Lorenzo Tancredi - Technical University Munich

COMETA Colloquium December 16th 2024

SCATTERING AMPLITUDES: FROM COLLIDER PHYSICS TO GEOMETRY

WHY (STILL) COLLIDERS? THE LHC (AND BEYOND)...

Future Circular Collider Circumference: 90 -100 km Energy: 100 TeV (pp) 90-350 GeV (e*e*)

Large Hadron Collider(LHC) Large Electron-Positron Collider (LEP)

Circumference: 27 km Energy: 14 TeV (pp) 209 GeV (e+e-)

Tevatron Circumference: 6.2 km Energy: 2 TeV(pp)

y p [fb¹] **Reference** pp = 96.07 [±] 0.18 [±] 0.91 mb (data) COMPETE HPR1R2 (theory) ⁵⁰⇥10⁸ PLB 761 (2016) 158 - LIU I*n*v devoiie a ine **Standard Model Total Production Cross Section Measurements** *ATLAS* Preliminary **THE LHC HAS BECOME A PRECISION MACHINE**

THE HIGGS BOSON: THE LAST MISSING PIECE The Higgs boson

of those famous CERN T-shirts

"understanding" = knowledge ?

Hints to answer these questions hidden in the **details of Higgs interactions to SM particles**

HIGGS INTERACTIONS AT THE LHC

HIGGS INT

Higgs discovery through its couplings to gauge sect

$F^{\mu\nu}$ $\frac{1}{4}$ $\frac{1}{4}$ Events / 2.5 GeV \Box \Box *ATLAS* $\overline{}$ 30 y وي
مون
مون 80 100 120 140 160 $m_{\scriptscriptstyle\rm 4l}^{\scriptscriptstyle\rm Constrained}$ [GeV]

 0.05 0.05 0.04 dil 0.15 0.2 oc understanding ³³ experience 2. Discovery couplings to **second family** (*μ* & *c*)

HIGGS INTERACTIONS THE YUKAWA SECTOR **CONSTRANTS ASSESSED ASSESS TO A 25.9-137 fb⁻¹ (13 TeV)**

with 2010 estimates 40 ATLAS+CMS

alge **With 2018 estimates 4***σ* **ATLAS+CMS**

Higgs self coupling extremely difficult to measure.

HIGGS SELF INTERACTIONS THE MOST MYSTERIOUS?

HL-LHC first to see the triple-H coupling

HIGGS SELF INTERACTIONS THE MOST MYSTERIOUS?

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Gavin Salam Precision Accepted 2023

HIGGS SELF INTERACTIONS THE MOST MYSTERIOUS?

HL-LHC first to see the triple-H coupling

FITE SCIENCE SHOWSDAIR DETA Multiposon film as probe of the electroweak sector of the construction of the state of the $\mathbf{W} \rightarrow$ \mathbf{W} \mathbf{W} \mathbf{W} \mathbf{W} \mathbf{W} \equiv <u>He Green Is this a</u> i **13 MS** 138 fb⁻¹ (13 TeV) \mathbf{r} \sum \mathbf{K} total (⇥2) inelastic *p* 100 GeV *p*^T >75 GeV dijets *p*^T > 70 GeV \cdot 2 *p*^T >100 GeV *E* \mathbf{E} 125 M *E* $\overline{\mathbf{1}}$ 25 GeV *nj* 1 $n_j \geq 2$ *nj* 3 *E* $\overline{}$ 100 GeV *nj* 0 *p***T** \overline{A} 30 GeV *nj* 1 *nj* 2 *n* 3 *nj* 4 <u>nj</u> 5 *nj* 6 *nj* 7 *p*^T > 10 G $n_j=1$ *nj* = 2 *nj* = 3 *nj* = 4 72 *nj* 6 *nj* 0 *p*^T > 30 GeV *nj* 1 *nj* 2 *nj* 3 *nj* 4 *nj* 5 *nj* 6 *nj* 7 total *p*^T > 25 GeV *nj* 4 *nj* 5 *i* 11 *nj* 7 *nj* ⁸ *tZj t*-chan *s*-chan *Wt ZZ ZZ WZ WW Z* ┌ $\overline{}$ *ZZ WZ WW Z* \overline{a} \bigstar *z* L *WZ WW Z W* Н Д. *H*!*WW*⇤ \blacktriangleright **H** \overline{H} \mathbf{R} $F - \epsilon$ **de tain disc early have a meet started for the compactive partial from a station of the simulation of the simulat** $\ddot{\mathbf{t}}$ *H* ! \mathbf{A} *H*!*ZZ*⇤ *H* ! ⌧⌧ $|$ \bullet - \pm \cdot *H*. *H*!*WW*⇤ \blacksquare *^H*!*bb*¯ ${\bf m}$ 42 *t*¯*tW*[±] *t*¯*tZ* **H** ! **WWW** *WWZ* tot. y. *WZ* ZYWY 144 *Wjj Zjj* Water a water to the there have the total water of the state of the water 5 |
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1 **Theory LAP** 21 *s* **= 13.6** TeV **LATE** *s* **= 13** TeV $\frac{1}{2}$ **LEGAL** *s* **= 8** TeV **LHC 15 P** *s* $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ TIKE 4.16 D H_{tot} *s* **= 5** TeV Data of the contract of the **Standard Model Production Cross Section Measurements** *Status: October 2023* **ATLAS PRELISES p**
d *s* = 5,7,8,13,13.6 TeV WW_γ **CMS WWW.LICOCLEUGHELORGIE EN 1999 FELSO CON PERSONALE DE LA PRODUCTIVA DE LA PRODUCTIVA DE LA PRODUCTIVA DE LA**
CAMBIOLORGIE EN 1999 FELSO DE LA PRODUCTIVA DE LA PRODUCTIVA DE LA PRODUCTIVA DE LA PRODUCTIVA DE LA PRODUCTI े प्राप्त किया । स्टूटर । स्क्री कुल्म क्रिया । स्वा ख्या ख्या स्पर्ध या प्रवृद्ध के प्राप्त प्राप्त का इस्त क
अपनी क्रिया सिंह निर्माण के स्वा पित्र किया है कि सिंह के प्राप्त कर किया है। स्वा कि स्वा के स्वा कि सिंह as we have the construction of scales, PDFs, and parton shower modeling from all simulations. \mathbf{a} 1500 Events 12 atchun
Brialaz d c
Ing $\frac{1}{2}$ V_{ICK} ∈ (20,150] *mll*^γ ∈ (150,250] *mll*^γ ∈ (250,∞) *mll*^γ **POISTER WWW.** CMS Text CM CHORO TO THE COLLER TO COLLER TO THE COLLER TO THE COLLER TO THE COLLER TO THE COLLECTION OF THE COLLE **negligible.** (2.4) and *p* <u>138 Teven of the set of</u> on jet multiplicity: 0 jet and 1 status based on the number of events in data and predictions after the fit to the data are listed in Table 1. The observed (expected) signal significance from the fit is 5.6 (5.1) standard deviations, corresponding to the observed distribution of the observed distributions, corresponding to the observed distribution of the observed distribution of the observed distributions to the fit to the data shown in Fig. 3. The observed signal strength, while the observed signal strength, and the observed signal strength, $\frac{1}{2}$ 0.11 GRSD **ta_teDabMsDathranic .ckeV1ch4000Sscrownessponneulpsitzen.print, iODSGTVf2Ct1Cl4Stf#Dult1100PSf2fu
16 (pp 1) 19 Full Region defined by a figure that with the signal region defined by a file of the signal regio
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Mushim the requirements of the requirements of the requirements of the requirements of the propriations. The th ore the Wight prediction for the Wight and Wight and Wight and Wight and Wight and Teptons or photons within jets,
Ore critical cross the Wight was the Wight and Wight and Wight and Press section is 5.34 (cold or the PDF) NLO QUE AS EVALUATED BY MADGE AS EVALUATED BY MADGE AND LOCAL AND ALCOHOLOGY CROSS SECTION FROM REPORT OF THE W the simultaneous fit with the simulation of the statistical of the statistical, and the uncertainties of the uncertai cal the components is the state of the component of the system of the section of the se The theoretical model in \mathbb{R} in \mathbb{R} for Based on the matching to the generator level, the t scales, POFS, POPS, POPS, POPS, AND PARTON STATISTICS, POPS, WWγ **MS** 13 **Events /bin Postfit WW**γ² SRR SPONE SRR SPONE SRR SPONE sol the productions in proceed we appenied for photons, the section shower model that productions. Predictions $\overline{1}$ $\frac{1}{1}$ **Events /bin** $\sqrt{2}$ $S_{\overline{Q}}$ ।
लेट Data **Poster WAY SEBIE Category 0 jet** The Contract in Ref. [52]. Based on the matching to the echerato **COMBET COMMOGETTE @ 13 GB-14 TEV COMMENTS** article and the submitted June **Folong de Hugh disampfe Kolone CMS WWγ @ 13 TeV [138 fb-1] Land of WWW. ANGELER HERE I SEARCH COMPUTER OF WARDERS OF WARDERS OF WARDERS OF WARDERS OF WARDERS OF WARDERS by hydroling the conduction of the conduction** art article in the second with the second control to the second control to the second control to the second co
The control of the second control to the second control to the second control to the second control to the seco scales, Particulation of the construction of the particular from a straight of the construction of the constructions. $V_γ$ **Postfit WW**_γ $MS - L$ $\frac{1}{2}$ $\frac{1$ **Events /bin** negligible. \blacksquare vvvv γ and \blacksquare must be present in the present in the present in the present in the present and the p two categories based on the stategories and interest multiplicity: 0 jet. The number of the number of events in
The number of the number o dictions after the fit to the data are listed in Table 1. The observed (expected) signal significance from the fit is 5.6 (5.1) standard deviation to the observed distribution of the observed distribution of the o
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C 11 p. 5 to 15 jet and 1 jet and pre-ata and perfect in the number of the number of events in data and pre-at dictions after the fit to the fit to the data are listed in Table 1. The observed (expected) signal sig from the fit is 5.6 (5.1) standard deviations, corresponding to the observed distribution of the observed distr
The fit is 6.6 (5.1) standard distributions after the observed distributions after the observed distributions the fit to the data shown in Fig. 3. The observed signal strength, and the observed signal strength, and the observed signal strength, $\frac{1}{4}$ 1. 15 (system and the system in a fiducial region defined by a fiducial region of the signal reg selection at particle level, with the requirements of the requirements on b jets and additional leptons. oretical prediction for the WWW. The WAG fiducial construction is 5.33 **\$2.33 ± 0.34 ± 0.34 (scale) fb at 199 NLO CONDUCTS AS EVALUATED CONDUCTS AND CONDUCTS AND MADE IN CONDUCTS AND CONDUCTS AND AND LOCAL CONDUCTS OF A REAL AND CONDUCTS ON A REAL AND CONDUCTS OF A REAL AND CONDUCTS OF A REAL AND CONDUCTS OF A REAL AND CONDUCTS O** the simultaneous fit with the uncertainties of the uncertainties of the uncertainties of the statistical, and control model to the component of the state of the system of the state of the system of the soft of the soft of the soft of the system of $\mathbf{MS} \rightarrow \mathbf{MSS}$ and the theoretical modeling uncertainty weighted the renormalization \mathbf{CMSS} \mathbf{v}_γ and parton shower modeling from all \mathbf{v}_γ and \mathbf{v}_γ are \mathbf{v}_γ spirande repriedente steffensten franczier partial trapel **Events /bin**

Category ≥ **1 jet**

Postfit WWW

$\sqrt{1677}$ mely hard to measure even at (HL-)LHC: $\overline{\text{max}}$ and $\overline{\text{max}}$

$b\bar{b}\tau\tau + b\bar{b}\gamma\gamma + b\bar{b}b\bar{b}$ **Indirect sensitivity through precision studies! M** *WW WW* **LHC pp** ^p *s* **= 5** TeV recision su

THE MNST C **PROBING H SELF INTERACTION** THE MOST CHALLENGING? **because** interference

Direct sensitivity in HH production: Progress, but extremely hard to measure even at (HL-)LHC $\overline{}$.
ctic

Investigate Quantum Field Theory at the highest energies!

BEYOND THE AIGGS: PROBING OM AT THE HIGHEST EXERGIES

jets of strongly interacting particles

BEYOND THE HIGGS: PROBING QM AT THE HIGHEST ENERGIES

ENERGY DISTRIBUTION OF THE UNIVERSE

Answering questions related to **Quantum Gravity** will require an entirely **point of view**.

o Big Bang? Black Holes? o DM? o DE? …?

DARK MATTER

Testing limitations of QFT *might* **suggest how to go beyond!**

For the first time in decades, we might not expect new particles ahead...

still thanks to the % precision physics program at colliders we have the change the change of the increasing physics more relationships we have the change of the increasing physics more relationships we have the change of support and investment in collider machines **"new types of interactions"**, and **scrutinize quantum field theory to the highest precisions** Still, thanks to the % precision physics program at colliders, we have the chance to **discover**

PRECISION STUDIES "OPPORTUNITIES" AND THE RECISION OF RECISION OF ALL ONES ALL ONES ALL ONES ALL ONES ALL ONES AND THE RECTION OF PARTY ALL ONES ARE ALL ONES ALL ONES ALL ONES ALL ONES ARE ALL ONES ARE ALL ONES ARE ALL O discrimination, CRs

<u>rd Model Production Cross Section Meas</u> **Standard Model Production Standard Model Production Cross Section Measurements** *Status: October 2023*

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Standard Model Production Cross Section Measurements

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% **PRECISION, HOW DO WE GET THERE? (AND WHEN SHOULD WE STOP?)**

FROM THEORY TO THEORY PREDICTIONS IT'S A LONG WAY!

 $L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$
+ iFBY $+ \chi_{i} y_{ij} \chi_{j} \phi + h_{c}$
+ $|\partial_{m} \phi|^{2} - V(\phi)$

PRECISION AT CONLIDERS: THE "STANDARD" FACTORIZATION PICTURE

QCD is everywhere:

strong interactions introduce extremely complex dynamics due to asymptotic freedom!

PRECISION AT COLLIDERS: THE "STANDARD" FACTORIZATION PICTURE

PRECISION AT COLLIDERS: THE "STANDARD" FACTORIZATION PICTURE

 $\boldsymbol{\Omega}$ d $\sigma = \int dx_1 dx_2 f(x_1) f(x_2) d\sigma_{part}$

$(x_1, x_2)(1 + \mathcal{O}(\Lambda_{QCD}^n/Q^n))$))

Here, we ignore all that and zoom in the so-called 'Hard Scattering'

% precision

PRECISION AT COLLIDERS: THE "STANDARD" FACTORIZATION PICTURE

Scattering Amplitude expanded in partial waves

FROM AMPLITUDES TO CROSS SECTION: IN QUANTUM MECHANICS

$$
f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l(k)} \sin \delta_l(k) P_l(\cos \theta)
$$

[Drawing from S. Gasiorowicz, Quantum Physics]

cross section $d\sigma = |f(\theta)|^2 d\Omega$

FROM AMPLITUDES TO CROSS SECTION: IN PERTURBATIVE QFT

 $\left[\mathrm{dPS} \right] |\mathcal{M}_{q\bar{q} \to gg}|$ 2

FROM AMPLITUDES TO CROSS SECTION: IN PERTURBATIVE OFT

 $\left[\mathrm{dPS} \right] |\mathcal{M}_{q\bar{q} \to gg}|$ 2

$$
\left|\mathcal{M}^{NLO}_{q\bar{q}\rightarrow gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2 \left|\mathcal{M}^{NNLO}_{q\bar{q}\rightarrow gg}\right|^2 + \ldots
$$

 $\left[\mathrm{dPS} \right] |\mathcal{M}_{q\bar{q} \to gg}|$ 2

FROM AMPLITUDES TO CROSS SECTION: IN PERTURBATIVE QFT

FROM AMPLITUDES TO CROSS SECTION: IN PERTURBATIVE QFT

$$
\sigma_{q\bar q\to gg} = \int
$$

$$
|\mathcal{M}_{q\bar{q}\to gg}|^2 = \left|\mathcal{M}_{q\bar{q}\to gg}^{LO}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)|\mathcal{M}_{q\bar{q}\to gg}^{NLO}|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2|\mathcal{M}_{q\bar{q}\to gg}^{NNLO}|^2 + \dots
$$
\n
$$
\frac{A_n^{ij, \text{ MHV}}}{\left|\frac{\alpha_n^{ij, \text{ MHV}}}{\alpha_n^{ij, \text{ MHV}}} = A_n^{\text{tree}}(1^+, 2^+, \dots, i^-, \dots, j^-, \dots, n^+)\right|}
$$
\n
$$
= \lim_{\substack{i \to \infty \\ i \to i \text{ odd} \\ j^-(i-1)^i}} \sum_{\substack{j \to \infty \\ j \to (i-1)^i \\ j^-(i-1)^i}}^{2} = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}
$$
\n
$$
= \lim_{\substack{i \to \infty \\ j \to (i-1)^i}} \sum_{\substack{j \to \infty \\ j \to (i-1)^i \\ \text{[slide from L. Dixon]}}}
$$

 $\left[\mathrm{dPS} \right] |\mathcal{M}_{q\bar{q} \to gg}|$ 2 $\int \left[\text{dPS} \right] \left| \mathcal{M}_{q\bar{q} \to gg} \right|^2$

 $\left[\mathrm{dPS} \right] |\mathcal{M}_{q\bar{q} \to gg}|$ 2

$|\mathcal{M}q\bar{q}\rightarrow gg|$ $^2 = |\mathcal{M}^{LO}_{q\bar{q} \rightarrow gg}$ $\begin{array}{c} \hline \end{array}$ $\begin{array}{c} \end{array}$ $\begin{array}{c} + \end{array}$

FROM AMPLITUDES TO CROSS SECTION: IN PERTURBATIVE QFT

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 $\sigma_{q\bar{q}\rightarrow gg} =$ z
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$|\mathcal{M}q\bar{q}\rightarrow gg|$ $^2 = |\mathcal{M}^{LO}_{q\bar{q} \rightarrow gg}$ $\begin{array}{c} \hline \end{array}$ $\begin{array}{c} \end{array}$ 2 $\begin{array}{c} + \end{array}$ $\sqrt{\alpha_s}$ 2π $\bigg)$

 $\left[\mathrm{dPS} \right] |\mathcal{M}_{q\bar{q} \to gg}|$ 2

$$
\left|\mathcal{M}^{NLO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2 \left|\mathcal{M}^{NNLO}_{q\bar{q}\to gg}\right|^2 + ...
$$

ville van die see talk by Samuel B **THE NEED OF PRECISION: TOWARDS THE** % **LEVEL**
ON AMPLITUDES AND LOOPS

0000

$$
\frac{1}{k^2(k-p_2)^2(k-p_2-p_3)^2(k-p_1-p_2-p_3)^2} = \mathcal{J}
$$

Strip it of "trivial" Lorentz and Dirac structures (dependence on spin & polarizations of external particles)

expanded in Feynman diagrams

Scalar Feynman Integrals!

For every **closed loop**, an **integral** over the **unconstrained virtual momentum** *k* of the particle circulating in the loop

ON AMPLITUDES AND LOOPS

Strip it of "trivial" Lorentz and Dirac structures (dependence on spin & polarizations of external particles)

expanded in Feynman diagrams

$$
4 \frac{k-p_1-p_2-p_3}{k} \frac{1}{k-p_2-p_3} \sim \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k-p_2)^2(k-p_2-p_3)^2(k-p_1-p_2-p_3)^2} = \mathcal{F}
$$

$$
k-p_2
$$

Scalar Feynman Integrals!

For every **closed loop**, an **integral** over the **unconstrained virtual momentum** *k* of the particle circulating in the loop

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rational functions

ON AMPLITUDES AND LOOPS

analytic structure reflects basic principles of **unitarity** and **causality**

rational functions

scalar Feynman integrals

"special functions and special numbers"

analytic structure reflects basic principles of **unitarity** and **causality**

expanded in Feynman diagrams

 $..., \pi$, $\log z$, ζ_3 , $\text{Li}_2(z)$, ...

position of **poles** and **branch cuts** dictated by *virtual particles going on-shell!*

ON AMPLITUDES AND LOOPS

AMPLITUDES FOR COLLIDERS: HOW DO WE THINK ABOUT THEM?

The integrand

Decomposition into building blocks

Decomposition into building blocks

computations of the building blocks

Decomposition into building blocks

Usually dealt with separately

Decomposition into building blocks

Connections among them largely to explore

MANY OPEN QUESTIONS AND SOME ANSWERS:

- What are general **numbers and functions** that can appear in the final result?
- How does **physics** constrain the **mathematical properties** of the result?
- What is the **"shortest"** path to the **"simplest"** form of the result?

- ……..

A possible **key to understanding these questions**:

explore interplay between mathematics of scattering amplitudes (**geometry**) and their physical properties (**singularities, discontinuities, soft/collinear limits…**)

WHAT IS AN AMPLITUDE? Dear Sir Or Madam, 1977

"just" a sum of Feynman diagrams

WHAT IS AN AMPLITUDE?

- = 50000 Feynman diagrams
- $= 10⁷$ Feynman integrals!

+ 500 more pages

gg → *gg* **@ 3 loops in QCD**

+ 500 more pages

- = 50000 Feynman diagrams
- $= 10⁷$ Feynman integrals!

 $S_i = \{$

FROM INTEGRAND TO SPECIA

$gg \rightarrow gg$ @ 3 loops in QCD

10.
$$
\frac{1}{2} \int_{e=1}^{1} \frac{1}{(2\pi)^{D}} \int_{e=1}^{e^{2}} \frac{1}{(2\pi)^{D}} \int_{e}^{e^{2}} \frac{e^{2}}{2} e^{2} \int_{e}^{
$$

FROM INTEGRAND TO SPECIAL

$gg \rightarrow gg$ @ 3 loops in QCD

- = 50000 Feynman diagrams
- $= 10⁷$ Feynman integrals!

+ 500 more pages

+ 500 more pages

- = 50000 Feynman diagrams
- $= 10⁷$ Feynman integrals!

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FROM INTEGRAND TO SPECIAL

$gg \rightarrow gg$ @ 3 loops in QCD

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w 3 loops in QCD \sim only 500 master integrals $gg \rightarrow gg$ @ 3 loops in QCD \sim "only" 500 master integrals

 \overline{z} ι *sa* @*k^µ j vµ j n*_m evnman integrals is a **finite-dimensional** vector space; master integrals are a basis in this space! where **v** *j* can be a smithov, it is possible to use the smithov. It is possible to use the u *z*-aimensional *f*(*d*; *{pⁱ · pj}*) *,* (4.4) *, (4.4) , (4.4)* *****, (4.4) , (4.4) , (4.4) , (4.4)* *****, (4.4) , (4.4) , (4.4) , (4.4)* *****, (4.4) , (4.4) , (4.4) , (4.4) , (4.4)* *****, (4.4) , (4.4) , (4.4)* Space of Feynman integrals is a *finite-dimensional vector space; master integrals are a basis in this space!* **[Smirnov, Petukhov '10]**

Master integrals can be conveniently *organized in a "tree"* depending on # of propagators

Box

Master integrals can be conveniently *organized in a "tree"* depending on # of propagators

Box

triangles (pinching 1 propagator)

Master integrals can be conveniently *organized in a "tree"* depending on # of propagators

Box

bubbles (pinching 2 propagators)

Master integrals can be conveniently *organized in a "tree"* depending on # of propagators

Box

(pinching 3 propagators)

Master integrals can be conveniently *organized in a "tree"* depending on # of propagators

Box

bubbles

(pinching 2 propagators)

tadpoles

(pinching 3 propagators)

At **1 loop** every graph has **1 master integral** *(at most)*

At *higher loops*: a "graph" can have more than one master integral

the equal-mass sunrise

At *higher loops*: a "graph" can have more than one master integral

the equal-mass sunrise

At *higher loops*: a "graph" can have more than one master integral

the equal-mass sunrise

"dot": 1 propagator squared

$$
\int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{1}{(k_1^2 - m^2)(k_2^2 - m^2)^2((k_1 - k_2 - p)^2 - m^2)}
$$

the equal-mass sunrise

double tadpole (pinching 1 propagator)

 T_{atm} T_{atm} T_{atm} of T_{atm} of the integral family that the electron self-energy at the el Fundamental difference with one loop \rightarrow hint that complexity of the problem "jumps"

FROM INTEGRAND TO SPECIAL FUNCTIONS The complexity of the necessary transformation to the canonical basis depends heavily on the chosen initial, pre-canonical integrals. We choose to start from

At *higher loops*: a "graph" can have more than one master integral

We can **differentiate** Feynman integrals w.r.t. the kinematical invariants

DIFFERENTIAL EQUATIONS FOR FEYNMAN INTEGRALS

$$
\frac{\partial}{\partial s_{ij}}\left[\int_{\ell=1}^{L}\frac{d^Dk_{\ell}}{(2\pi)^D}\frac{S_1^{a_1}\dots S_{\sigma}^{a_{\sigma}}}{D_1^{b_1}\dots D_n^{b_n}}\right]=
$$

$$
\forall s_{ij} = \{p_i \cdot p_j, m_k^2\}
$$

DIFFERENTIAL EQUATIONS FOR FEYNMAN INTEGRALS

$$
\frac{\partial}{\partial s_{ij}}\left[\int \prod_{\ell=1}^{L} \frac{d^{D}k_{\ell}}{(2\pi)^{D}} \frac{S_{1}^{a_{1}}\dots S_{\sigma}^{a_{\sigma}}}{D_{1}^{b_{1}}\dots D_{n}^{b_{n}}}\right] = \sum_{I} c_{I} \int \prod_{\ell=1}^{L} \frac{d^{D}k_{\ell}}{(2\pi)^{D}} \frac{S_{1}^{a_{1}}\dots S_{\sigma}^{a_{\sigma}}}{D_{1}^{b_{1}}\dots D_{n}^{b_{n}}} \qquad \forall s_{ij} = \{p_{i} \cdot p_{j}, m_{k}^{2}\}
$$

[Kotikov '93; Remiddi '97; Gehrmann, Remiddi '99]

We can **differentiate** Feynman integrals w.r.t. the kinematical invariants

DIFFERENTIAL EQUATIONS FOR FEYNMAN INTEGRALS

Rational functions , $A(s_{ij}, D)$

$$
\frac{\partial}{\partial s_{ij}}\left[\int_{\ell=1}^{L} \frac{d^D k_{\ell}}{(2\pi)^D} \frac{S_1^{a_1} \dots S_{\sigma}^{a_{\sigma}}}{D_1^{b_1} \dots D_n^{b_n}}\right] = \sum_{I} c_I \int_{\ell=1}^{L} \frac{d^D k_{\ell}}{(2\pi)^D} \frac{S_1^{a_1} \dots S_{\sigma}^{a_{\sigma}}}{D_1^{b_1} \dots D_n^{b_n}} \qquad \forall s_{ij} = \{p_i \cdot p_j, m_k^2\}
$$

$$
\frac{\partial}{\partial s_{ij}}\vec{I} = A(s_{ij}, D)\vec{I},
$$

[Kotikov '93; Remiddi '97; Gehrmann, Remiddi '99]

We can **differentiate** Feynman integrals w.r.t. the kinematical invariants

$$
\forall s_{ij} = \{p_i \cdot p_j, m_k^2\}
$$

$$
\frac{\partial}{\partial s_{ij}}\vec{I} = A(s_{ij}, D)\vec{I},
$$

DIFFERENTIAL EQUATIONS FOR FEYNMAN INTEGRALS *∂si* \overline{a} *k* (**A***si*)*jk*(*{sn}*, *d*)*Ik*, (2.22) where *k* runs over master integrals in the same sector and subtopologies. It is convenient

We can differentiate Feynman integrals w.r.t. the kinematical invariants

$$
\frac{\partial}{\partial s_{ij}}\left[\int_{\ell=1}^{L} \frac{d^D k_{\ell}}{(2\pi)^D} \frac{S_1^{a_1} \dots S_{\sigma}^{a_{\sigma}}}{D_1^{b_1} \dots D_n^{b_n}}\right] = \sum_{I} c_I \int_{\ell=1}^{L} \frac{d^D k_{\ell}}{(2\pi)^D} \frac{S_1^{a_1} \dots S_{\sigma}^{a_{\sigma}}}{D_1^{b_1} \dots D_n^{b_n}} \qquad \forall s_{ij} = \{p_i \cdot p_j, m_k^2\}
$$

block-triangular:

integrals with more propagators depend on ones with fewer

∂ ~ *I* = **A***si* (*{sn}*, *^d*)[~] *I*. (2.23) **DIFFERENTIAL EQUATIONS FOR FEYNMAN INTEGRALS** \overline{M} **D5** \overline{M} Table 5.1.: Definition of the integral family that the electron self-energy at two loops at two loops at two lo

 \sum coupied $n \times n$ bioens correspond \sim to graphs with *n* master coupled $n \times n$ blocks correspond to graphs with *n* master integrals

 1×1 blocks correspond to single master integrals

blocks True for general values of dimensions D ! inter sonct interview since intervention of the same sector couple to each other in general. True for *general values of dimensions D* !

∂ ~ *I* = **A***si* (*{sn}*, *^d*)[~] *I*. (2.23) **DIFFERENTIAL EQUATIONS FOR FEYNMAN INTEGRALS** \overline{M} **D5** \overline{M} Table 5.1.: Definition of the integral family that the electron self-energy at two loops at two loops at two lo

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blocks True for general values of dimensions D ! inter sonct interview since intervention of the same sector couple to each other in general. True for *general values of dimensions D* !

Equations might "decouple" close to $D = 4$ space-time dimensions a_{ij} are rational functions \rightarrow solution written iteratively in e as *iterated integrals of rational functions!* other space-time dimensions. Thus there are eight master integrals. Thus there are eight master in the second master In iteratively in ϵ as iterated integrals of rational functions! Facton willicen fictually in a do not the theory rule of the sixter functions.

OFFICIAL LEAL THE STATE OF
The story might change for **FROM DIFFERENTIAL EQUATIONS TO GEOMETRY** *^D*⁵ = (*k*² *^p*)²

The story might change for $D \to 4-2\epsilon$

MULTIPLE POLYLOGS AND THE RIEMANN SPHERE

If we integrate a rational function on \mathbb{CP}^1 **Only non-trivial thing:**

$$
\log(1 - x/a) = \int_0^x \frac{dt}{t - a}
$$

MULTIPLE POLYLOGS AND THE RIEMANN SPHERE

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$$

MULTIPLE POLYLOGS AND THE RIEMANN SPHERE

$$
G(c_1, c_2, ..., c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n)
$$

=
$$
\int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2}
$$

If we integrate a rational function on \mathbb{CP}^1 **Only non-trivial thing:**

Generalisation: **Multiple PolyLogarithms (MPLs)**

$$
\log(1 - x/a) = \int_0^x \frac{dt}{t - a}
$$

POLYLOGARITHMS, DLOG FORMS AND AMPLITUDES

Rational functions:

encode poles (*single particles going on shell*)

Generalization of MPLs:

iterated integrals over d-log forms

All information on branch cuts \rightarrow unitarity !

Enormously simplify computation of all "polylogarithmic" amplitudes → **Bootstrap program**!
Ri (*sij*) ∫*γ* d log *f* Gehrmann, Jacubčík, Mella, Syrrakos, Tancredi

Lee, 1 and 1 Steinhauser (2022) from these developments:

Collider physics profited enormously
from these developments:
calculations for the production of
1,2,3 massless particles up to **four,**
three and two loops respectively calculations for the production of **1,2,3** massless particles up to **four**, **three** and **two** loops respectively

$\overline{\mathbf{V}}$ **POLYLOGARITHMS, DLOG FORMS AND AMPLITUDES**

(production of jets at LHC) modeling of **QCD dynamics**

Ri (*sij*) ∫*γ* d log *f* Gehrmann, Jacubčík, Mella, Syrrakos, Tancredi

3-jet production in **NNLO QCD**

$\overline{\mathbf{V}}$ **POLYLOGARITHMS, DLOG FORMS AND AMPLITUDES**

[Czakon, Mitov, Poncelet '22, '23] FIG. 5: The three panels show *R*3*/*2(*H^T , y* ⇤), in each panel a **[Czakon, Mitov, Poncelet '22, '23]**

HOW GENERAL IS THIS PICTURE?

OPPE DIE ENLISHERTER
Again, look at limit $D \rightarrow 4$. **FROM DIFFERENTIAL EQUATIONS TO GEOMETRY**

Again, look at limit $D \to 4-2\epsilon$

⇤ If equation does not decouple, there is an intrinsic "higher-order equation" *(2nd order Picard-Fuchs equation)*

FROM DIFFERENTIAL EQUATIONS TO GEOMETRY WHILA I FOILATIONS TO SEONAFTINY IN THE SUNRISE GRAPH HAS A length in the literature [4, 17, 49–52], we recall here some of its properties for convenience

 $\det D \to 4-2\epsilon$ Again, look at limit $D \to 4-2\epsilon$

where we have used the dimensionless ratio \mathcal{L} defined in eq. (2.1). It is well known that close \mathcal{L}

THE TWO-LOOP ELECTRON PROPAGATOR Figure 1: *The Feynman graphs contributing to the two-loop electron self-energy.*

The electron propagator in QED; A. Sabri 1962

THE TWO-LOOP ELECTRON PROPAGATOR Figure 1: *The Feynman graphs contributing to the two-loop electron self-energy.* **002 ELECTRON** UUF LLLUINUN P \blacksquare

FROM DIFFERENTIAL EQUATIONS TO GE0METRY

\n
$$
\sum_{x} \sum_{x} \sum_{x} \sum_{y} \left(x \frac{d}{dx} \right)^2 + \left(\frac{1}{x-1} + \frac{9}{x-9} + 2 \right) \left(x \frac{d}{dx} \right) + \frac{27}{4(x-9)} + \frac{1}{4(x-1)} + \sqrt{\frac{\omega(x)}{4(x-1)}} = 0
$$

TRUM DISCUSSES

ELLIPTIC CURVES AND COMPLEX TORI

Elliptic curve given by an algebraic equation $y = \pm \sqrt{(x - a_1)(x - a_2)(x - a_3)(x - a_4)}$

ELLIPTIC CURVES AND COMPLEX TORI

Elliptic curve given by an algebraic equation

Torus is the **Riemann surface** associated to the square root with 4 branching points

$$
y = \pm \sqrt{(x - a_1)(x - a_2)(x - a_3)(x - a_4)}
$$

DIFFERENTIAL FORMS ON ELLIPTIC GEOMETRIES

entire space of functions spanned by single poles

$$
\log(1 - x/a) = \int_0^x \frac{dt}{t - a}
$$

entire space of functions spanned by single poles

single poles *^g* [∼] [∫] *dx* $(x - c_i)y$

$$
\log(1 - x/a) = \int_0^x \frac{dt}{t - a}
$$

Third kind

genus 1, elliptic curve; $y = \sqrt{P_3(x)}$

DIFFERENTIAL FORMS ON ELLIPTIC GEOMETRIES

entire space of functions spanned by single poles

$$
\log(1 - x/a) = \int_0^x \frac{dt}{t - a}
$$

single poles *^g* [∼] [∫] *dx* $(x - c_i)y$

First kind

No poles
$$
\omega \sim \int \frac{dx}{y}
$$

Third kind

Second kind

double poles
$$
\eta \sim
$$

$$
\frac{dx x}{y}
$$

genus 1, elliptic curve; $y = \sqrt{P_3(x)}$

DIFFERENTIAL FORMS ON ELLIPTIC GEOMETRIES

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY

2 "top graphs"

[**Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]**

 $\hat{p} \Sigma_{V}(p^2, m^2) + m \Sigma_{S}(p^2, m^2)$

 Σ_V & Σ_S expressed in terms of $O(50)$ Masters Integrals \vec{J}

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY

Fig. 1. Examples of the spannix property of all intigerant property corresponding to the election of the election of the spannix of the spannix spanni same elliptic curve as 2loop sunrise graph massive banana graph.

Figure 2: Planar and non-planar top sector diagrams relevant for the calculation of the L oun, sasparor. From the graphs of L and L , L and L [**Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]**

 $\hat{p} \Sigma_{V}(p^2, m^2) + m \Sigma_{S}(p^2, m^2)$

 Σ_V & Σ_S expressed in terms of $\mathcal{O}(50)$ Masters Integrals *J*

d*J* \bar{J}

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY d
G_z
*G*_{*i*} *f*
*G*_{*i*} *G*_{*I*} *G*_{*D*} *G*_{*B*} JULAI IUN: THE THREE-LOOP QED SELF-ENERGY

[Görges, Nega, Tancredi, Wagner '23] The geometrical picture allows us to solve differential equations

 $= \epsilon$ $\sqrt{}$ *i* $G_i \, \omega_i$! *J .* \vec{J} \longleftrightarrow $f_i(x)dx = \omega_i$ such that the system of differential equations takes the form

the above topologies through 18 master integrals. Most of the course Differential forms $\overline{}$ p \overline{a} X ! **Differential forms** can be classified using properties of **the underlying geometry**

[Duhr, Gasparotto, Nega, Tancredi, Wei [**Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]**

 $\hat{p} \Sigma_{V}(p^2, m^2) + m \Sigma_{S}(p^2, m^2)$

 Σ_V & Σ_S expressed in terms of $\mathcal{O}(50)$ Masters Integrals \vec{J} $\Delta V \propto \Delta g$ capitssed in terms of $\sigma(\omega)$ masters integrals J

angulation calculation of the electron $\mathcal{L}_{S,0}$ (101 s \mathcal{L}_{S}). **Figure 4:** Real and imaginary part of $\Sigma_{S,0}^{(3)}$ (for $\xi = 0$).

Full analytic and numerical control of result across all values of the momentum p^2

Figure 6: Real and imaginary part of $\Sigma_{S,0}^{(3)}$ close to $x_0 = 9$ (for $\xi = 0$). **Figure 6**: Real and imaginary part of $\Sigma_{S,0}^{(3)}$ close to $x_0 = 9$ (for $\xi = 0$).

$$
(p^2, m^2) + m \Sigma_S(p^2, m^2)
$$

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY

[**Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]**

[Forner Nega, Tancredi '24]

IA ¹¹¹¹¹¹⁰⁰⁰ *, I^A* ⁰¹²¹¹⁰¹¹⁰ *, I^A* ⁰¹¹⁰¹¹²¹⁰ *, I^A* ¹¹¹⁰¹¹¹¹⁰ *, I^A* $Im \sim \left(p^2 g_{\mu\nu} - p^{\mu} p^{\nu} \right) \Pi(p^2)$ *IB* ⁰⁰¹¹¹⁰¹¹⁰ *, I^B* ¹⁰¹¹¹⁰¹¹⁰ *, I^B* ¹¹⁰¹¹⁰²¹⁰ *, I^B* ²¹⁰¹¹⁰²¹⁰ *, I^B* $\left(p^2 g_{\mu\nu} - p^{\mu} p^{\nu}\right) \Pi(p^2)$

 $\prod(n^2)$ or proceed in energy up to the three loops. In the area one, there are one, there are one, then \mathcal{L} ⁰¹¹¹⁰⁰²¹¹ *, I^B* ⁰¹¹¹⁰⁰¹²¹ *, I^B* ⁰¹¹¹⁰⁰³¹¹ *, I^B* ¹¹¹¹¹⁰¹¹¹ *,* $\Pi(p^2)$ expressed in terms of $\mathcal{O}(36)$ Masters Integrals \bar{J}

AN EXAMPLE CALCULATION: BEYOND FILIPTIC PHOTONS

2 "top graphs"

AN EXAMPLE CALCULATION: BEYOND ELLIPTIC PHOTONS IEVANN ELLIPTIC PHATANS

³ *^m*² *^k*² ³ *k*² 3 This time there is a **three-massive particle cut** \rightarrow three-loop banana associated to a K3 geometry *p* (*k*₂ *b*)² (*k*₂ *p*)² (*k*₃ (*k*)² (*k*)² (*k*)² (*k*)² (*k*)² (*k*) can be thought of as a higher dimensional elliptic curv d ime energiare el elliptic guyma $\frac{1}{\sqrt{5}}$ can be thought of as a higher dimensional elliptic curve

[Forner Nega, Tancredi '24] (*a*² *a*1)(*a*⁴ *a*3) (*a*) *ancreal* 24

IA ¹¹¹¹¹¹⁰⁰⁰ *, I^A* ⁰¹²¹¹⁰¹¹⁰ *, I^A* ⁰¹¹⁰¹¹²¹⁰ *, I^A* ¹¹¹⁰¹¹¹¹⁰ *, I^A* $\lim_{M \to \infty} \lim_{M \to \infty}$ *IB* ⁰⁰¹¹¹⁰¹¹⁰ *, I^B* ¹⁰¹¹¹⁰¹¹⁰ *, I^B* ¹¹⁰¹¹⁰²¹⁰ *, I^B* ²¹⁰¹¹⁰²¹⁰ *, I^B* $\left(p^2 g_{\mu\nu} - p^{\mu} p^{\nu}\right) \Pi(p^2)$ λ \mathcal{L} $\begin{pmatrix} 1 & \text{O}\mu\text{V} & 1 & 1 \end{pmatrix}$ and the this section might seem somewhat redundant, as the theory of elliptic integrals has been very well understood for a long time. Nevertheless, when considering the

 $\prod(n^2)$ or proceed in energy up to the three loops. In the area one, there are one, there are one, then \mathcal{L} α $\Pi(p^2)$ expressed in terms of $\mathcal{O}(36)$ Masters Integrals \vec{J}

AN EXAMPLE CALCULATION: BEYOND ELLIPTIC PHOTONS

 $\left(p^2 g_{\mu\nu} - p^{\mu} p^{\nu}\right) \Pi(p^2)$ $\left\{\begin{array}{c}1 \end{array}\right\}$ self-energy develops a square-root branch cut from the threshold *x* = 4 to *x* = 1. Before

[Forner Nega, Tancredi '24]

Eollowing same approach: derive and solve differential equation three loops, respectively. The contract velocity of \mathcal{C} wing same approach: derive and solve differential Following same approach: derive and solve differential equations using properties of K3 geometry

ELLIPTIC AND BEYOND: HIGHER GENUS AND HIGHER DIMENSION seit the the dedications, along with a british with a brief and with a brief and α

This picture holds for **many other elliptic cases** (more to appear hopefully soon!) micture holds for many other elliptic cases (more to annear hopefully finite reminder for the form $\frac{1}{2}$ alongside with $\frac{1}{2}$ files with $\frac{1}{2}$ files $\frac{1}{2}$ and $\frac{1}{2}$

-
- And for **more general geometries**, with obvious generalizations: higher-order eqs, more "solutions"...

aamm

BEYOND ELLIPTICS "CALABI-YAUS IN THE SKY"

Recently, CY geometries have been shown to be indispensable to **model gravitation waves** in **Post-Minkoskian expansion**

[Klemm, Nega, Sauer, Plefka '24; Frellesvig, Morales, Willhelm '23] [Bern, Parra-Martinez, Roiban, Ruf, Shen '21,…'24]

CONCLUSIONS AND OUTLOOK

- amplitudes are *fundamental building blocks in QFT*, for precision collider physics and beyond
- complexity of the calculations is often matched by **unexpected simplicity in final results**
- searching for a way to make **simplicity manifest** informs on how to compute amplitudes more efficiently *(language of differential forms on complex varieties is an example!)*
- what we learnt in past 10 years is finally **bearing fruit**: the first realistic "correlators" and amplitudes under *analytic and numerical control*
- same structures observed in *gravitational waves calculations* and *cosmological corrrelators!*

THANK YOU VERY MUCH!

BACK-UP SLIDES

From the case, you might be likely enough to be a singularities, we can use a particular way to find the following equations:																																			
$S^2 = S^2$	$S^3 = S^2$	$S^3 = S^2$	$S^3 = S^3$ </td																																

 \sim 1. To any regular singular point, this equation admits two solutions, a regular one and one and

second independent solution from second integration contour **[Primo, Tancredi '16,'17]**

 Σ_V & Σ_S expressed in terms of $\mathcal{O}(50)$ Masters Integrals \bar{J} *,*

 $\varpi_0(x)$ is the first elliptic period $\omega_0(x)$ is the modernique period

$$
(p^2, m^2) + m \Sigma_S(p^2, m^2)
$$

$$
f_i \in \left\{ \frac{1}{x(x-1)(x-9)\varpi_0(x)^2}, \varpi_0(x), \frac{\varpi_0(x)}{x-1}, \frac{(x-3)\varpi_0(x)}{\sqrt{(1-x)(9-x)}}, \frac{(x+3)^4\varpi_0(x)^2}{x(x-1)(x-9)}, \frac{(x+3)(x-1)\varpi_0(x)^2}{x(x-9)}, \frac{\varpi_0(x)^2}{(x-1)(x-9)} \right\}
$$
 for $i = 10, ..., 16$,

 \overline{D} ^p(3 + *^x*)(1 *^x*) *,* $\overline{1}$ *,* rol over iter ^p(1 *^x*)(9 *^x*) **integrals** over these forn 7 (independent) elliptic differential forms: full analytic control over **iterated integrals** over these forms

ancillary file to this manuscript. WAMPLE UALUULAHUN: THE THREE-LOOP QED SELF-ENERGY **AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY**

of 16 differential forms !*ⁱ* = *fi*(*x*)d*x*. 9 of these differential forms are dlog forms, and the remaining 7 involvements 7 involvements 7 involvements and Picard-Fuchs equation (2.11). By using the notation
The notation of the notation (2.11). By using the notation (2.11). By using the notation (2.11). By using the [**Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]** Σ_V & Σ_S expressed in terms of $\mathcal{O}(50)$ Masters Integrals \bar{J} *,*

the above topologies through 18 master integrals. Most of these integrals have been calculated in the course of the

analytical calculation of the electron anomalous magnetic moment [10] and can be taken from there. It is remarkable

$$
(p^2, m^2) + m \Sigma_S(p^2, m^2)
$$

$$
f_i \in \left\{ \frac{1}{x(x-1)(x-9)\varpi_0(x)^2}, \varpi_0(x), \frac{\varpi_0(x)}{x-1}, \frac{(x-3)\varpi_0(x)}{\sqrt{(1-x)(9-x}}, \frac{(x+3)^4\varpi_0(x)^2}{x(x-9)}, \frac{(x+3)(x-1)\varpi_0(x)^2}{x(x-9)}, \frac{(x+3)(x-1)\varpi_0(x)^2}{(x-1)(x-9)} \right\}
$$

3 of the terms drop in the physical amplitude they are related to forms of the second kind with

 \overline{D} ^p(3 + *^x*)(1 *^x*) *,* $\overline{1}$ *,* rol over iter ^p(1 *^x*)(9 *^x*) **integrals** over these forn 7 (independent) elliptic differential forms: full analytic control over **iterated integrals** over these forms

> α double poles" \rightarrow a hint for bootstrap program? 3 of the kernels drop in the physical amplitude: they are related to forms of the second kind with

the regular singular point *x*0. Explicitly, they are given in eqs. (2.12) to (2.15). To simplify

ancillary file to this manuscript. WAMPLE UALUULAHUN: THE THREE-LOOP QED SELF-ENERGY **AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY**

of 16 differential forms !*ⁱ* = *fi*(*x*)d*x*. 9 of these differential forms are dlog forms, and the remaining 7 involvements 7 involvements 7 involvements and Picard-Fuchs equation (2.11). By using the notation
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[Duhr, Gaspa [**Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]**

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY

$$
\Sigma_{V, \text{res}}^{(3)} = \left[-\frac{27}{128\epsilon^3} - \frac{673}{1152\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right) \right] + (1 - x)^{-2\epsilon} \left[\frac{27}{64\epsilon^3} + \frac{27}{32\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right) \right] \n+ (1 - x)^{-4\epsilon} \left[-\frac{27}{128\epsilon^3} - \frac{27}{128\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right) \right] + \mathcal{O}(1 - x), \n\Sigma_{S, \text{res}}^{(3)} = \left[-\frac{653}{1152\epsilon^3} + \frac{1447}{6912\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right) \right] + (1 - x)^{-2\epsilon} \left[\frac{1}{x - 1} \left(\frac{27}{32\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right) \right) \n+ \frac{9}{16\epsilon^3} - \frac{91}{256\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right) \right] + (1 - x)^{-4\epsilon} \left[\frac{27}{128\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right) \right] + \mathcal{O}(1 - x),
$$

$$
(p^2, m^2) + m \Sigma_S(p^2, m^2)
$$

wip requiremed requise close to an shall limit $n^2 - m^2$ required for LIV reportion One can obtain resummed results close to on-shell limit $p^2 = m^2$ required for UV renormalization

This picture holds for **many other elliptic cases: Bhabha scattering**

Bhabha is a "standard-candle" process in Quantum Electrodynamics: $e^+ e^- \rightarrow e^+ e^-$

At leading order (tree level) just two diagrams **Classical calculation in QFT 1 courses**

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)} = \frac{\pi\alpha^2}{s}\left(u^2\left(\frac{1}{s} + \frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right)
$$

Cross section *(for massless electrons)* on Wikipedia!

MORE ELLIPTICS ELECTRON-ELECTRON SCATTERING

MORE ELLIPTICS ELECTRON-ELECTRON SCATTERING there are a total of 47 Feynman diagrams; however, many of these diagrams generate identical results. Of the 47 diagrams, 35 contain no fermion loop, 11 contain one fermion loop, and 1 contains correct UV and IR behavior, as illustrated above. In addition, we consider the bare and the bare and the finite remaining \sim FI FOTDAU COATTEDINO display the various orders of the series for the two-

This picture holds for many other elliptic cases: 2loop Bhabha scattering \mathcal{L} Ecases: 2100p bilabila scattering

