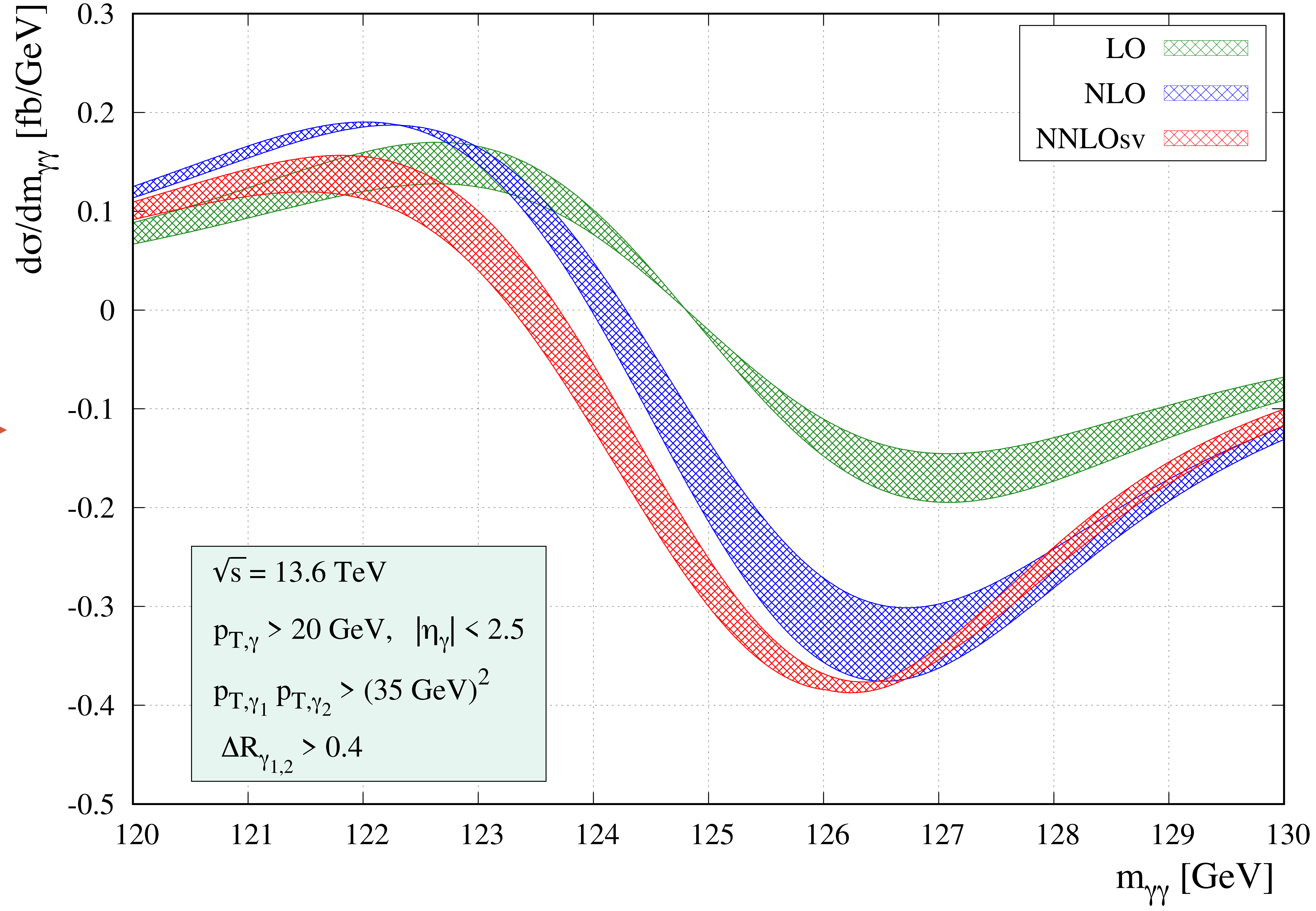
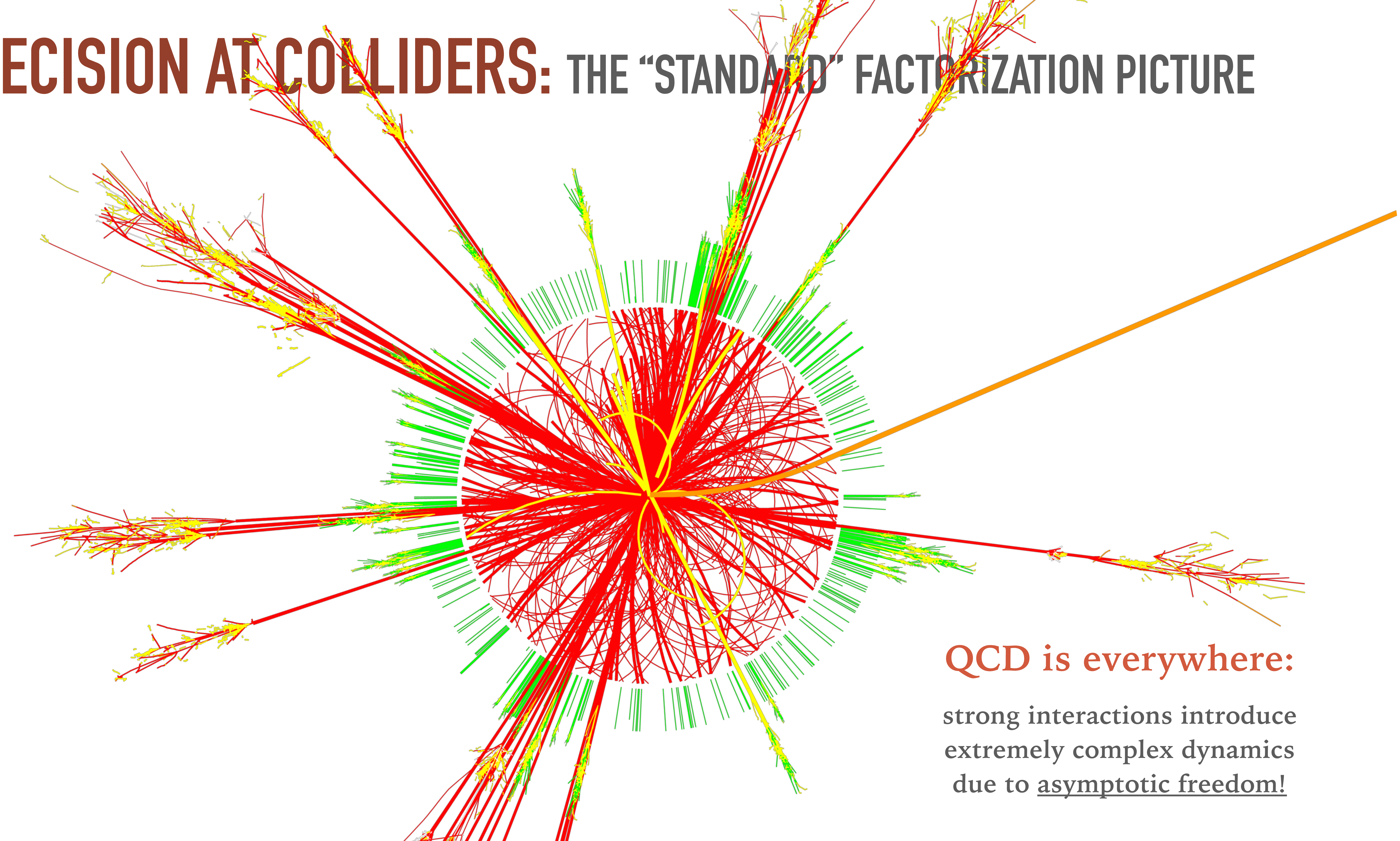


Signal to BKG interference for $gg \to H \to \gamma\gamma$

[Bargiela, Buccioni, Caola, Devoto, Manteuffel, Tancredi '22]

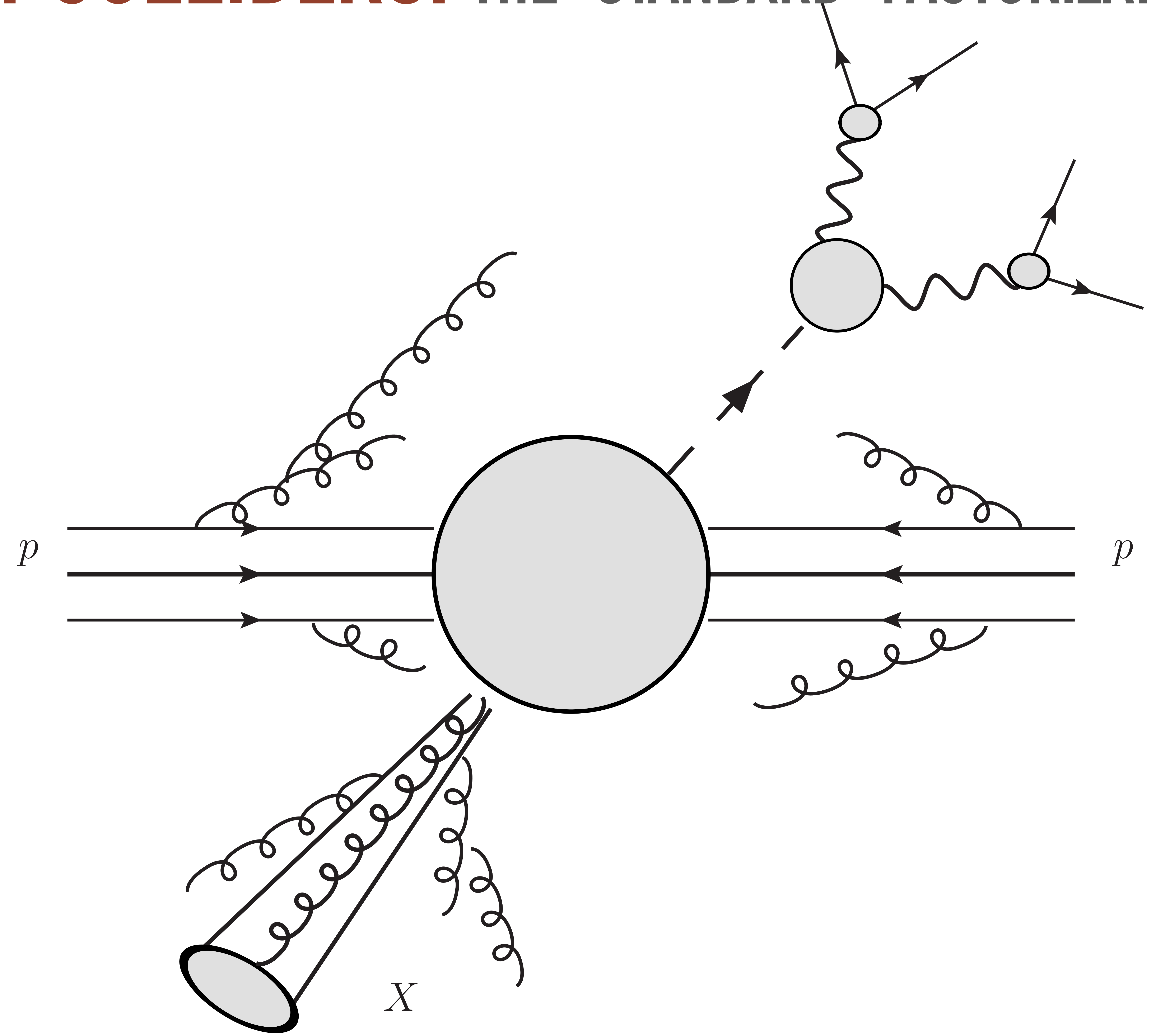
PRECISION AT COLLIDERS: THE "STANDARD" FACTORIZATION PICTURE



QCD is everywhere:

strong interactions introduce extremely complex dynamics due to <u>asymptotic freedom!</u>

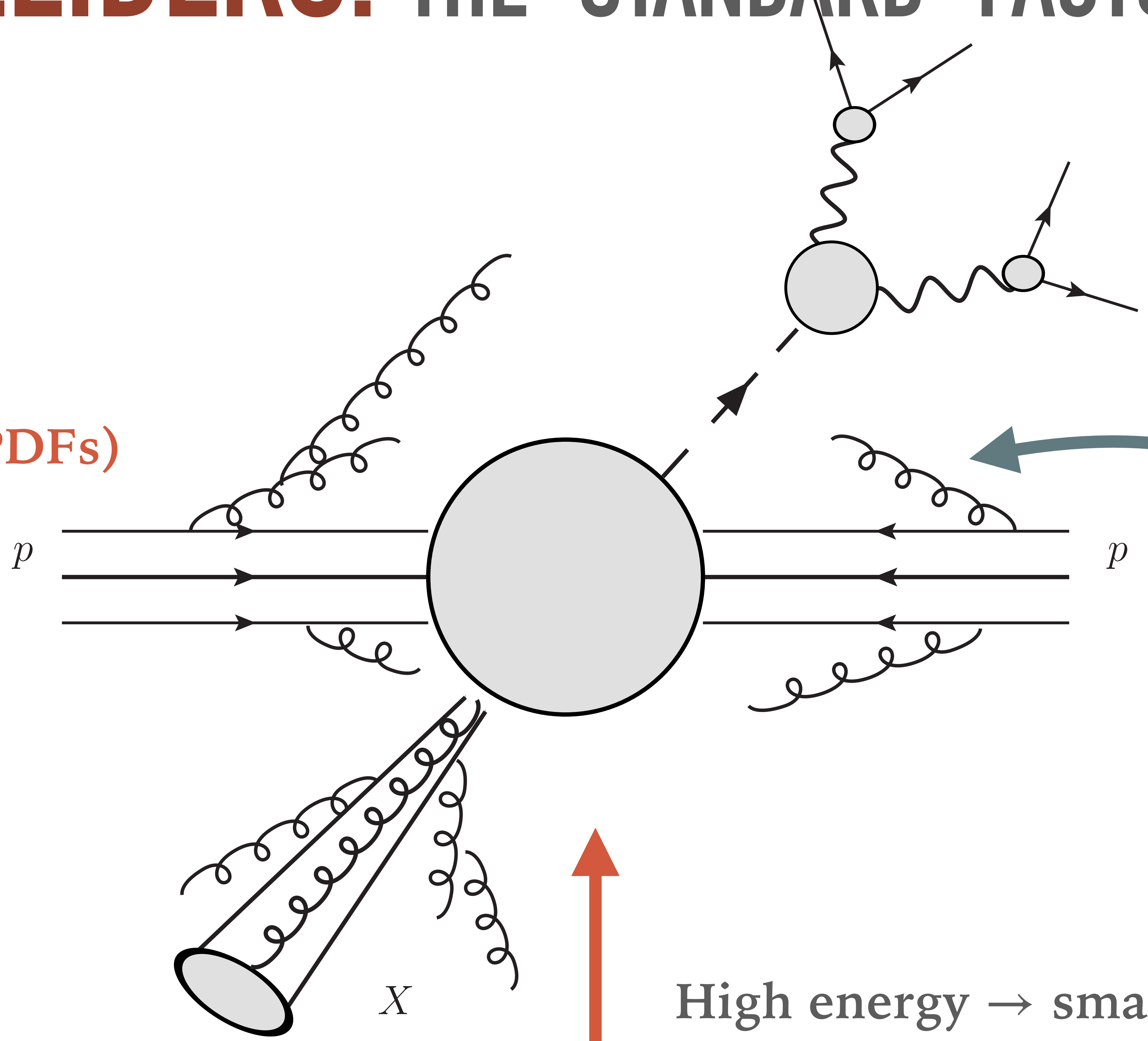
PRECISION AT COLLIDERS: THE "STANDARD" FACTORIZATION PICTURE



PRECISION AT COLLIDERS: THE "STANDARD" FACTORIZATION PICTURE

Non-perturbative:

Parton Distribution Functions (PDFs)



Non perturbative

"soft/collinear" physics

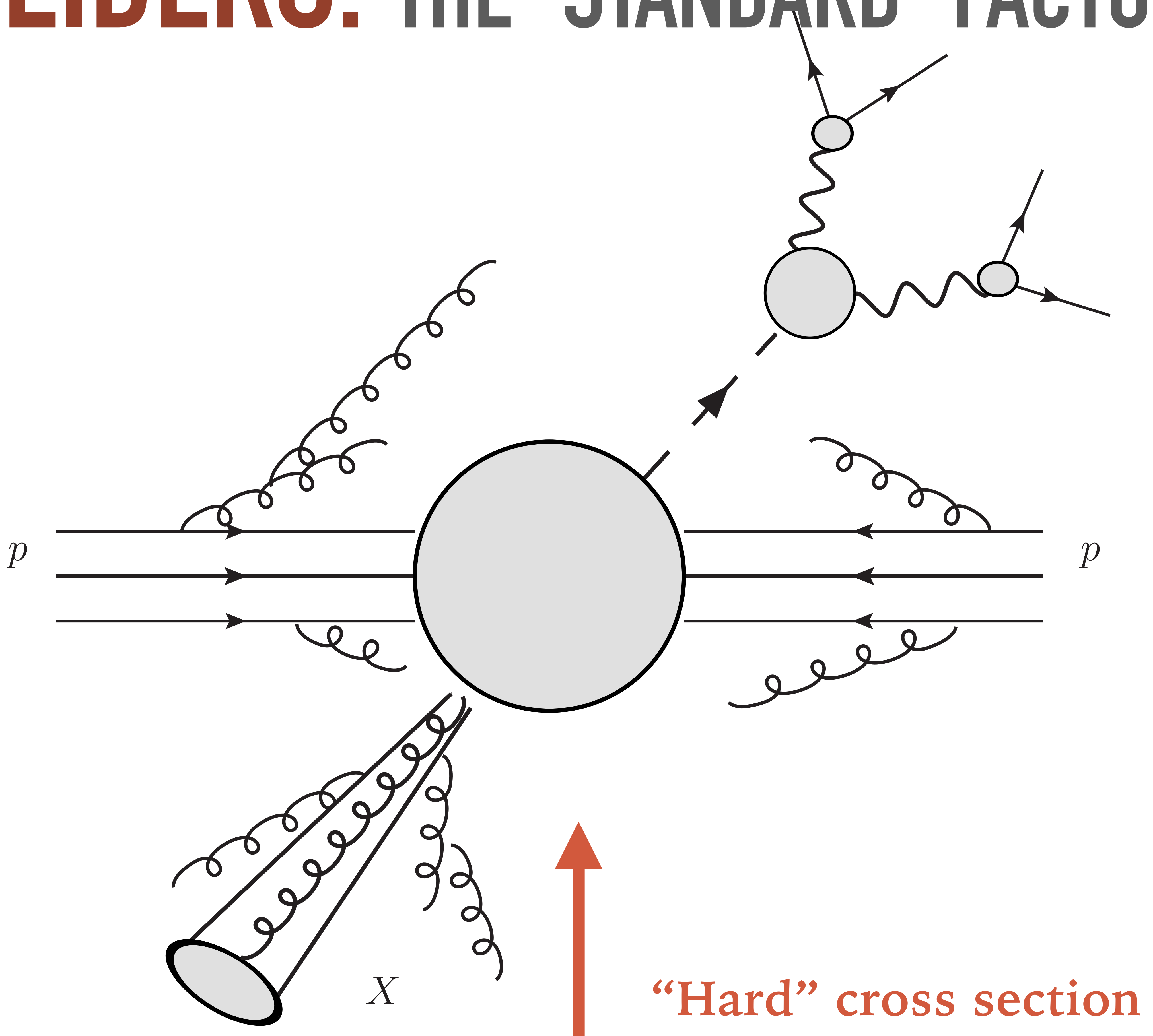
High energy → small coupling

"Hard" cross section

$$\mathrm{d}\sigma = \int dx_1 dx_2\, f(x_1) f(x_2)\, \mathrm{d}\sigma_{part}(x_1, x_2)(1 + \mathcal{O}(\Lambda^n_{\mathrm{QCD}}/Q^n))$$

PRECISION AT COLLIDERS: THE “STANDARD” FACTORIZATION PICTURE

Here, we ignore all that and zoom in the so-called ‘Hard Scattering’

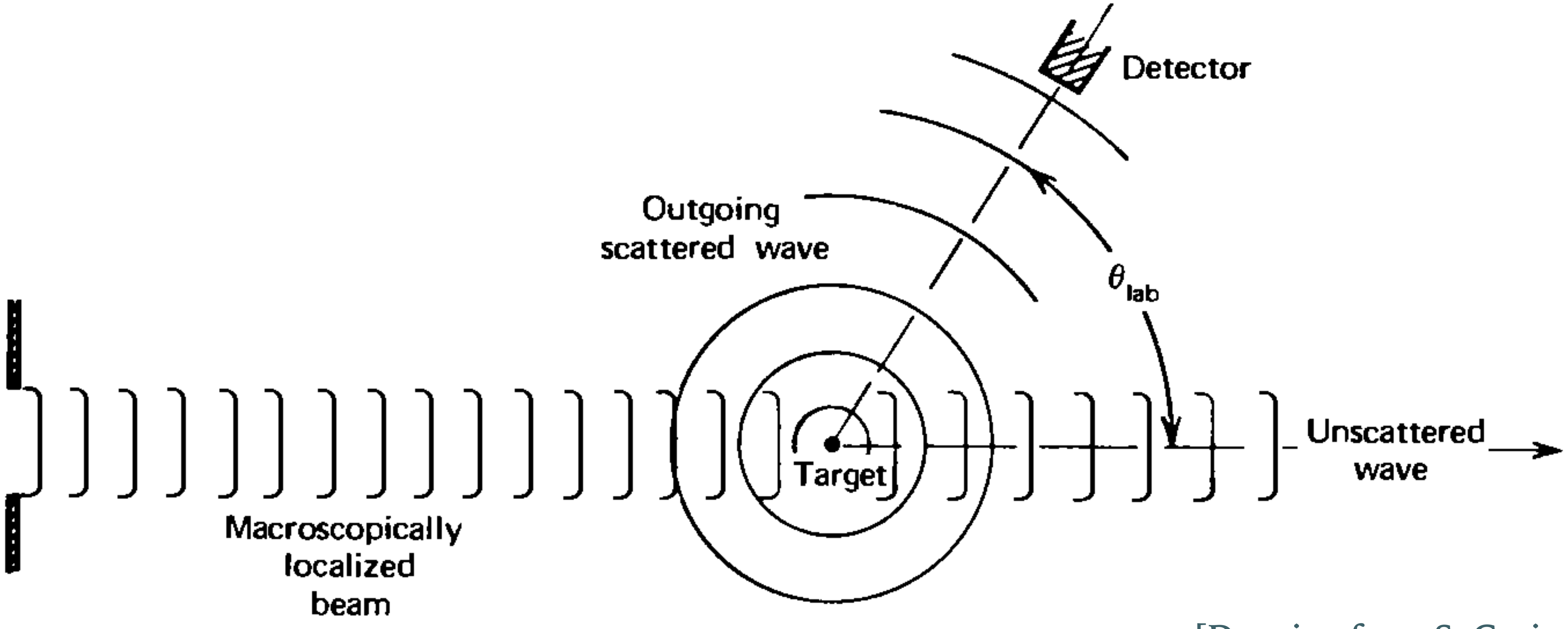


“Hard” cross section

% precision

$$\mathrm{d}\sigma = \int dx_1 dx_2\, f(x_1) f(x_2)\, \mathrm{d}\sigma_{part}(x_1, x_2)(1 + \mathcal{O}(\Lambda^n_{QCD}/Q^n))$$

FROM AMPLITUDES TO CROSS SECTION: IN QUANTUM MECHANICS



[Drawing from S. Gasiorowicz, Quantum Physics]

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l + 1)\, e^{i\delta_l(k)} \sin \delta_l(k)\, P_l(\cos \theta)$$

Scattering Amplitude expanded in partial waves

→ cross section $d\sigma = |f(\theta)|^2\, d\Omega$

FROM AMPLITUDES TO CROSS SECTION: IN PERTURBATIVE QFT

$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}]\, |\mathcal{M}_{q\bar{q}\to gg}|^2$$

FROM AMPLITUDES TO CROSS SECTION: IN PERTURBATIVE QFT

$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}] \left|\mathcal{M}_{q\bar{q}\to gg}\right|^2$$

small "coupling constant" ~ 0.1

$$\left|\mathcal{M}_{q\bar{q}\to gg}\right|^2 = \left|\mathcal{M}^{LO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)\left|\mathcal{M}^{NLO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2\left|\mathcal{M}^{NNLO}_{q\bar{q}\to gg}\right|^2 + \ldots$$

FROM AMPLITUDES TO CROSS SECTION: IN PERTURBATIVE QFT

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$\sim$ 1% ?

FROM AMPLITUDES TO CROSS SECTION: IN PERTURBATIVE QFT

$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}]\, \left|\mathcal{M}_{q\bar{q}\to gg}\right|^2$$

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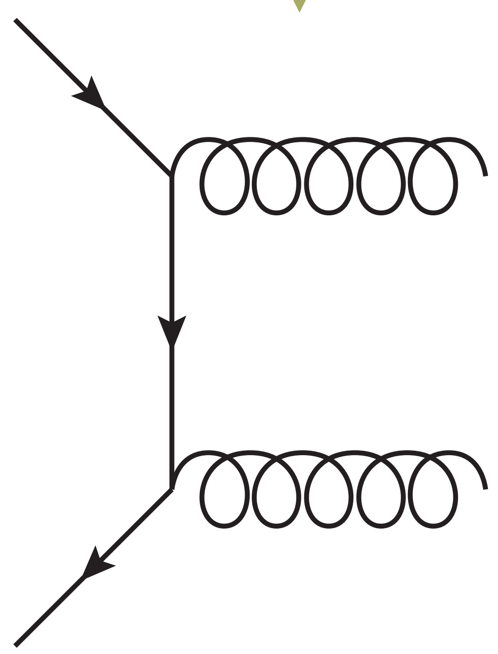
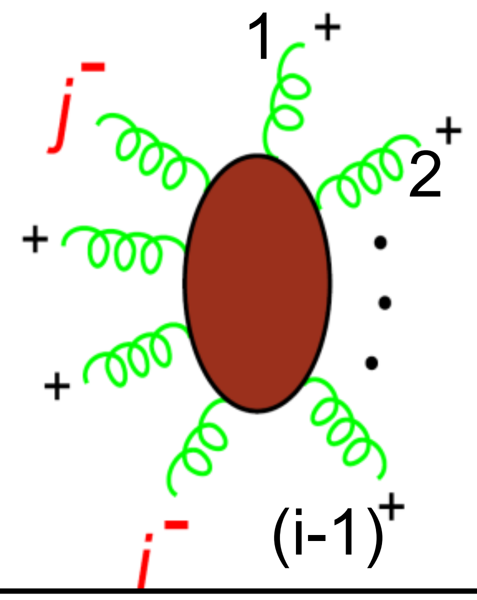


~ O(100%-50%) precision

$$A_n^{ij,\ \mathrm{MHV}} = A_n^{\mathrm{tree}}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)$$

$$=$$



$$= \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$$

Parke-Taylor formula (1986)

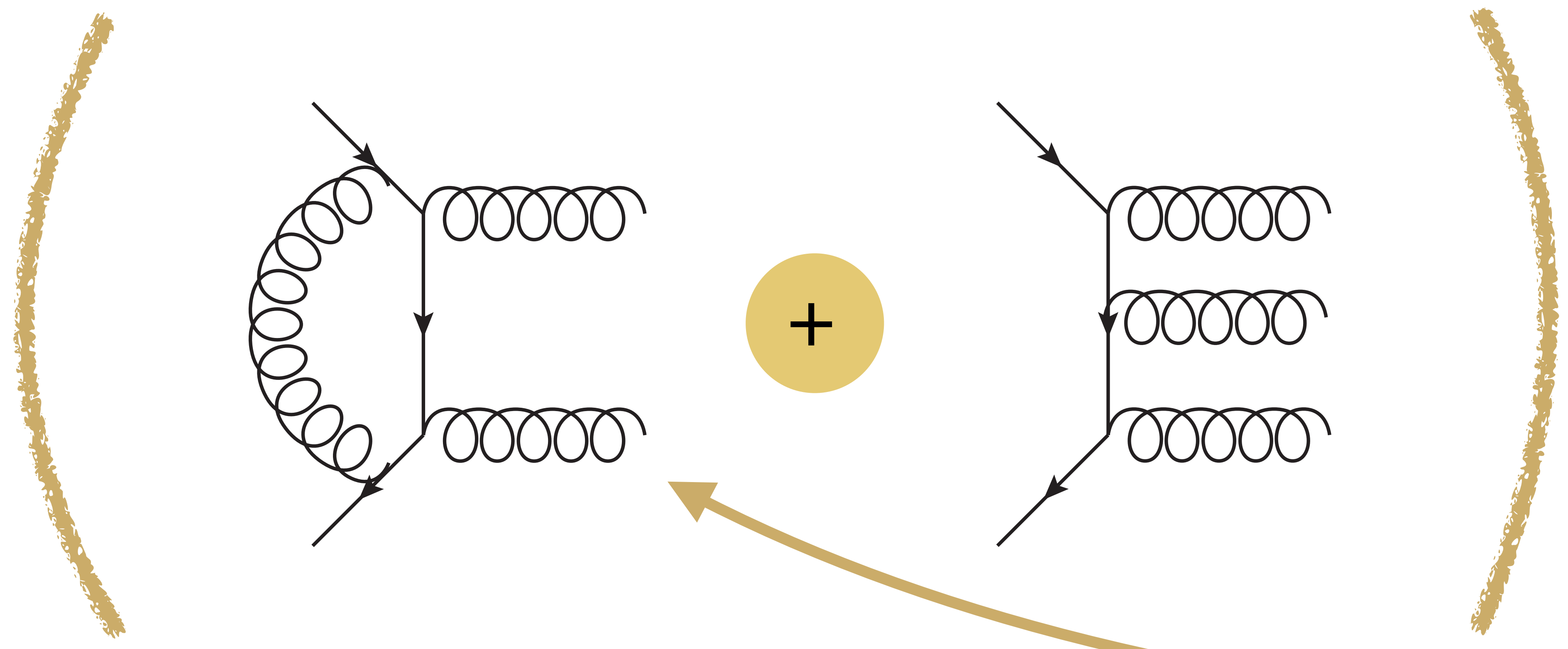
[slide from L. Dixon]

Tree-level Amplitudes

FROM AMPLITUDES TO CROSS SECTION: IN PERTURBATIVE QFT

$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}] \left|\mathcal{M}_{q\bar{q}\to gg}\right|^2$$

$$\left|\mathcal{M}_{q\bar{q}\to gg}\right|^2 = \left|\mathcal{M}^{LO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)\left|\mathcal{M}^{NLO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2\left|\mathcal{M}^{NNLO}_{q\bar{q}\to gg}\right|^2 + \ldots$$



Virtual

Real

~ O(30%-10%) precision

One-loop Amplitudes

FROM AMPLITUDES TO CROSS SECTION: IN PERTURBATIVE QFT

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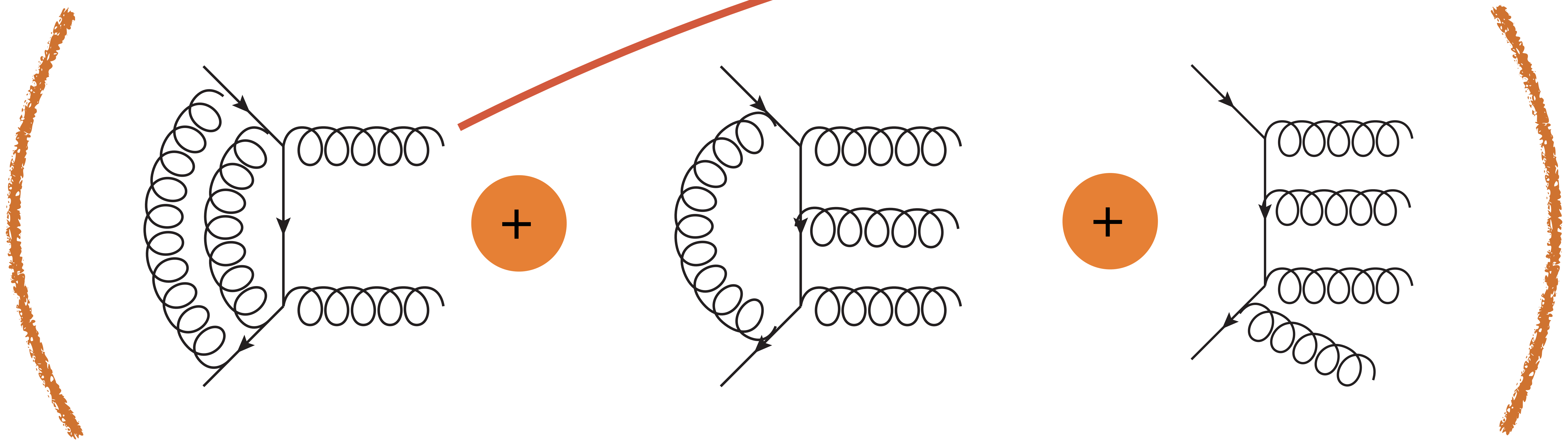
Two-loop amplitudes



Double Virtual

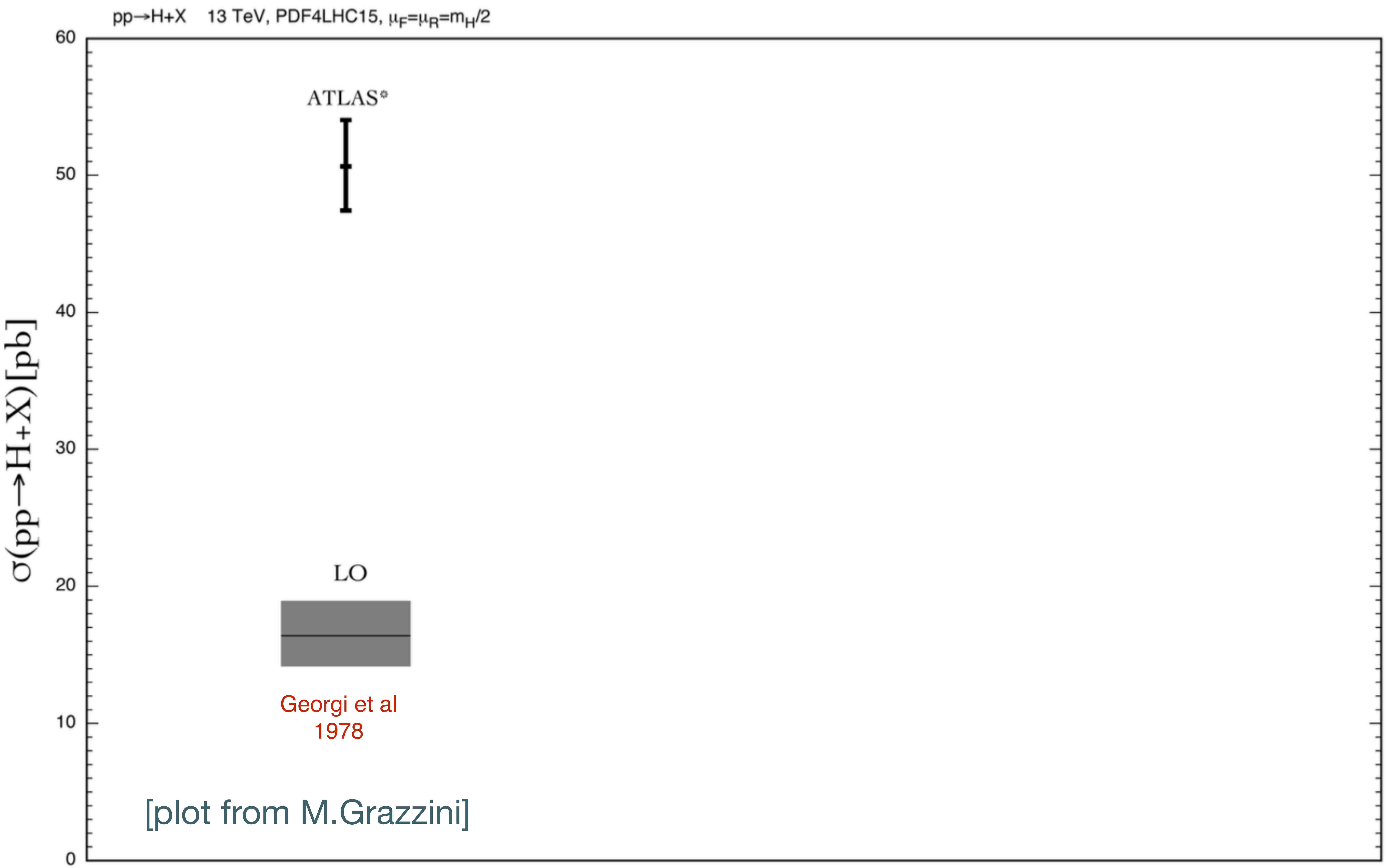
Real Virtual

Double Real

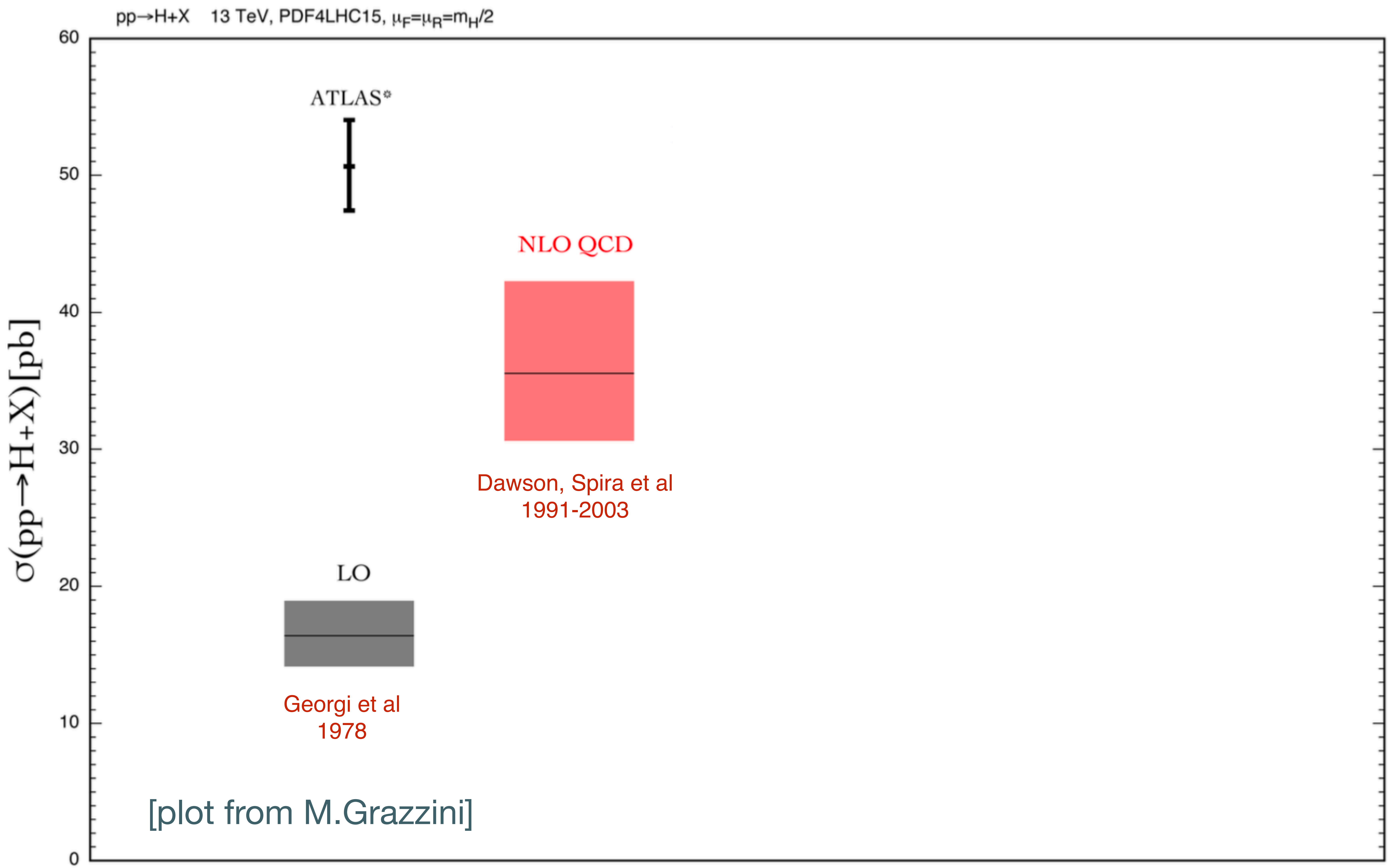
~ O(5%) precision

Often not enough!

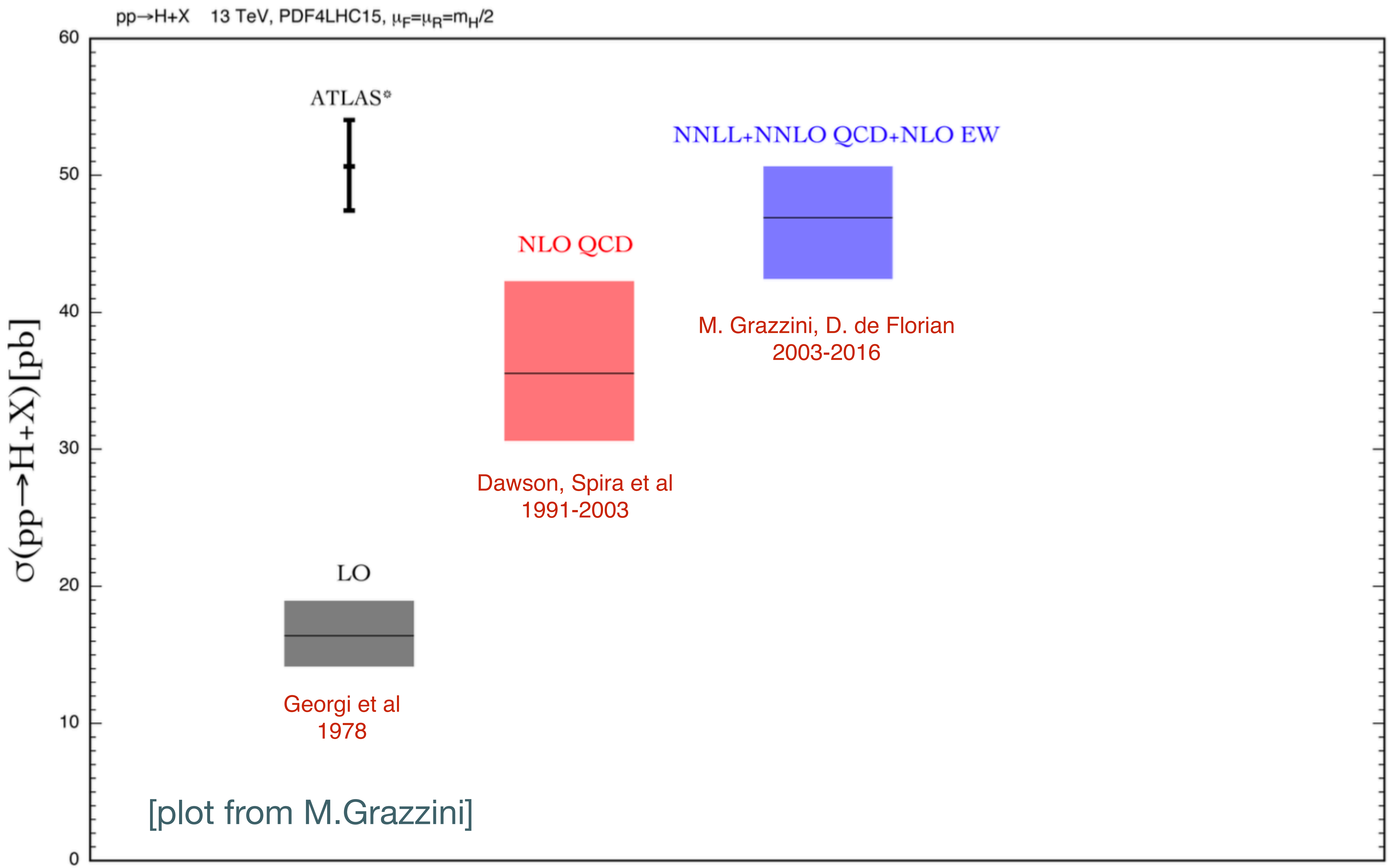
THE NEED OF PRECISION: TOWARDS THE % LEVEL



THE NEED OF PRECISION: TOWARDS THE % LEVEL



THE NEED OF PRECISION: TOWARDS THE % LEVEL



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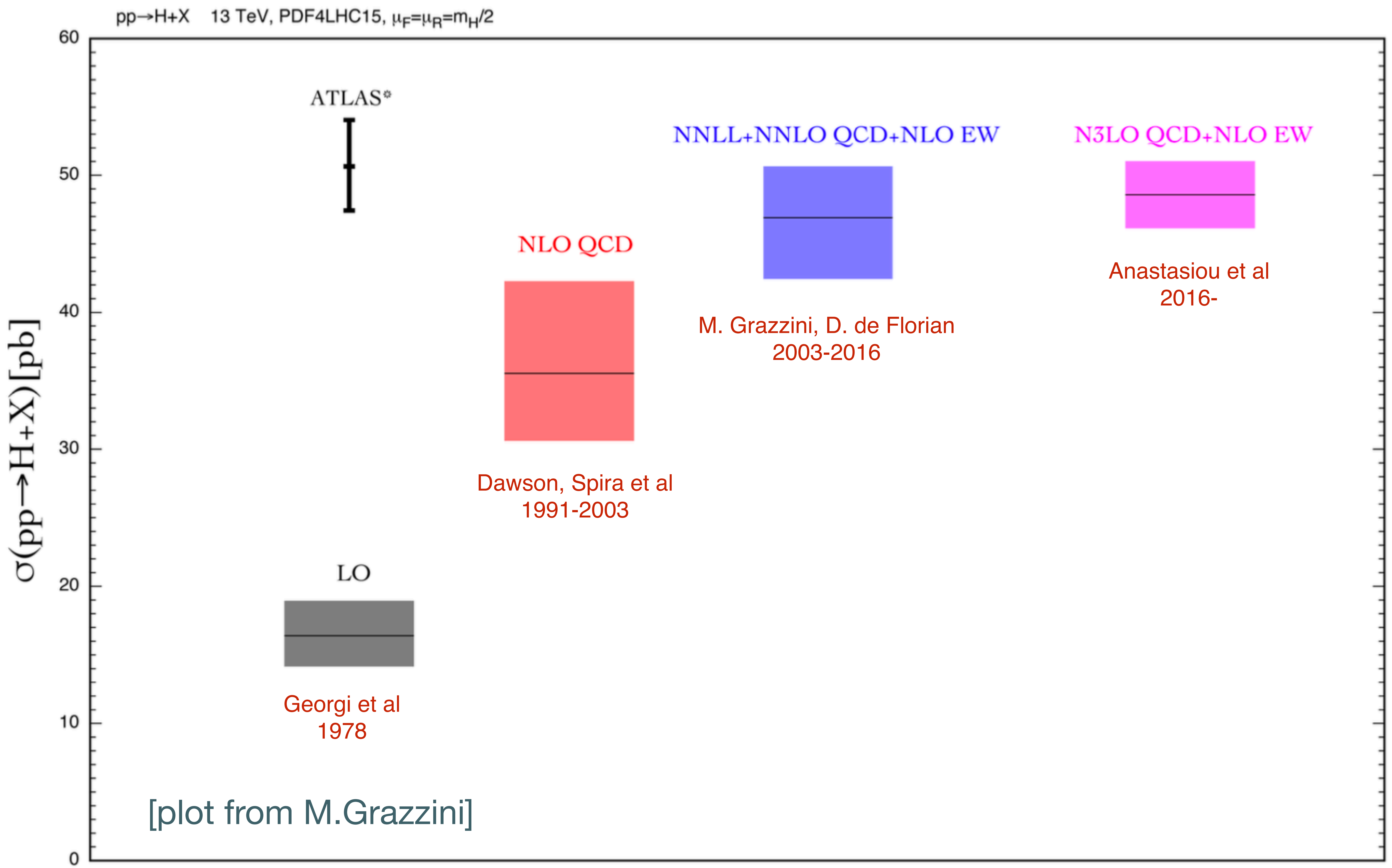

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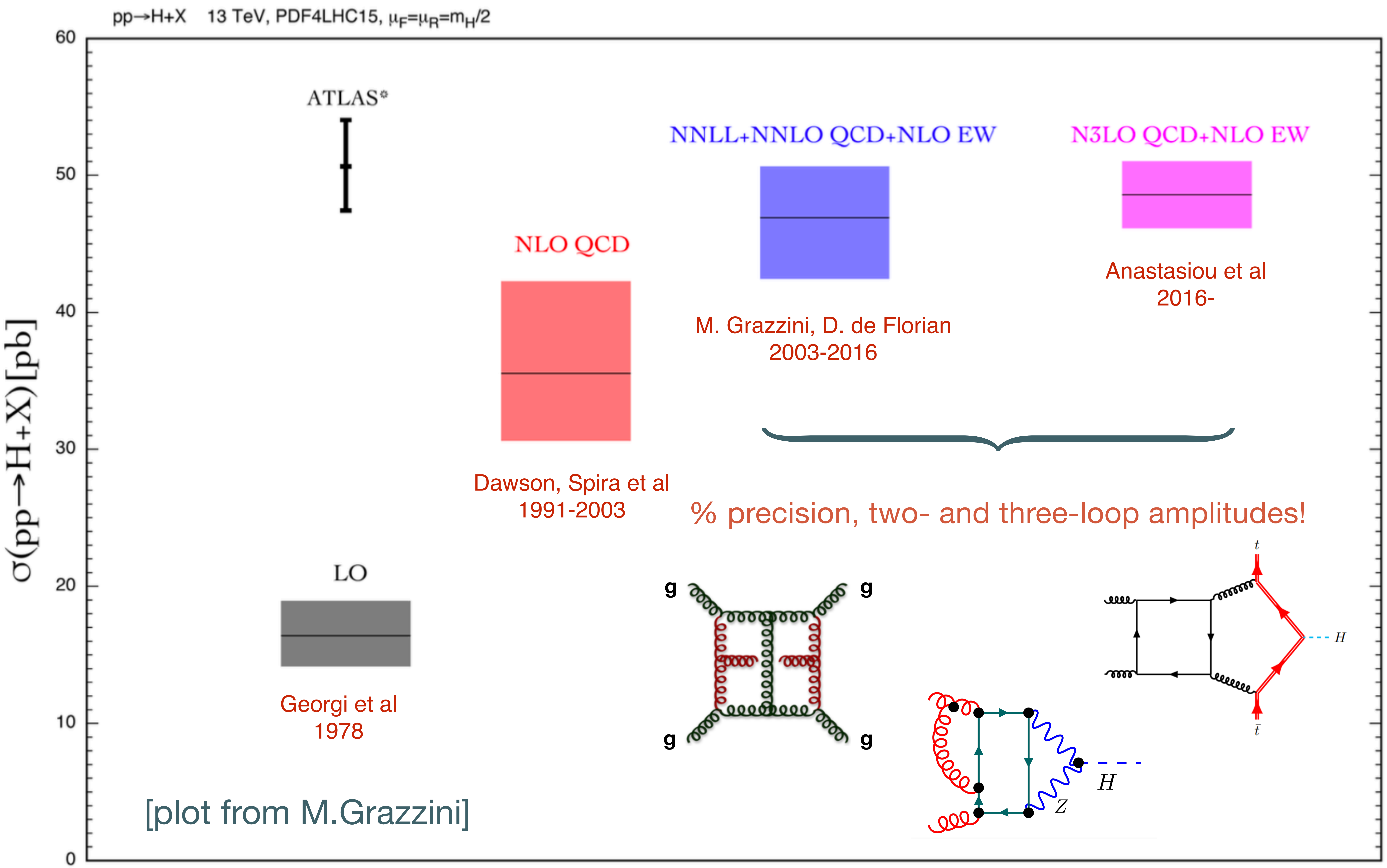
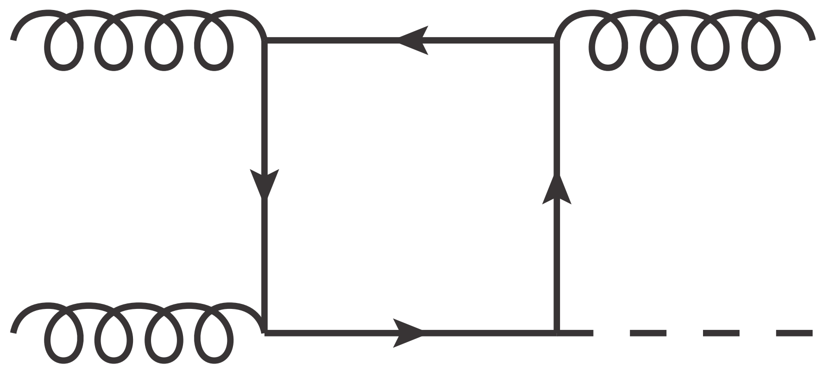
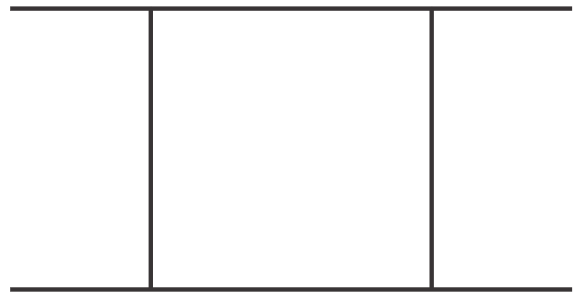


ON AMPLITUDES AND LOOPS

$\mathscr{A} \longrightarrow$  expanded in Feynman diagrams

Strip it of "trivial" Lorentz and Dirac structures (dependence on spin & polarizations of external particles)



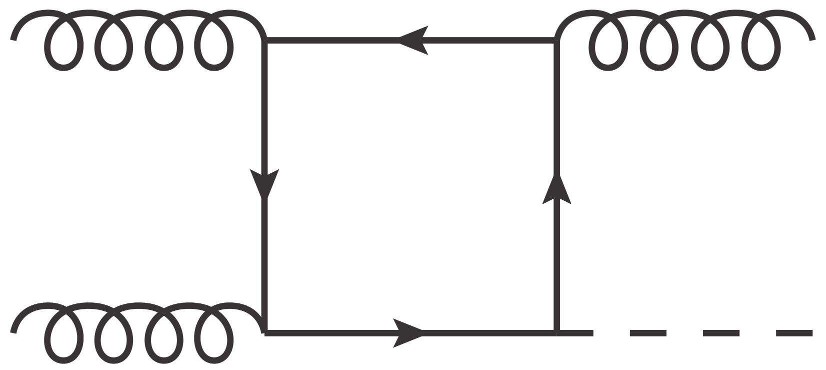
$$\sim \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k-p_2)^2(k-p_2-p_3)^2(k-p_1-p_2-p_3)^2} = \mathscr{I}$$

Scalar Feynman Integrals!

For every **closed loop**, an **integral** over the **unconstrained virtual momentum** k of the particle circulating in the loop

ON AMPLITUDES AND LOOPS

$\mathscr{A} \longrightarrow$



expanded in Feynman diagrams

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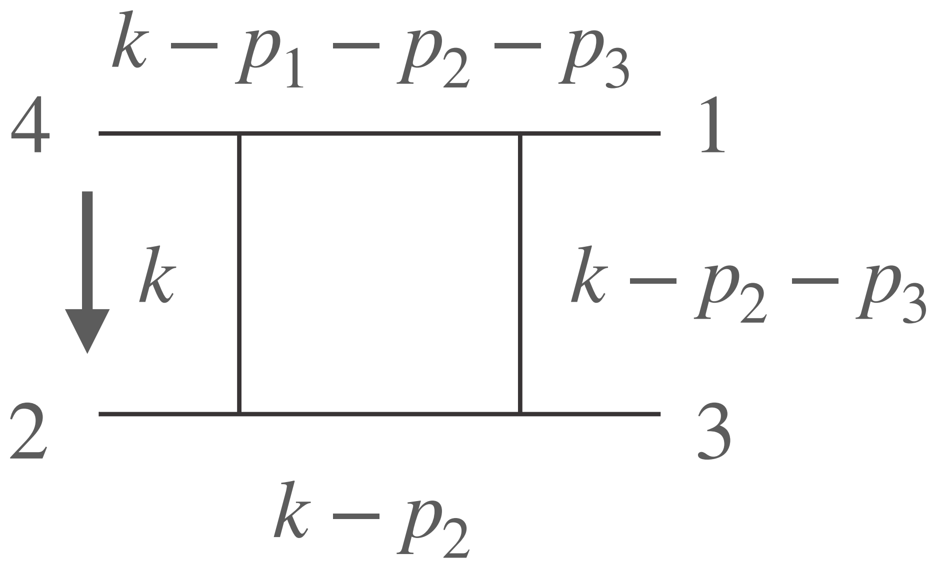

$$\sim \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p_2)^2 (k-p_2-p_3)^2 (k-p_1-p_2-p_3)^2} = \mathscr{I}$$

Scalar Feynman Integrals!

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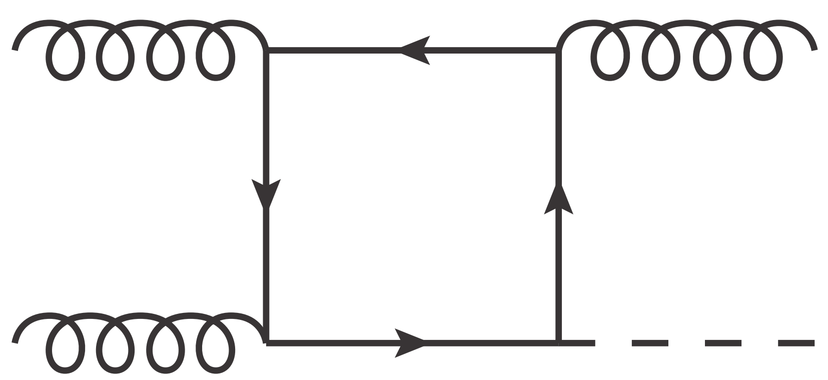
ON AMPLITUDES AND LOOPS

$\mathcal{A}$ →



expanded in Feynman diagrams

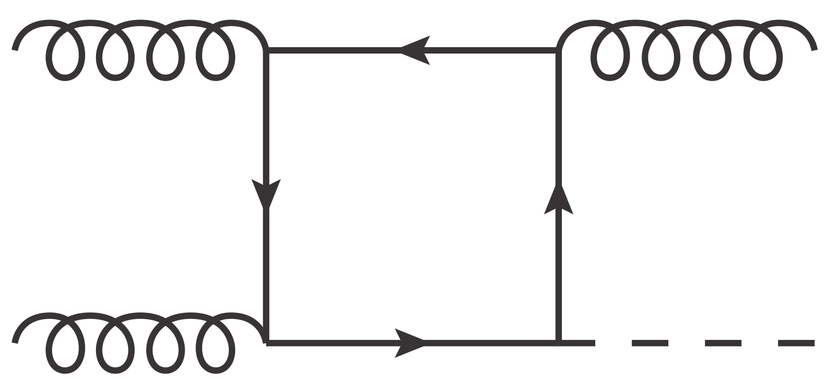
$$= \sum_{j=1}^{N} R_j \mathcal{I}_j$$

→ scalar Feynman integrals

↓ rational functions

analytic structure reflects basic principles of **unitarity** and **causality**

ON AMPLITUDES AND LOOPS

$\mathscr{A}$ →  expanded in Feynman diagrams

$$= \sum_{j=1}^{N} R_j \mathscr{I}_j$$

→ scalar Feynman integrals

$\ldots, \pi, \log z, \zeta_3, \mathrm{Li}_2(z), \ldots$

“special functions and special numbers”

↓ rational functions

analytic structure reflects basic principles of **unitarity** and **causality**

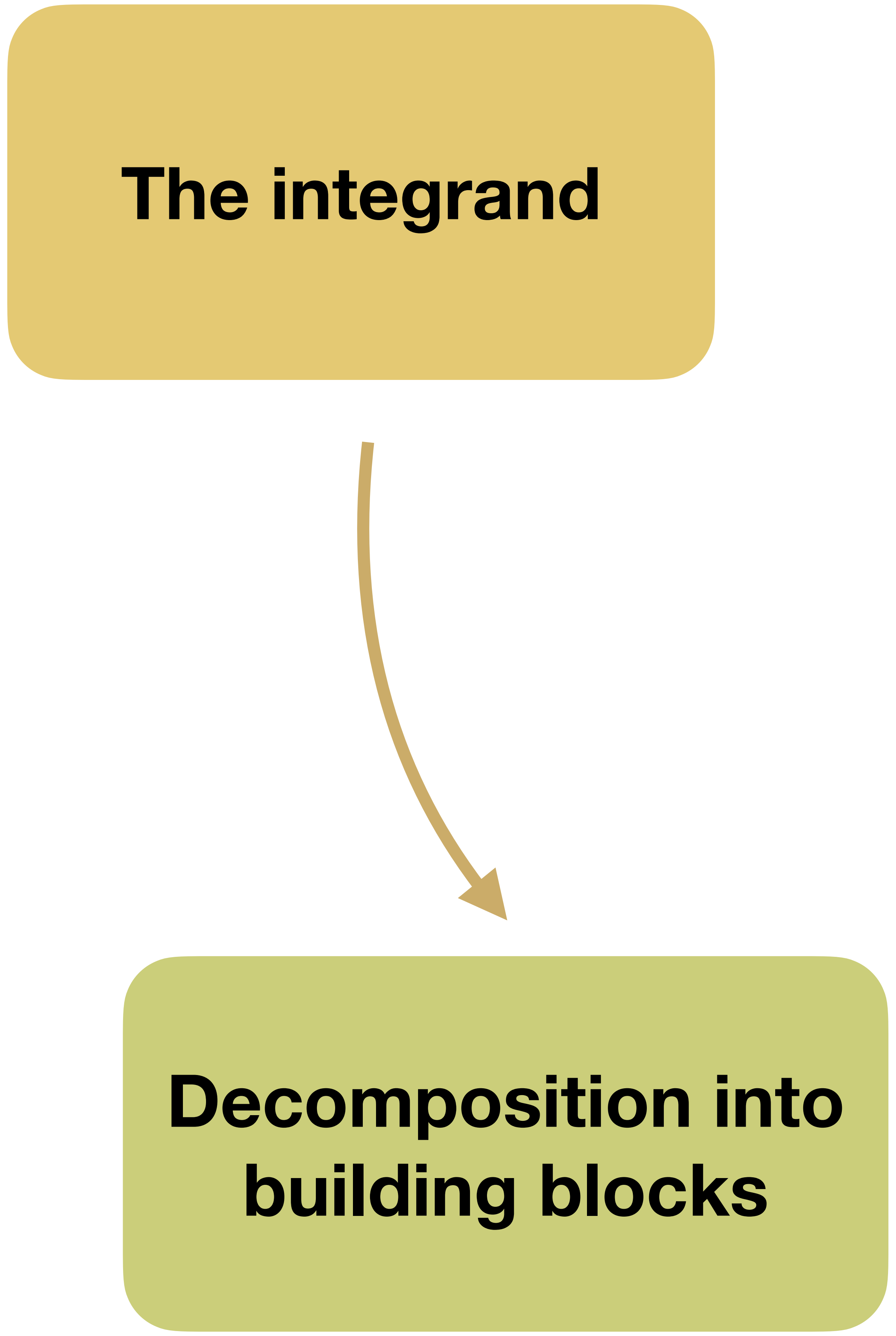
↓ position of **poles** and **branch cuts** dictated by ***virtual particles going on-shell!***

AMPLITUDES FOR COLLIDERS: HOW DO WE THINK ABOUT THEM?

The integrand

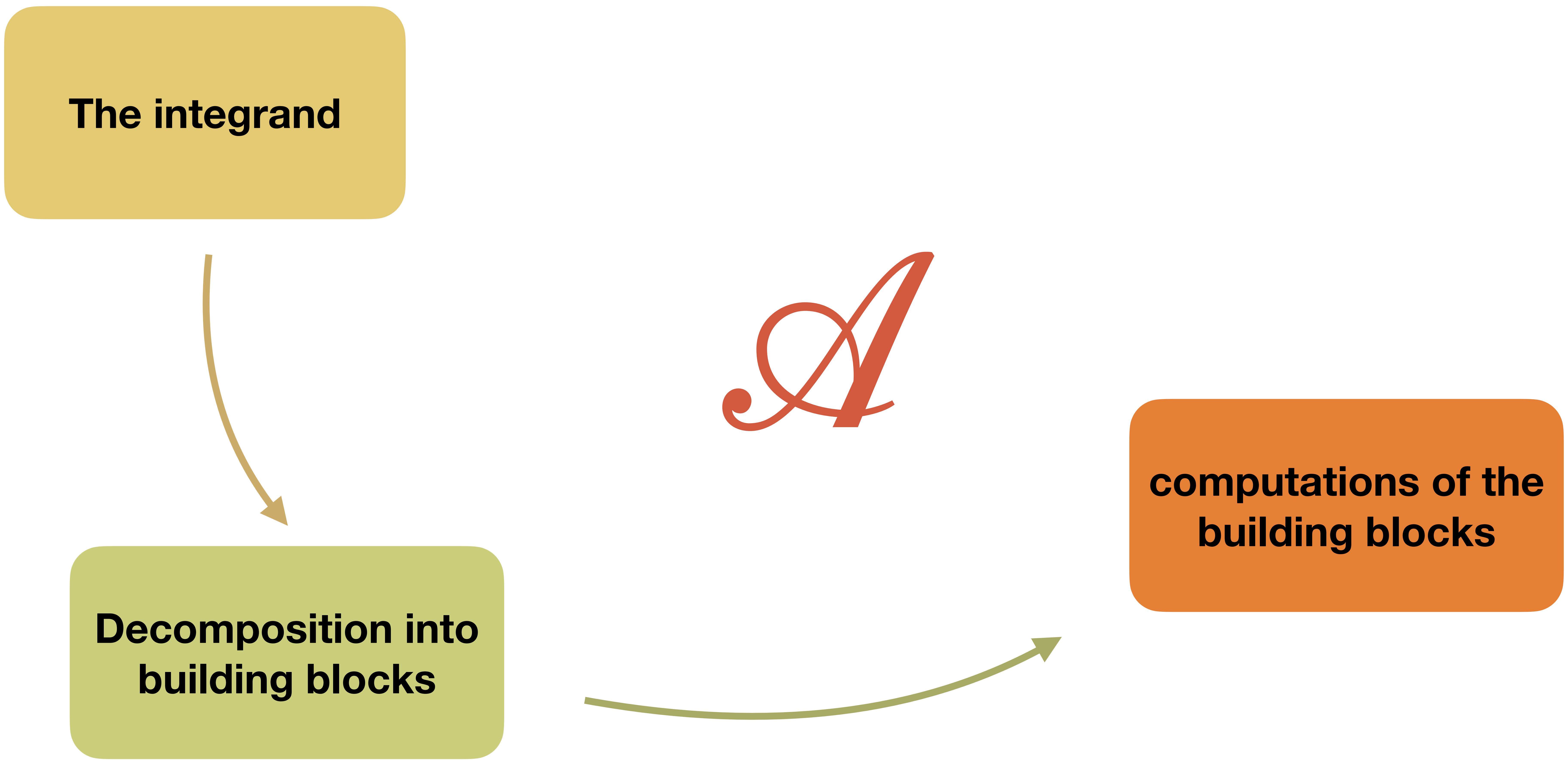
$$\mathcal{A}$$

AMPLITUDES FOR COLLIDERS: HOW DO WE THINK ABOUT THEM?



$\mathscr{A}$

AMPLITUDES FOR COLLIDERS: HOW DO WE THINK ABOUT THEM?

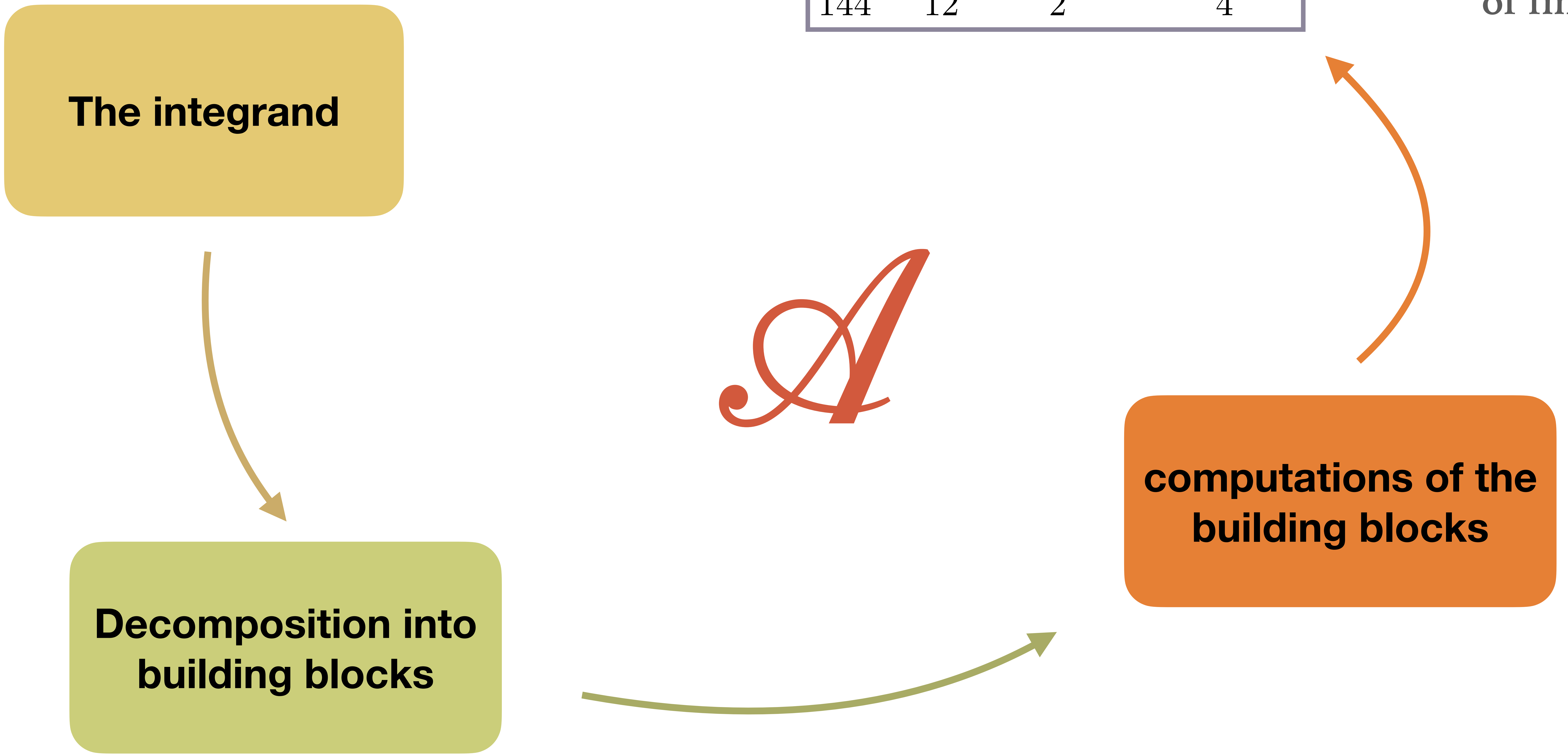


AMPLITUDES FOR COLLIDERS: HOW DO WE THINK ABOUT THEM?

2-loop electron g-2 in QED

$$\frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3)$$

Often: unexpected simplicity of final results



Usually dealt with separately

AMPLITUDES FOR COLLIDERS: HOW DO WE THINK ABOUT THEM?

2-loop electron g-2 in QED

$$\frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3)$$

Often: unexpected simplicity of final results

The integrand

$\mathcal{A}$

computations of the building blocks

Decomposition into building blocks

Connections among them largely to explore

MANY OPEN QUESTIONS AND SOME ANSWERS:

- What are general **numbers and functions** that can appear in the final result?
- How does **physics** constrain the **mathematical properties** of the result?
- What is the **"shortest"** path to the **"simplest"** form of the result?
- ....

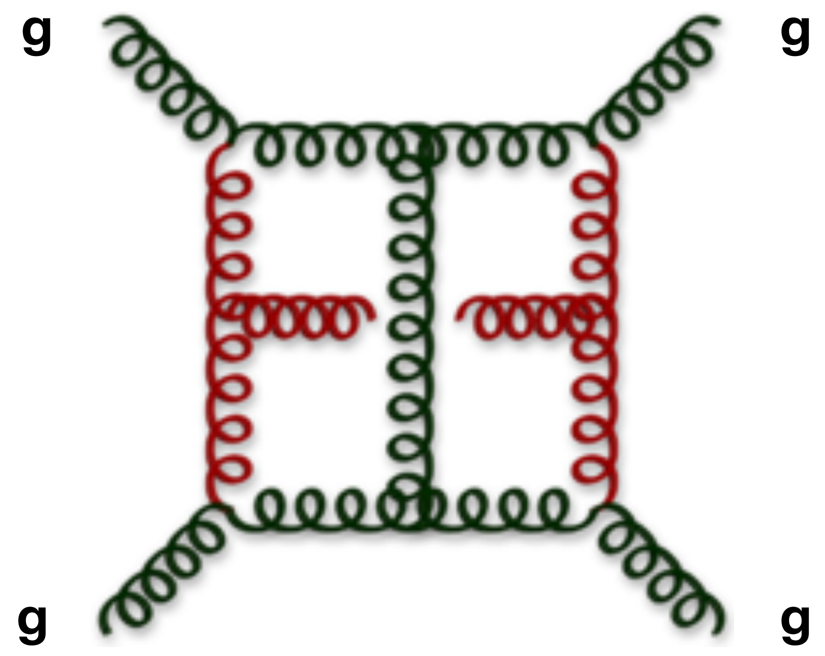
↓

A possible **key to understanding these questions**:

explore interplay between mathematics of scattering amplitudes (**geometry**) and their physical properties (**singularities, discontinuities, soft/collinear limits…**)

WHAT IS AN AMPLITUDE?

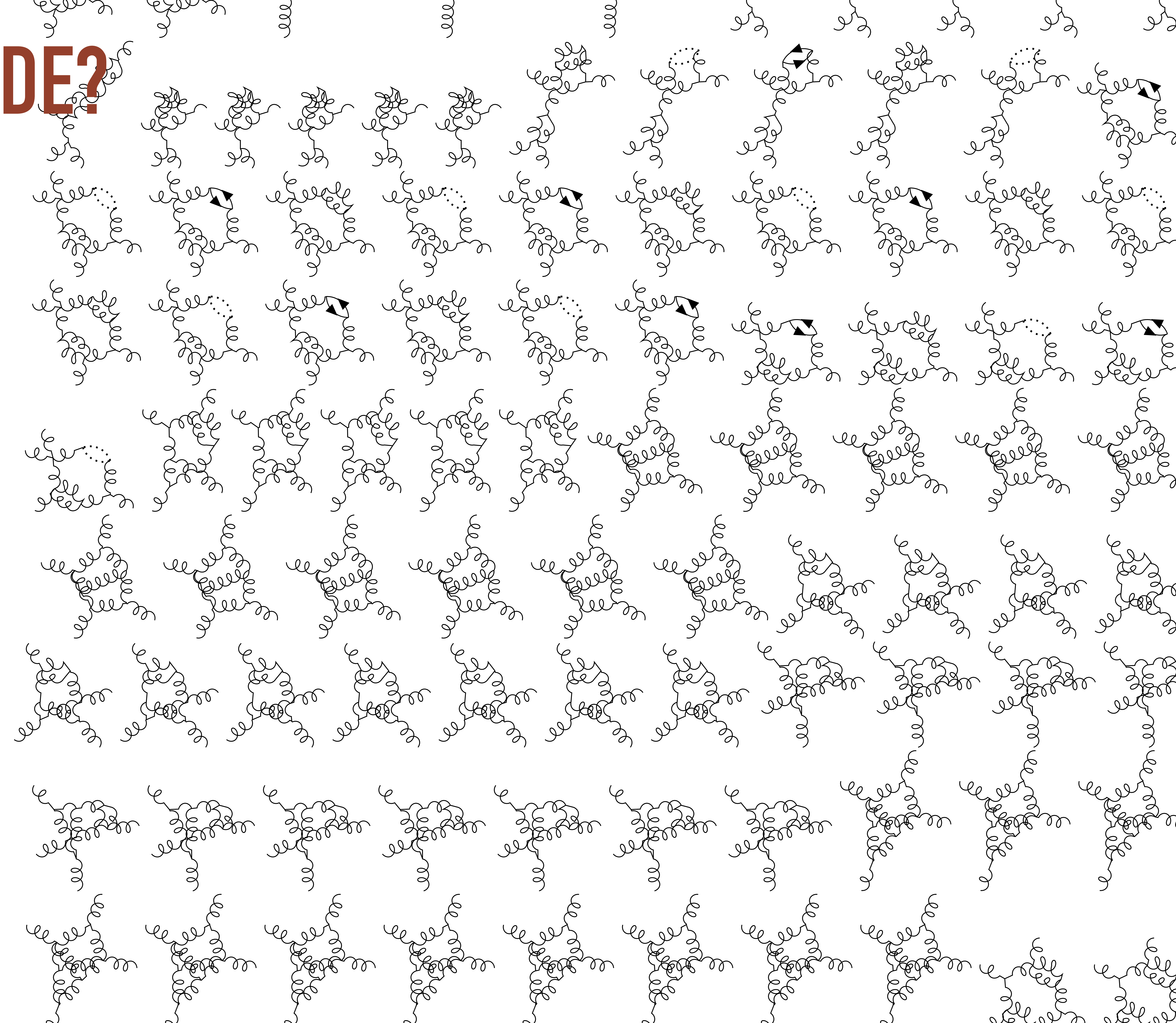
$\mathscr{A}$ ⟶



“just” a sum of Feynman diagrams

WHAT IS AN AMPLITUDE?

$\mathscr{A} \longrightarrow$



gg → *gg* @ 3 loops in QCD

+ 500 more pages

= 50000 Feynman diagrams

= 10^7 Feynman integrals!

FROM INTEGRAND TO SPECIAL FUNCTIONS

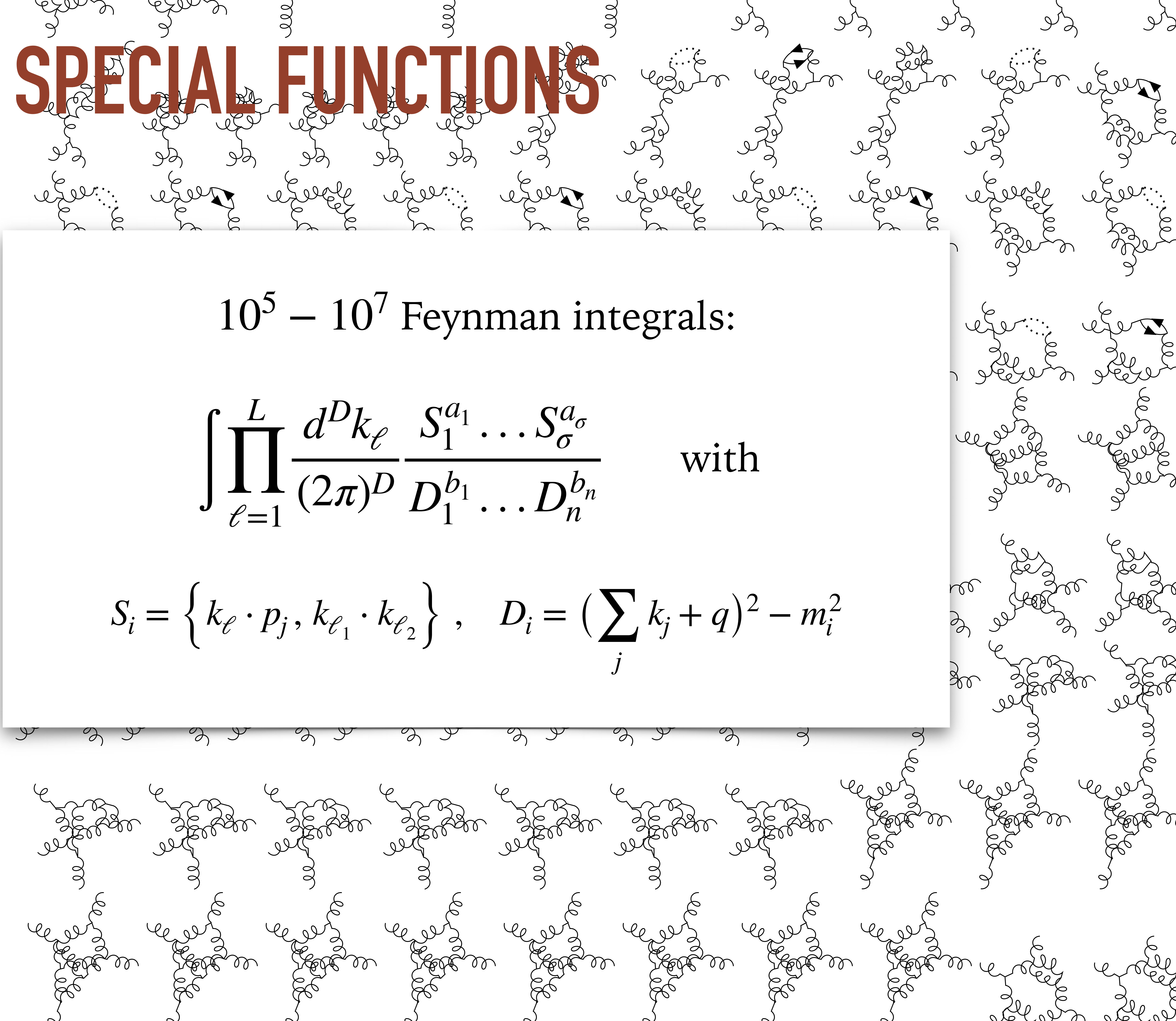
$\mathscr{A} \longrightarrow$

$gg \to gg$ @ 3 loops in QCD

+ 500 more pages

= 50000 Feynman diagrams

= 10^7 Feynman integrals!



$10^5 - 10^7$ Feynman integrals:

$$\int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D} \frac{S_1^{a_1} \ldots S_\sigma^{a_\sigma}}{D_1^{b_1} \ldots D_n^{b_n}} \qquad \text{with}$$

$$S_i = \left\{ k_\ell \cdot p_j \,, k_{\ell_1} \cdot k_{\ell_2} \right\} \,, \quad D_i = \Big(\sum_j k_j + q\Big)^2 - m_i^2$$

FROM INTEGRAND TO SPECIAL FUNCTIONS

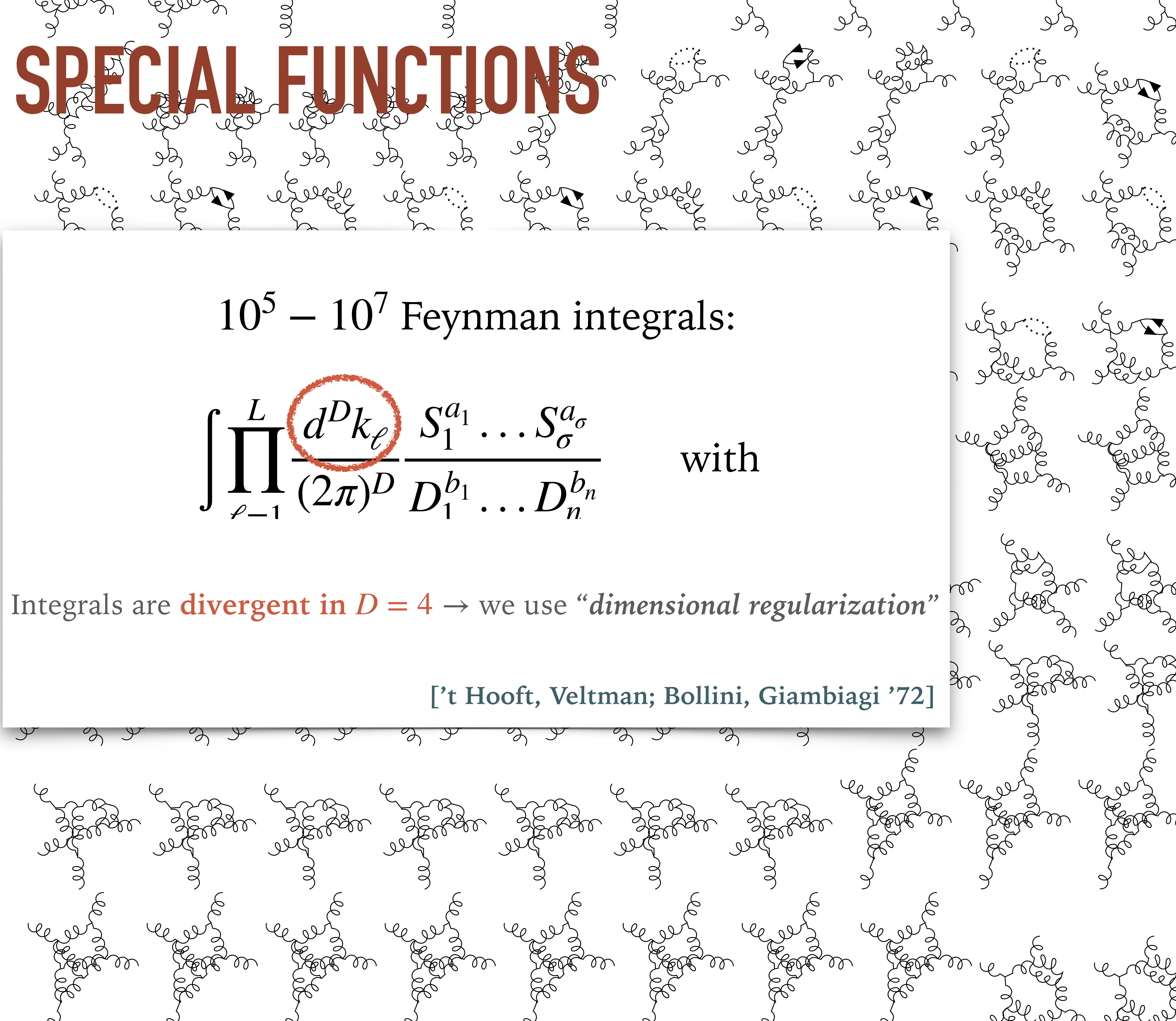
$\mathcal{A}$ →

$gg \to gg$ @ 3 loops in QCD

+ 500 more pages

= 50000 Feynman diagrams

= 10^7 Feynman integrals!



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Integrals are **divergent in $D = 4$** → we use “***dimensional regularization***”

['t Hooft, Veltman; Bollini, Giambiagi '72]

FROM INTEGRAND TO SPECIAL FUNCTIONS

$\mathcal{A}$ $\longrightarrow$

$gg \to gg$ @ 3 loops in QCD

+ 500 more pages

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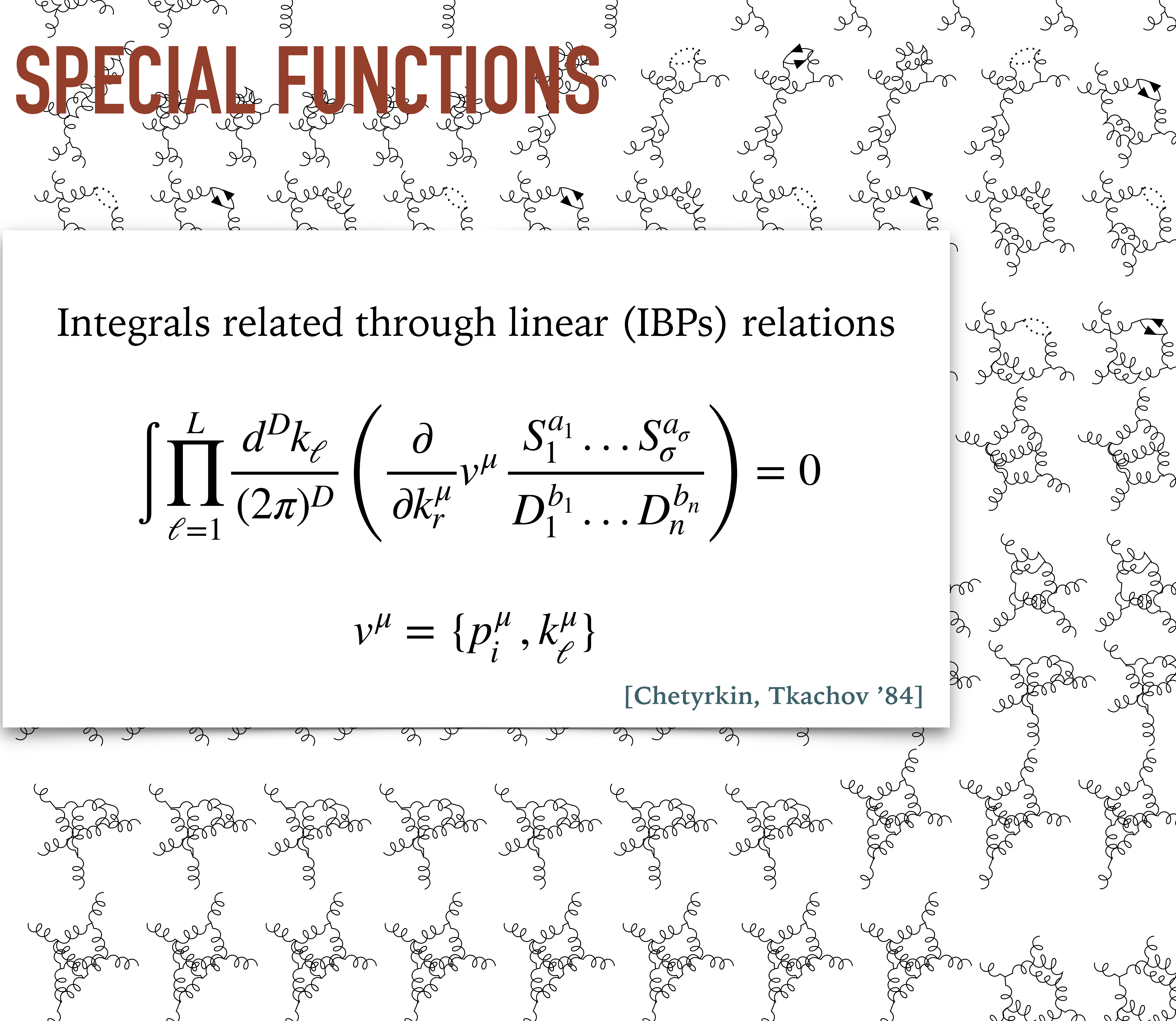


Integrals related through linear (IBPs) relations

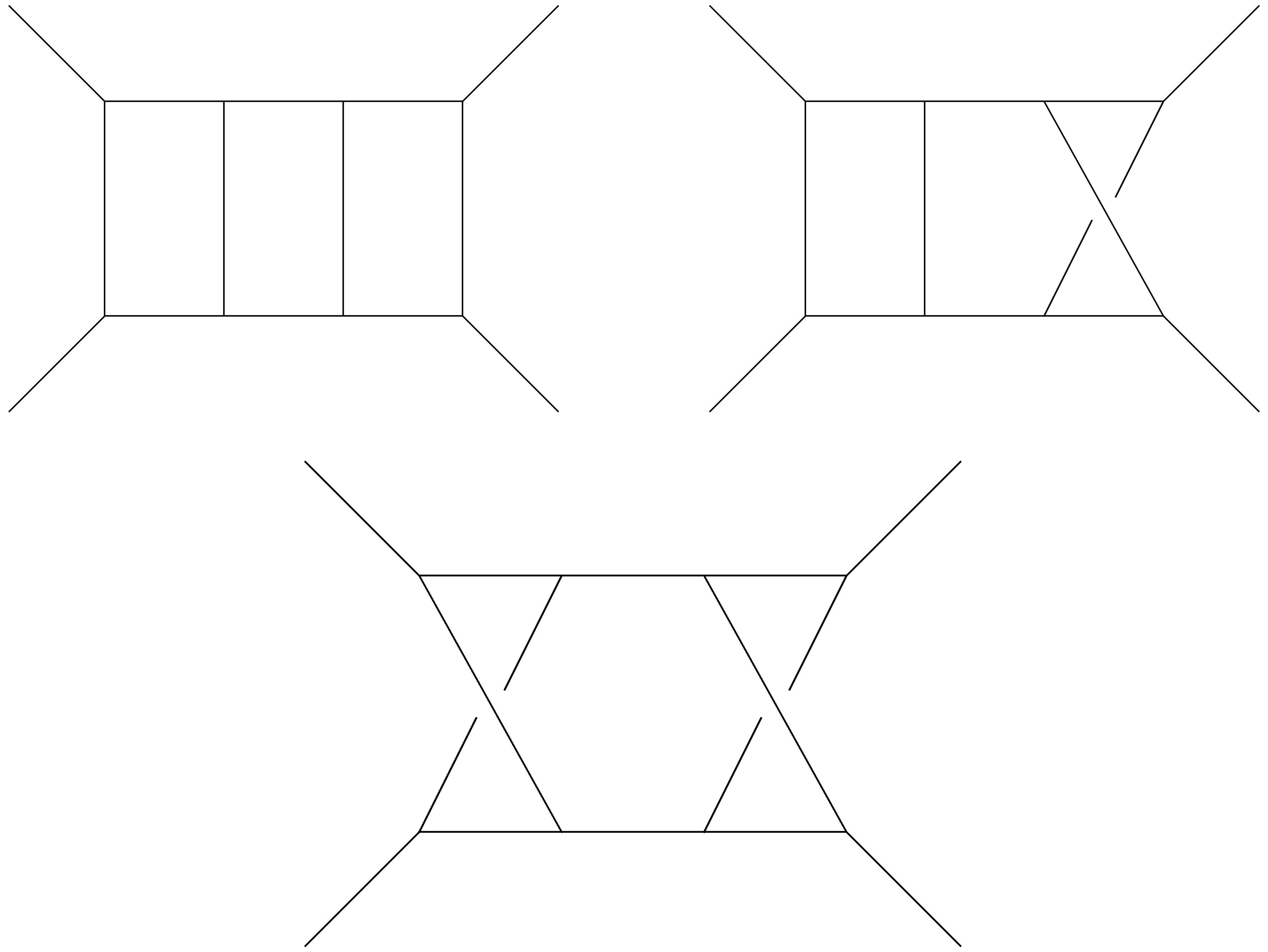
$$\int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D} \left(\frac{\partial}{\partial k_r^\mu} v^\mu \frac{S_1^{a_1} \ldots S_\sigma^{a_\sigma}}{D_1^{b_1} \ldots D_n^{b_n}} \right) = 0$$

$$v^\mu = \{p_i^\mu, k_\ell^\mu\}$$

[Chetyrkin, Tkachov '84]

FROM INTEGRAND TO SPECIAL FUNCTIONS

$$\mathscr{A} \longrightarrow \sum_i$$



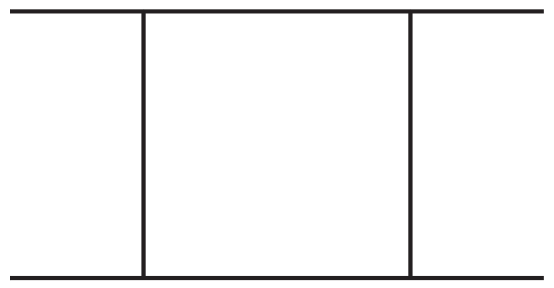
$gg \to gg$ @ 3 loops in QCD ~ "only" 500 master integrals

Space of Feynman integrals is a *finite-dimensional vector space; master integrals are a basis in this space!*

[Smirnov, Petukhov '10]

FROM INTEGRAND TO SPECIAL FUNCTIONS

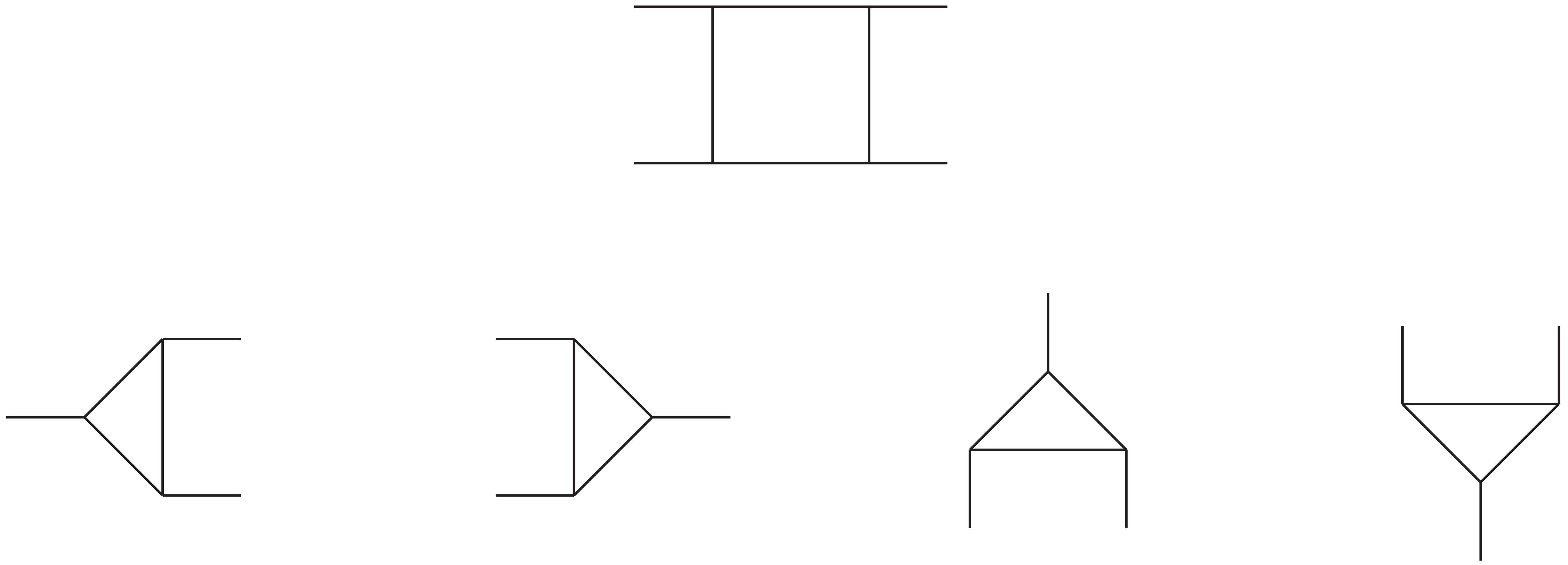
Master integrals can be conveniently ***organized in a "tree"*** depending on # of propagators



Box

FROM INTEGRAND TO SPECIAL FUNCTIONS

Master integrals can be conveniently ***organized in a "tree"*** depending on # of propagators



Box

triangles

(pinching 1 propagator)

FROM INTEGRAND TO SPECIAL FUNCTIONS

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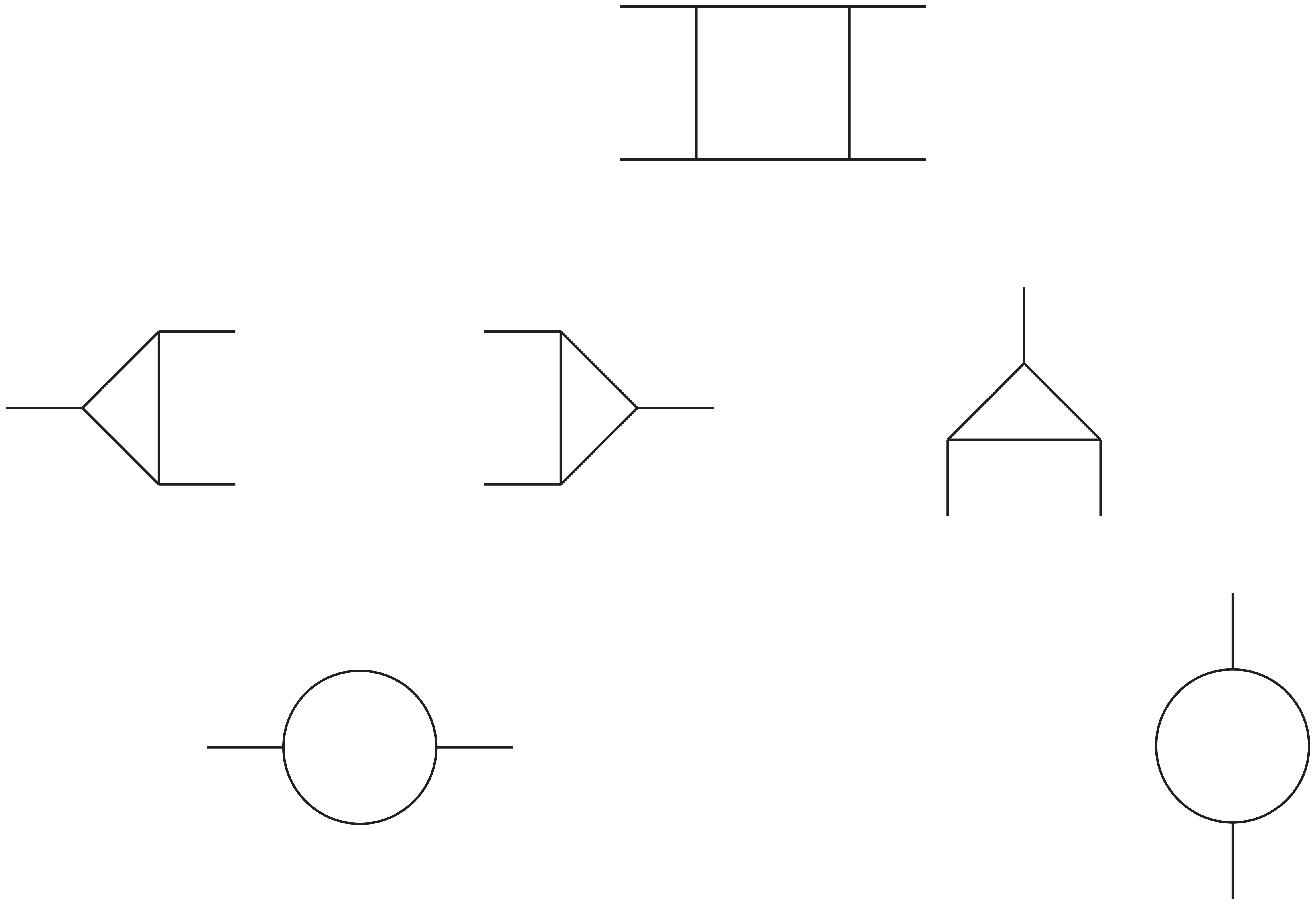


Box

triangles

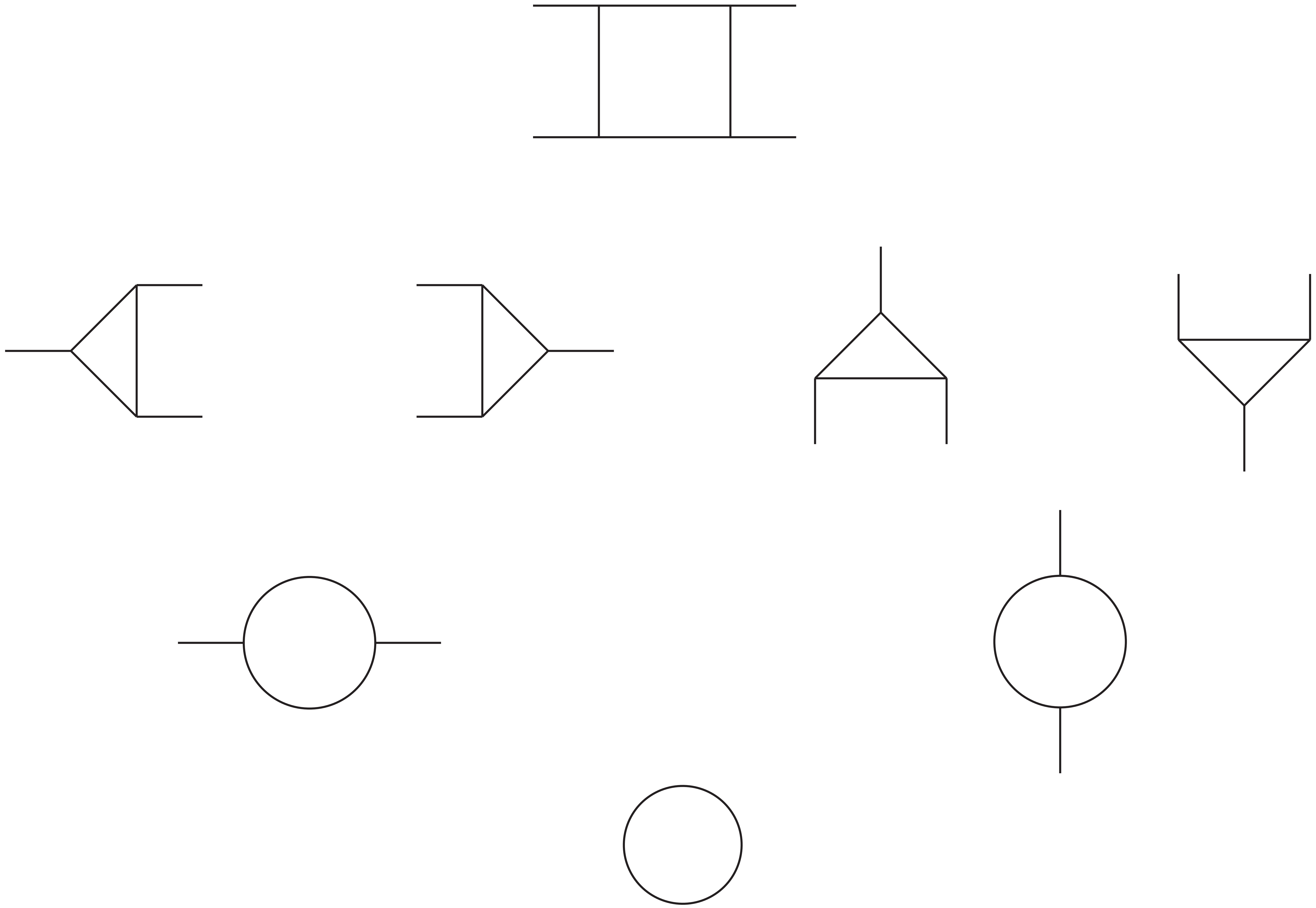
(pinching 1 propagator)

bubbles

(pinching 2 propagators)

FROM INTEGRAND TO SPECIAL FUNCTIONS

Master integrals can be conveniently ***organized in a "tree"*** depending on # of propagators



Box

triangles

(pinching 1 propagator)

bubbles

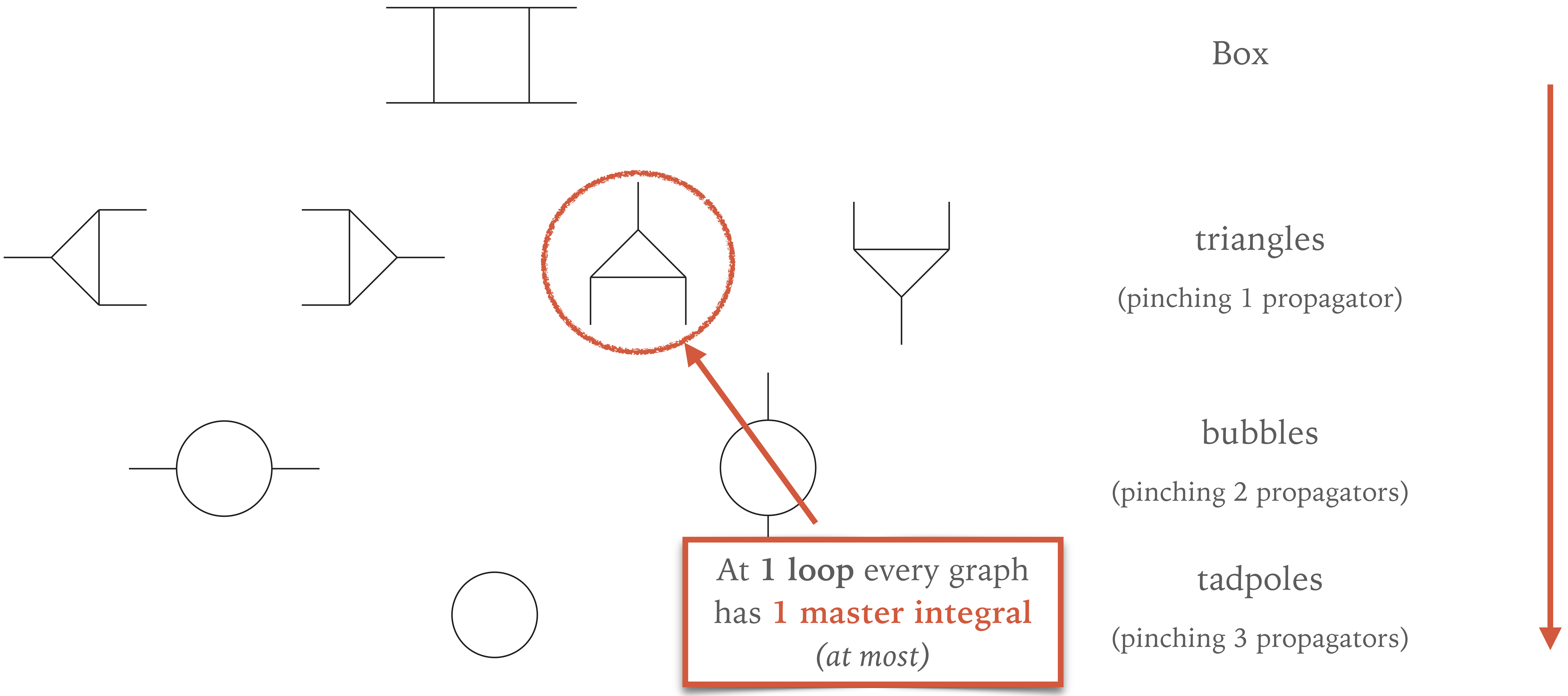
(pinching 2 propagators)

tadpoles

(pinching 3 propagators)

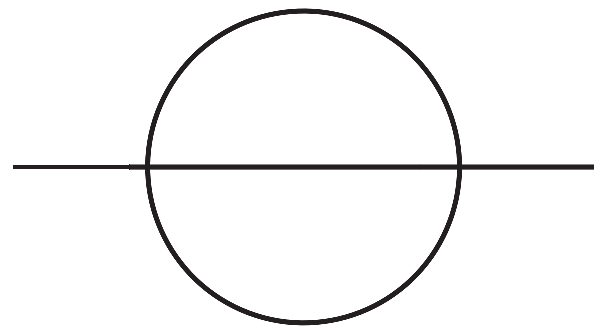
FROM INTEGRAND TO SPECIAL FUNCTIONS

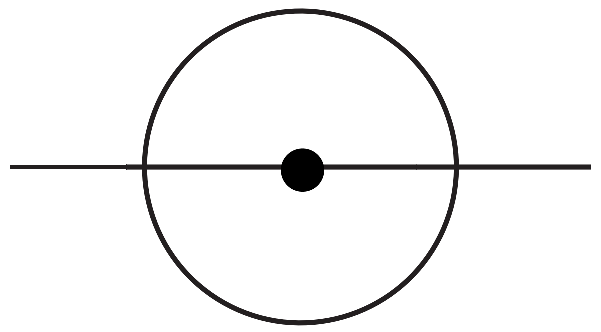
Master integrals can be conveniently ***organized in a "tree"*** depending on # of propagators



FROM INTEGRAND TO SPECIAL FUNCTIONS

At ***higher loops***: a “graph” can have more than one master integral





the equal-mass sunrise

FROM INTEGRAND TO SPECIAL FUNCTIONS

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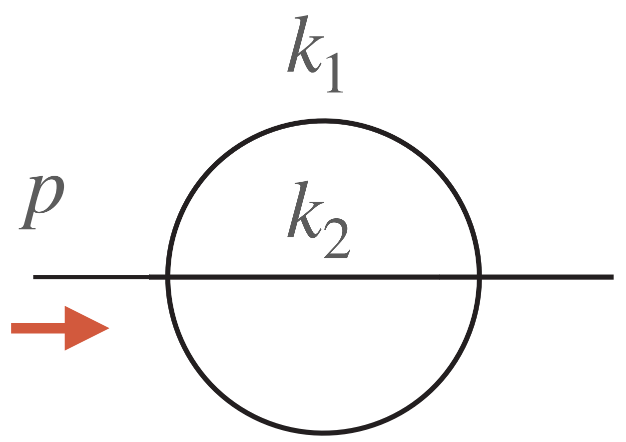

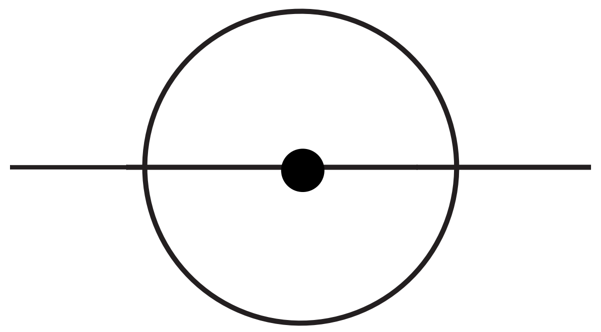

the equal-mass sunrise

$$\int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{1}{(k_1^2 - m^2)(k_2^2 - m^2)((k_1 - k_2 - p)^2 - m^2)}$$

FROM INTEGRAND TO SPECIAL FUNCTIONS

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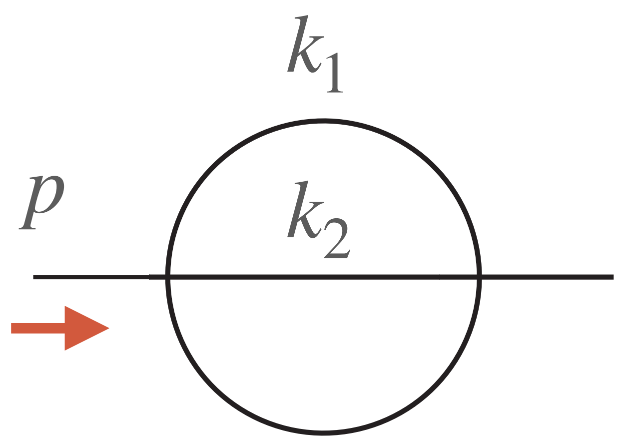

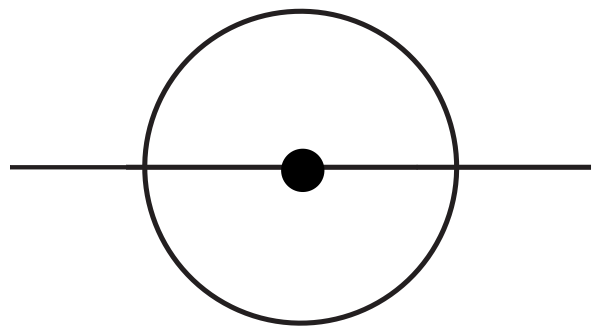

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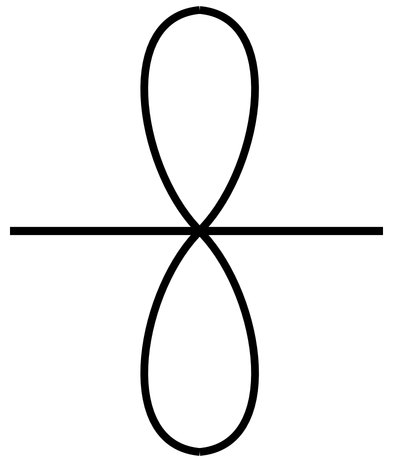
$$\int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{1}{(k_1^2 - m^2)(k_2^2 - m^2)^2((k_1 - k_2 - p)^2 - m^2)}$$

"dot": 1 propagator squared

FROM INTEGRAND TO SPECIAL FUNCTIONS

At ***higher loops***: a “graph” can have more than one master integral

the equal-mass sunrise



double tadpole

(pinching 1 propagator)

Fundamental difference with one loop → hint that complexity of the problem “jumps”

DIFFERENTIAL EQUATIONS FOR FEYNMAN INTEGRALS

We can **differentiate** Feynman integrals w.r.t. the kinematical invariants

$$\frac{\partial}{\partial s_{ij}}\left[\int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D} \frac{S_1^{a_1} \dots S_\sigma^{a_\sigma}}{D_1^{b_1} \dots D_n^{b_n}}\right] = \qquad \forall s_{ij} = \{p_i \cdot p_j \, , \, m_k^2\}$$

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[Kotikov '93; Remiddi '97; Gehrmann, Remiddi '99]

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[Kotikov ’93; Remiddi ’97; Gehrmann, Remiddi ’99]

↓

$$\frac{\partial}{\partial s_{ij}}\vec{I}=A(s_{ij},D)\vec{I}\,,\qquad A(s_{ij},D)$$

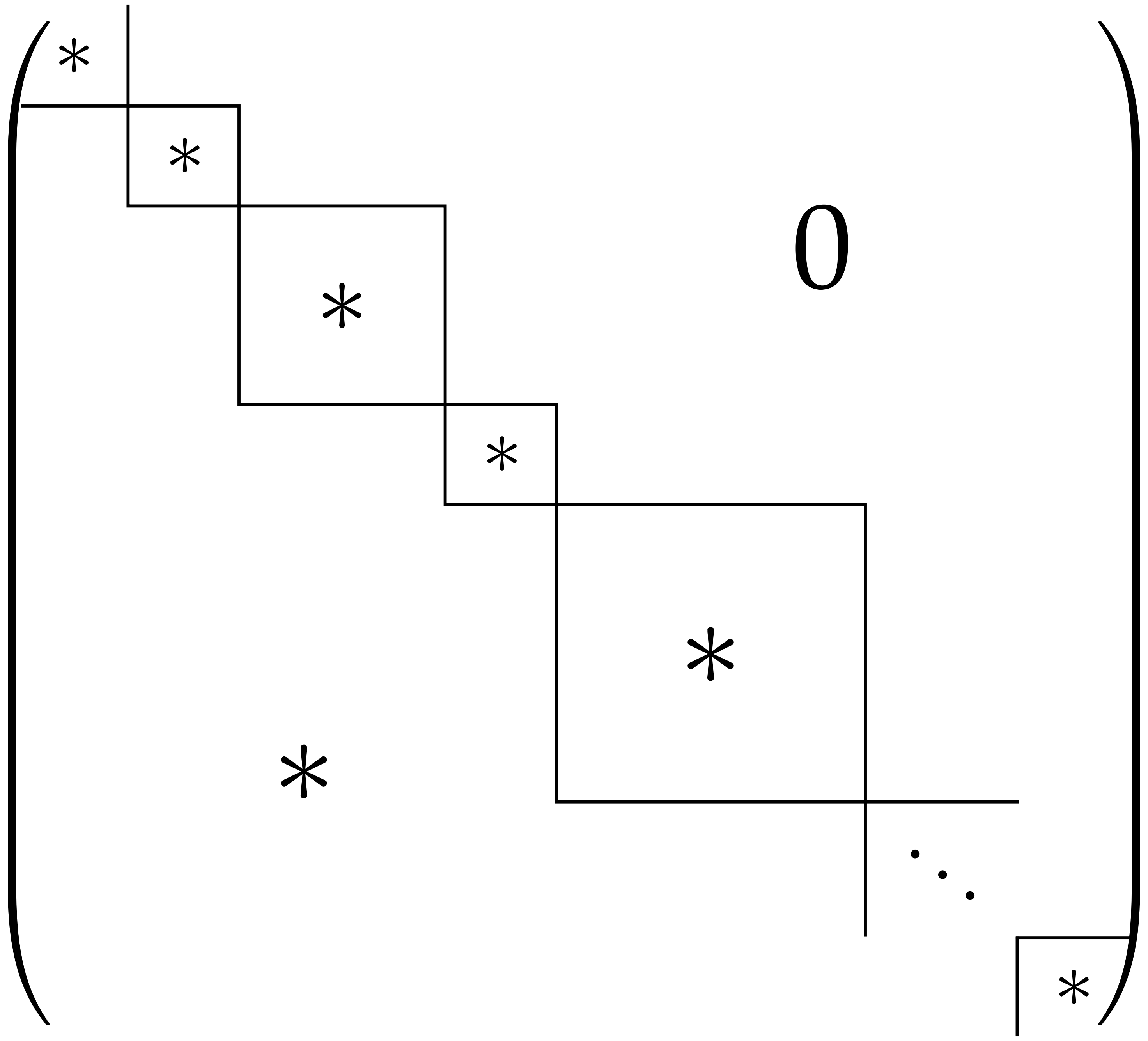
Rational functions

DIFFERENTIAL EQUATIONS FOR FEYNMAN INTEGRALS

We can **differentiate** Feynman integrals w.r.t. the kinematical invariants

$$\frac{\partial}{\partial s_{ij}}\left[\int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D} \frac{S_1^{a_1} \dots S_\sigma^{a_\sigma}}{D_1^{b_1} \dots D_n^{b_n}}\right] = \sum_I c_I \int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D} \frac{S_1^{a_1} \dots S_\sigma^{a_\sigma}}{D_1^{b_1} \dots D_n^{b_n}} \qquad \forall s_{ij} = \{p_i \cdot p_j\,,\, m_k^2\}$$

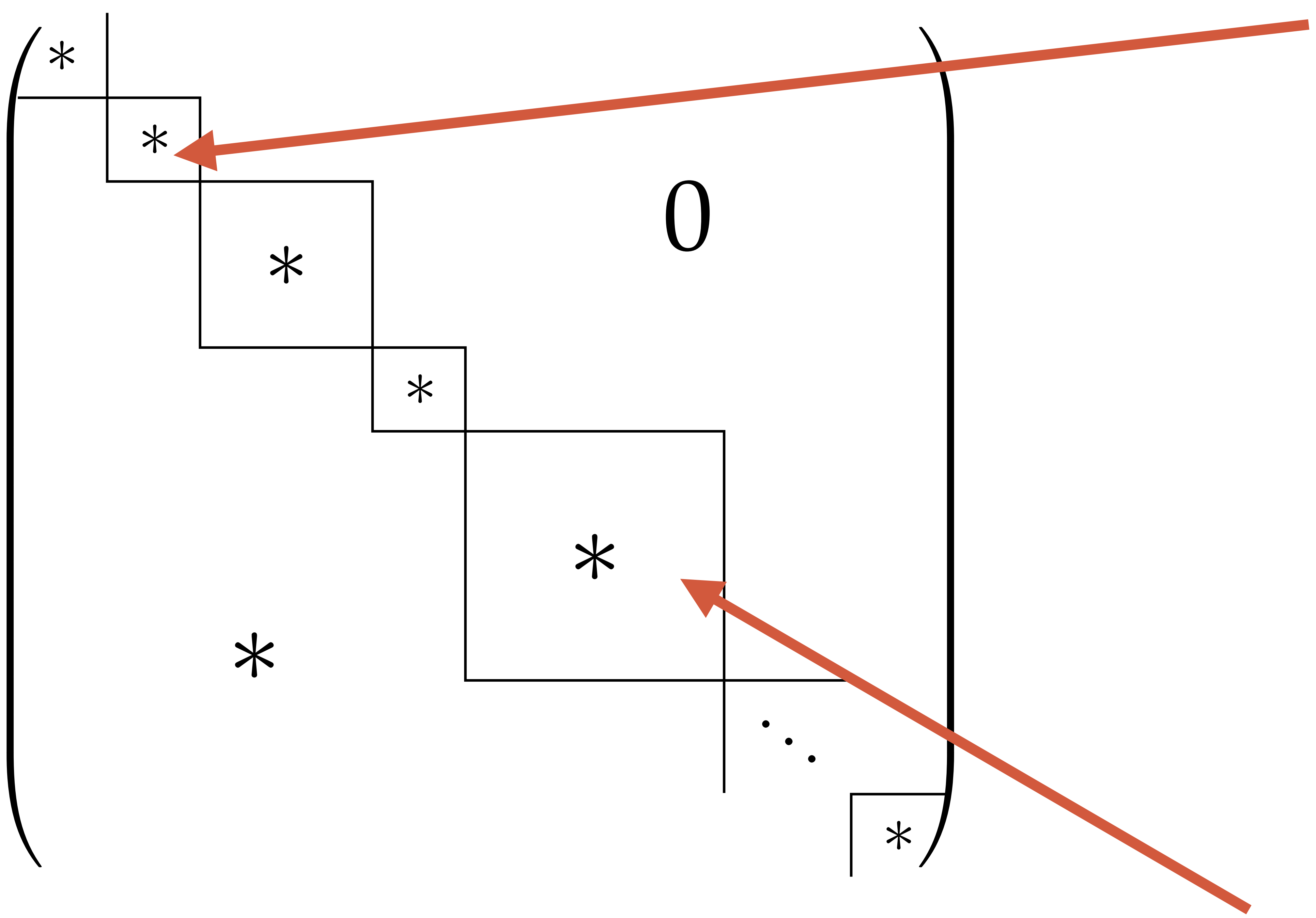
$$\frac{\partial}{\partial s_{ij}} \vec{I} = A(s_{ij}, D)\vec{I}\,, \qquad A(s_{ij}, D) =$$

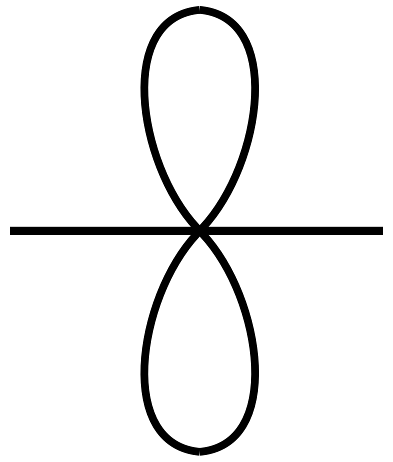


block-triangular:

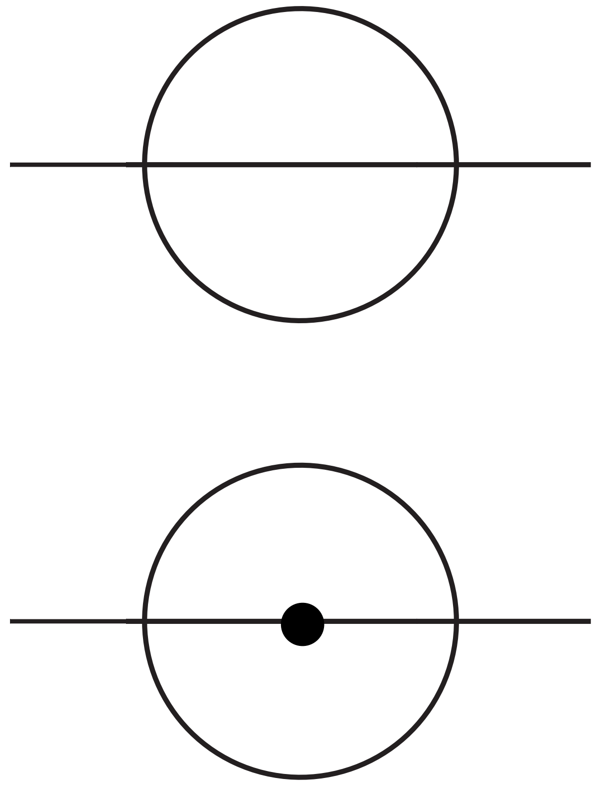
integrals with more propagators depend on ones with fewer

DIFFERENTIAL EQUATIONS FOR FEYNMAN INTEGRALS





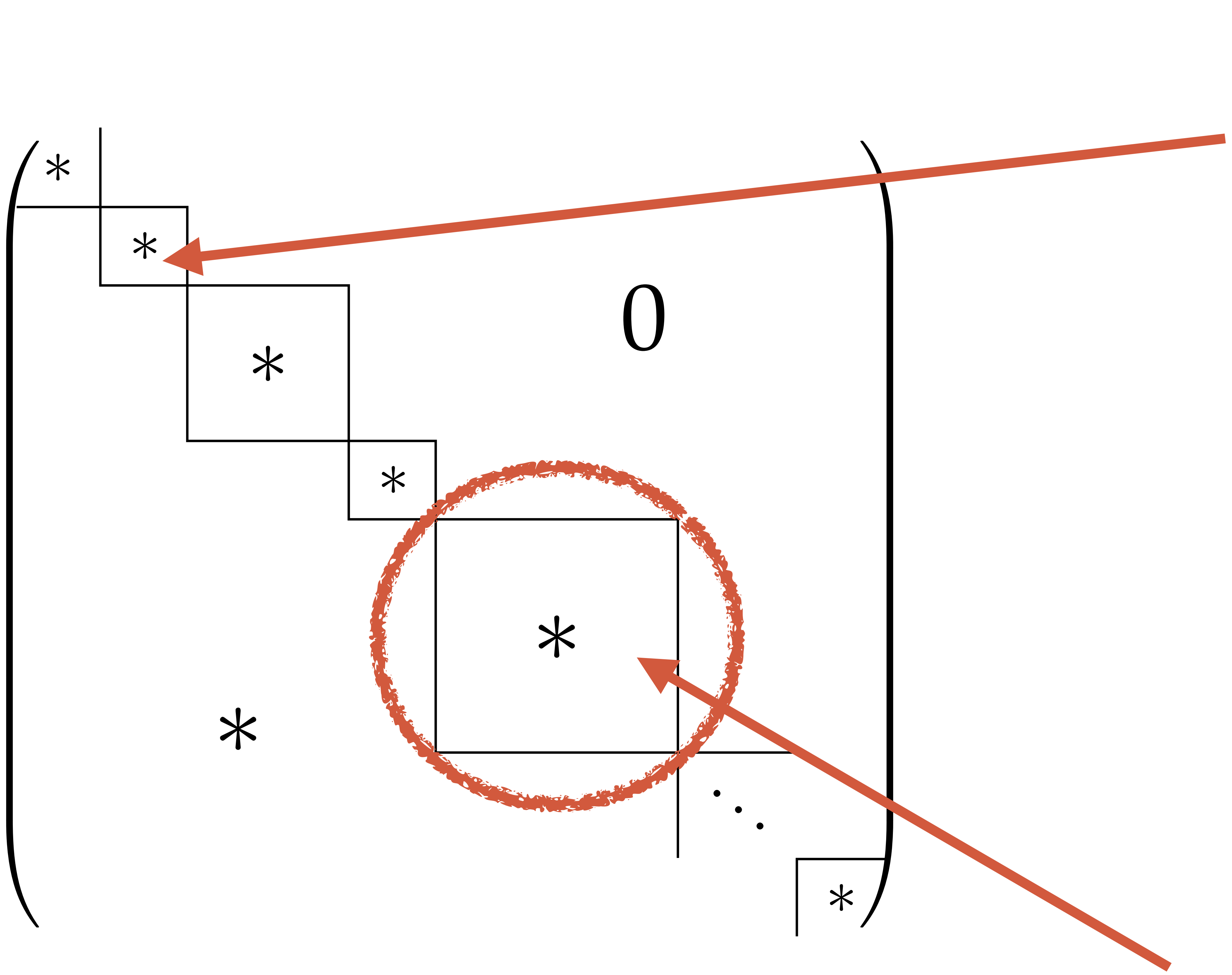
1×1 blocks correspond to single master integrals

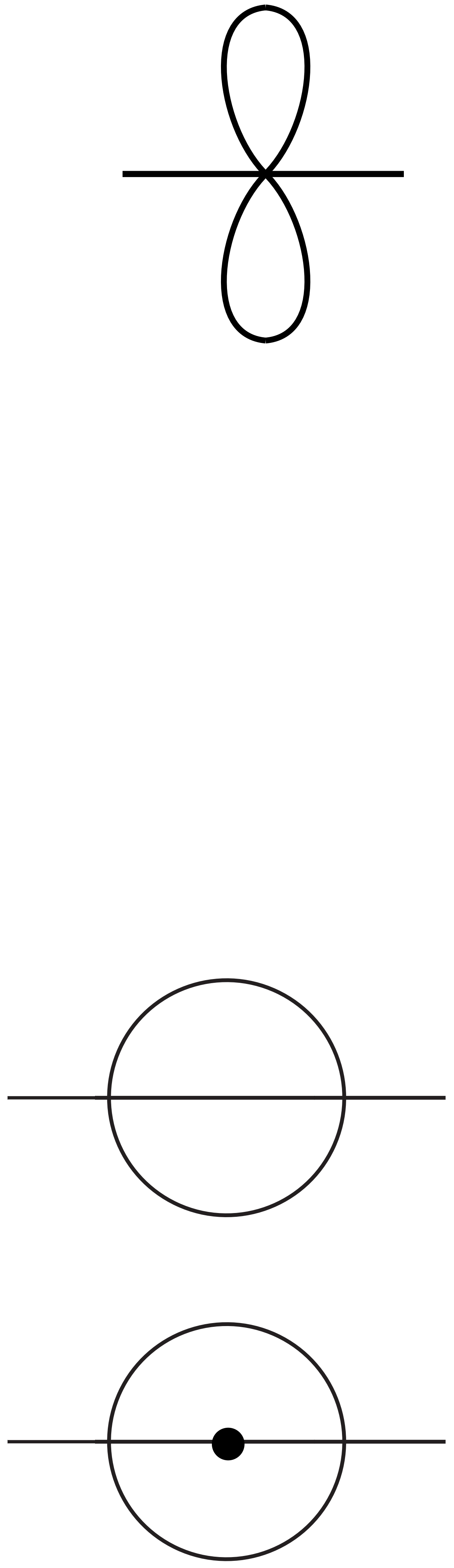


coupled $n \times n$ blocks correspond to graphs with n master integrals

True for *general values of dimensions D* !

DIFFERENTIAL EQUATIONS FOR FEYNMAN INTEGRALS





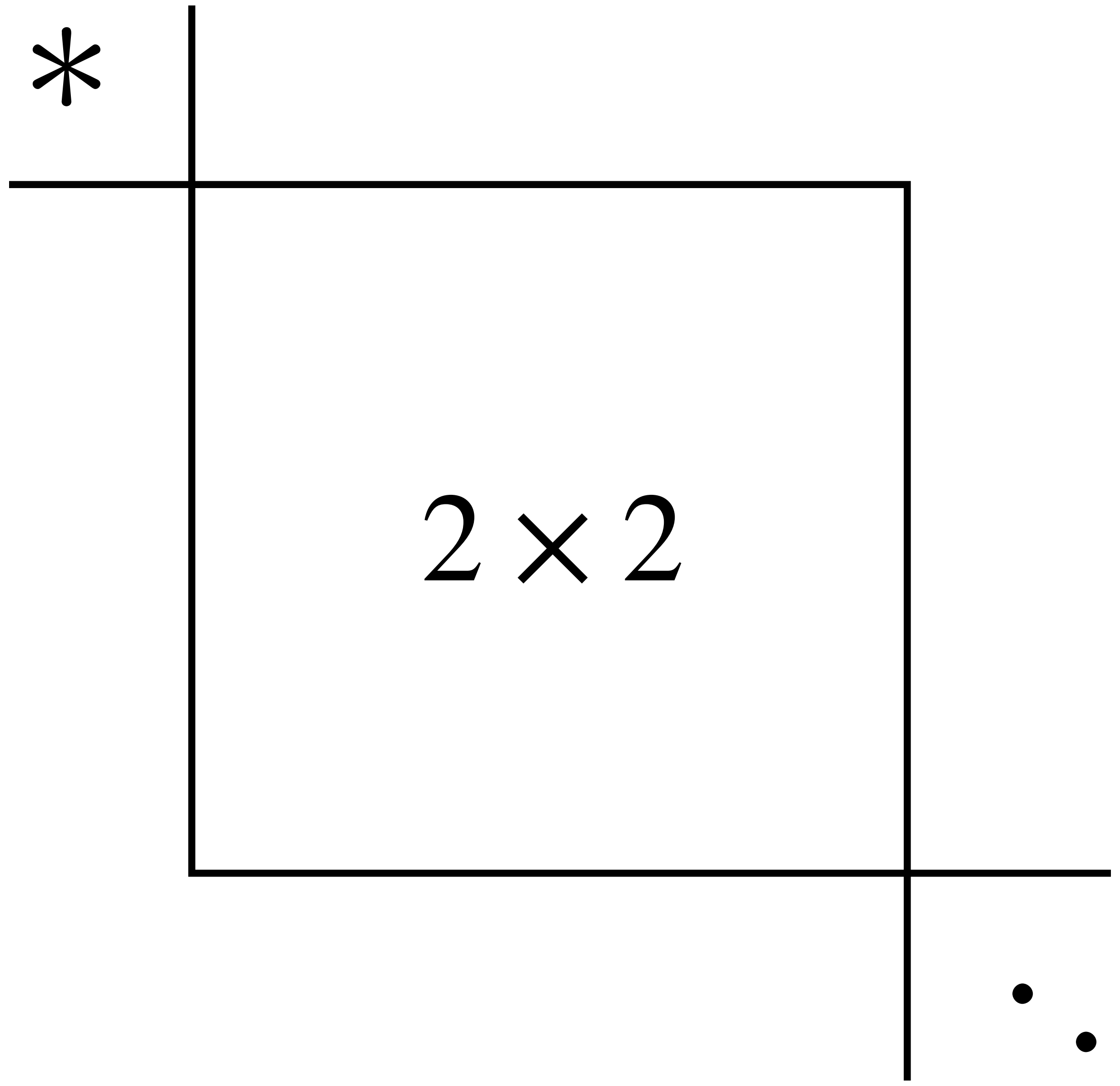
1×1 blocks correspond to single master integrals

coupled $n \times n$ blocks correspond to graphs with n master integrals

True for *general values of dimensions D* !

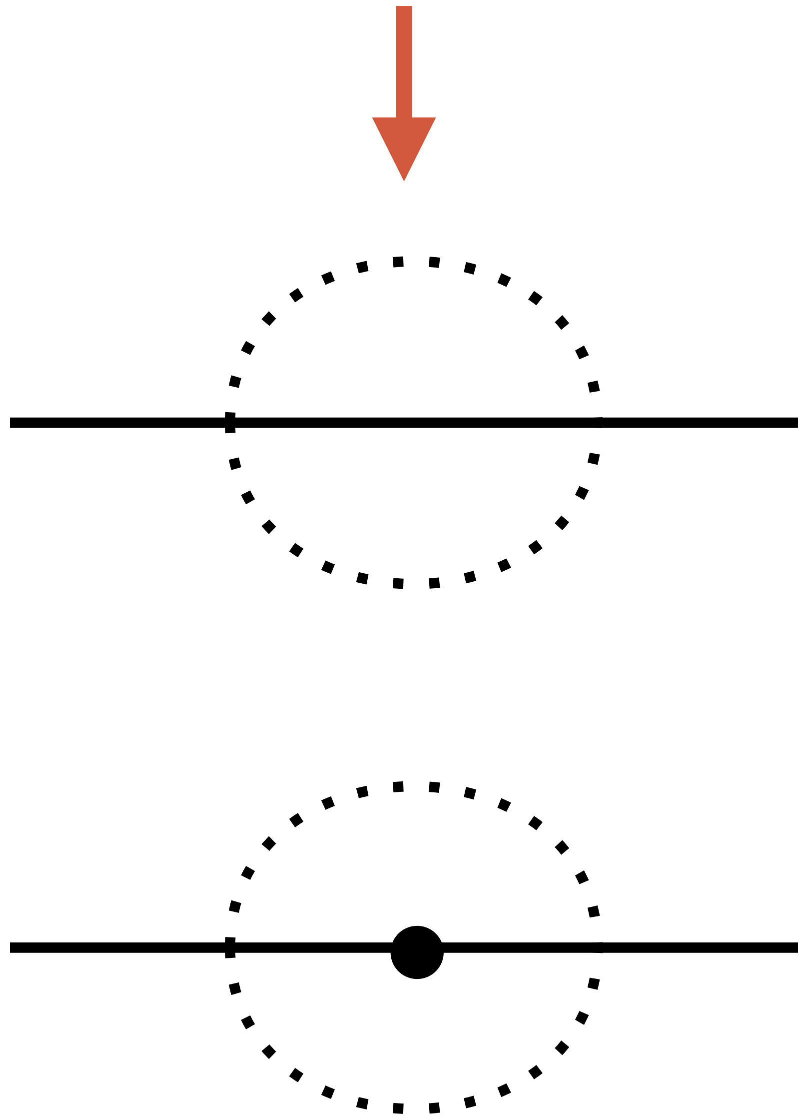
FROM DIFFERENTIAL EQUATIONS TO GEOMETRY

The story might change for $D \to 4 - 2\epsilon$



$$\to \begin{bmatrix} a_{11}(s_{ij}) & a_{12}(s_{ij}) \\ 0 & a_{22}(s_{ij}) \end{bmatrix} + \mathcal{O}(\epsilon)$$

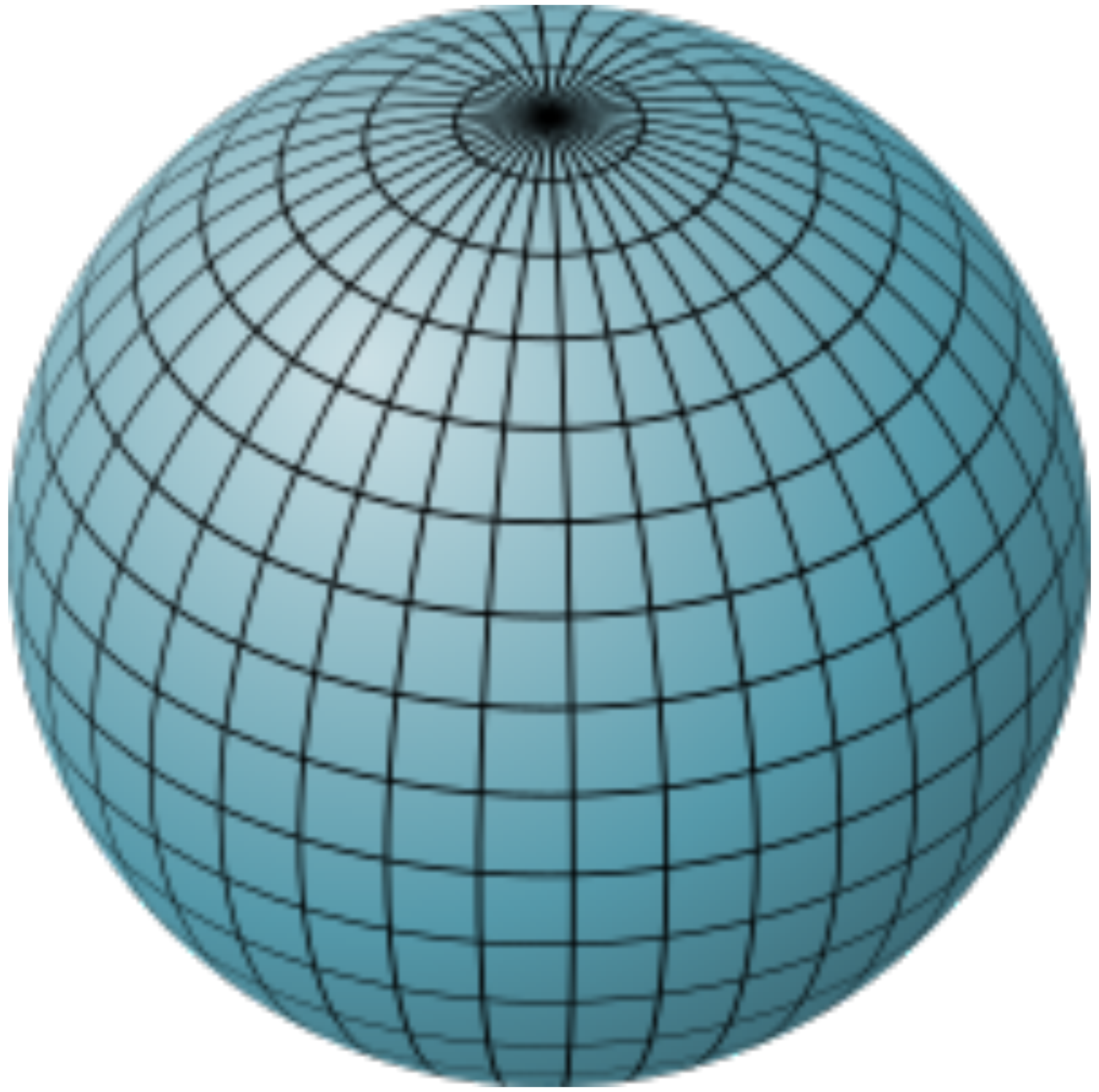
massless propagator: “photon”



Equations might “decouple” close to $D = 4$ space-time dimensions

a_{ij} are rational functions → solution written iteratively in ϵ as ***iterated integrals of rational functions!***

MULTIPLE POLYLOGS AND THE RIEMANN SPHERE

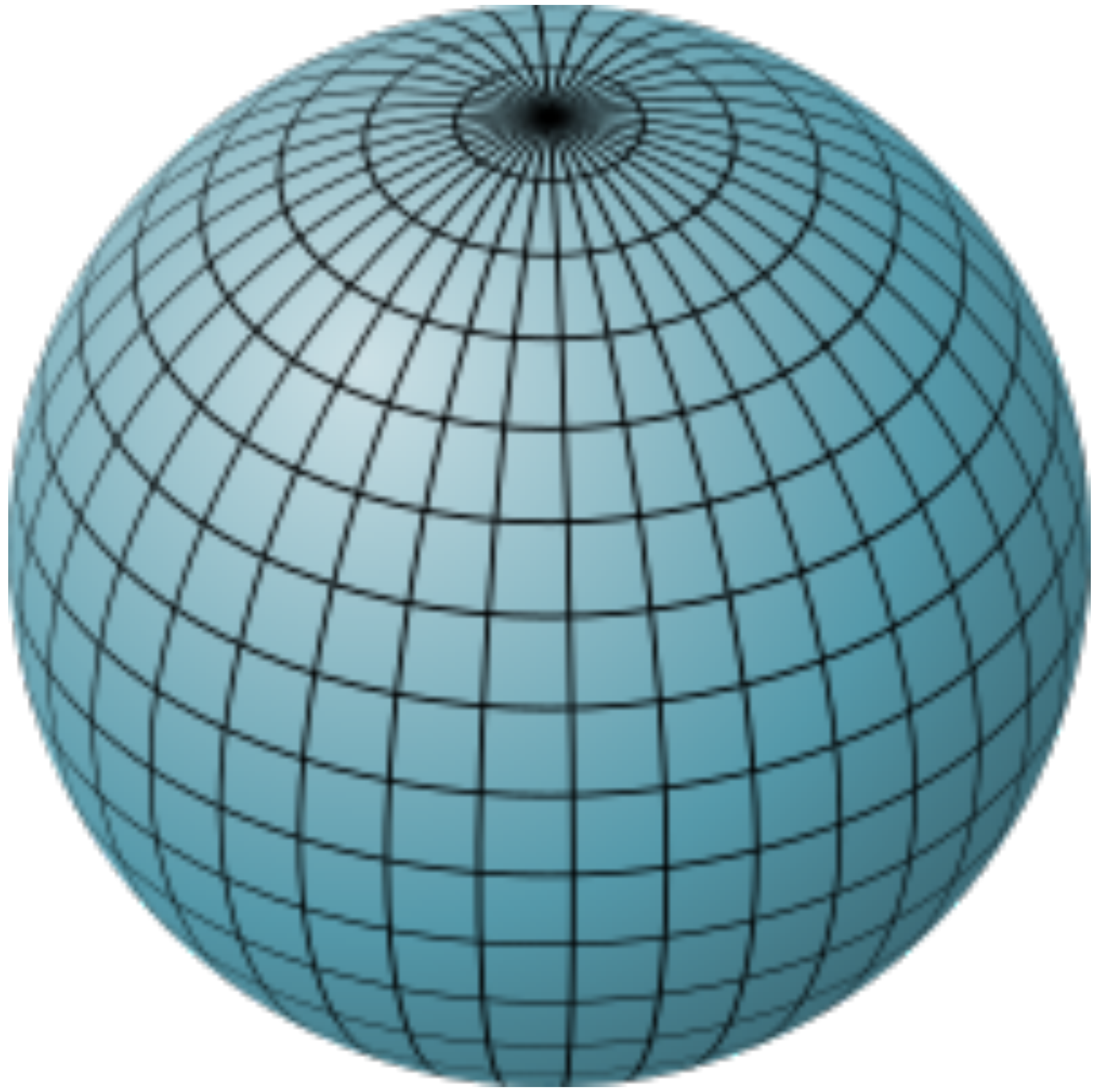


If we integrate a rational function on $\mathbb{CP}^1$

Only non-trivial thing:

$$\log(1 - x/a) = \int_0^x \frac{dt}{t-a}$$

MULTIPLE POLYLOGS AND THE RIEMANN SPHERE



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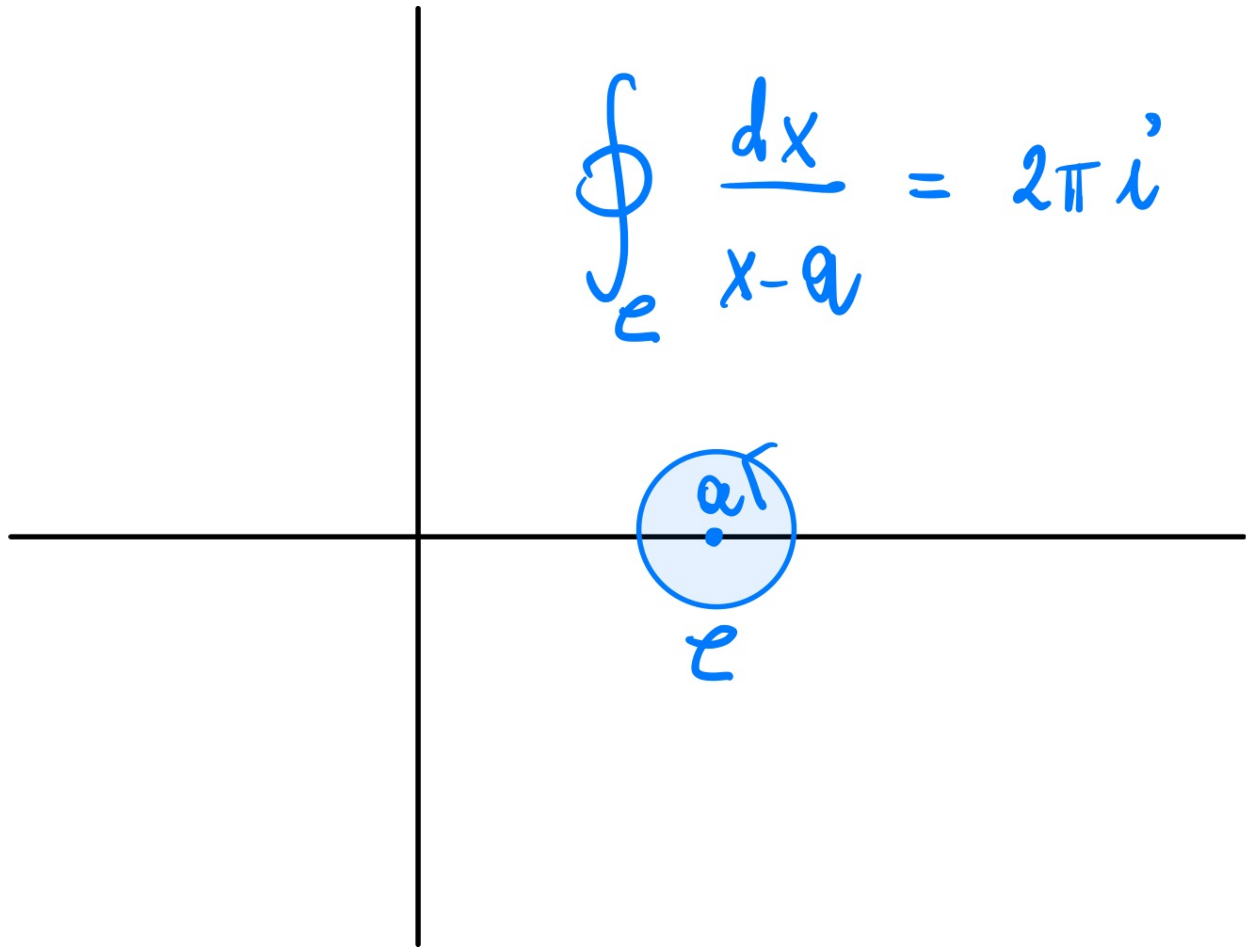

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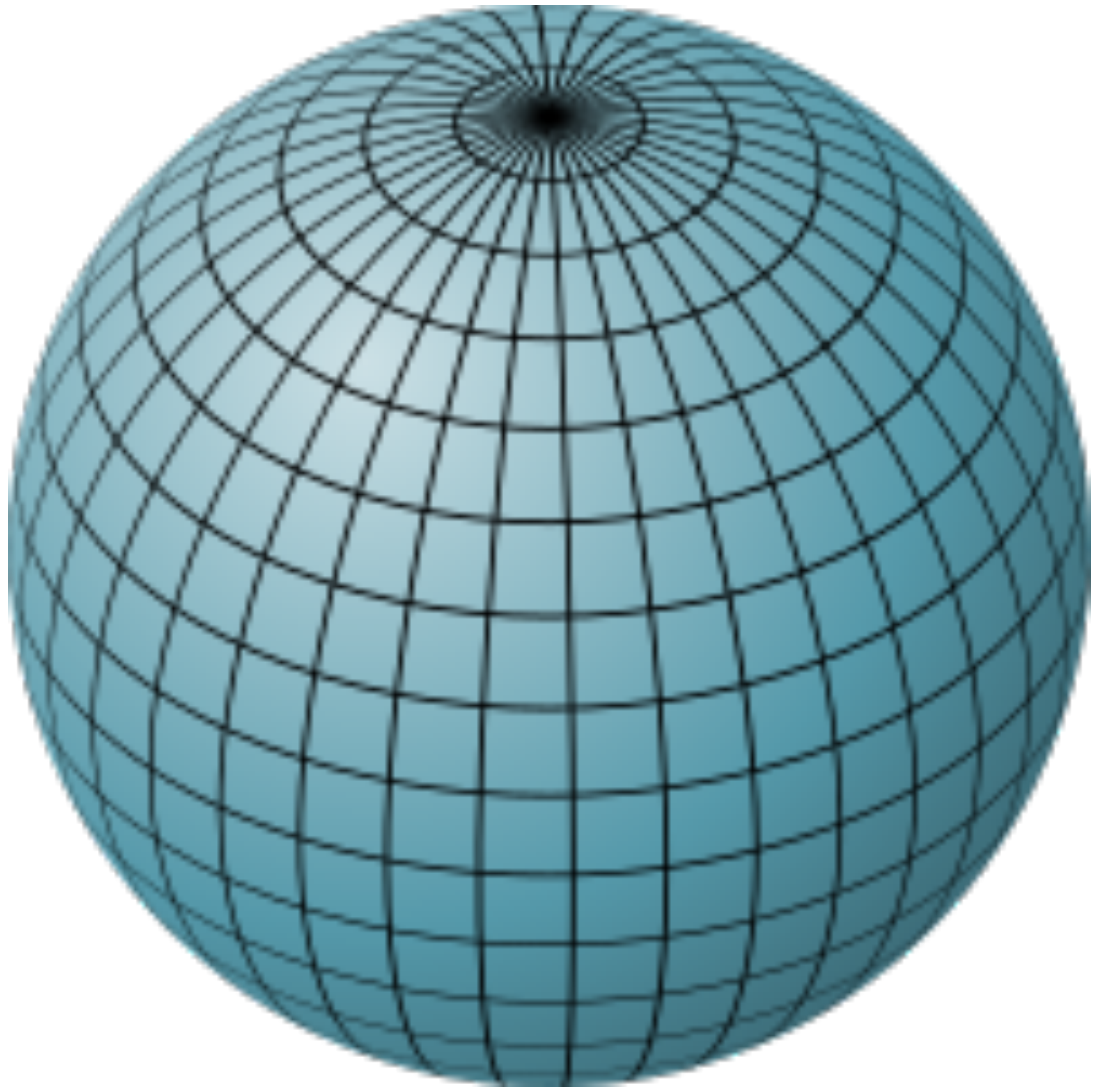

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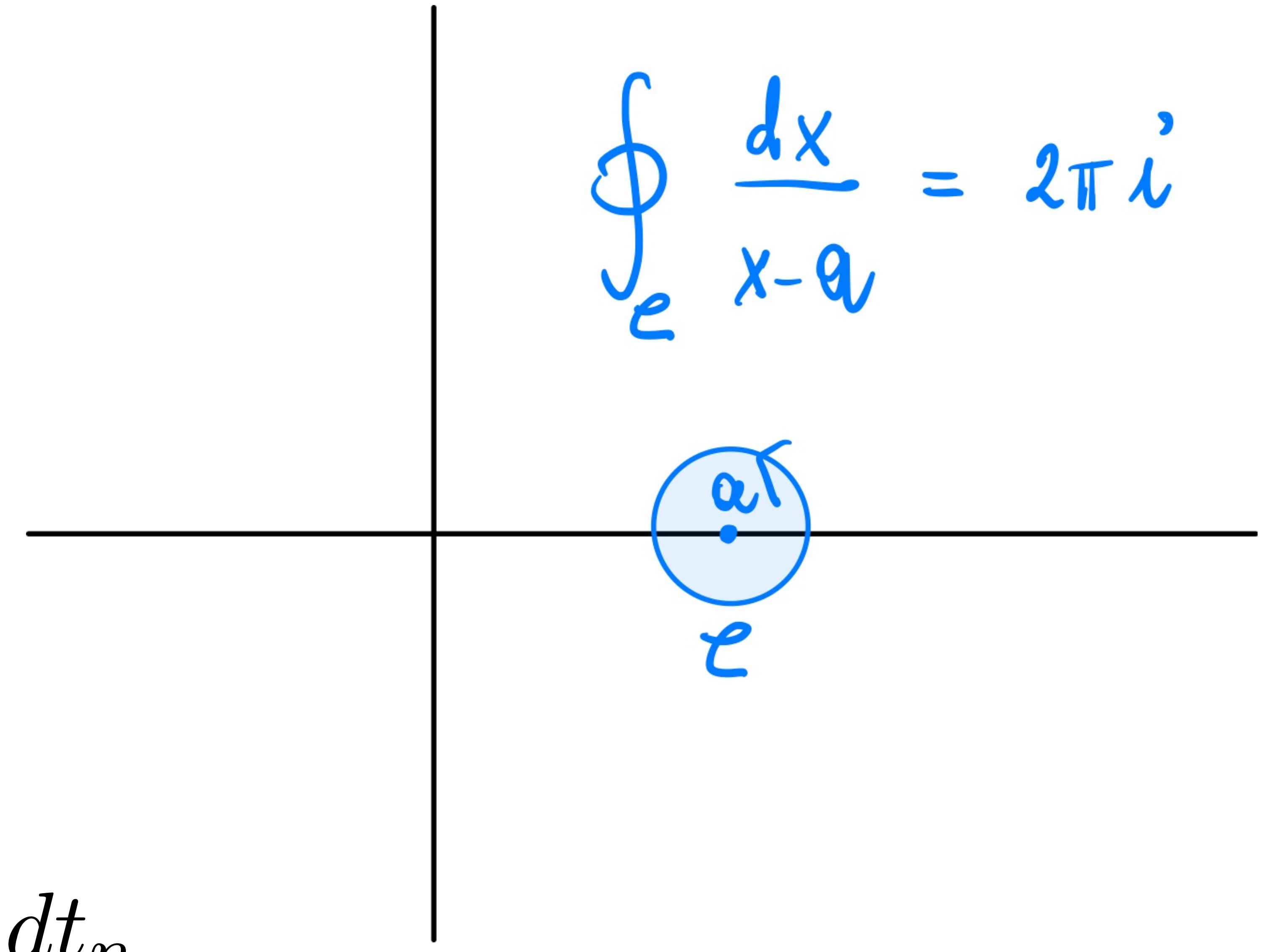


Generalisation: **Multiple PolyLogarithms (MPLs)**

$$G(c_1, c_2, \ldots, c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, \ldots, c_n, t_1)$$

$$= \int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} \ldots \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n}$$

POLYLOGARITHMS, DLOG FORMS AND AMPLITUDES

$$\mathscr{A} \longrightarrow \sum_i R_i(s_{ij}) \int_\gamma d\log f_n \wedge \ldots \wedge d\log f_1$$

Rational functions:

encode poles (***single particles going on shell***)

Generalization of MPLs:

iterated integrals over d-log forms

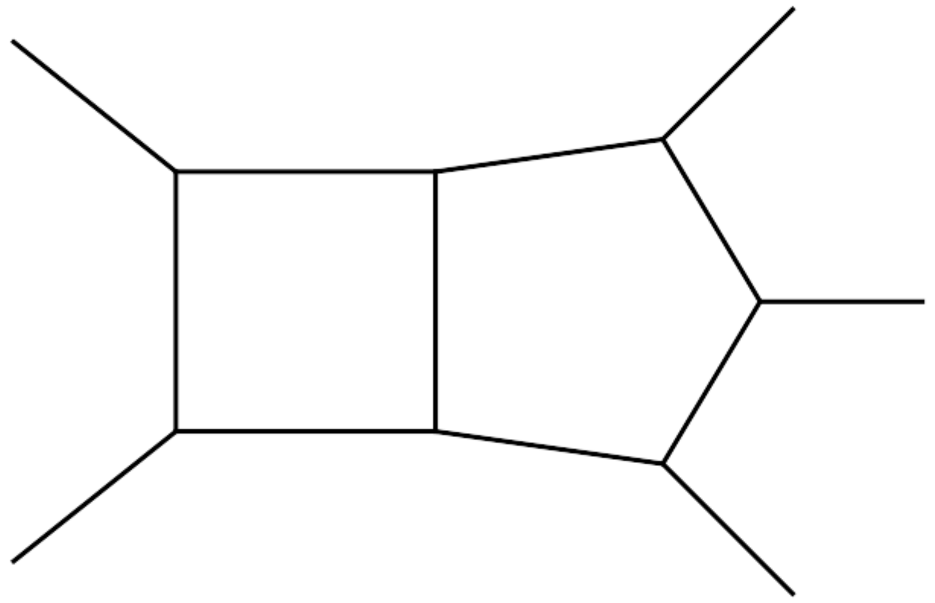
All information on branch cuts → unitarity !

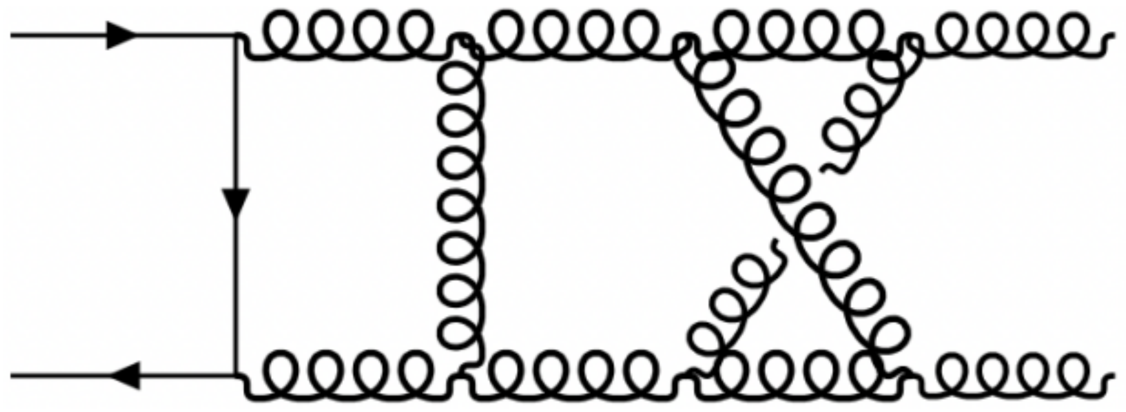
Enormously simplify computation of all "polylogarithmic" amplitudes

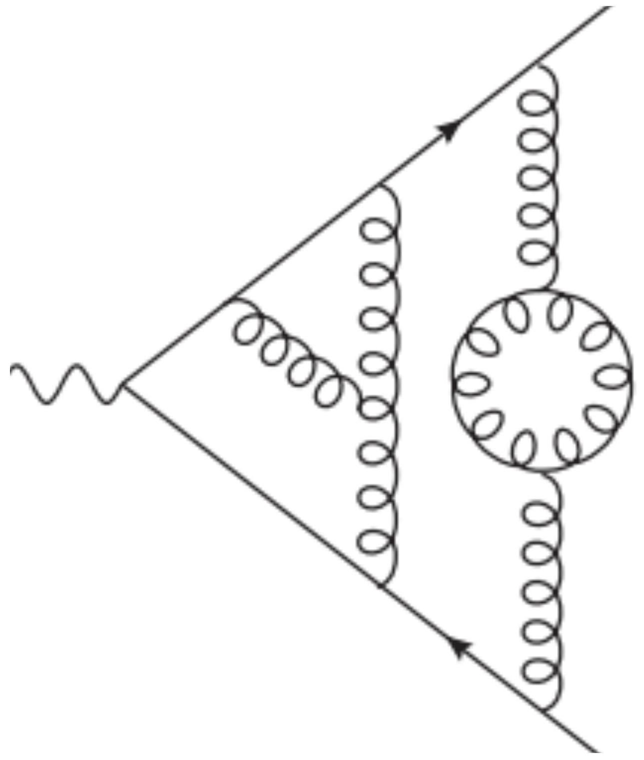
→ Bootstrap program!

POLYLOGARITHMS, DLOG FORMS AND AMPLITUDES

$$\mathcal{A} \longrightarrow \sum_i R_i(s_{ij}) \int_\gamma \mathrm{d}\log f_n \wedge \ldots \wedge \mathrm{d}\log f_1$$







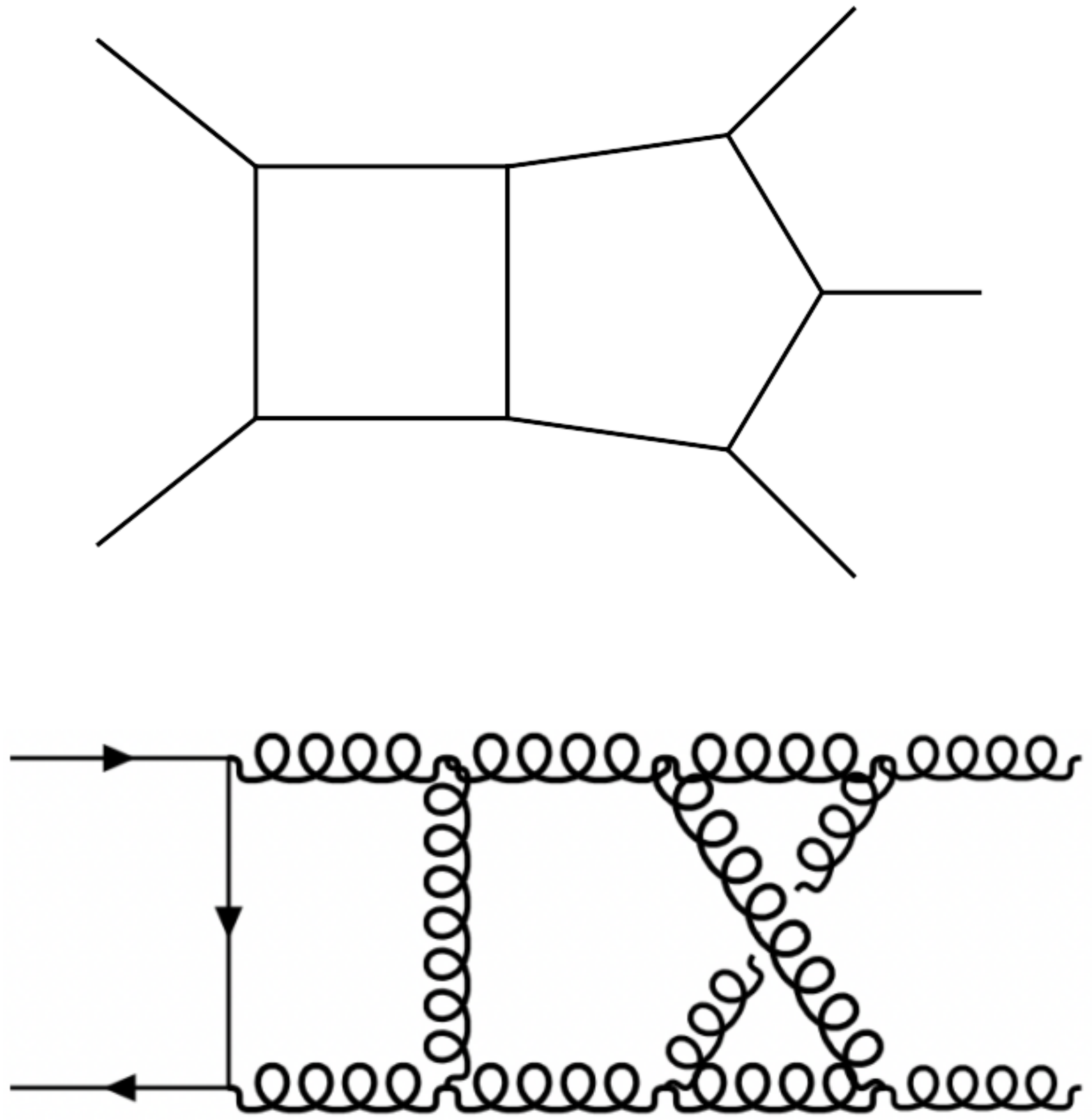
Collider physics profited enormously from these developments:

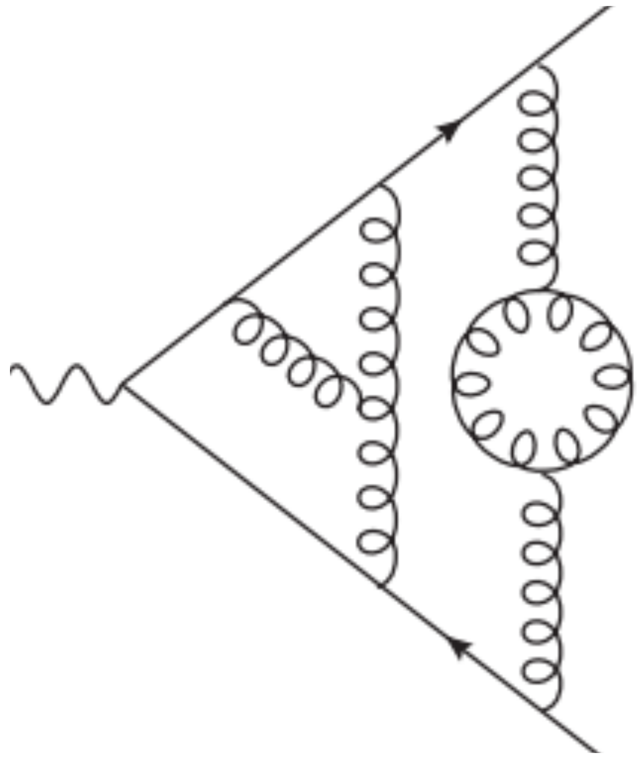
calculations for the production of 1,2,3 massless particles up to four, three and two loops respectively

→ modeling of **QCD dynamics** (production of jets at LHC)

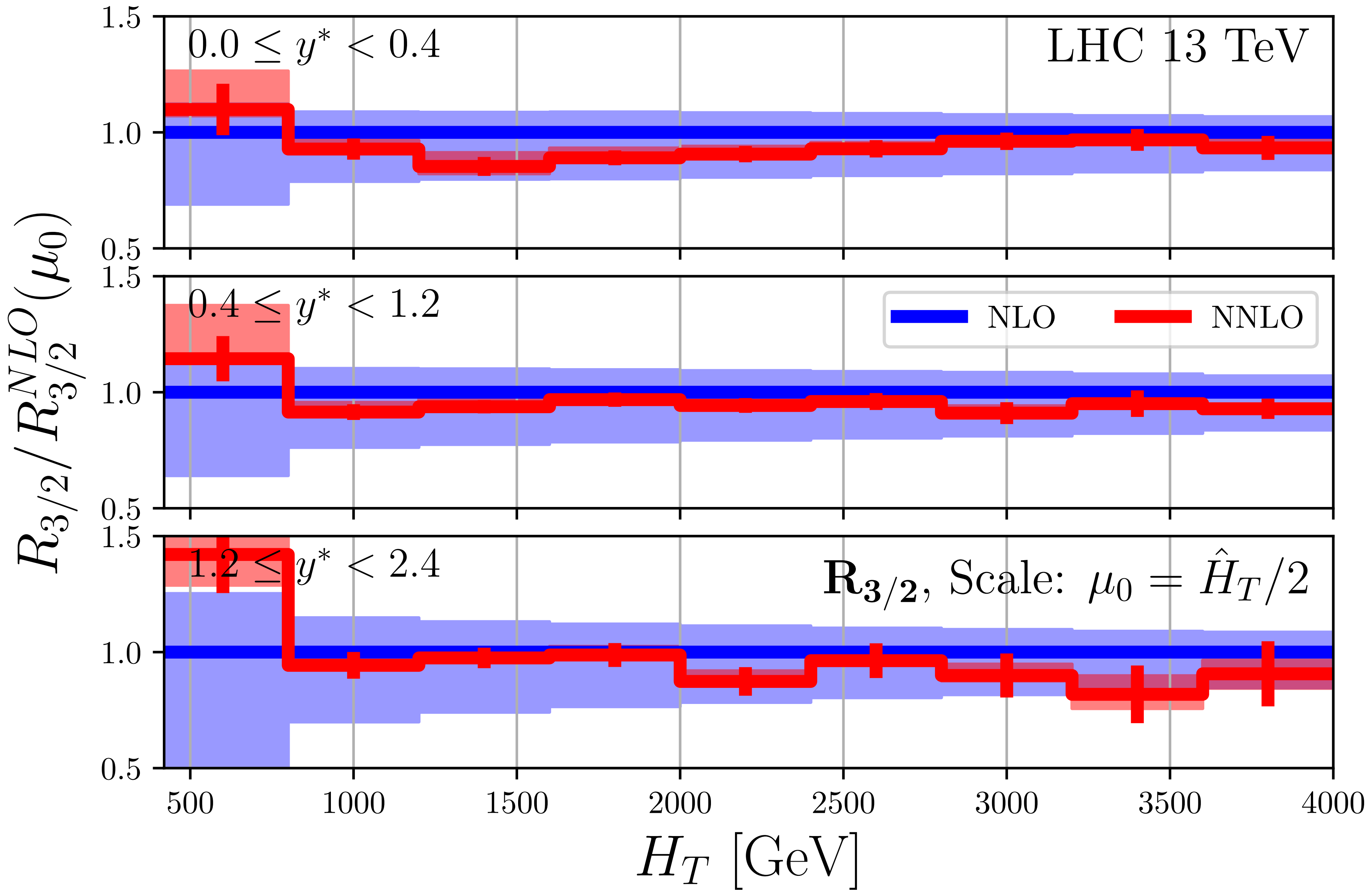
POLYLOGARITHMS, DLOG FORMS AND AMPLITUDES

$$\mathscr{A} \longrightarrow \sum_i R_i(s_{ij}) \int_\gamma \mathrm{d}\log f_n \wedge \ldots \wedge \mathrm{d}\log f_1$$





3-jet production in **NNLO QCD**



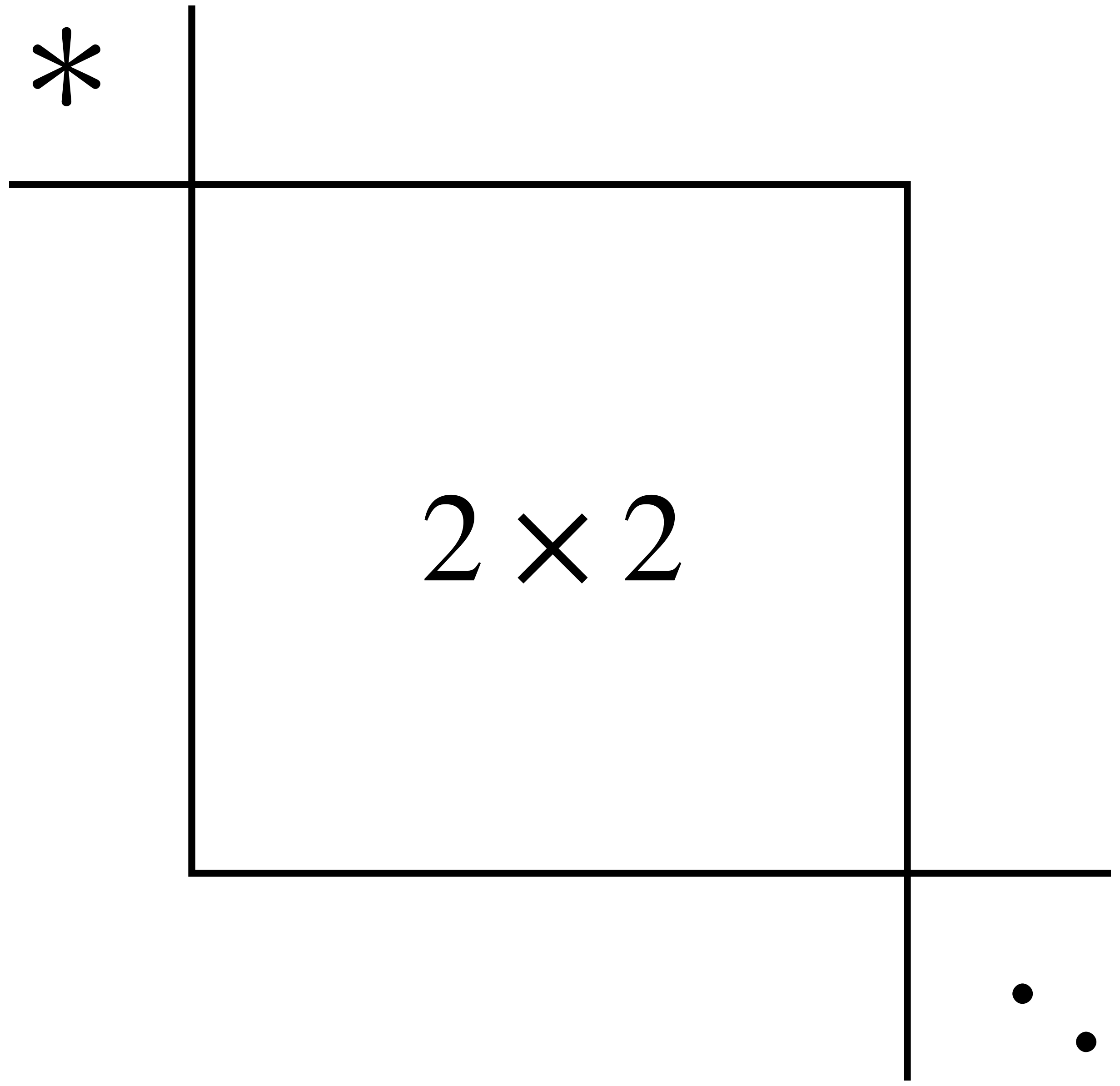
[Czakon, Mitov, Poncelet ’22, ’23]

HOW GENERAL IS THIS PICTURE?

FROM DIFFERENTIAL EQUATIONS TO GEOMETRY

Again, look at limit $D \to 4 - 2\epsilon$

all massive propagators



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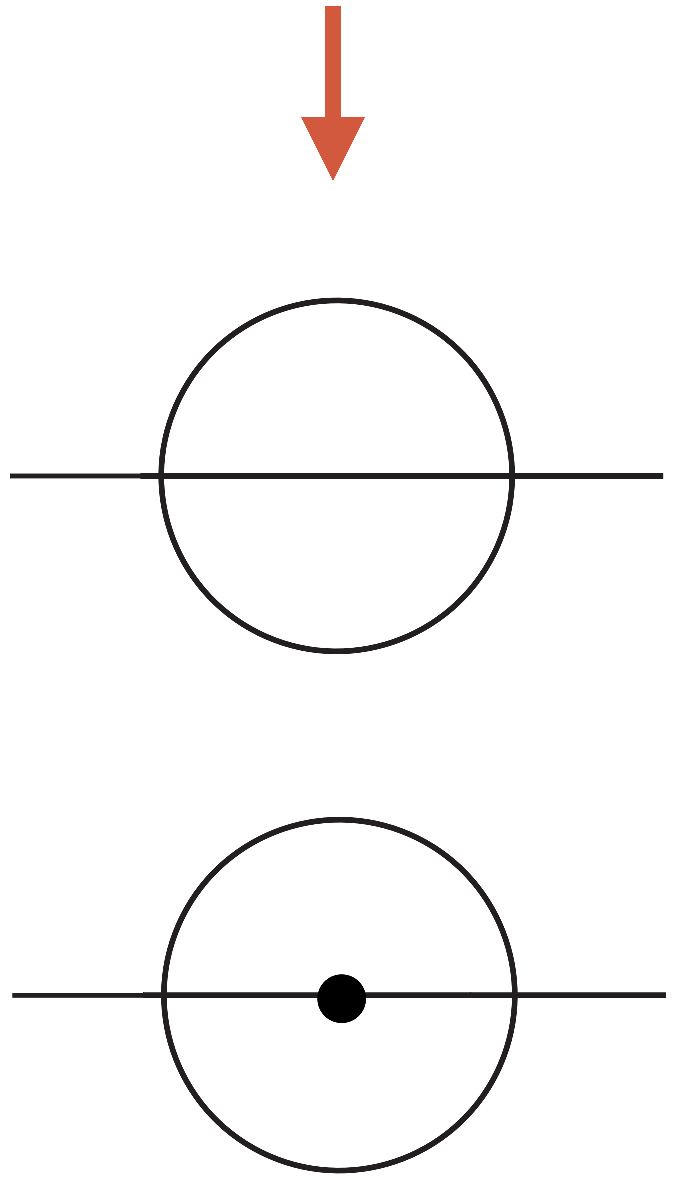

If equation does not decouple, there is an intrinsic “higher-order equation” *(2nd order Picard-Fuchs equation)*

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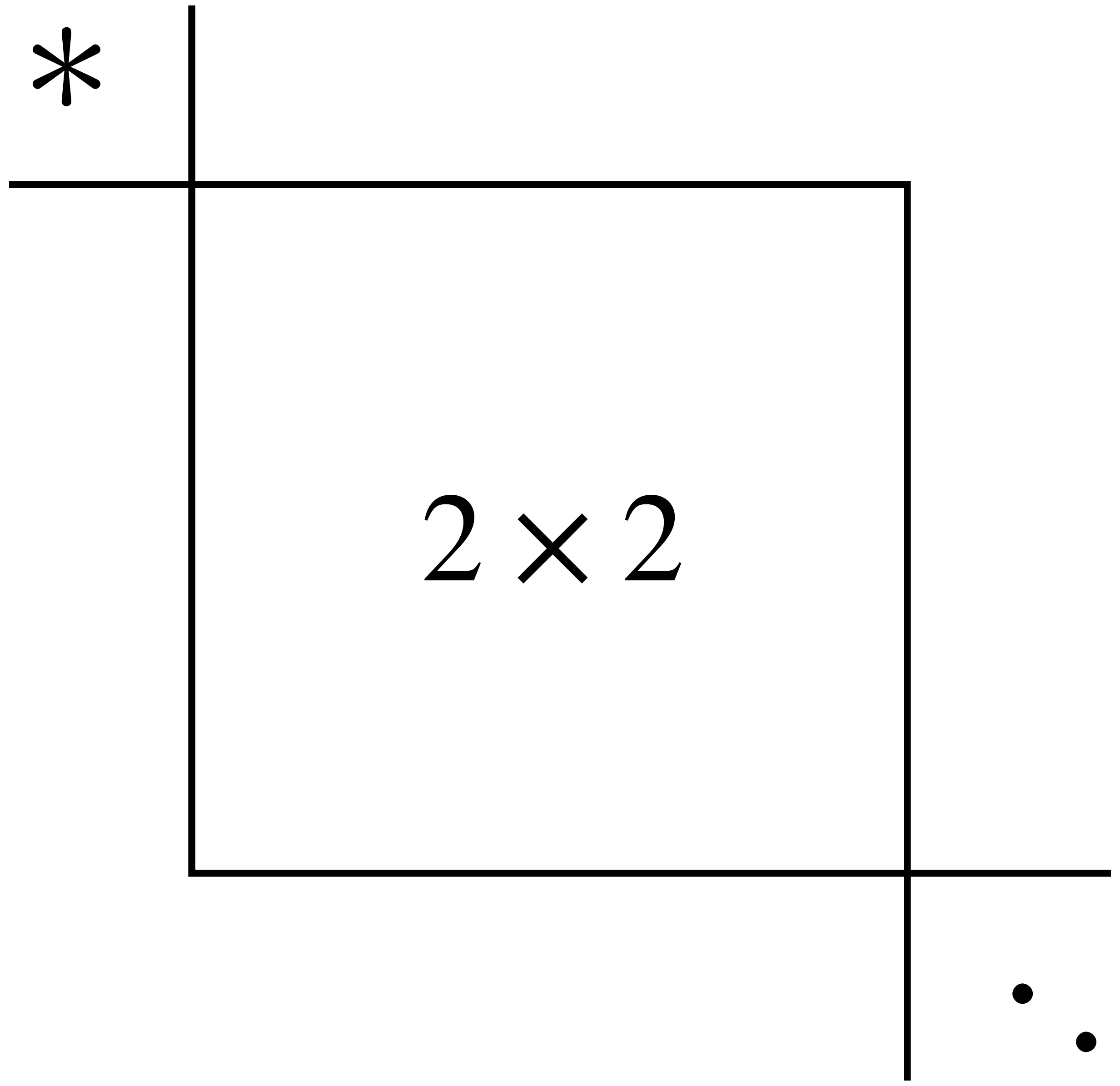
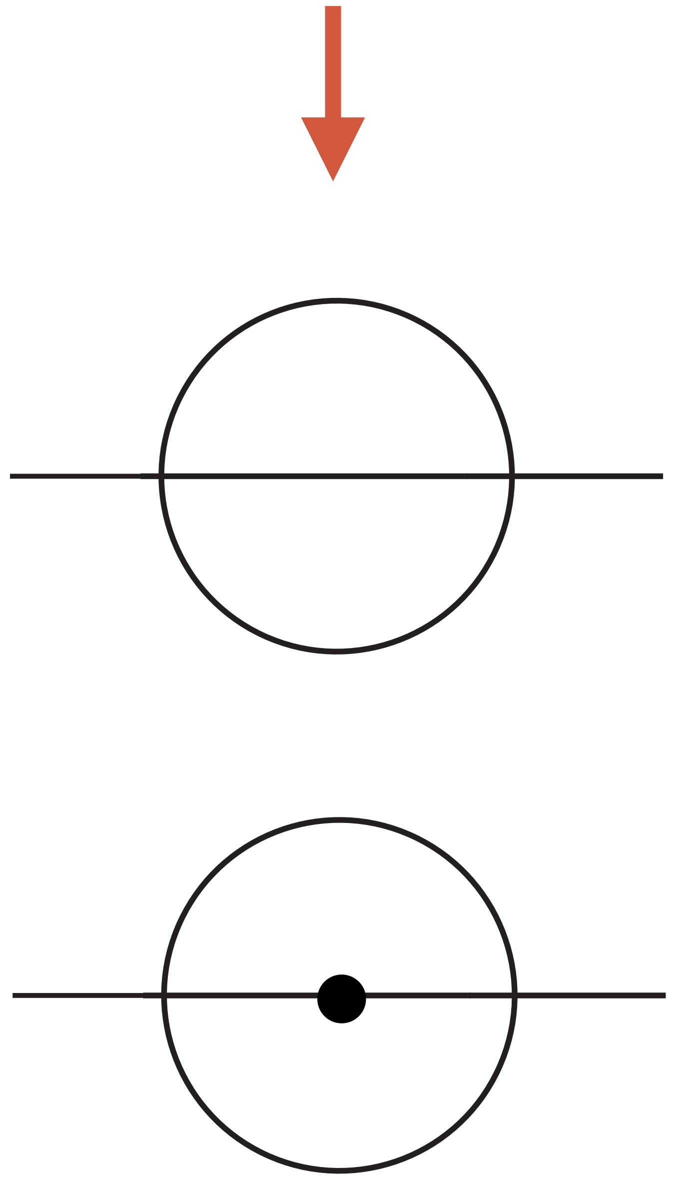


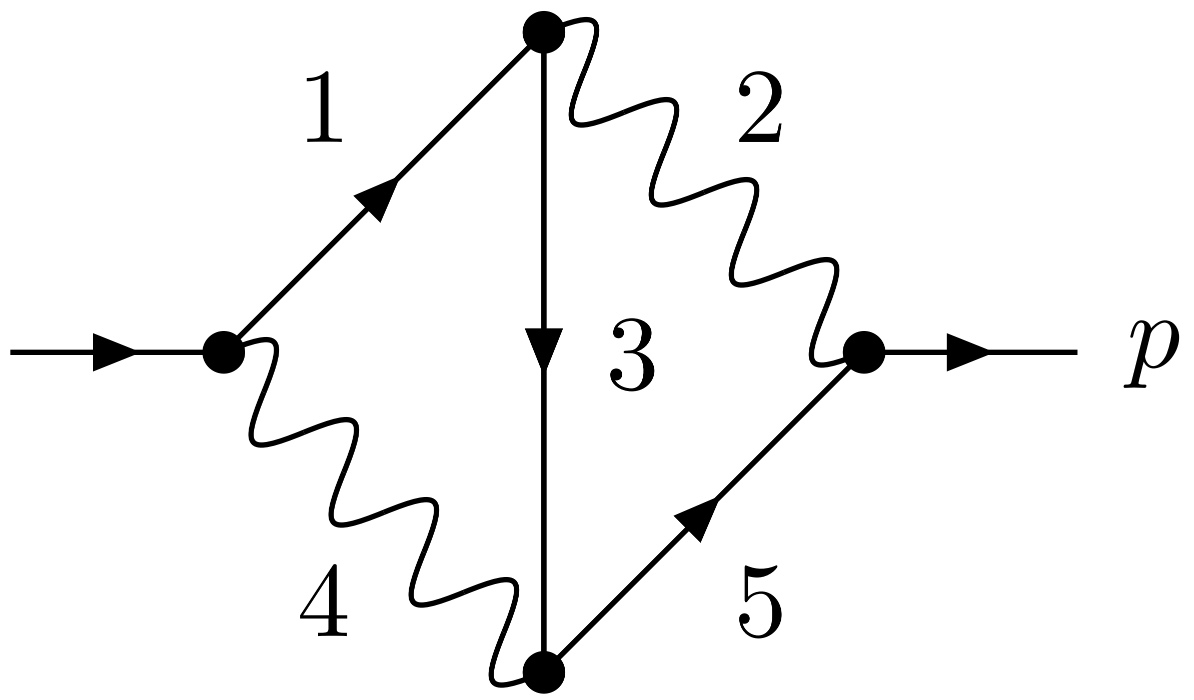
$$\rightarrow \begin{bmatrix} a_{11}(s_{ij}) & a_{12}(s_{ij}) \\ a_{21}(s_{ij}) & a_{22}(s_{ij}) \end{bmatrix} + \mathcal{O}(\epsilon)$$



If equation does not decouple, there is an intrinsic "higher-order equation" *(2nd order Picard-Fuchs equation)*

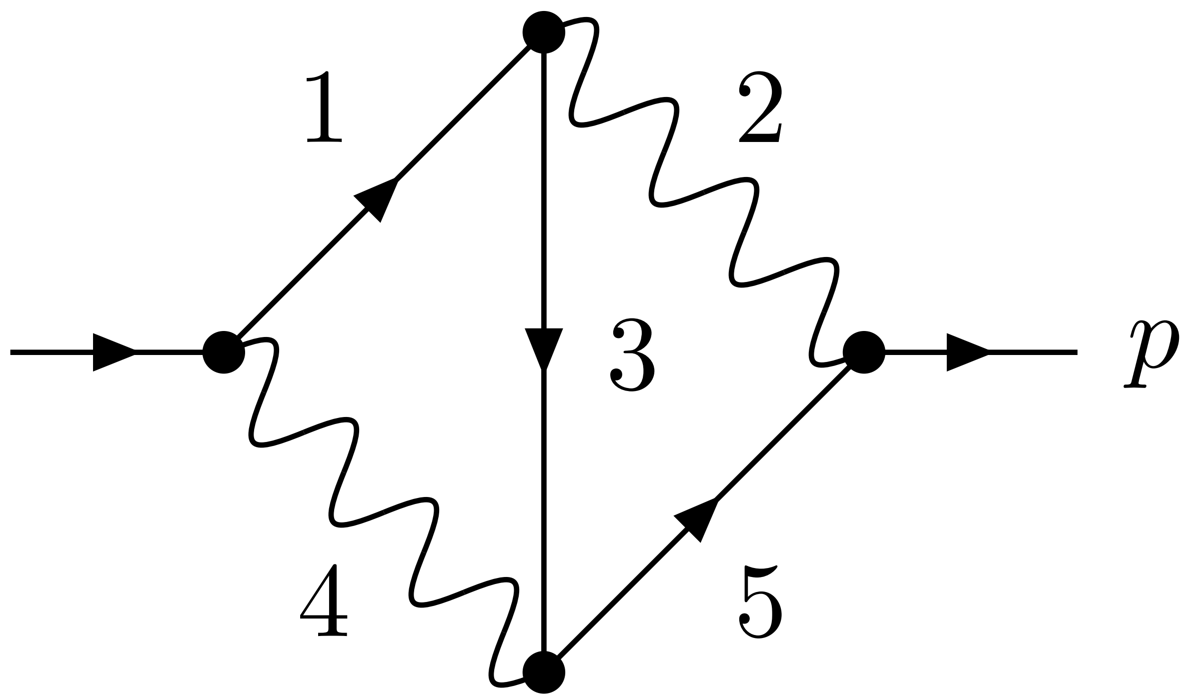
$$x = \frac{p^2}{m^2} \qquad \left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \left(\frac{1}{x-1} + \frac{9}{x-9} + 2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) + \frac{27}{4(x-9)} + \frac{1}{4(x-1)} + 1\right]\varpi(x) = 0$$

THE TWO-LOOP ELECTRON PROPAGATOR



The electron propagator in QED; A. Sabri 1962

THE TWO-LOOP ELECTRON PROPAGATOR



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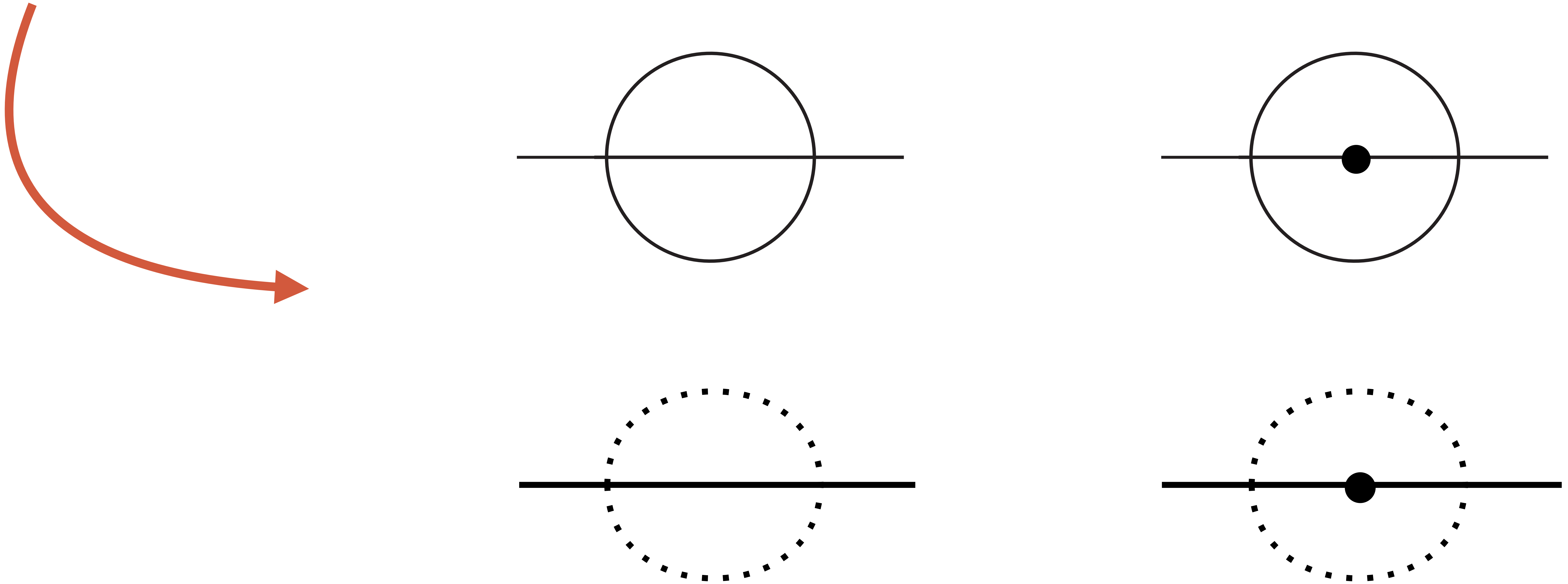
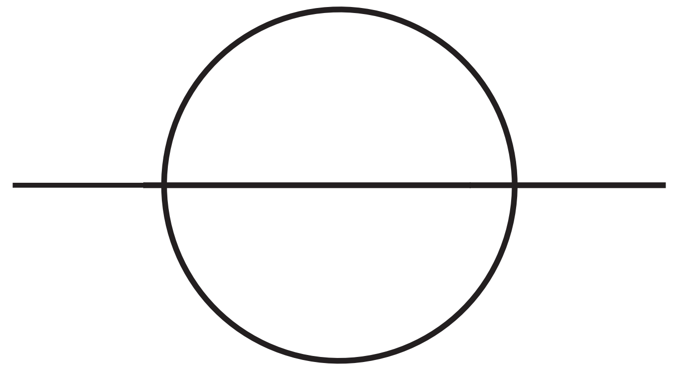
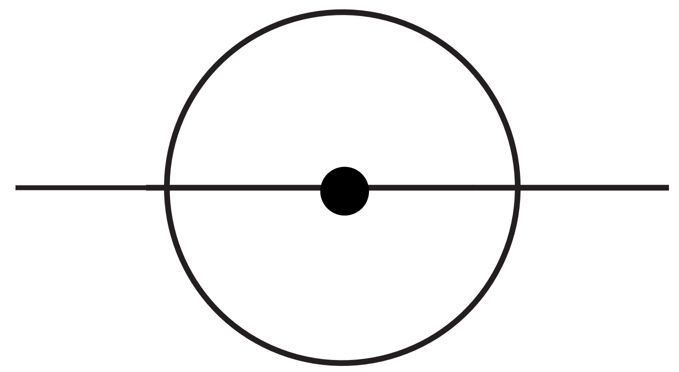


FROM DIFFERENTIAL EQUATIONS TO GEOMETRY

$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2+\left(\frac{1}{x-1}+\frac{9}{x-9}+2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)+\frac{27}{4(x-9)}+\frac{1}{4(x-1)}+1\right]\varpi(x)=0$$



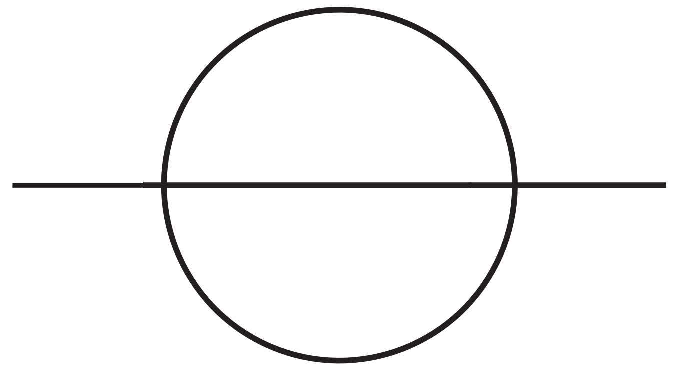


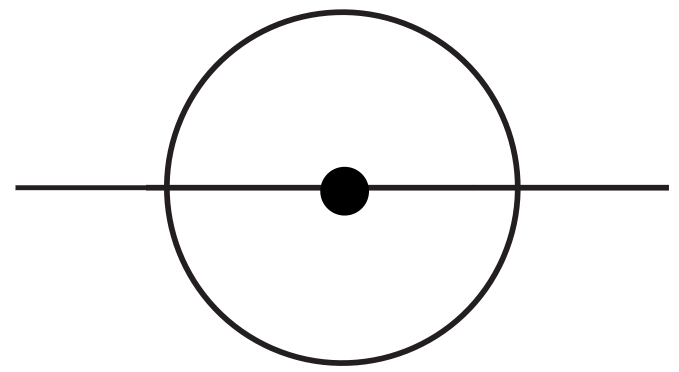
Solutions: **periods of an elliptic curve.** Obvious?

In some cases, you might be lucky enough to find the diff equation in some list of known ones…

FROM DIFFERENTIAL EQUATIONS TO GEOMETRY

$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2+\left(\frac{1}{x-1}+\frac{9}{x-9}+2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)+\frac{27}{4(x-9)}+\frac{1}{4(x-1)}+1\right]\varpi(x)=0$$

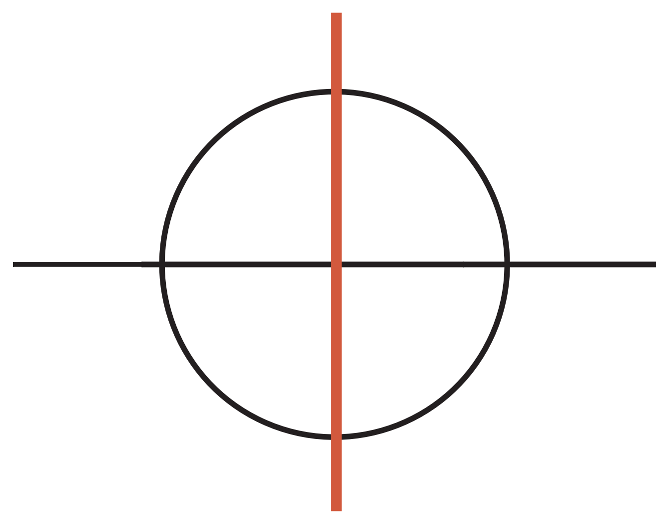




Solutions: **periods of an elliptic curve.** Obvious!!!

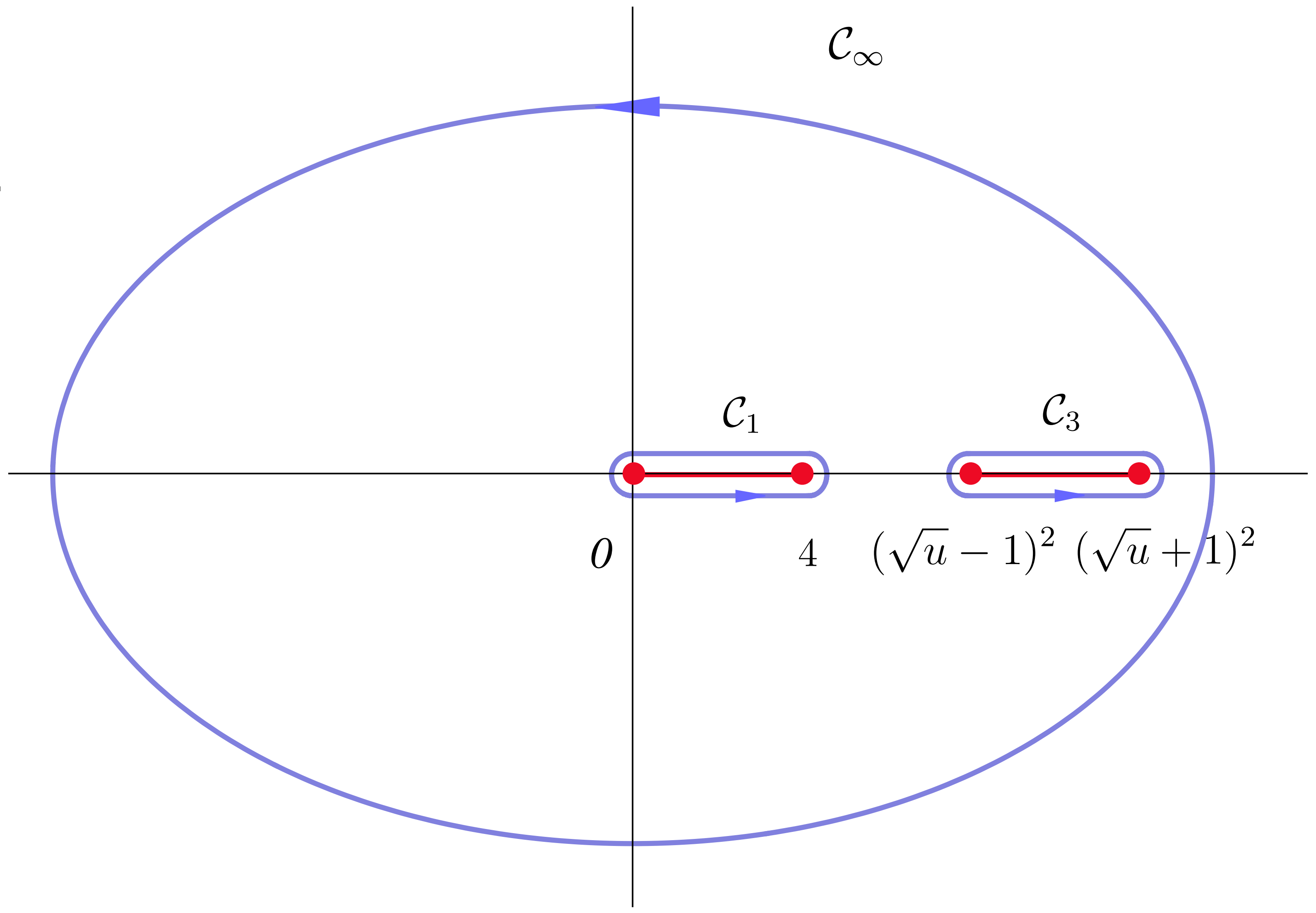
Information again from "**unitarity**" → study what happens when virtual particles go on shell!

[Laporta, Remiddi '04; Primo, Tancredi '16,17]



$$=\int_{4m^2}^{(\sqrt{s}-m)^2}\frac{dt}{Y}\qquad Y=\sqrt{t(t-4m^2)(t-(\sqrt{s}-m)^2)(t-(\sqrt{s}+m)^2)}$$

$$=\frac{1}{\sqrt{(3m-\sqrt{s})(\sqrt{s}+m)^3}}\,\mathrm{K}\left(\frac{16m^3\sqrt{s}}{(3m-\sqrt{s})(\sqrt{s}+m)^3}\right)$$



ELLIPTIC CURVES AND COMPLEX TORI

Elliptic curve given by an algebraic equation $y = \pm\sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}$

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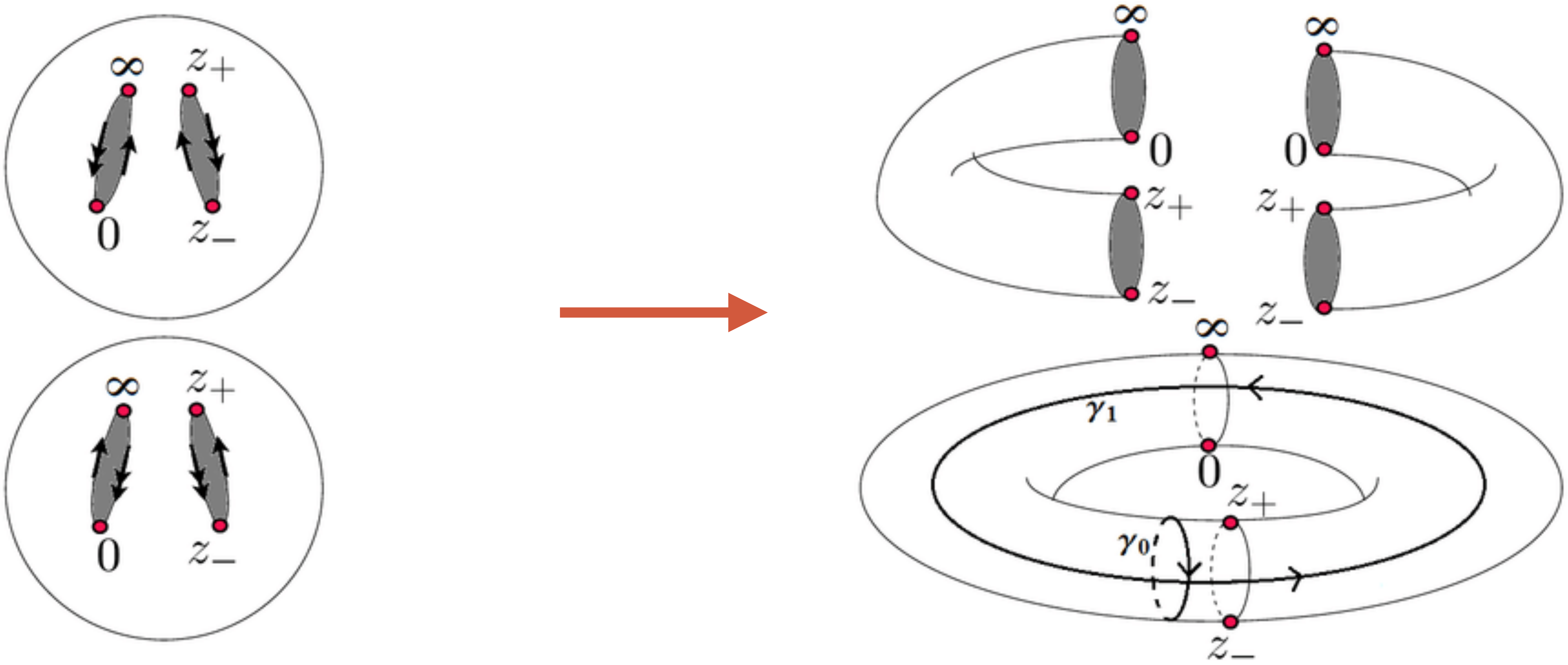
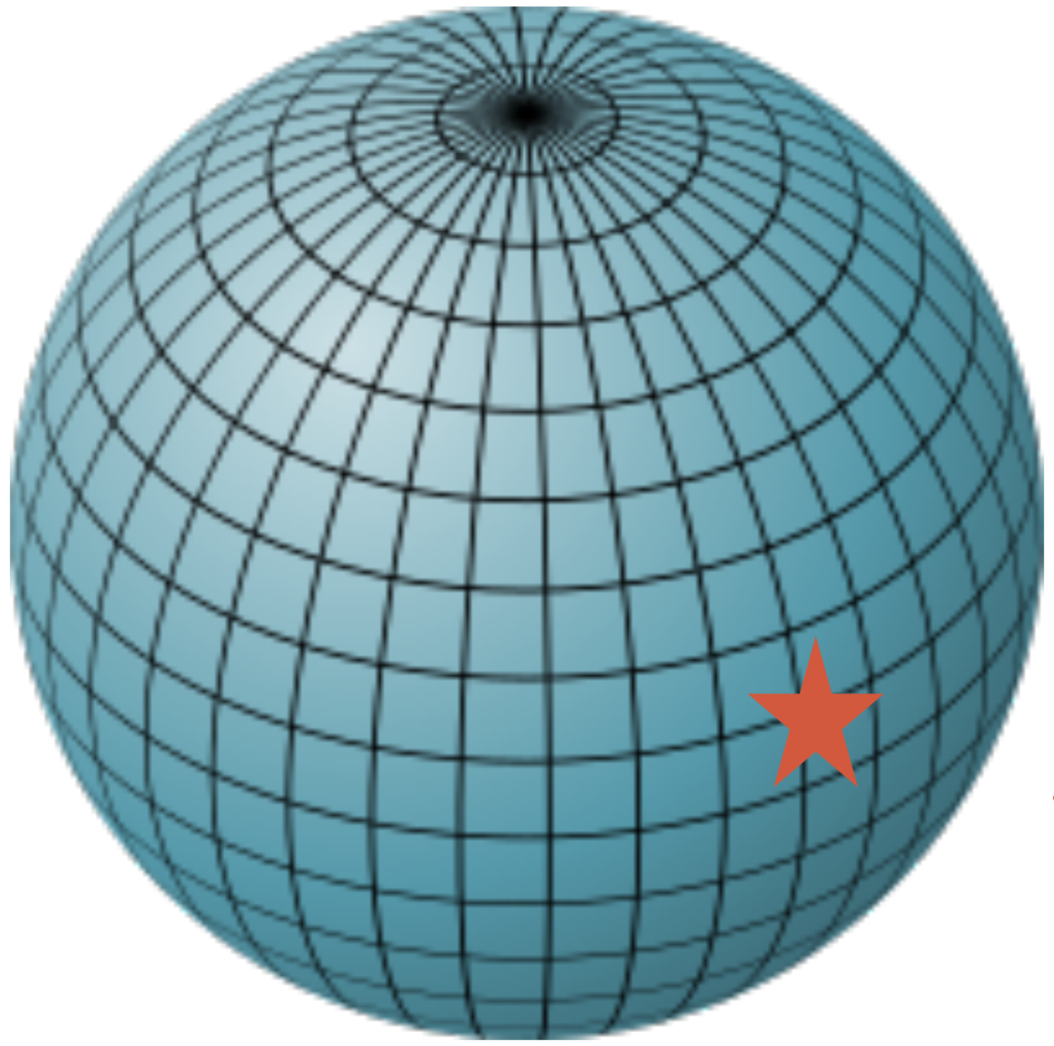
Torus is the **Riemann surface** associated to the square root with 4 branching points



DIFFERENTIAL FORMS ON ELLIPTIC GEOMETRIES

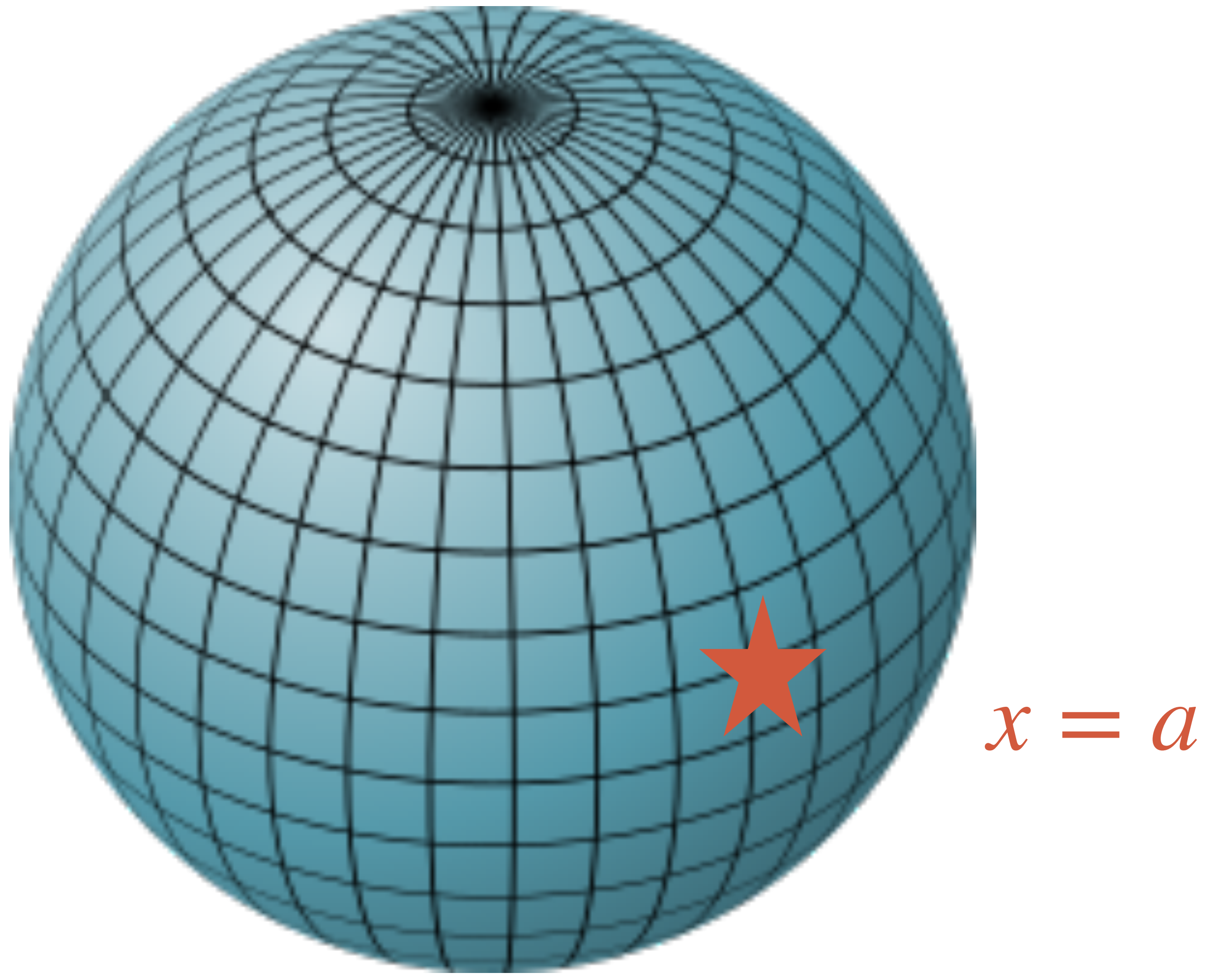


$x = a$

entire space of functions spanned by single poles

$$\log(1 - x/a) = \int_0^x \frac{dt}{t - a}$$

DIFFERENTIAL FORMS ON ELLIPTIC GEOMETRIES



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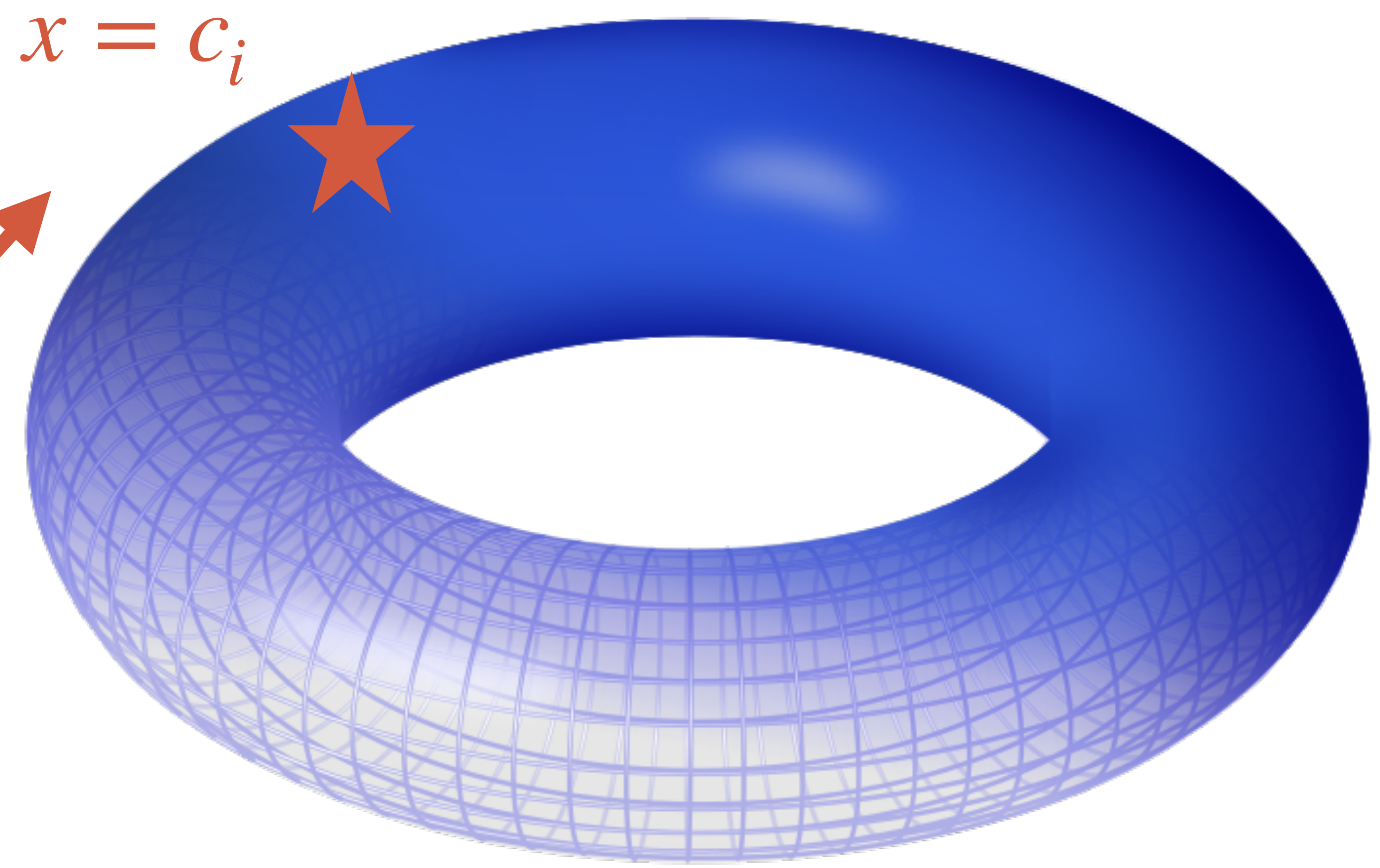
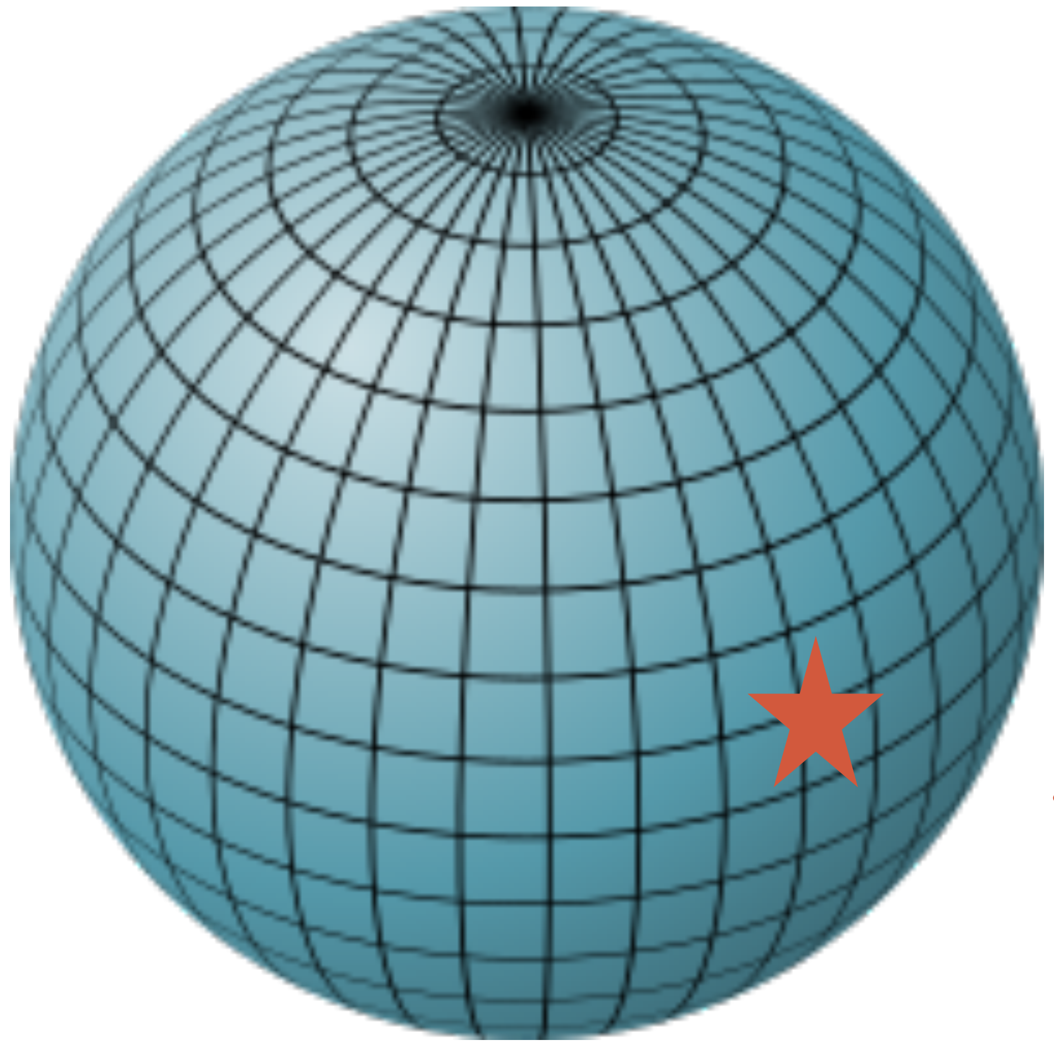


genus 1, elliptic curve; $y = \sqrt{P_3(x)}$

Third kind

single poles $g \sim \int \frac{dx}{(x-c_i)y}$

DIFFERENTIAL FORMS ON ELLIPTIC GEOMETRIES



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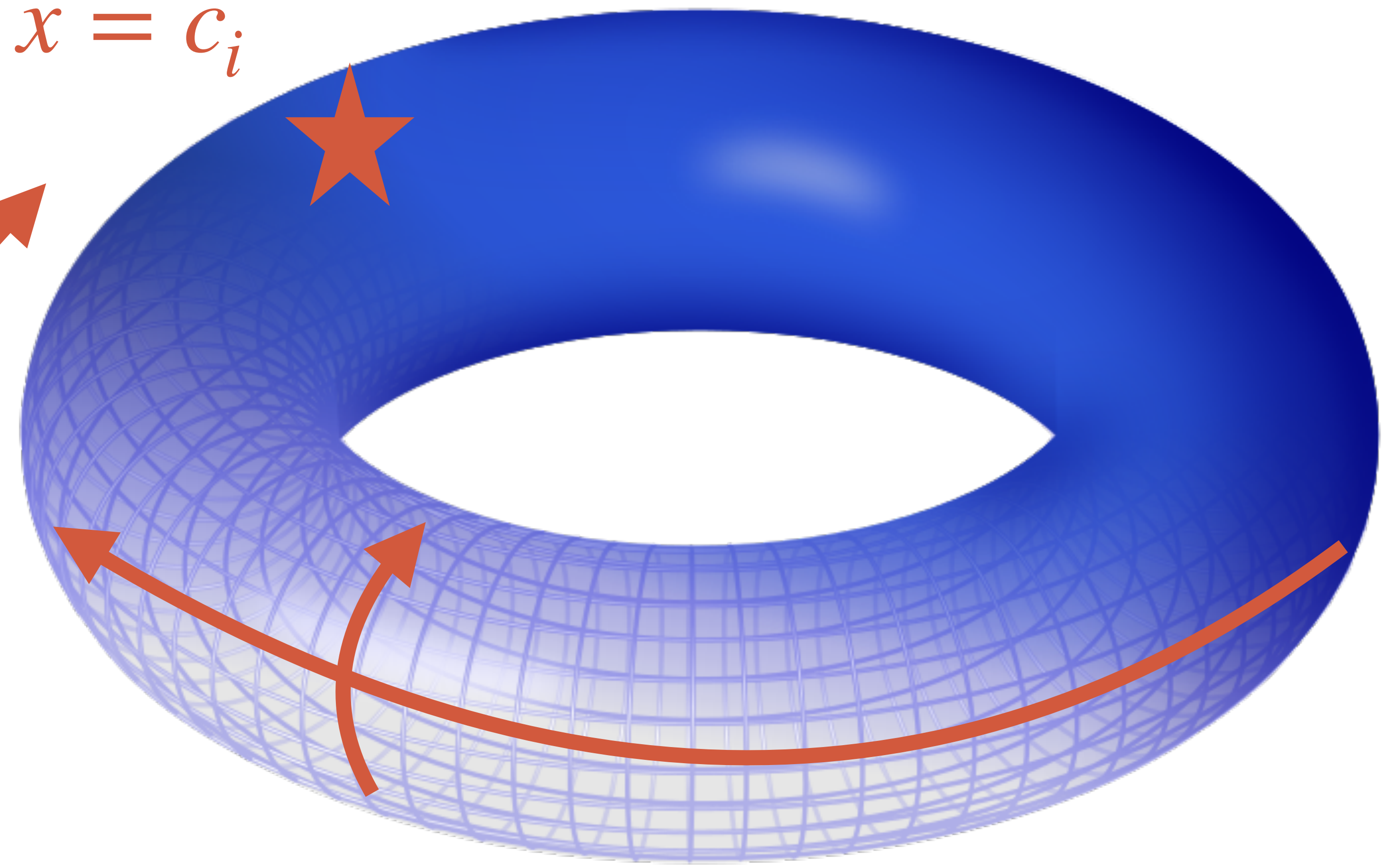


genus 1, elliptic curve; $y = \sqrt{P_3(x)}$

First kind

No poles $\omega \sim \int \frac{dx}{y}$

Second kind

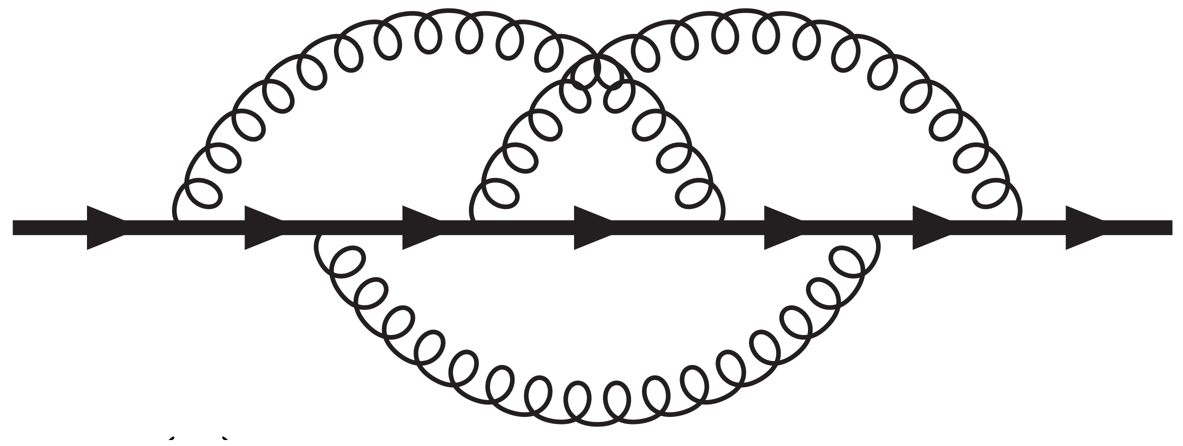
double poles $\eta \sim \int \frac{dx\, x}{y}$

Third kind

single poles $g \sim \int \frac{dx}{(x - c_i)y}$

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY

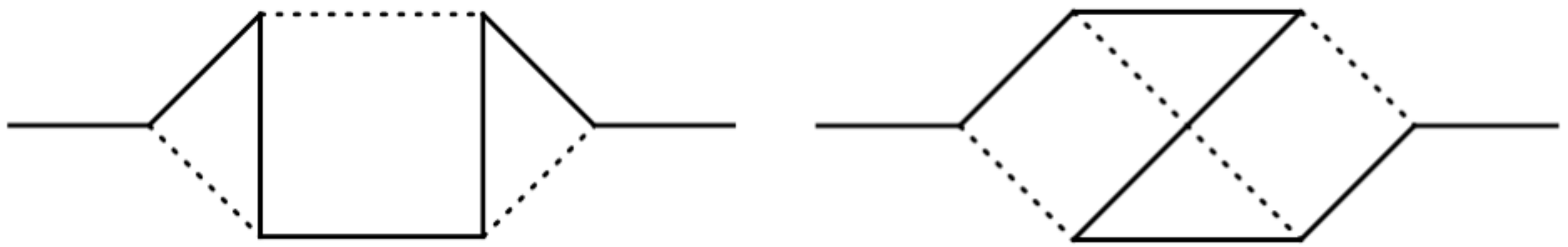
[Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]



$$\hat{p}\ \Sigma_V(p^2, m^2) + m\ \Sigma_S(p^2, m^2)$$

Σ_V & Σ_S expressed in terms of $\mathcal{O}(50)$ Masters Integrals $\vec{J}$

2 "top graphs"



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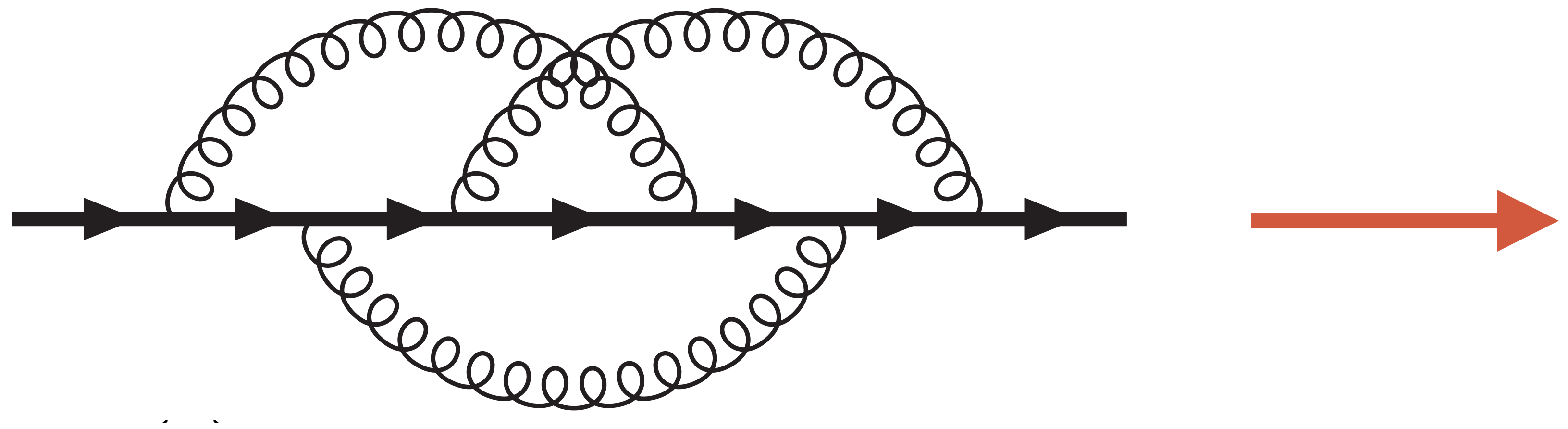

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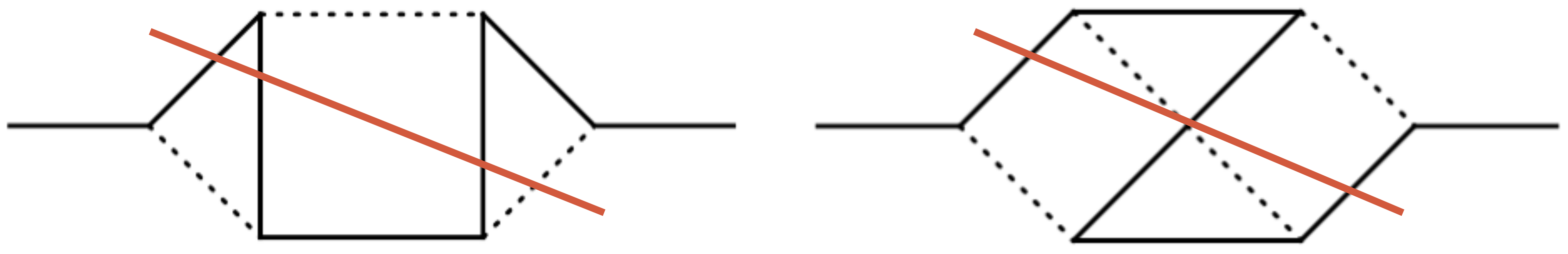
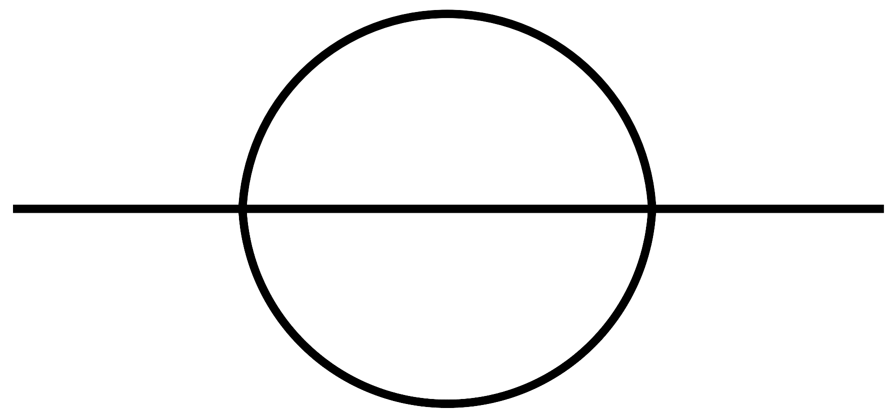


mix of elliptic and logarithmic sectors

same elliptic curve as 2loop sunrise graph



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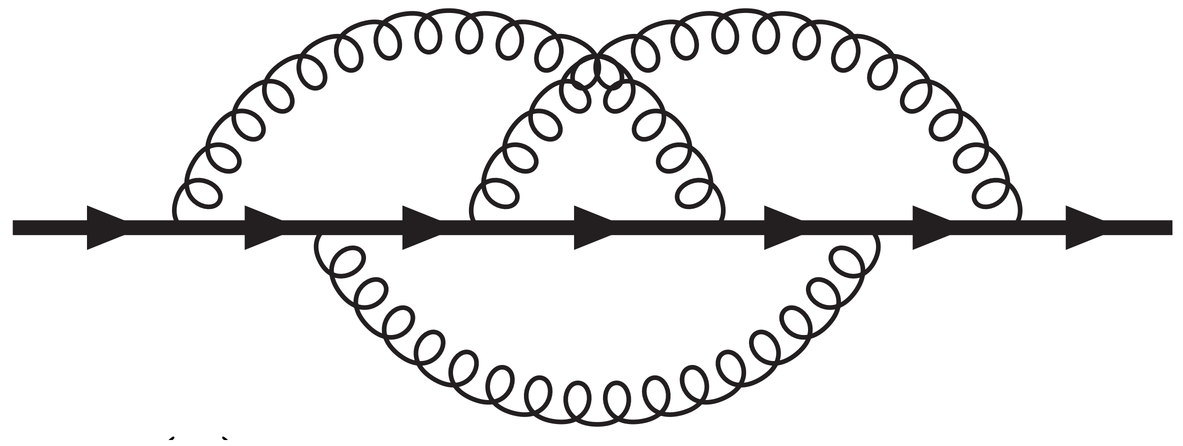


$$\hat{p}\,\Sigma_V(p^2,m^2) + m\,\Sigma_S(p^2,m^2)$$

Σ_V & Σ_S expressed in terms of $\mathcal{O}(50)$ Masters Integrals $\vec{J}$

The geometrical picture allows us to solve differential equations

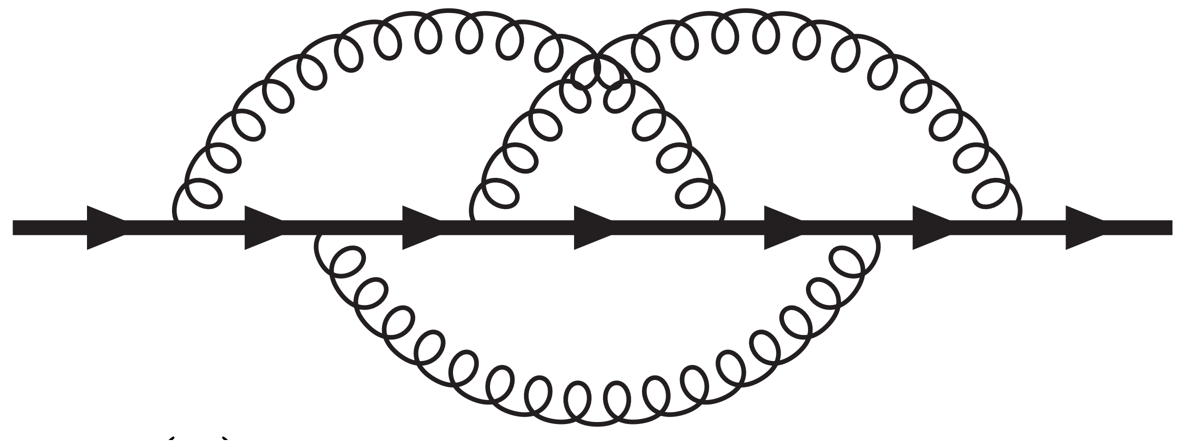
[Görges, Nega, Tancredi, Wagner '23]

$$\mathrm{d}\vec{J} = \epsilon \left(\sum_i G_i\,\omega_i \right) \vec{J} \quad \longleftrightarrow \quad f_i(x)\mathrm{d}x = \omega_i$$

Differential forms can be classified using properties of **the underlying geometry**

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY

[Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]



$$\hat{p}\,\Sigma_V(p^2, m^2) + m\,\Sigma_S(p^2, m^2)$$

Full analytic and numerical control of result across all values of the momentum p^2

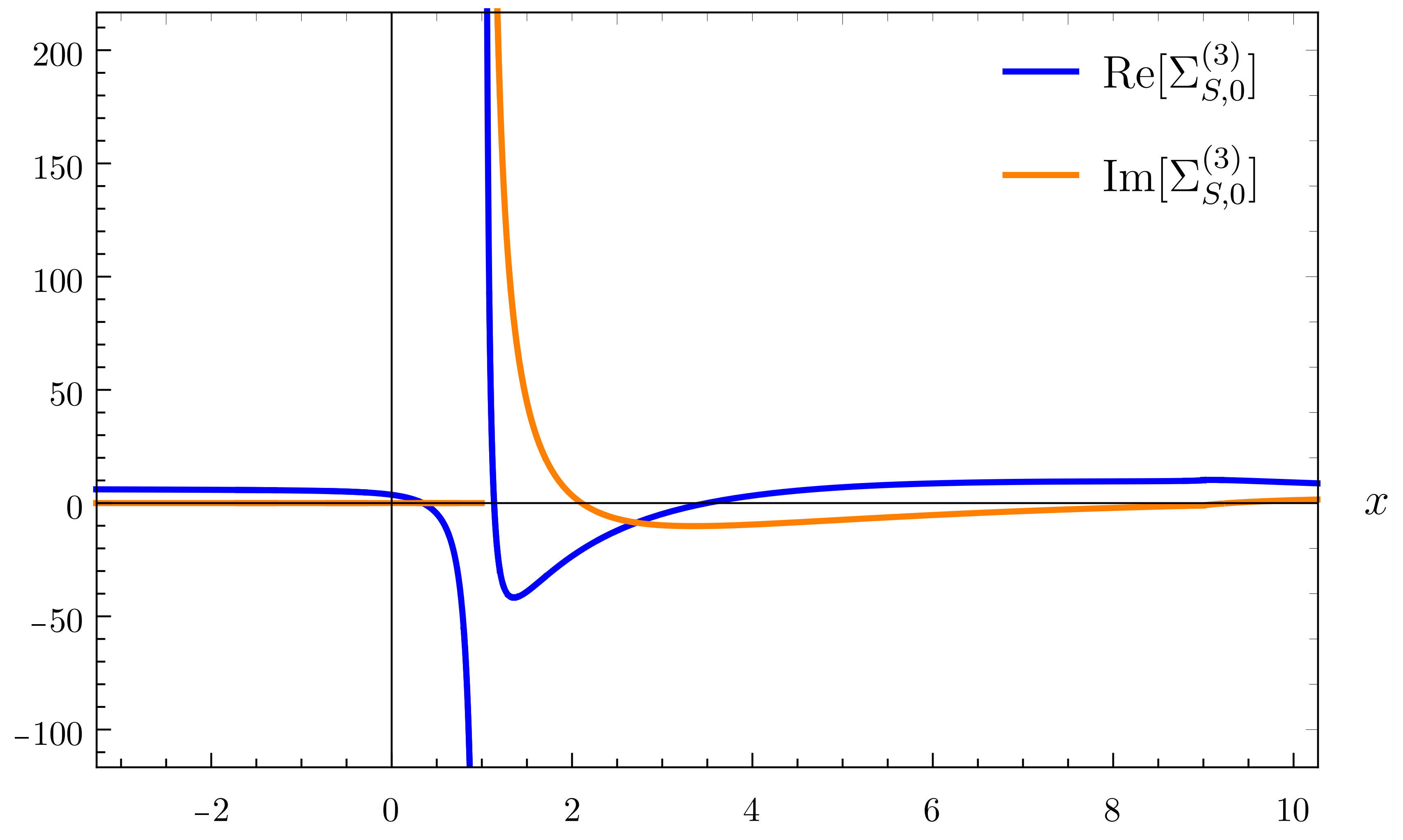


Figure 4: Real and imaginary part of $\Sigma^{(3)}_{S,0}$ (for $\xi = 0$).

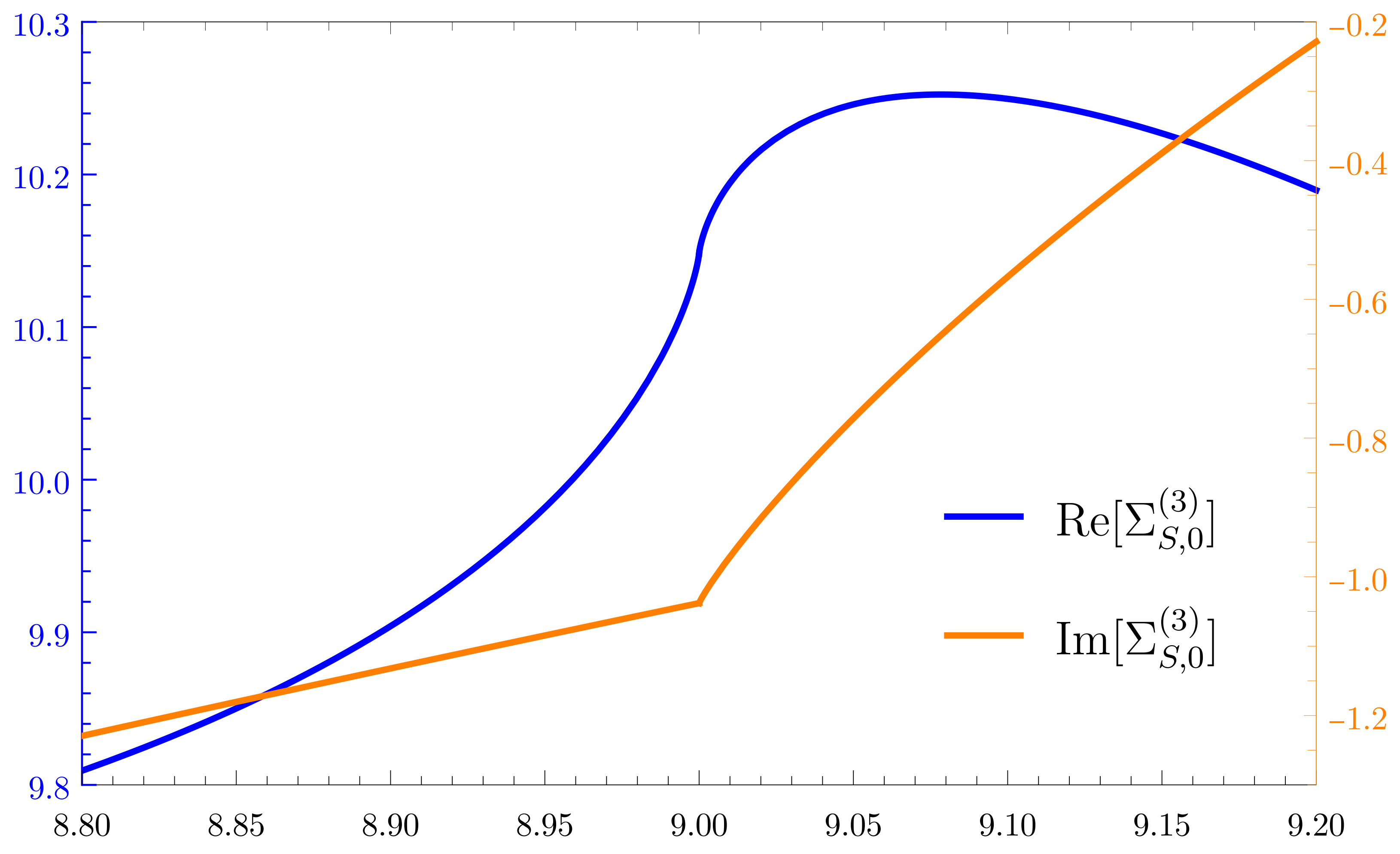
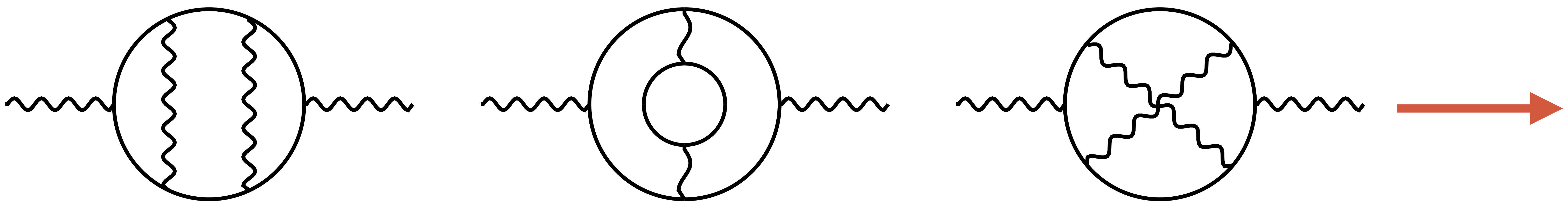


Figure 6: Real and imaginary part of $\Sigma^{(3)}_{S,0}$ close to $x_0 = 9$ (for $\xi = 0$).

AN EXAMPLE CALCULATION: BEYOND ELLIPTIC PHOTONS

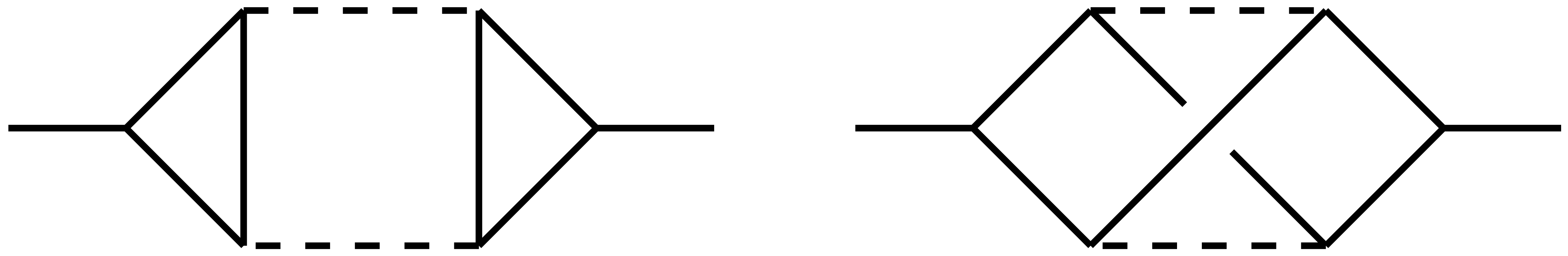
[Forner Nega, Tancredi '24]



$$\left(p^2 g_{\mu\nu} - p^\mu p^\nu\right) \Pi(p^2)$$

$\Pi(p^2)$ expressed in terms of $\mathcal{O}(36)$ Masters Integrals $\vec{J}$

2 "top graphs"



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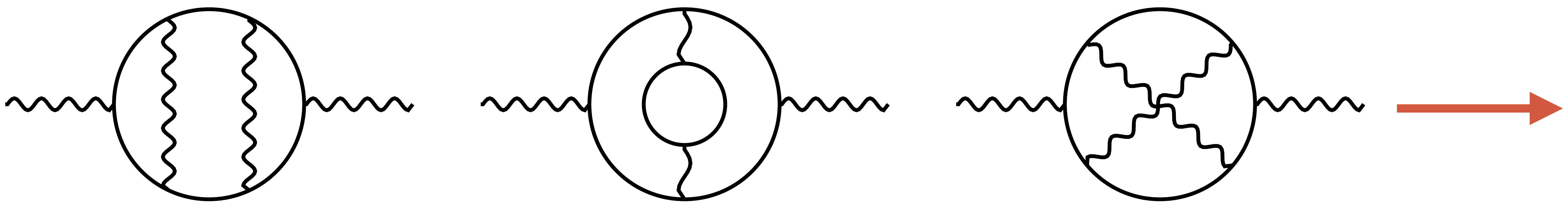

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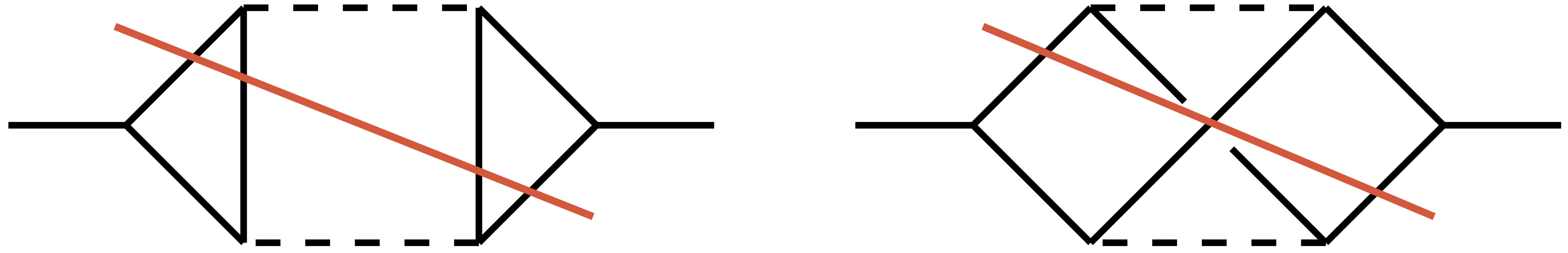
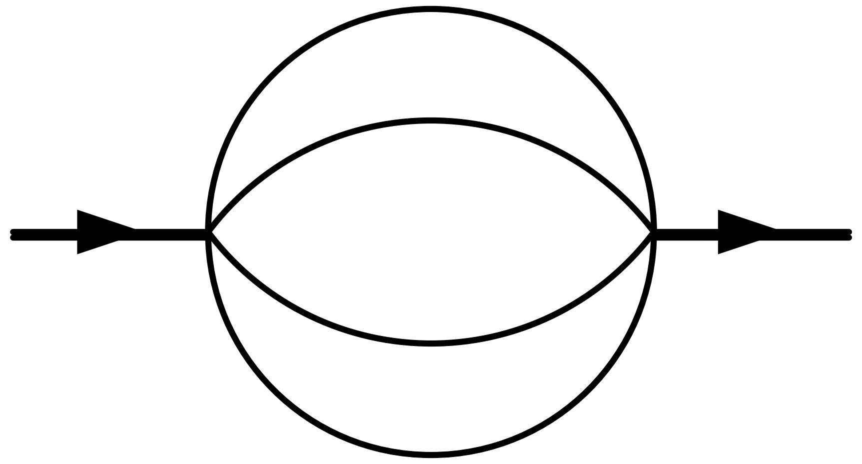


This time there is a **three-massive particle cut**

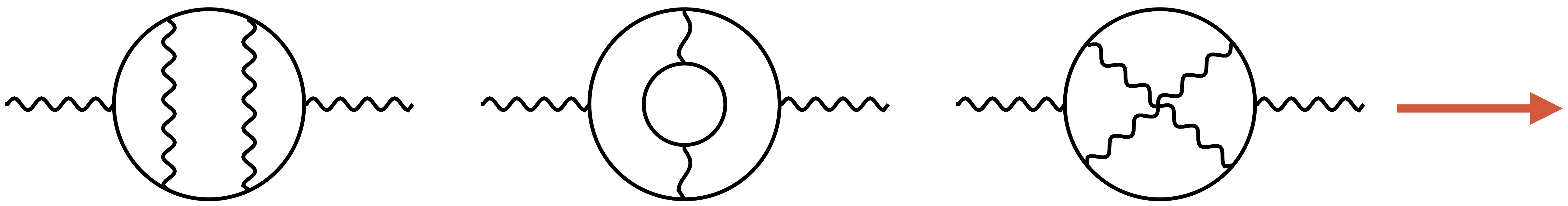
→ three-loop banana associated to a **K3 geometry**

can be thought of as a higher dimensional elliptic curve



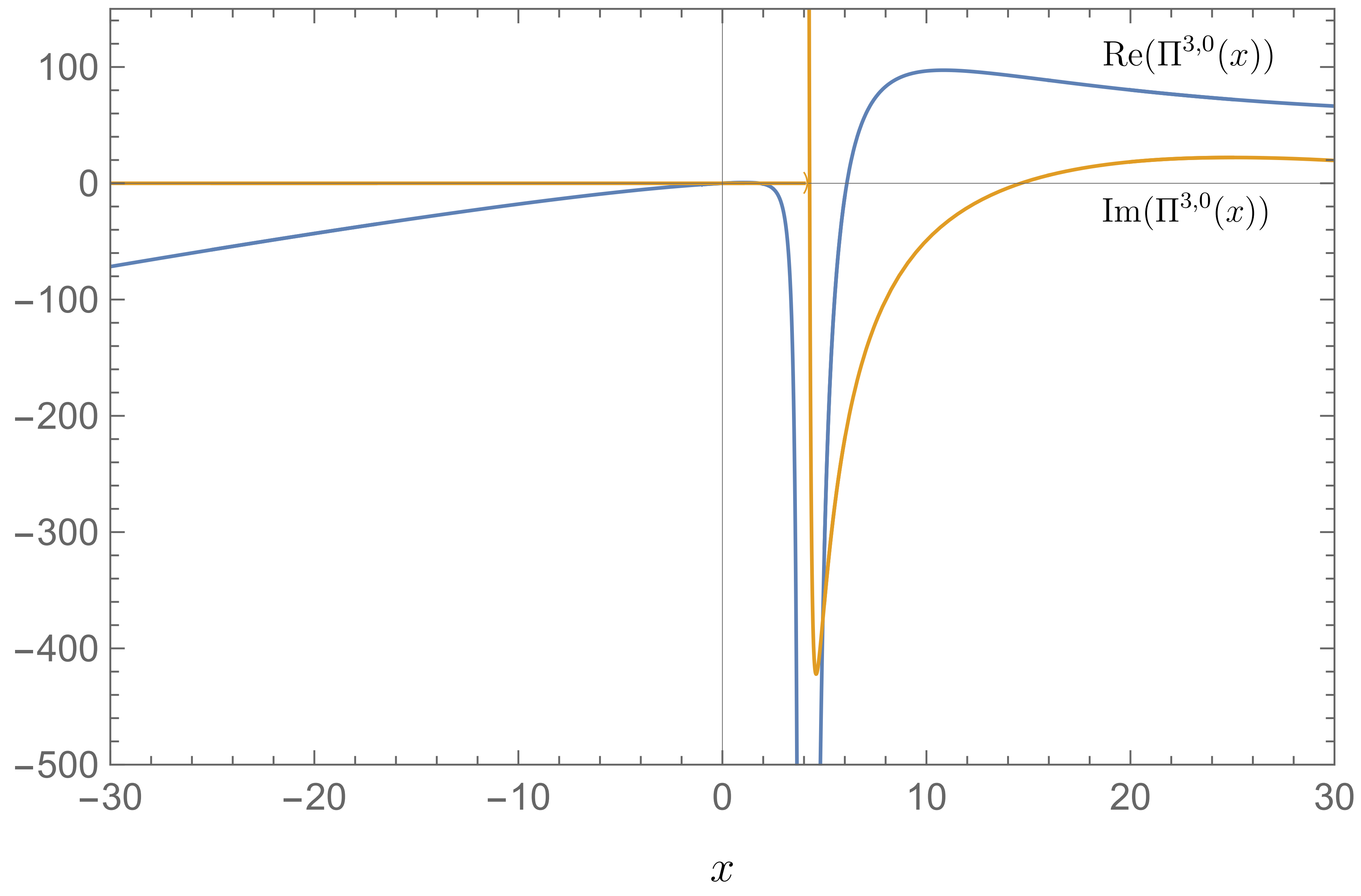
AN EXAMPLE CALCULATION: BEYOND ELLIPTIC PHOTONS

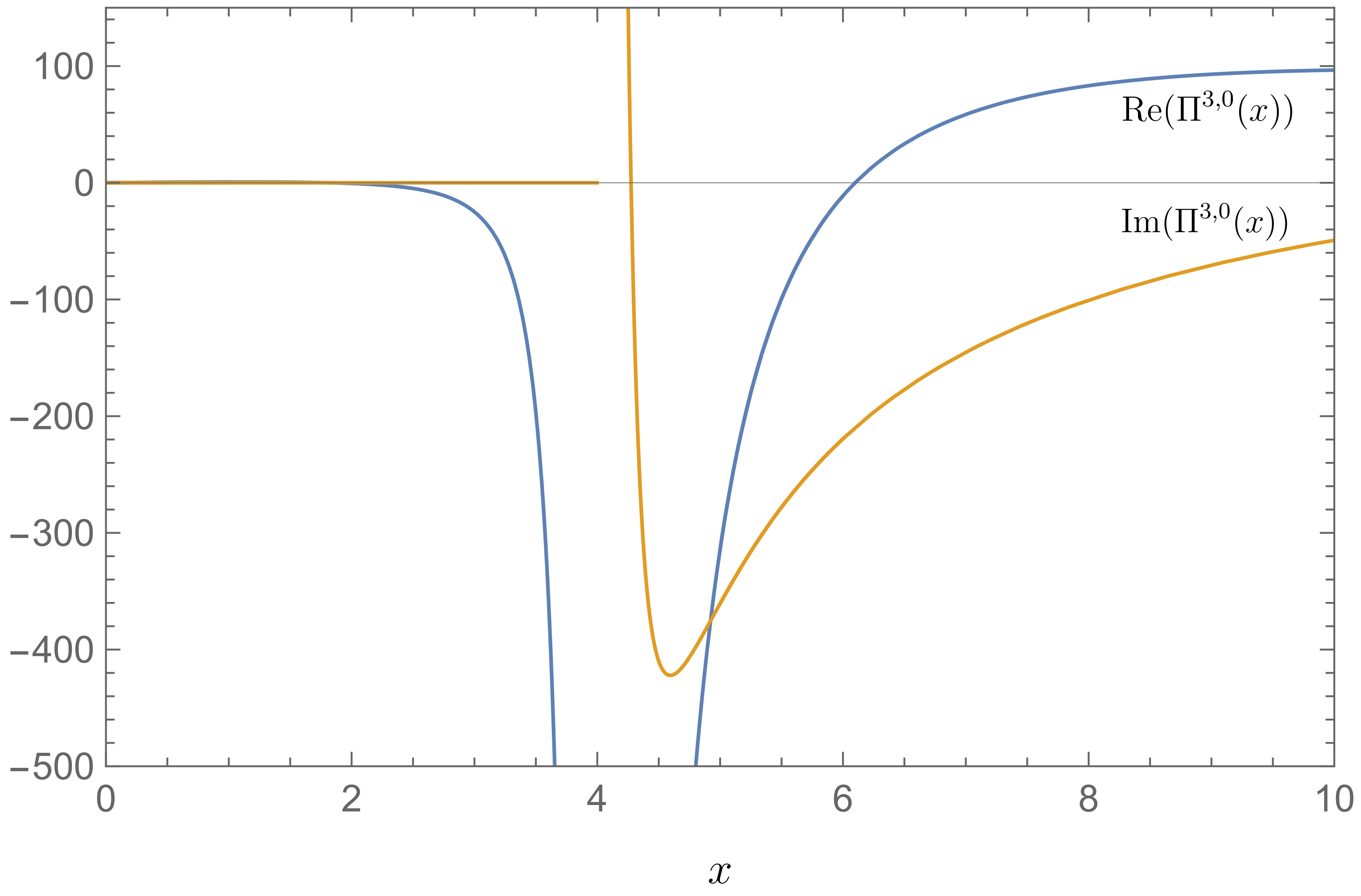
[Forner Nega, Tancredi '24]



$$\left(p^2 g_{\mu\nu} - p^{\mu} p^{\nu}\right) \Pi(p^2)$$

Following same approach: derive and solve differential equations using properties of K3 geometry

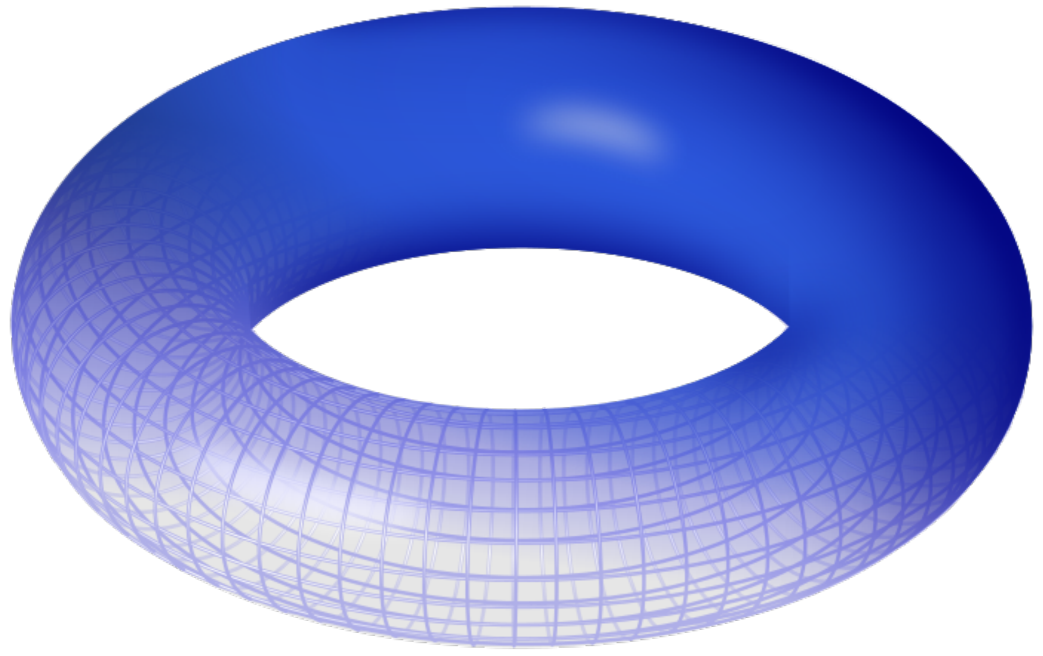


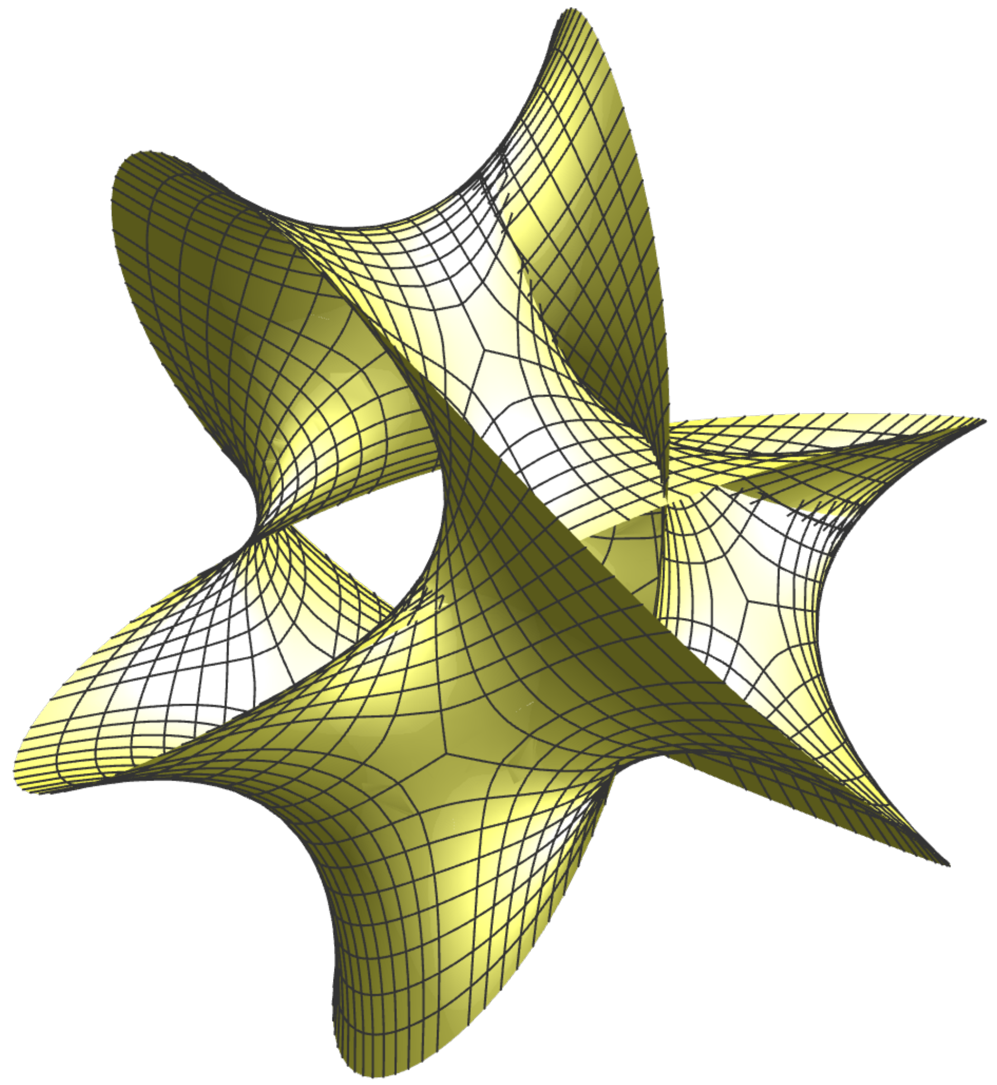


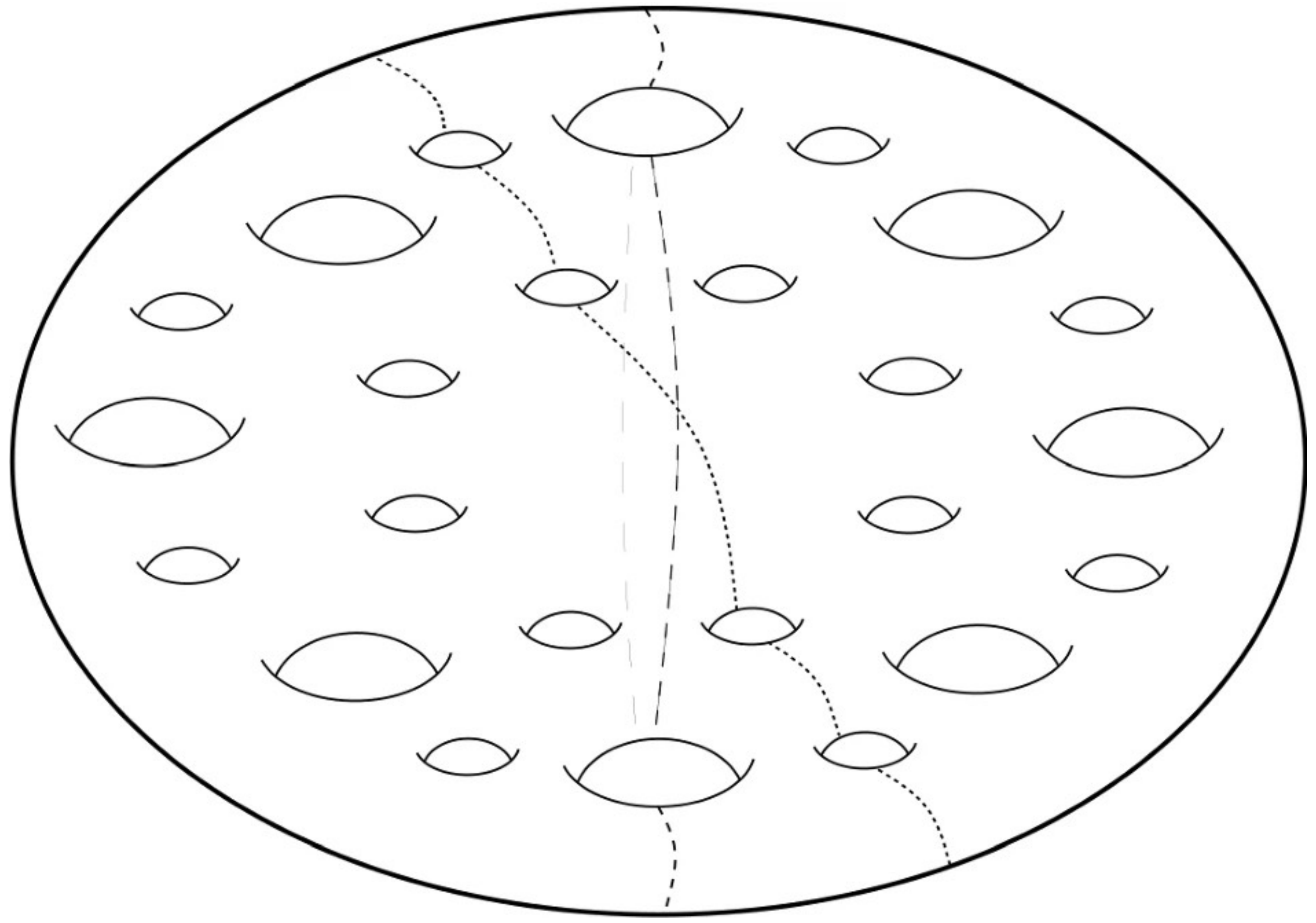
ELLIPTIC AND BEYOND: HIGHER GENUS AND HIGHER DIMENSION

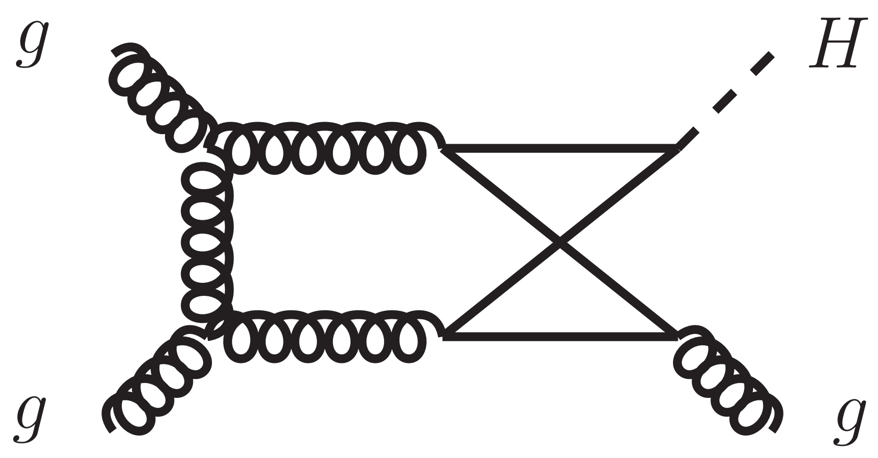
This picture holds for **many other elliptic cases** (more to appear hopefully soon!)

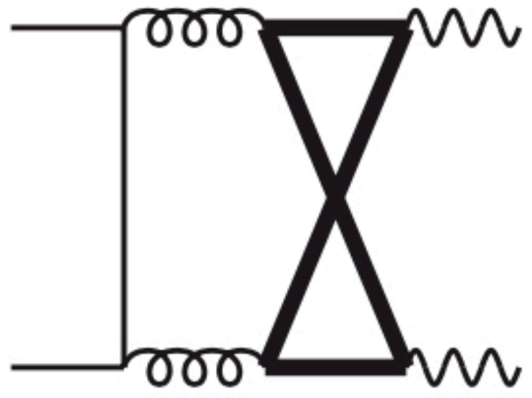
And for **more general geometries**, with obvious generalizations: higher-order eqs, more “solutions”…

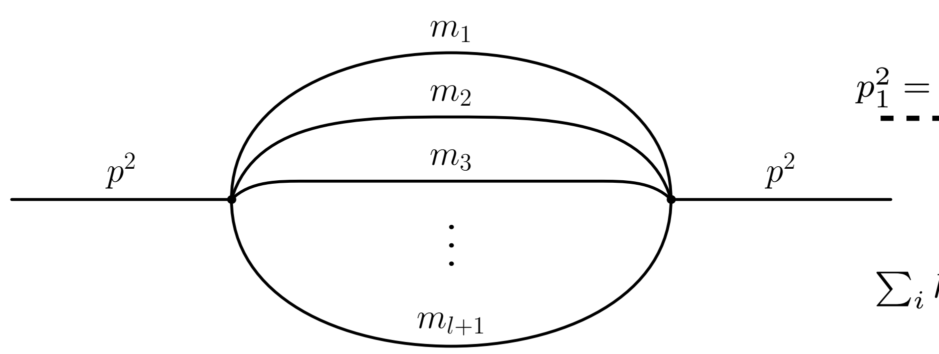


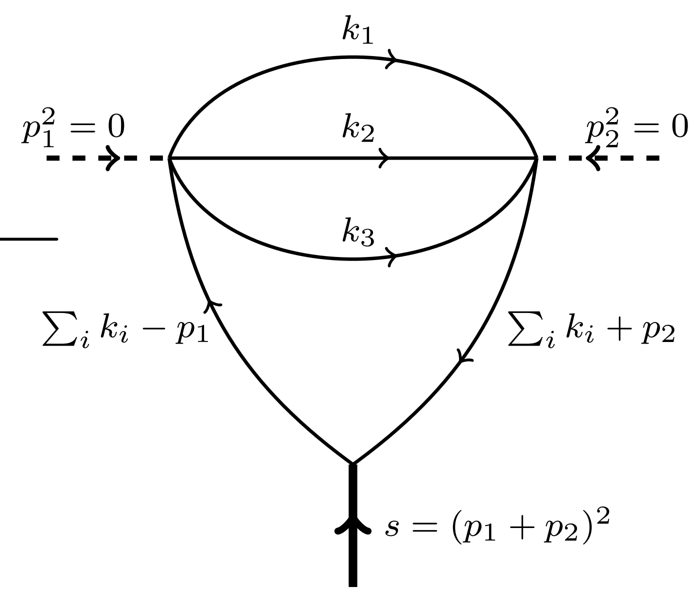


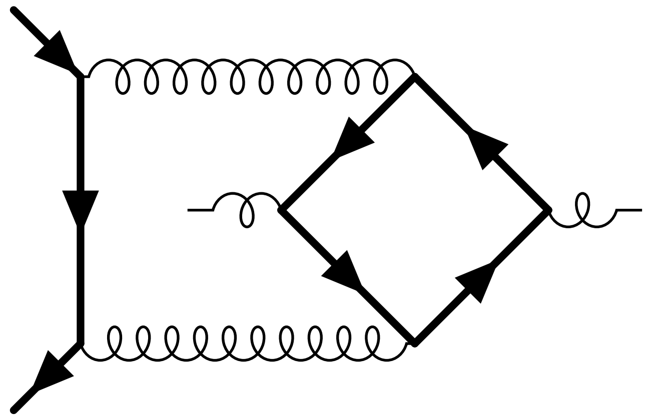


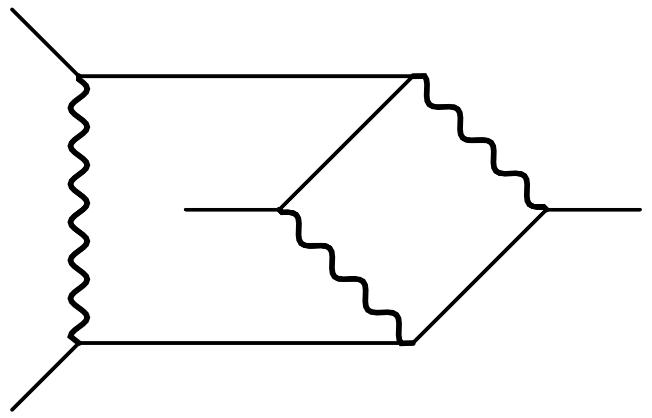




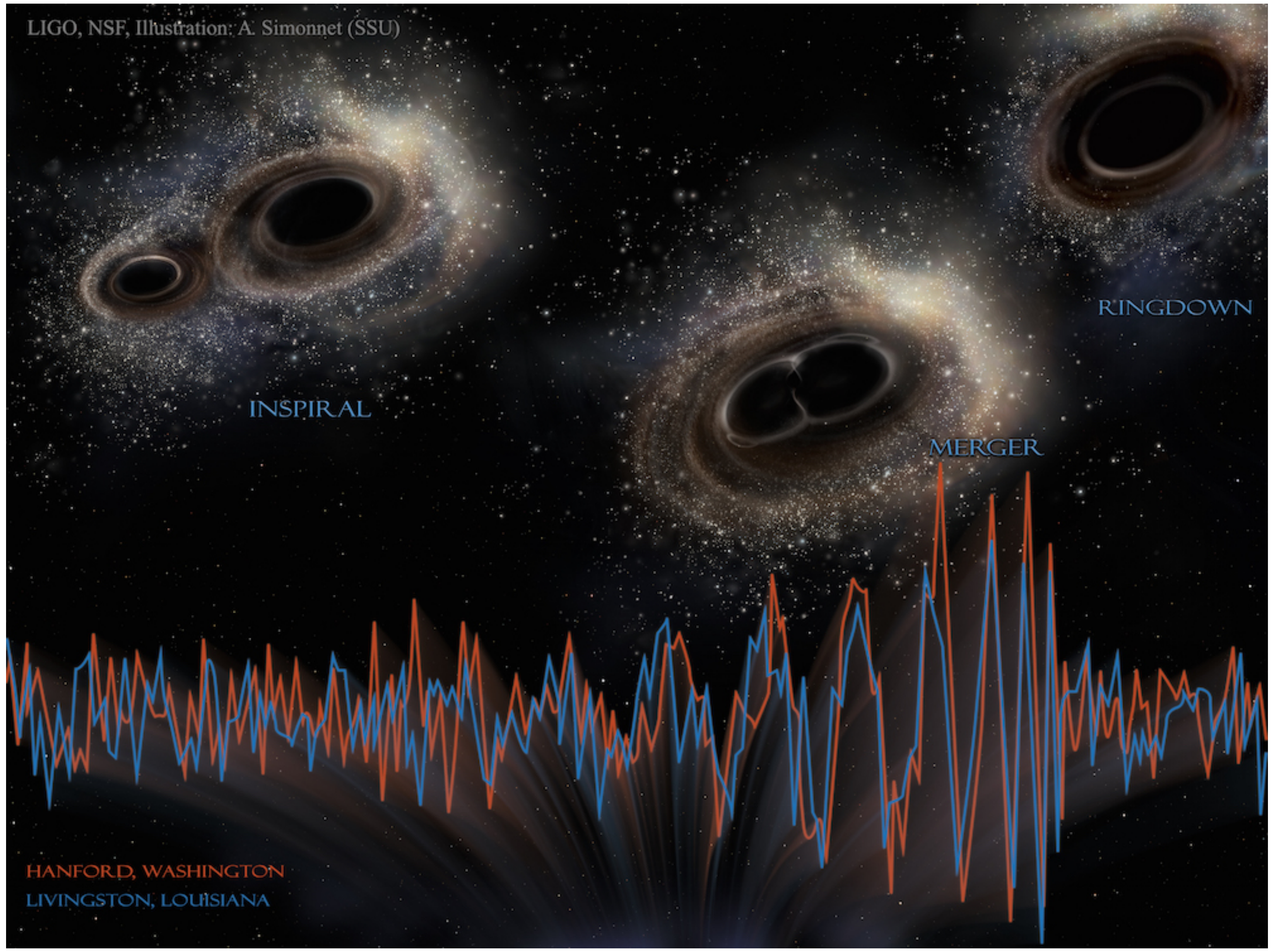


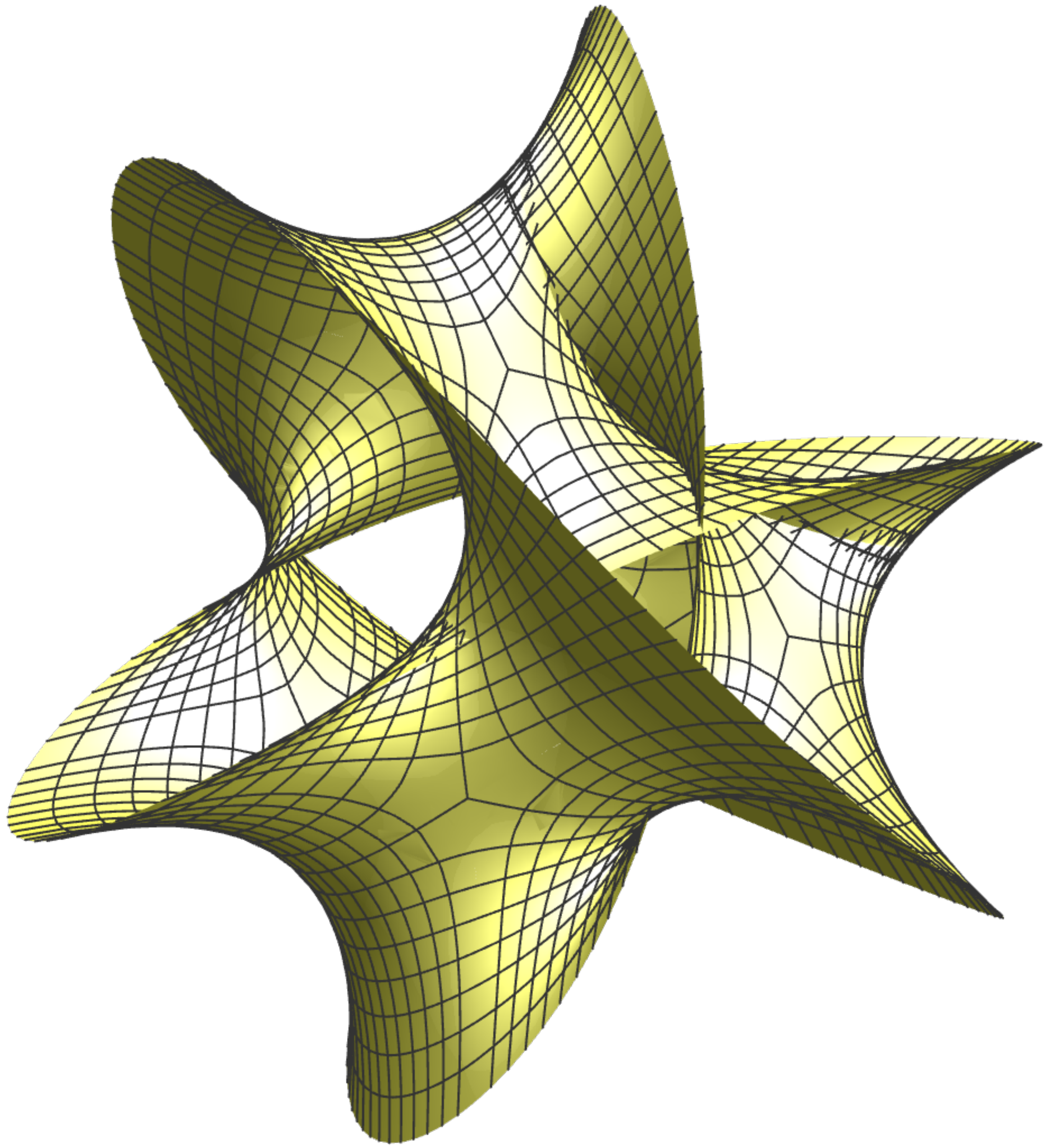






BEYOND ELLIPTICS "CALABI-YAUS IN THE SKY"





Recently, CY geometries have been shown to be indispensable to **model gravitation waves** in **Post-Minkoskian expansion**

[Klemm, Nega, Sauer, Plefka ’24; Frellesvig, Morales, Willhelm ’23]

[Bern, Parra-Martinez, Roiban, Ruf, Shen ’21,…’24]

CONCLUSIONS AND OUTLOOK

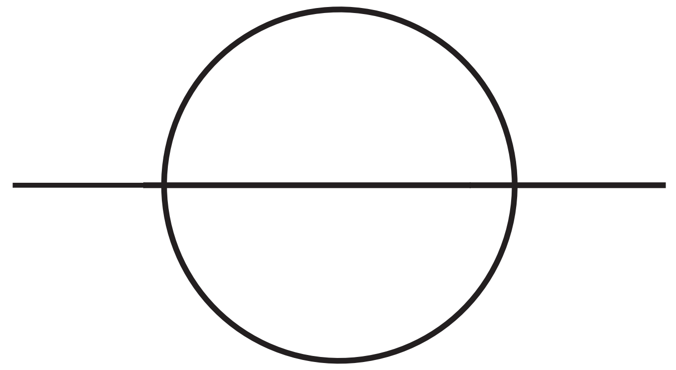
- amplitudes are *fundamental building blocks in QFT*, for precision collider physics and beyond
- complexity of the calculations is often matched by **unexpected simplicity in final results**
- searching for a way to make **simplicity manifest** informs on how to compute amplitudes more efficiently ***(language of differential forms on complex varieties is an example!)***
- what we learnt in past 10 years is finally **bearing fruit**: the first realistic "correlators" and amplitudes under ***analytic and numerical control***
- same structures observed in *gravitational waves calculations* and *cosmological corrrelators!*

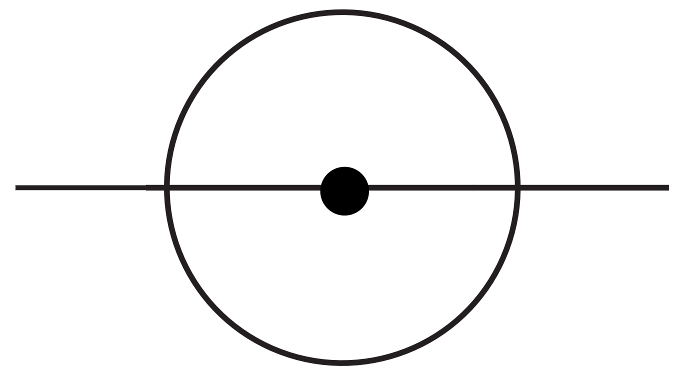
THANK YOU VERY MUCH!

BACK-UP SLIDES

FROM DIFFERENTIAL EQUATIONS TO GEOMETRY

$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \left(\frac{1}{x-1} + \frac{9}{x-9} + 2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) + \frac{27}{4(x-9)} + \frac{1}{4(x-1)} + 1\right]\varpi(x) = 0$$





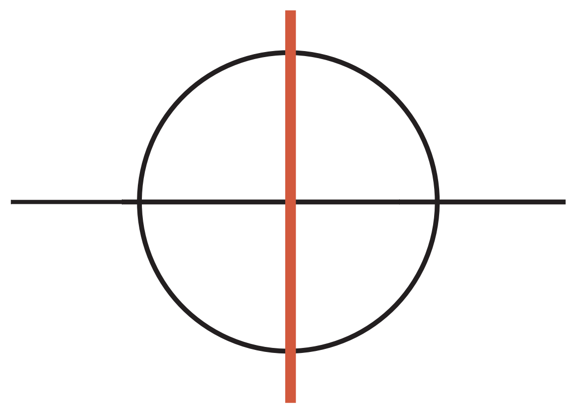
Solutions: **periods of an elliptic curve.** Obvious?

In some cases, you might be lucky enough to find the diff equation in some list of known ones…

By cutting all propagators (and continuing down to leading singularities) we can "expose" simplest integral which fulfils the **homogeneous differential equation**

[Primo, Tancredi '16,'17]

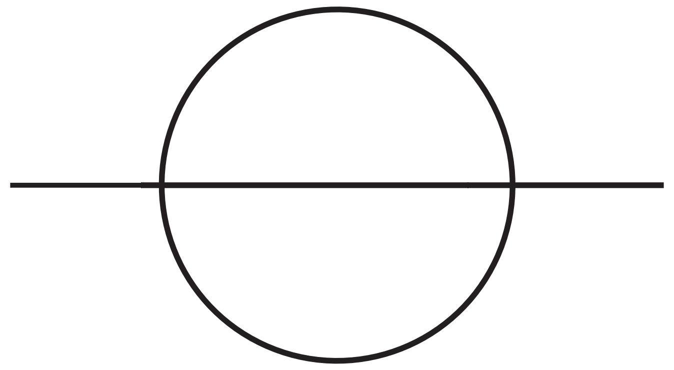
$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \left(\frac{1}{x-1} + \frac{9}{x-9} + 2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) + \frac{27}{4(x-9)} + \frac{1}{4(x-1)} + 1\right]$$

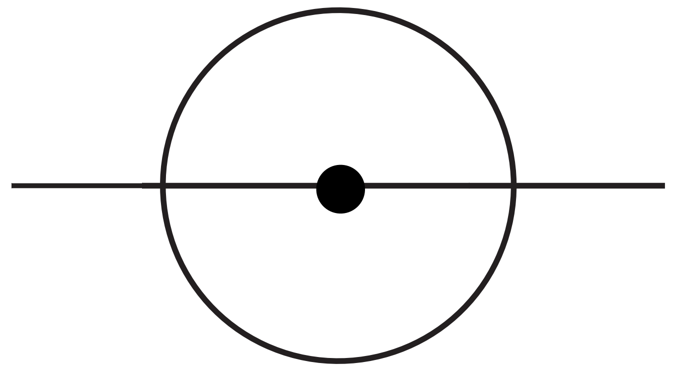


$= 0$

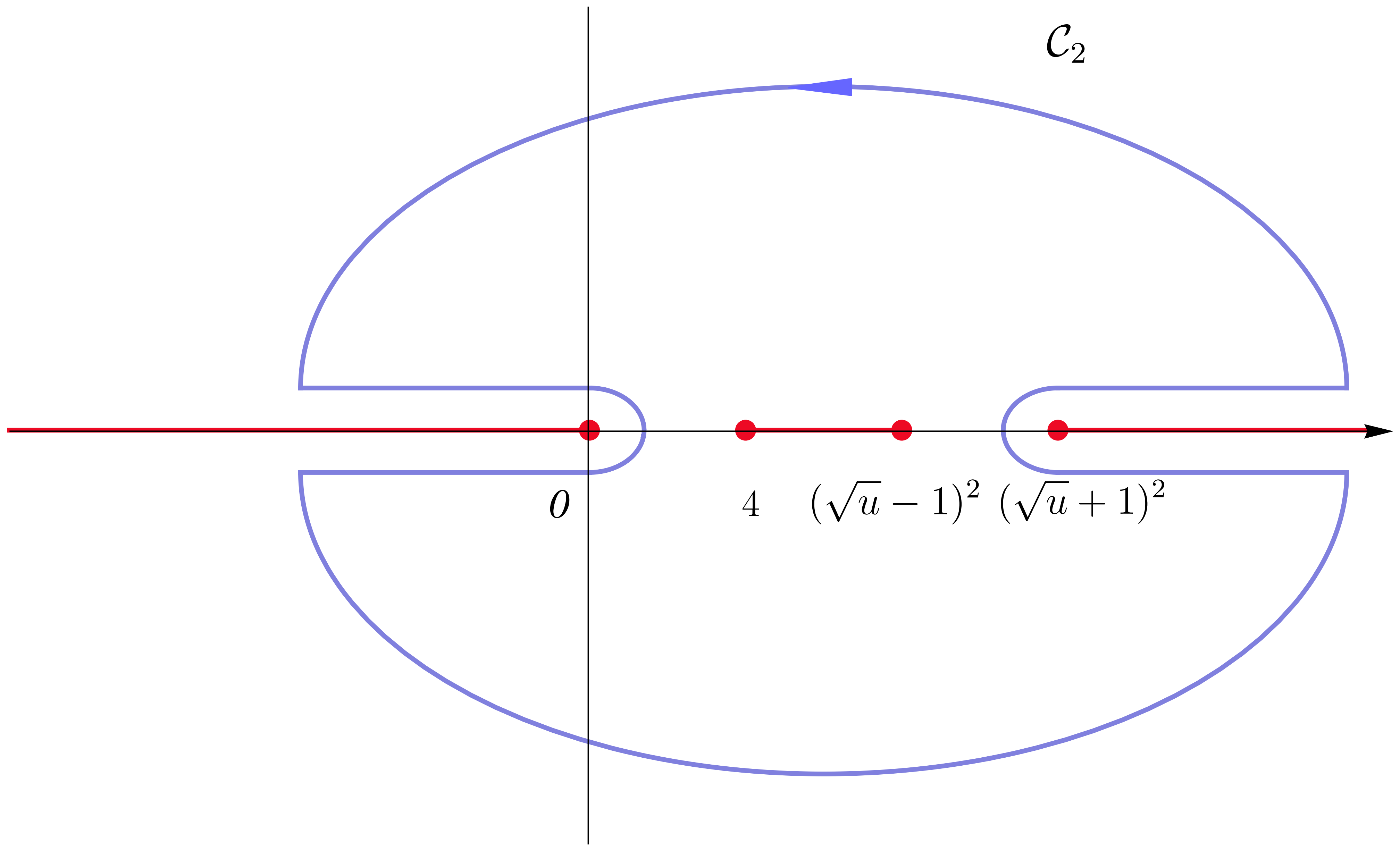
FROM DIFFERENTIAL EQUATIONS TO GEOMETRY

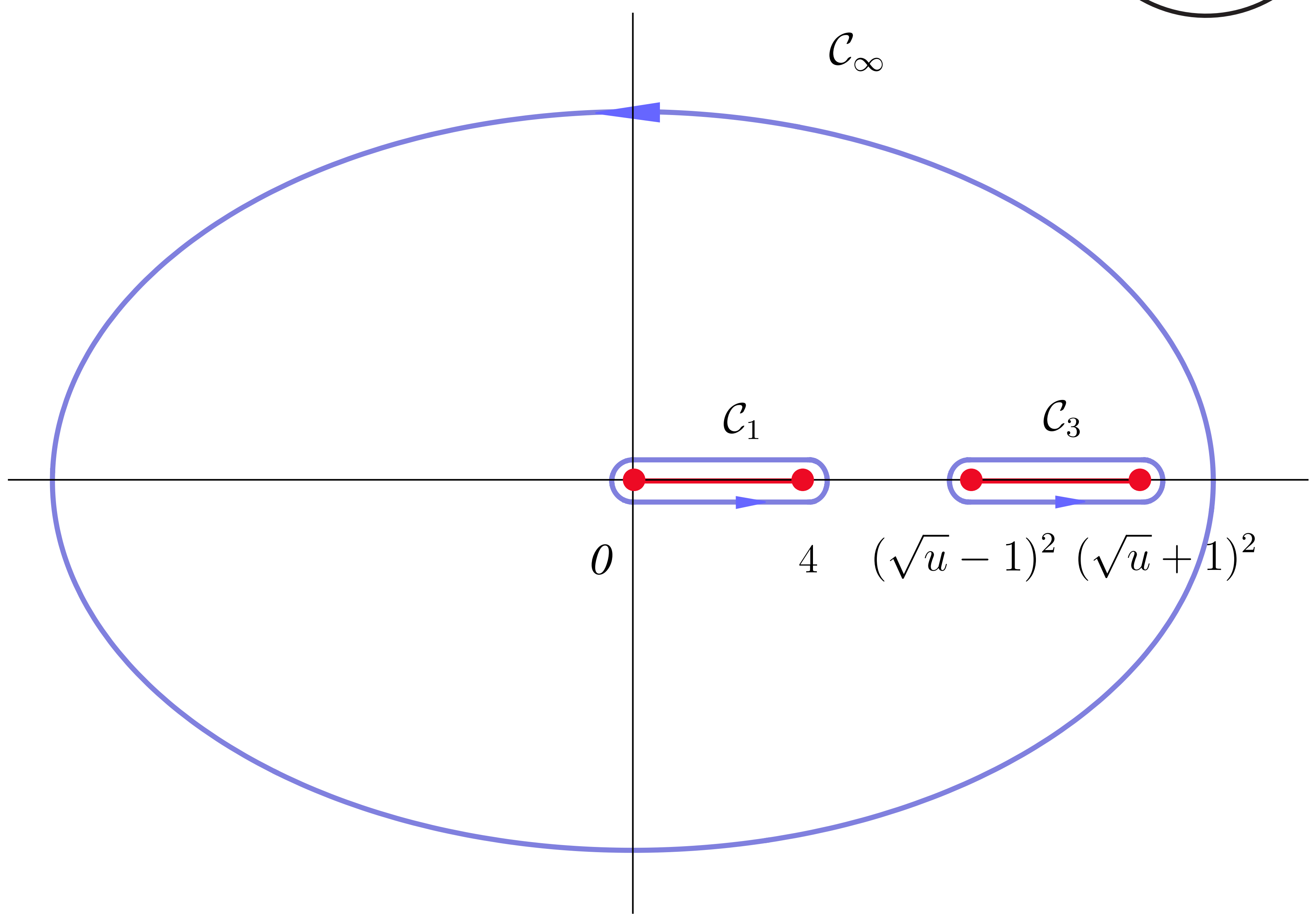
$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2+\left(\frac{1}{x-1}+\frac{9}{x-9}+2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)+\frac{27}{4(x-9)}+\frac{1}{4(x-1)}+1\right]\varpi(x)=0$$





Solutions: **periods of an elliptic curve.** Obvious!!!



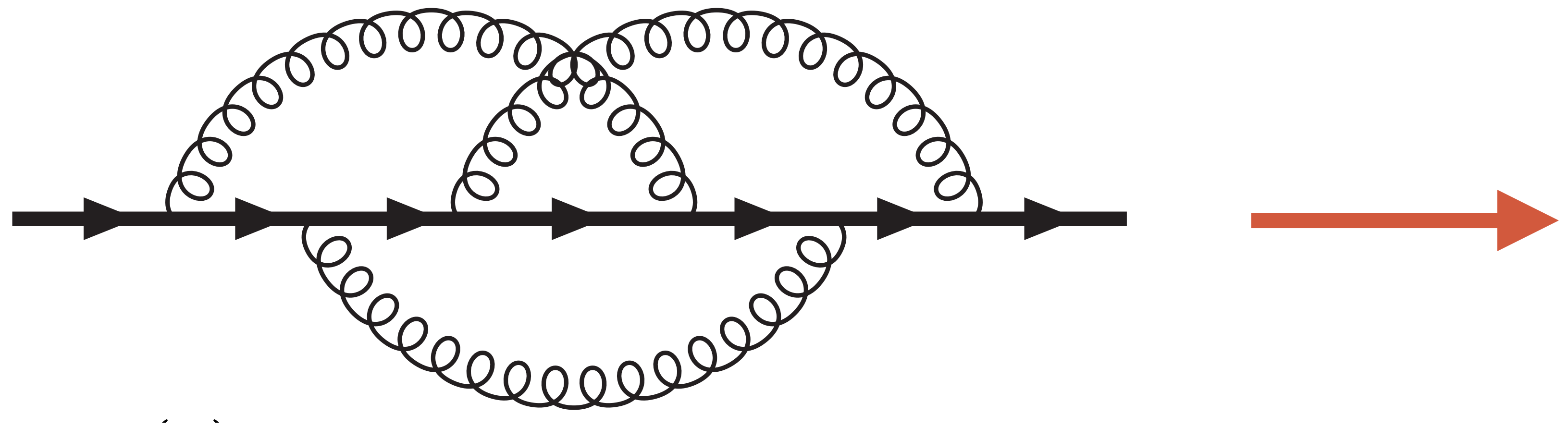


second independent solution from second integration contour

[Primo, Tancredi ’16,’17]

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY

[Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]



$\hat{p}\, \Sigma_V(p^2, m^2) + m\, \Sigma_S(p^2, m^2)$

Σ_V & Σ_S expressed in terms of $\mathcal{O}(50)$ Masters Integrals $\vec{J}$

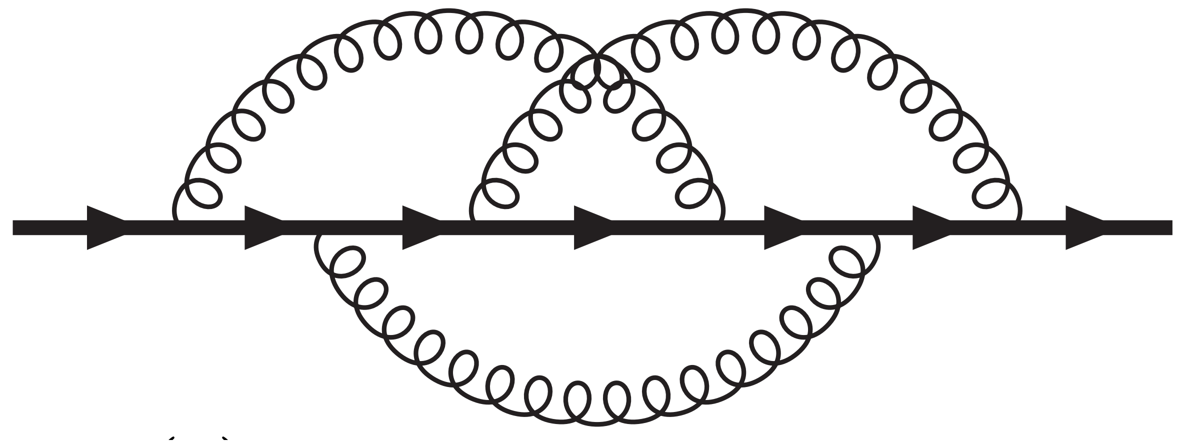
7 (independent) elliptic differential forms: full analytic control over **iterated integrals** over these forms

$$f_i \in \left\{ \frac{1}{x(x-1)(x-9)\varpi_0(x)^2}, \varpi_0(x), \frac{\varpi_0(x)}{x-1}, \frac{(x-3)\varpi_0(x)}{\sqrt{(1-x)(9-x)}}, \frac{(x+3)^4\varpi_0(x)^2}{x(x-1)(x-9)}, \frac{(x+3)(x-1)\varpi_0(x)^2}{x(x-9)}, \frac{\varpi_0(x)^2}{(x-1)(x-9)} \right\} \quad \text{for } i = 10, \ldots, 16\,,$$

$\varpi_0(x)$ is the first elliptic period

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY

[Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]



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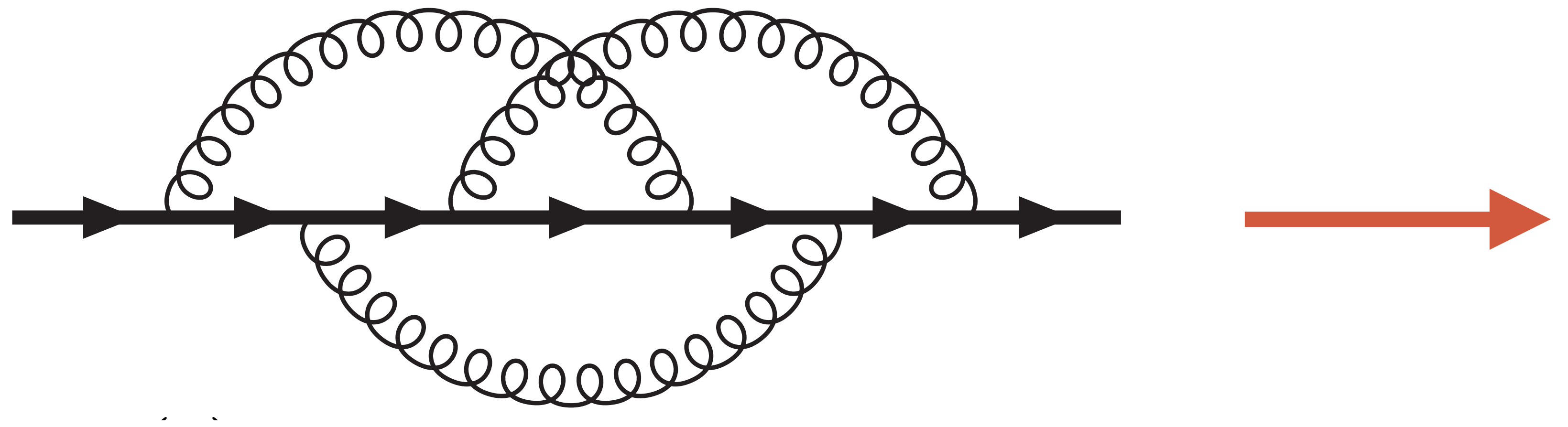
3 of the kernels drop in the physical amplitude: they are related to forms of the second kind with "double poles" → a hint for bootstrap program?

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY

[Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]



$$\hat{p}\, \Sigma_V(p^2, m^2) + m\, \Sigma_S(p^2, m^2)$$

One can obtain resummed results close to on-shell limit $p^2 = m^2$ required for UV renormalization

$$\Sigma^{(3)}_{V,\mathrm{res}} = \left[-\frac{27}{128\epsilon^3} - \frac{673}{1152\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right] + (1-x)^{-2\epsilon}\left[\frac{27}{64\epsilon^3} + \frac{27}{32\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right]$$
$$+ (1-x)^{-4\epsilon}\left[-\frac{27}{128\epsilon^3} - \frac{27}{128\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right] + \mathcal{O}(1-x)\,,$$
$$\Sigma^{(3)}_{S,\mathrm{res}} = \left[-\frac{653}{1152\epsilon^3} + \frac{1447}{6912\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right] + (1-x)^{-2\epsilon}\left[\frac{1}{x-1}\left(\frac{27}{32\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right)\right.$$
$$\left. + \frac{9}{16\epsilon^3} - \frac{91}{256\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right] + (1-x)^{-4\epsilon}\left[\frac{27}{128\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right] + \mathcal{O}(1-x)\,,$$

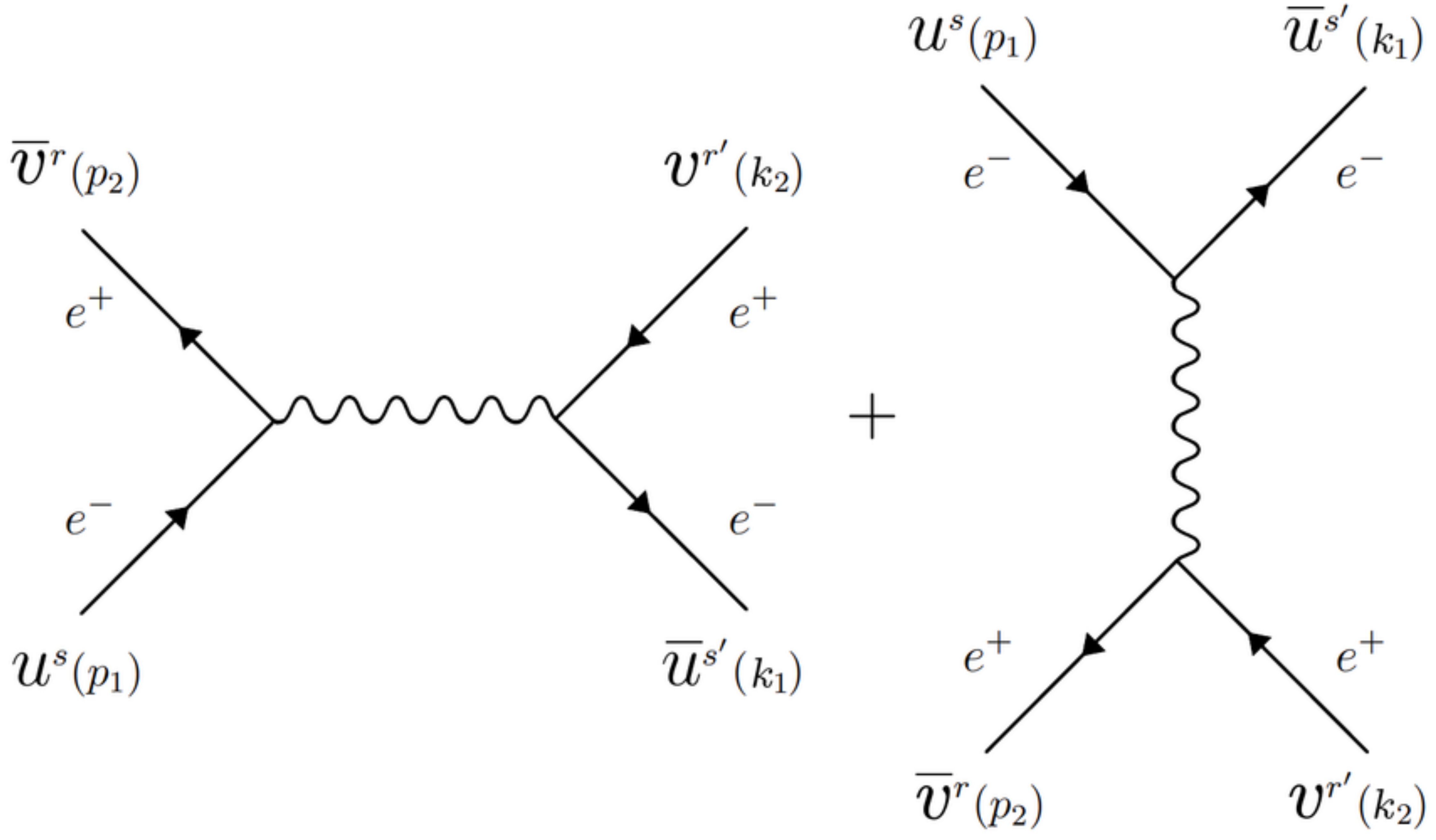
MORE ELLIPTICS ELECTRON-ELECTRON SCATTERING

This picture holds for **many other elliptic cases:** **Bhabha scattering**

Bhabha is a “standard-candle” process in Quantum Electrodynamics: $e^+ e^- \to e^+ e^-$

At leading order (tree level) just two diagrams

Classical calculation in QFT 1 courses



$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)} = \frac{\pi\alpha^2}{s}\left(u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right)$$

Cross section *(for massless electrons)* on Wikipedia!

MORE ELLIPTICS ELECTRON–ELECTRON SCATTERING

This picture holds for **many other elliptic cases:** 2loop Bhabha scattering

At two loops, some Feynman diagrams require dealing with **underlying Elliptic geometry** [Delto, Duhr, Zhu, Tancredi '23]