Second order transport coefficients for realistic equations of state with chemical potential

Outline

- Introduction and motivations
- Quasiparticles and their link to the Wigner formalism in quantum field theories
- Kinetic like method to extract second order viscous hydrodynamics, and its transport coefficients

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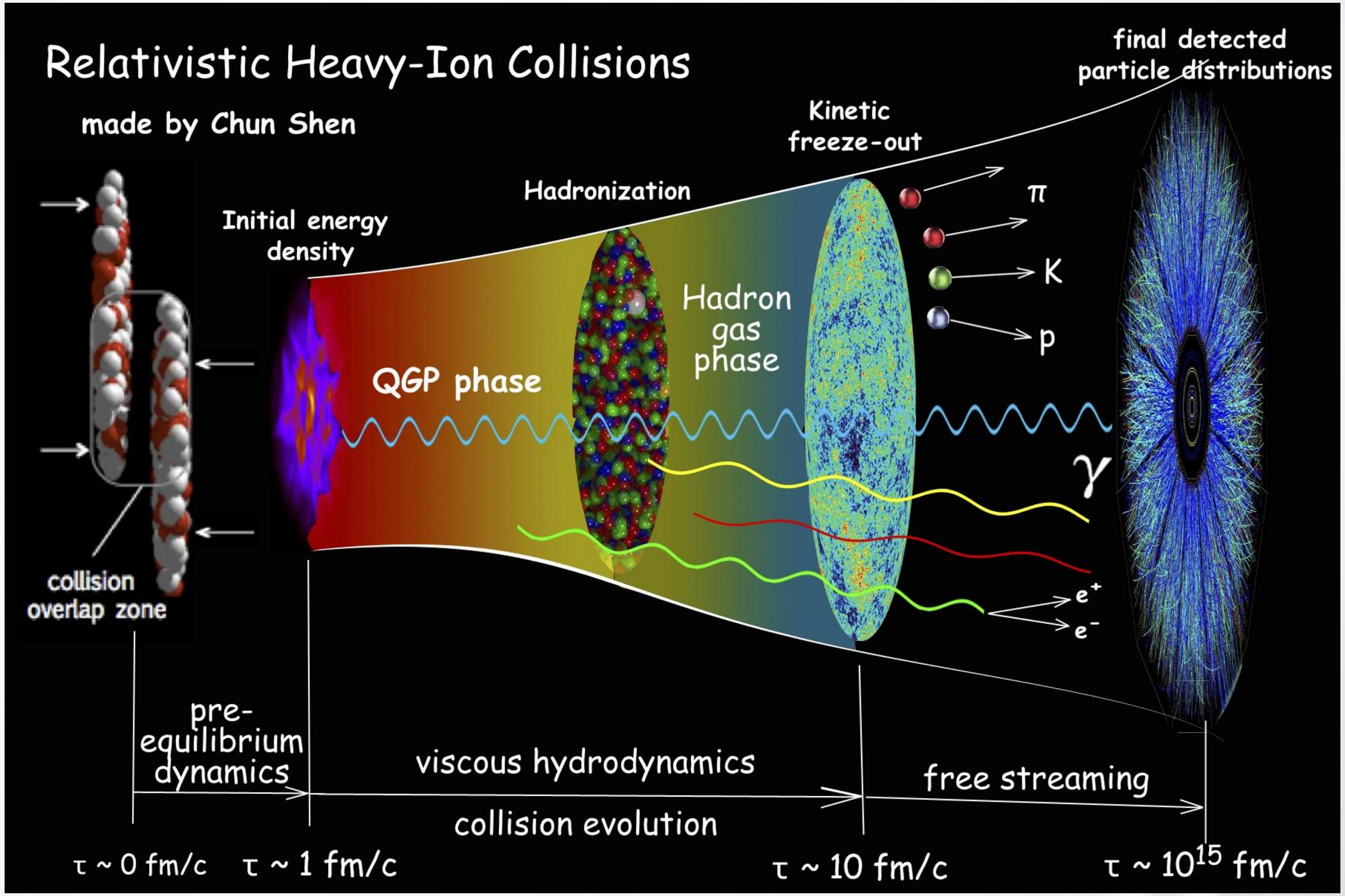
Motivations

What we do (now, mostly)

- Initial conditions
  (Monte Carlo Glauber, color glass condensate, etc...)
- Pre-hydro smoothening
  (gaussians, free-streaming partons, etc...)
- Hydrodynamics
  (ideal, second-order, aHydro, etc...)
- Hadronization
  (direct freeze-out or rescattering)



Motivations

Hydrodynamics as an intermediate step between the initial and final stages

- The main equations are rather solid:

$$\partial_\mu T^{\mu\nu} = 0$$

(also $\partial_\mu J^\mu = 0$, BES high density systems?)

$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$

- The equation of state is enough for ideal hydrodynamics:

$$T^{\mu\nu} = \mathcal{E}\, u^\mu u^\nu - \mathcal{P}\Delta^{\mu\nu}$$

$$J^\mu = \rho u^\mu$$

(6 degrees of freedom, 5 conservation equations, 1 EOS)

- The viscous corrections are still needed (AdS-CFT, experiments…)

Motivations

It would be nice to have a single, consistent way to extract hydrodynamics

$$u_\mu T^{\mu\nu} \stackrel{\text{def}}{=} \mathcal{E}\, u^\nu$$

- General decomposition (ideal and non-ideal part):

$$T^{\mu\nu} = \mathcal{E}\, u^\mu u^\nu - (\mathcal{P} + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$J^\mu = \rho\, u^\mu + \nu^\mu$$

- Hydrodynamics ⇒ how to treat the rest,

eg $\tau_\pi\, \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\, \sigma^{\mu\nu} + \cdots$ (other second order terms)

- Complications using the same framework as the EOS (integrals of commutators)

How to fix the transport coefficients?

(kinetic theory would be handy)

The relativistic Boltzmann equation

$$p \cdot \partial f = \mathcal{C}[f] = -\frac{(p \cdot u)}{\tau_{eq}} \delta f$$

RTA

$$\delta f \simeq -\frac{\tau_{eq}}{(p \cdot u)} (p \cdot \partial f_0)$$

covariant momentum integral

$$\frac{g}{(2\pi)^3} \int d^4p \; 2\, \Theta(p_0) \delta(p^2 - m^2) \stackrel{\text{def}}{=} \int_{\boldsymbol{p}}$$

also

$$T^{\mu\nu} = \int_{\boldsymbol{p}} p^\mu p^\nu f$$

after some algebra

$$\Rightarrow \pi^{\mu\nu} = \int_{\boldsymbol{p}} p^{\langle\mu} p^{\nu\rangle} \delta f \simeq -\tau_{eq} \int_{\boldsymbol{p}} \frac{p^{\langle\mu} p^{\nu\rangle}}{(p \cdot u)} [p \cdot \partial (e^{-\beta(p \cdot u)})] = 2\, (\tau_{eq}\, \beta_\pi)\, \sigma^{\mu\nu}$$

many ways to extend, see G Denicol, J.Phys. G41 (2014) no.12, 124004

$$u \cdot \partial f = \dot{f} = -\frac{p \cdot \nabla f}{(p \cdot u)} - \frac{C[f]}{(p \cdot u)}$$

exact equations

How to fix the transport coefficients?

(kinetic theory would be handy)

$$\mathcal{O}^{\langle\mu_1\rangle\cdots\langle\mu_l\rangle} = \Delta^{\mu_1}_{\alpha_1}\cdots\Delta^{\mu_l}_{\alpha_l}\mathcal{O}^{\alpha_1\cdots\alpha_l}$$

a convenient basis

$$\mathfrak{f}^{\mu_1\cdots\mu_l}_r = \int_p (p\cdot u)^r\, p^{\langle\mu_1\rangle\cdots\langle\mu_l\rangle} f$$

for instance

$$\mathcal{E} = \mathfrak{f}_2, \qquad \mathcal{P}^{\langle\mu\rangle\langle\nu\rangle} = -(\mathcal{P}+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} = \mathfrak{f}^{\mu\nu}_0,$$

a popular decomposition of the degrees of freedom

$$\partial_\mu u_\nu = u_\mu \dot{u}_\nu + \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3}\theta\Delta_{\mu\nu}$$

lots of self interactions in the exact evolution

$$\dot{\mathcal{P}}^{\langle\mu\rangle\langle\nu\rangle} + C^{\langle\mu\rangle\langle\nu\rangle}_{-1} = 2(\mathcal{P}+\Pi)\sigma^{\mu\nu} + \frac{5}{3}\theta(\mathcal{P}+\Pi)\Delta^{\mu\nu} - \frac{5}{3}\theta\pi^{\mu\nu} - 2\pi^{(\mu}_{\alpha}\sigma^{\nu)\alpha} + 2\pi^{(\mu}_{\alpha}\omega^{\nu)\alpha}$$
$$-\nabla_\alpha \mathfrak{f}^{\alpha\langle\mu\rangle\langle\nu\rangle}_{-1} - \left(\sigma_{\alpha\beta} + \frac{1}{3}\theta\Delta_{\alpha\beta}\right)\mathfrak{f}^{\alpha\beta\mu\nu}_{-2}$$

L T, G Vujanovich, J Noronha, U Heinz, Phys. Rev. D 99, 016009

A Jaiswal, R Ryblewski, M Strickland, Phys. Rev. C 90, 044908

Shortcomings of the relativistic kinetic theory

(thermodynamic consistency)

Ideal equation of state $\mathcal{P} = \sum_i \mathcal{P}_i = T \sum_i \mathcal{N}_i$ (in the Boltzmann limit)

Quasiparticles instead (*a historic look*)

$$T^{\mu\nu} = T^{\mu\nu}_{\text{kin}} + B^{\mu\nu}$$

$$p^\mu \partial_\mu f_i + \frac{1}{2} \partial_\mu M_i^2 \frac{\partial f}{\partial p_\mu} = -\frac{(p_\mu u^\mu)}{\tau_{eq}} \delta f_i$$

- Medium dependent mass(-es)
- Needs a bag (to fit the EOS)
- Non-equilibrium bag too (local conservation of charges)
- Misunderstandings? (positivity of the f_i, $\int_p p^\mu \sum_i \mathcal{C}_i = 0$)

strongly interacting liquid??

L T, A Jaiswal, R Ryblewski, Phys. Rev. D 95, 054007

Digression about quantum field theory

(and how kinetic theory stems from it)

Quantum operators

$$T^{\mu\nu} = \mathrm{tr}\left(\hat{\rho}\, \hat{\mathrm{T}}^{\mu\nu}\right), \qquad \mathrm{J}^{\mu} = \mathrm{tr}\left(\hat{\rho}\, \hat{J}^{\mu}\right)$$

From the Lagrangian density

$$\hat{\mathcal{L}} = \sum_i \hat{\mathcal{L}}_{0,i} + \hat{\mathcal{L}}_{\mathrm{int}}$$

one has

$$T^{\mu\nu} = \sum_i T^{\mu\nu}_{0,i} + T^{\mu\nu}_{\mathrm{int}}$$

for scalars

$$T_0^{\mu\nu} = \int d^4p\, p^\mu p^\nu\, W(x,p), \qquad J^\mu = q \int d^4p\, p^\mu \mathrm{W}(\mathrm{x},\mathrm{p}),$$

with

$$W(x,p) = \frac{2}{(2\pi)^4} \int d^4v\, e^{-ip\cdot v}\, \mathrm{tr}\left(\hat{\rho}\, \hat{\Phi}^\dagger(x+v/2)\hat{\Phi}(x-v/2)\right)$$

- Relativistic Kinetic Theory. Principles and Applications - De Groot, S.R. et al. Amsterdam, Netherlands: North-Holland (1980)

Digression about quantum field theory

(and how kinetic theory stems from it)

$$T^{\mu\nu} = \sum_i T^{\mu\nu}_{0,i} + T^{\mu\nu}_{\text{int}}$$

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overdetermined system of equations

From the Klein-Gordon equation

$$\left[\frac{1}{4}\hbar^2\Box - \left(k^2 - m^2c^2\right) + i\hbar k\cdot\partial\right] W(x,k) = \cdots$$

- T. S. Biro and A. Jakovac, *Emergence of Temperature in Examples and Related Nuisances in Field Theory*, Springer Briefs in Physics (2019)
- Relativistic Kinetic Theory. Principles and Applications - De Groot, S.R. et al. Amsterdam, Netherlands: North-Holland (1980)

Better to introduce quasiparticles here

(without assuming the kinetic limit)

Single weight for the current

$$W_b(x,p) = \frac{g_q}{(2\pi)^3} 2\Theta(p_0)\delta(p^2 - M^2(x))\, f^q(x,p) + \frac{g_q}{(2\pi)^3} 2\Theta(-p_0)\delta(p^2 - M^2(x))\, f^{\bar{q}}(x,-p)$$

no approximation!

therefore

$$J^\mu = q\int d^4p\, p^\mu W_b = q\,\frac{g_q}{(2\pi)^3}\int \frac{d^3p}{E_p}\, p^\mu f^-, \qquad f^- = f^q - f^{\bar{q}}.$$

Ansatz

$$p\cdot\partial f^\pm + \frac{1}{2}\partial_\mu M^2 \frac{\partial f^\pm}{\partial p_\mu} = -\frac{(p\cdot u)}{\tau_{eq}}\delta f^\pm$$

the first approximation

From the baryon number conservation

$$\int d^4p\, p^\mu C_b^- = 0$$

Better to introduce quasiparticles here

(without assuming the kinetic limit)

not only baryon carriers, also

$$W(x,p) = \frac{g}{(2\pi)^3} 2\Theta(p_0)\delta(p^2 - m^2(x))\, f^1(x,p) + \frac{g}{(2\pi)^3} 2\Theta(-p_0)\delta(p^2 - m^2(x))\, f^2(x,-p)$$

convenient, non necessary

$$T^{\mu\nu} = q\int d^4p\, p^\mu p^\nu (W + W_b) + B^{\mu\nu} = \int_{\boldsymbol{p}} p^\mu p^\nu f + \int_{\boldsymbol{q}} p^\mu p^\nu f^+ + B^{\mu\nu}$$

and also

$$p\cdot\partial f + \frac{1}{2}\partial_\mu m^2 \frac{\partial f}{\partial p_\mu} = -\frac{(p\cdot u)}{\tau_{eq}}\delta f$$

the second one

now, instead

$$\partial_\mu B^{\mu\nu} + \frac{u_\mu}{\tau_{eq}}\delta B^{\mu\nu} + m\partial^\nu m\int_{\boldsymbol{p}} f + M\partial^\nu M\int_{\boldsymbol{q}} f^+ = 0$$

Thermodynamics fixes the equilibrium bag

$$f_0^{\pm} = (e^{q\alpha} \pm e^{-q\alpha})e^{-\beta(p\cdot u)}$$
$$f_0^{\pm} = e^{-\beta(p\cdot u)}$$

density fixes one mass

$$\rho = \rho_0 = \frac{q g_q}{\pi^2}\,\sinh(\mathrm{q}\alpha)\frac{\beta^2 \mathrm{M}^2(\alpha,\beta)}{\beta^3}\mathrm{K}_2\big(\beta M(\alpha,\beta)\big) = \rho_{eq}(\alpha = \mu/T, \beta = 1/T)$$

the sum of energy and pressure fixes the other, their subtraction fixes the equilibrium bag

$$\mathcal{E}_0(\alpha,\beta) + \mathcal{P}_0(\alpha,\beta) = \mathcal{E}_{eq}(\alpha,\beta) + \mathcal{P}_{eq}(\alpha,\beta)$$

$$B_{eq}^{\mu\nu} = B_0(\alpha,\beta) g^{\mu\nu}$$

$$\mathcal{E}_{eq}(\alpha,\beta) - \mathcal{P}_{eq}(\alpha,\beta) = \mathcal{E}_0(\alpha,\beta) - \mathcal{P}_0(\alpha,\beta) + 2\,B_0(\alpha,\beta)$$

after the mases are fixed

L T, A Jaiswal, R Ryblewski, Phys. Rev. D 95, 054007

Dynamics fixes the non-equilibrium bag

$$\partial_\mu B^{\mu\nu} + \frac{u_\mu}{\tau_{eq}} \delta B^{\mu\nu} + m\partial^\nu m \int_{\boldsymbol{p}} f + M\partial^\nu M \int_{\boldsymbol{q}} f^+ = 0$$

four-momentum conservations

while, from the Gibbs-Duhem relations

$$\partial^\nu B_0 + m\partial^\nu m \int_{\boldsymbol{p}} f_0 + M\partial^\nu M \int_{\boldsymbol{q}} f_0^+ = 0$$

choosing the specific non-equilibrium bag

$$\delta B^{\mu\nu} = b_0 g^{\mu\nu} + b^\mu u^\nu + b^\nu u^\mu, \qquad b \cdot u = 0$$

$$\dot{b}_0 + \frac{b_0}{\tau_{eq}} = b \cdot \dot{u} - (\partial \cdot b) + m\dot{m} \int_{\boldsymbol{p}} \delta f + M\dot{M} \int_{\boldsymbol{p}} \delta f^+ = 0$$

$$\dot{b}^{\langle\mu\rangle} + \frac{b^\mu}{\tau_{eq}} = -\nabla^\mu b_0 - \theta b^\mu - (b \cdot \partial) u^\mu + m\,\nabla^\mu m \int_{\boldsymbol{p}} \delta f + M\nabla^\mu M \int_{\boldsymbol{p}} \delta f^+ = 0$$

second order

L T, A Jaiswal, R Ryblewski, Phys. Rev. D 95, 054007

Second order viscous hydrodynamics
(like the previous paper)

Keeping all the native self-interactions from the generalization of

$$\dot{\mathcal{P}}^{\langle\mu\rangle\langle\nu\rangle} + C_{-1}^{\langle\mu\rangle\langle\nu\rangle} = 2(\mathcal{P}+\Pi)\sigma^{\mu\nu} + \frac{5}{3}\theta(\mathcal{P}+\Pi)\Delta^{\mu\nu} - \frac{5}{3}\theta\pi^{\mu\nu} - 2\pi_{\alpha}^{(\mu}\sigma^{\nu)\alpha} + 2\pi_{\alpha}^{(\mu}\omega^{\nu)\alpha}$$
$$-\nabla_{\alpha}\mathfrak{f}_{-1}^{\alpha\langle\mu\rangle\langle\nu\rangle} - \left(\sigma_{\alpha\beta} + \frac{1}{3}\theta\Delta_{\alpha\beta}\right)\mathfrak{f}_{-2}^{\alpha\beta\mu\nu}$$

as well as the ν^{μ} evolution, plugging an approximation for the f and $f^{\pm}$ in the non-hydrodynamic tensors. Namely

- First order approximation in the gradients (from $\delta f \simeq -\tau_{eq}[p\cdot\partial f_0 + m\partial m\partial_{(p)} f_0](p\cdot u)$)
- Make the substitution (first order equations) $$\sigma^{\mu\nu} \to \frac{\pi^{\mu\nu}}{2\eta}, \qquad \theta \to -\frac{\Pi}{\zeta}, \qquad \nabla^{\mu}\alpha \to \frac{\nu^{\mu}}{\kappa_b}.$$

(the latter is to avoid mathematical instabilities)

L T, A Jaiswal, R Ryblewski, Phys. Rev. D 95, 054007

Second order viscous hydrodynamics

Obtaining

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{1}{\tau_{eq}}\pi^{\mu\nu} = \frac{2\eta}{\tau_{eq}}\sigma^{\mu\nu} - 2\pi_{\lambda}^{\langle\mu}\omega^{\nu\rangle\lambda} + \tau_{\pi\pi}\,\pi_{\lambda}^{\langle\mu}\sigma^{\nu\rangle\lambda} + \delta_{\pi\pi}\,\theta\,\pi^{\mu\nu} + \lambda_{\pi\Pi}\,\Pi\,\sigma^{\mu\nu}$$
$$+\tau_{\pi\nu}\,\nu^{\langle\mu}\dot{u}^{\nu\rangle} - \gamma_{\pi\nu}\,\nu^{\langle\mu}\,\nabla^{\nu\rangle}\alpha - \Delta^{\mu\nu}_{\alpha\beta}\nabla_{\lambda}\left[l_{\pi\nu}\left(\Delta^{\lambda\alpha}\,\nu^{\beta} + \Delta^{\lambda\beta}\,\nu^{\alpha}\right)\right]$$

$$\dot{\Pi} + \frac{1}{\tau_{eq}}\Pi = -\frac{\zeta}{\tau_{eq}}\theta + \delta_{\Pi\Pi}\,\theta\Pi + \lambda_{\Pi\pi}\,(\sigma:\pi) - \tau_{\Pi\nu}\,(\dot{u}\cdot\nu) + \mathrm{l}_{\Pi\nu}\,(\partial\cdot\nu)$$
$$+n_{\Pi\nu}\,(\nu\cdot\nabla)\alpha + \frac{5}{3}\nabla\cdot(l_{\pi\nu}\,\nu)$$

$$\dot{\nu}^{\langle\mu\rangle} + \frac{1}{\tau_{eq}}\nu^{\mu} = -\frac{\kappa_b}{\tau_{eq}}\nabla^{\mu}\alpha + \tau_{\nu\Pi}\,\Pi\dot{u}^{\mu} + c_{\pi\Pi}\left(\nabla^{\mu}\Pi - \Delta^{\mu}_{\alpha}\partial_{\beta}\pi^{\alpha\beta}\right) + \delta_{\nu\nu}\,\theta\nu^{\mu}$$
$$+c_{\nu\Pi}\,\Pi\nabla^{\mu}\alpha + \Delta^{\mu}_{\alpha}\nabla_{\beta}\left[l_{\nu\Pi}\left(\Pi\Delta^{\alpha\beta} - \pi^{\mu\nu}\right)\right] - \lambda_{\nu\nu}\,\sigma^{\mu\lambda}\nu_{\lambda} + \omega^{\mu\lambda}\nu_{\lambda}$$

Summary and outlook

- We generalized the quasiparticle treatment for $\mu \neq 0$
- Second order transport coefficients, thermodynamic consistency
- Link to quantum field theory, and possible further generalizations

Thank you for your attention!

Back up slides

Simplest case: free streaming

Exact solutions in 1+1 dimensions

$w = zk^0 - tk^z$

$$W(t,z;k^0,k_T,k^z) = \delta(k^0)\delta(k^z)\int d\xi\left[e^{-i\left(t\sqrt{4m_T^2+\xi^2}-z\,\xi\right)}\mathcal{A}(\xi;k_T) + e^{i\left(t\sqrt{4m_T^2+\xi^2}-z\,\xi\right)}\mathcal{A}^*(\xi;k_T)\right]$$

$$+\cos\left(2w\sqrt{\frac{k^2-m^2}{(k^0)^2-(k^z)^2}}\right)\mathcal{F}_{\rm even}(k_0,k_T,k^z) + \sqrt{\frac{(k^0)^2-(k^z)^2}{k^2-m^2}}\,\sin\left(2w\sqrt{\frac{k^2-m^2}{(k^0)^2-(k^z)^2}}\right)\mathcal{F}_{\rm odd}(k_0,k_T,k^z)$$

Proper classical limit

$$T^{\mu\nu}(x) = \int d^4k\; k^\mu k^\nu W(x,k) \xrightarrow{\text{small } \hbar} \int \frac{d^3p}{(2\pi\hbar)^3 E_{\mathbf{p}}/c}\, p^\mu p^\nu \left[f(x,\mathbf{p}) + \bar{f}(x,\mathbf{p})\right],$$

$$J^\mu = \int d^4k\; k^\mu W(x,k) \xrightarrow{\text{small } \hbar} \int \frac{d^3p}{(2\pi\hbar)^3 E_{\mathbf{p}}/c}\, p^\mu \left[f(x,\mathbf{p}) - \bar{f}(x,\mathbf{p})\right].$$

10.1103/PhysRevD.108.076022

Simplest case: free streaming

Classical limit of the exact solutions

$$\varepsilon = \frac{\hbar}{A} \qquad \tilde{w} = \frac{w}{A}$$

$$\frac{\theta(\chi)\theta(2-\chi)}{(2\pi\varepsilon)} \cos\left(\tilde{w}\frac{\chi}{\varepsilon}\right)\ \tilde{f}_{\rm even}\left(\frac{\chi}{\varepsilon}; k^0, k_T, k^z\right) \xrightarrow{\varepsilon\to 0^+} \frac{1}{2}\delta(\chi) \int \frac{d\chi'}{(2\pi)} \cos\left(\tilde{w}\chi'\right) \tilde{f}_{\rm even}\left(\chi'; \sqrt{m_T^2 + (k^z)^2}, k_T, k^z\right)$$

$$\frac{\theta(\chi)\theta(2-\chi)}{(2\pi\varepsilon)} \sin\left(\tilde{w}\frac{\chi}{\varepsilon}\right)\ \tilde{f}_{\rm odd}\left(\frac{\chi}{\varepsilon}; k^0, k_T, k^z\right) \xrightarrow{\varepsilon\to 0^+} \frac{1}{2}\delta(\chi) \int \frac{d\chi'}{(2\pi)} \sin\left(\tilde{w}\chi'\right) \tilde{f}_{\rm odd}\left(\chi'; \sqrt{m_T^2 + (k^z)^2}, k_T, k^z\right)$$

Proportional to the real (hence even in $\tilde{w}$) and imaginary (odd) part of the Fourier transform

$$\tilde{f}_{\rm even}\left(\frac{\chi}{\varepsilon}; k_T, k^z\right) = 2{\rm Re}\left[\int d\tilde{w}'\ f\left(\tilde{w}'; k_T, k^z\right)\ e^{-i\tilde{w}'\frac{\chi}{\varepsilon}}\right]$$

$$\tilde{f}_{\rm odd}\left(\frac{\chi}{\varepsilon}; k_T, k^z\right) = 2{\rm Im}\left[\int d\tilde{w}'\ f\left(\tilde{w}'; k_T, k^z\right)\ e^{-i\tilde{w}'\frac{\chi}{\varepsilon}}\right]$$

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Simplest case: free streaming

Classical limit of the exact solutions

$$\lim_{\hbar\to 0}\left[(2\pi\hbar)^3\, W(x,k)\right] \propto \delta(k^2-m^2)$$

$$\chi = 2\sqrt{\frac{k^2-m^2}{k^2-m_T^2}}$$

Particles

(similar for the antiparticles)

$$(2\pi\hbar)^3\, W^+ = \theta(k^0)\theta(k^2-m^2c^2)\frac{(4-\chi^2)^2}{4m_T^2\chi}\left[\cos\left(\frac{w\chi}{\hbar}\right)\tilde{f}_{\rm even}(k^0,k_T,k^z)+\sin\left(\frac{w\chi}{\hbar}\right)\tilde{f}_{\rm odd}(k^0,k_T,k^z)\right]\frac{A}{2\pi\hbar}$$

$$\varepsilon = \frac{\hbar}{A}$$

$$\int\frac{dx}{\varepsilon}\, g\left(\frac{x}{\varepsilon};x,p_1\cdots\right)\psi(x) = \int dy\; g\left(y;y\varepsilon,p_1\cdots\right)\psi(y\varepsilon) \xrightarrow{\varepsilon\to 0} \psi(0)\int dy\; g(y;0,p_1,\cdots),$$

$$\Rightarrow \qquad \frac{1}{\varepsilon}\, g\left(\frac{x}{\varepsilon};x,p_1\cdots\right) \xrightarrow{\varepsilon\to 0} \delta(x)\int dy\; g(y;0,p_1,\cdots).$$

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Simplest case: free streaming

Numerical results

$$\chi = 2\sqrt{\frac{k^2 - m^2}{k^2 - m_T^2}}\,, \quad A = T_0 \tau_0\,, \quad \varepsilon = \frac{\hbar}{A}, \qquad \tilde{w} = \frac{w}{A}$$

$$f(w, k_T) = \frac{\pi^4}{30} \exp\left\{-\frac{k_T^2}{2T_0^2} - \frac{w^2}{2T_0^2\tau_0^2}\right\}$$

$$\tilde{f}_{\rm even} = 2\sqrt{2\pi}\,\frac{\pi^4}{30} \exp\left\{-\frac{k_T^2}{2T_0^2}\frac{4}{4-\chi^2} - \frac{\chi^2}{2\varepsilon^2}\right\}$$

$$\mathcal{P}_L =$$

$$= \frac{T_0^4}{(2\pi\hbar)^3}\frac{\pi^5}{15}\left(\frac{\tau_0}{\tau}\right)^3 \int_{-\infty}^{\infty} d\tilde{w}\,\sqrt{\frac{\pi}{2}}\,\exp\left\{\frac{\tilde{w}^2\tau_0^2}{2\tau^2}\right\} \mathrm{Erfc}\left(\frac{\tilde{w}^2\tau_0^2}{2\tau^2}\right)\tilde{w}^2 \left\{\left[1 + \frac{1}{4}\frac{\partial^2}{\partial\tilde{w}^2}\right]\left(\exp\left\{\frac{-\tilde{w}^2}{2}\right\}\mathrm{Re}\left[\mathrm{Erf}\left(\frac{2 - i\varepsilon\tilde{w}}{\varepsilon\sqrt{2}}\right)\right]\right)\right\}$$

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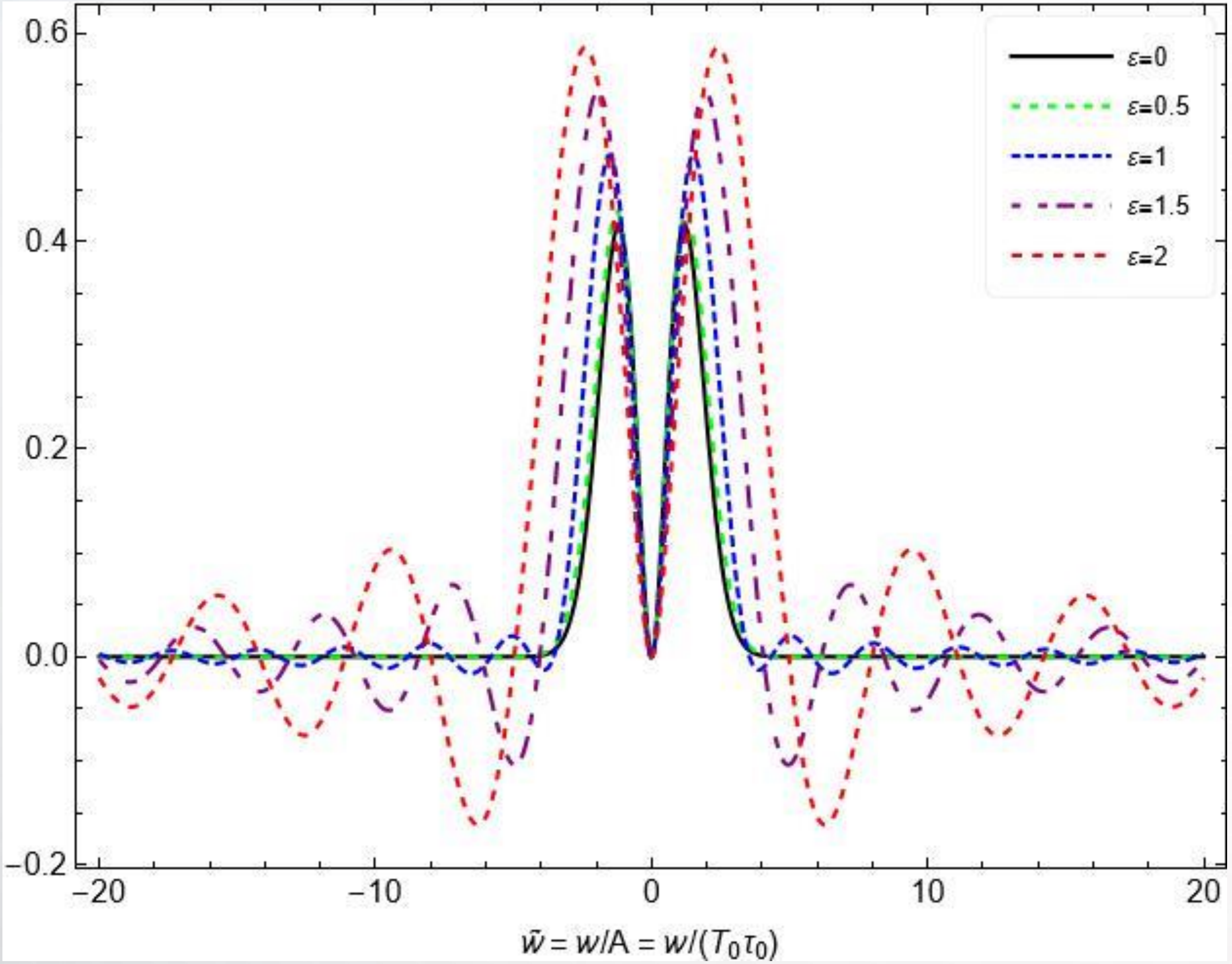
Simplest case: free streaming

Numerical results



The (non-trivial part of the) integrand of $\mathcal{P}_L$

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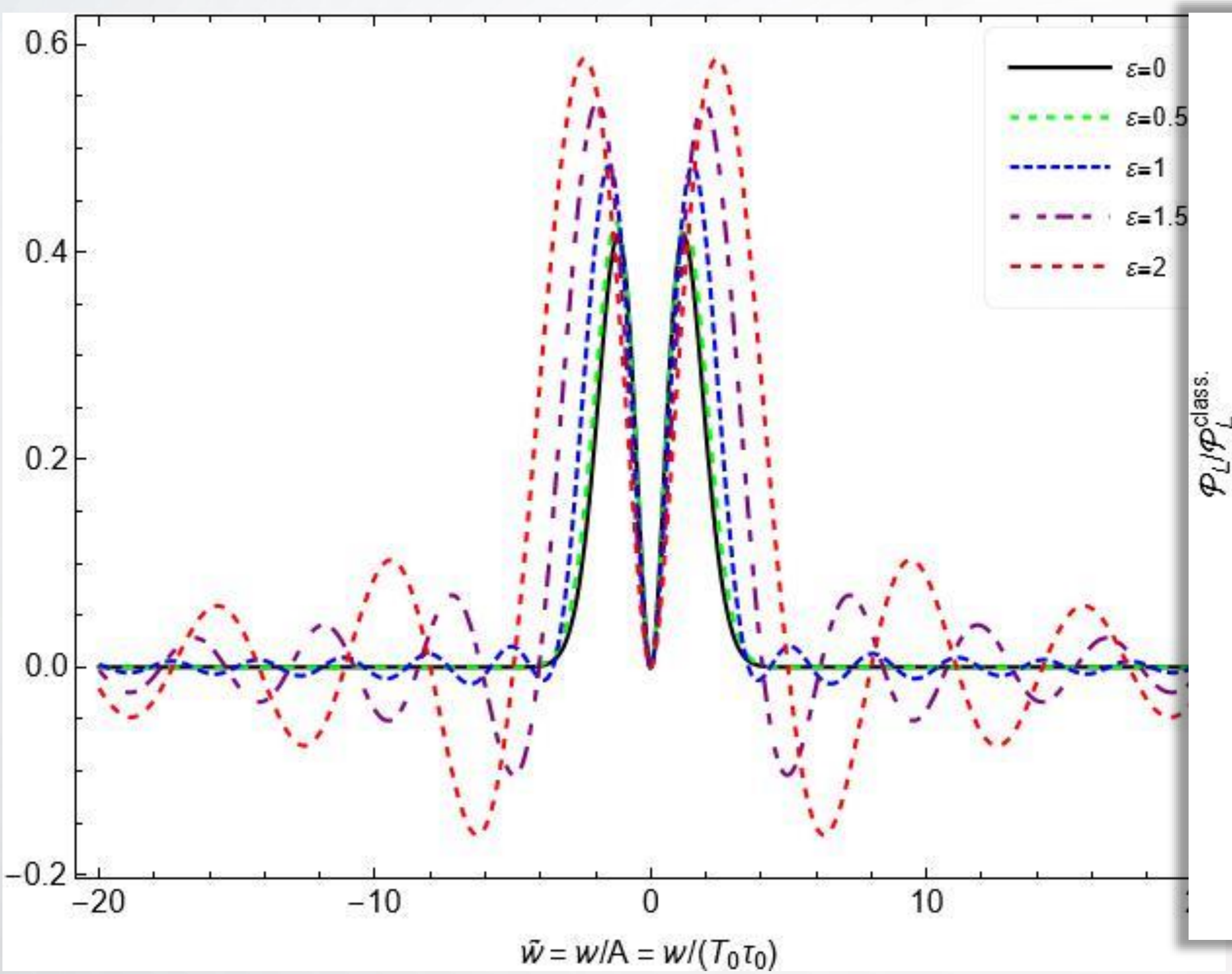
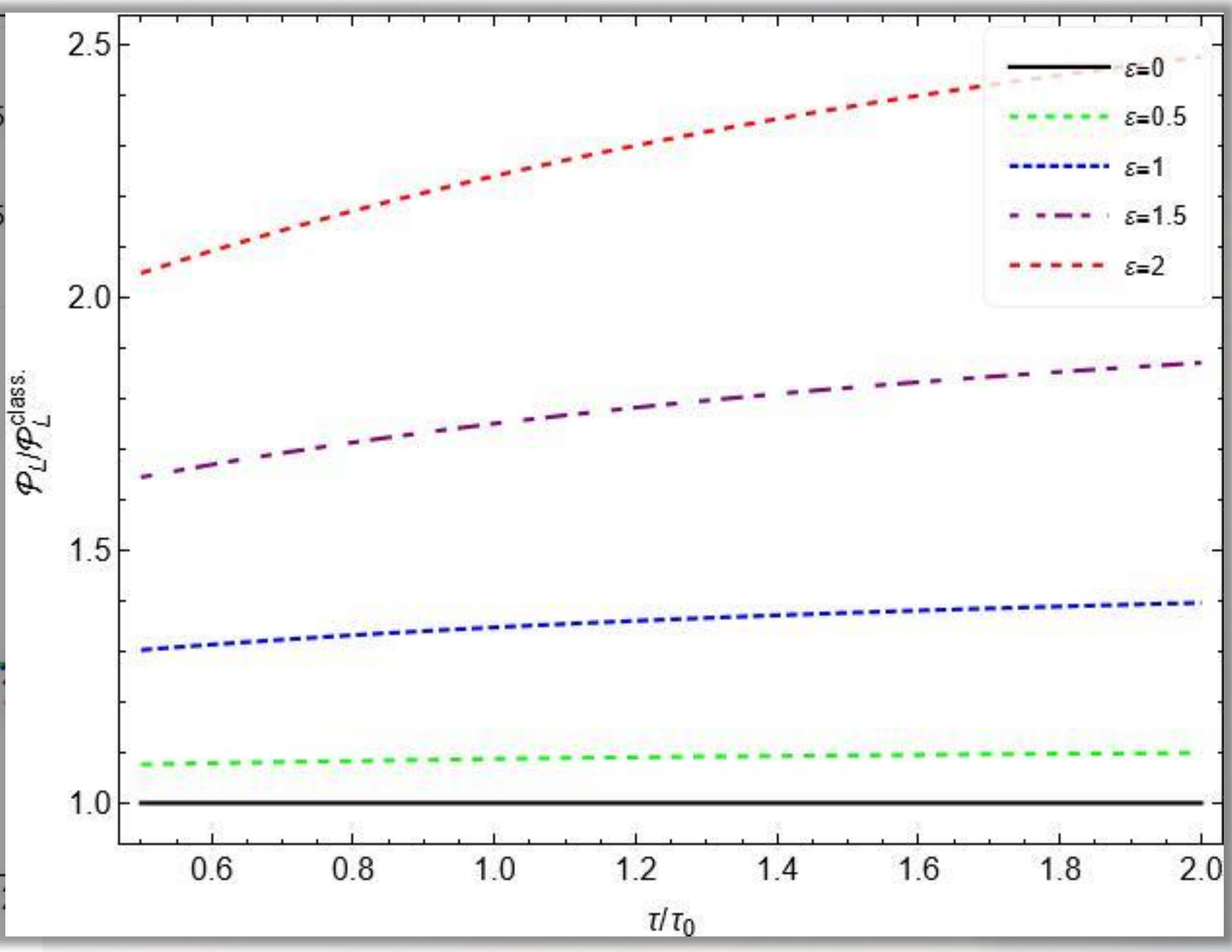
Simplest case: free streaming

Numerical results





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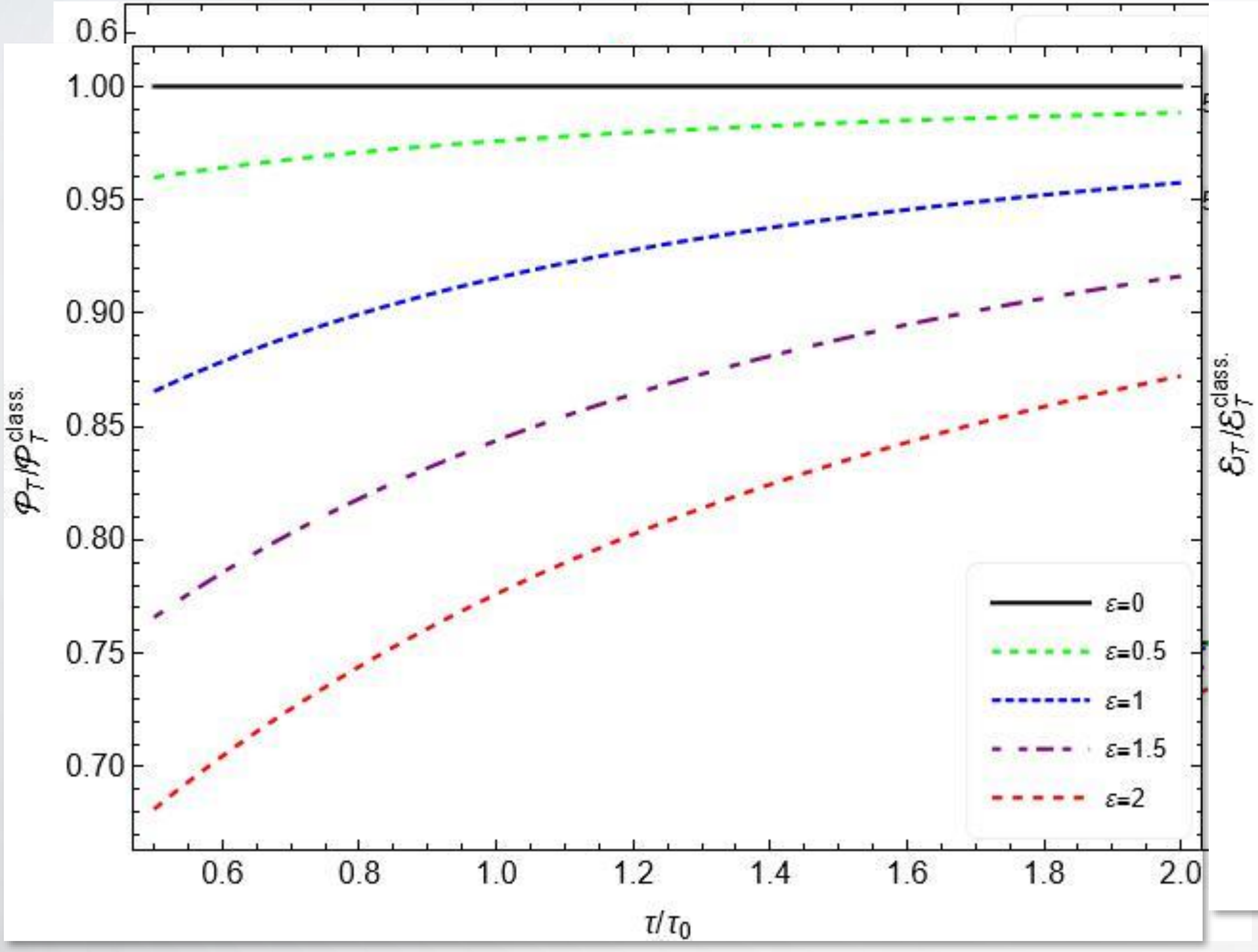

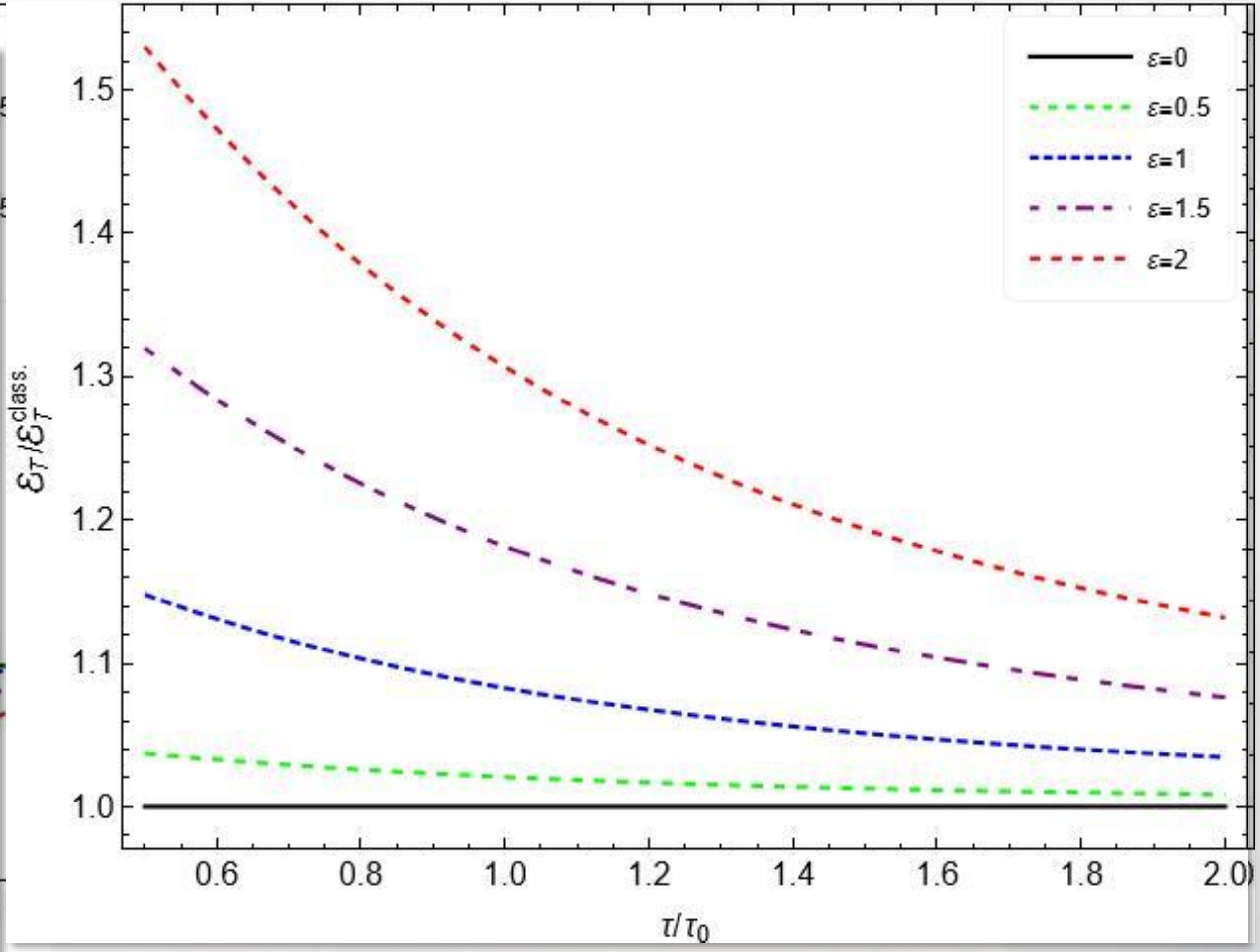

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Exact solutions for the Wigner distribution

- Conformal equation of state (equilibrium), $W_{eq.} = \frac{2\delta(k^2)}{(2\pi)^3} e^{-\frac{1}{T(\tau)}\sqrt{k_T^2+\frac{w^2}{\tau^2}}}$
- Constant shear-viscosity over entropy ratio: $\tau_R = 5\bar{\eta}/T$
- $\bar{\eta} = 3/(4\pi)$
- $\tau_0 = 1/4$ fm/c, $T_0 = 0.6$ GeV, two possible initial conditions:

$$W_0^{iso} = \frac{2}{(2\pi)^3\sqrt{2\pi\sigma}} e^{-\frac{v^2}{2\tau_0^2\sigma}} e^{-\frac{1}{T_0}\sqrt{\sigma = k_T^2+\frac{w^2}{\tau_0^2}}} \longrightarrow \mathcal{P}_0 = \mathcal{P}_{eq.} = \frac{1}{3}\mathcal{E}$$

$$W_0^{a} = \frac{2}{(2\pi)^3\sqrt{2\pi\sigma}} e^{-\frac{v^2}{2\tau_0^2\sigma}} e^{-\frac{1}{T_0}\sqrt{\sigma = k_T^2+\frac{w^2}{\tau_0^2}}} [1 - 3P_2\left(\frac{w}{\tau_0\sqrt{\sigma}}\right)] \longrightarrow \mathcal{P}_T^0 = \frac{8}{5}\mathcal{P}_{eq.}, \quad \mathcal{P}_L^0 = -\frac{1}{5}\mathcal{P}_{eq.}$$

10.1103/PhysRevD.108.036015

Resummed moments

Making use of regularized moments

$$\phi_n^{\mu_1\cdots\mu_s}(x,\zeta) = \int \frac{d^4k}{(2\pi)^4}\,(k\cdot u)^n\, e^{-\zeta(k\cdot u)^2} k^{\langle\mu_1\rangle}\dots k^{\langle\mu_s\rangle} W(x,k)$$

⇨ **well defined set of equations**

Particularly convenient, their version in the Bjorken (0+1)-d symmetric expansion, with RTA $k\cdot\partial\, W = -(k\cdot u)/\tau_R\ \delta W$

$$\hat{\mathcal{L}}\,[f] = 2\zeta\, f(\zeta) - \int_\zeta^\infty d\zeta' f(\zeta')$$

$$L_n = \phi_2^{\mu_1\cdots\mu_{2n}} z_{\mu_1}\cdots z_{\mu_{2n}}, \qquad T_n = \phi_2^{\mu_1\cdots\mu_{2n}\alpha\beta} z_{\mu_1}\cdots z_{\mu_{2n}}\, x_\alpha x_\beta$$

$$\dot{L}_n + \frac{1}{\tau_R}(L_n - L_n^{eq.}) = -\frac{2n+1}{\tau} L_n + \frac{1}{\tau}\hat{\mathcal{L}} L_{n+1}$$

$$\dot{T}_n + \frac{1}{\tau_R}(T_n - T_n^{eq.}) = -\frac{2n+1}{\tau} T_n + \frac{1}{\tau}\hat{\mathcal{L}} T_{n+1}$$

one can integrate the equations in ζ

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Hydrodynamic expansion

$$\hat{\mathcal{L}}\,[f] = 2\zeta\, f(\zeta) - \int_{\zeta}^{\infty} d\zeta' f(\zeta')$$

Hydrodynamics

$$\dot{\mathcal{E}} = -\frac{\mathcal{E}+\mathrm{P}_L}{\tau}$$

$$\dot{\mathrm{P}}_L + \frac{1}{\tau_R}(\mathrm{P}_L - \mathrm{P}) = -\frac{3}{\tau}\mathrm{P}_L + \frac{1}{\tau}\mathcal{R}_L^{(1)}$$

$$\dot{\mathrm{P}}_T + \frac{1}{\tau_R}(\mathrm{P}_T - \mathrm{P}) = -\frac{1}{\tau}\mathrm{P}_T + \frac{1}{\tau}\mathcal{R}_T^{(1)}$$

systematically improvable set of scalar equations…

$$\mathcal{E} = L_0(\tau, \zeta = 0)$$

$$\mathrm{P}_L = \int_{\zeta}^{\infty} d\zeta'\, L_1(\tau, \zeta')$$

$$\mathrm{P}_T = \int_{\zeta}^{\infty} d\zeta'\, T_0(\tau, \zeta')$$

$$\mathcal{R}_T^{(n)} = \int_0^{\infty} d\zeta\, (\hat{\mathcal{L}})^n T_n\,, \qquad \mathcal{R}_L^{(n)} = \int_0^{\infty} d\zeta\, (\hat{\mathcal{L}})^n L_{n+1}$$

$$\dot{\mathcal{R}}_T^{(n)} + \frac{1}{\tau_R}\delta\mathcal{R}_T^{(n)} = -\frac{2n+1}{\tau}\mathcal{R}_T^{(n)} + \frac{1}{\tau}\mathcal{R}_T^{(n+1)}$$

$$\dot{\mathcal{R}}_L^{(n)} + \frac{1}{\tau_R}\delta\mathcal{R}_T^{(n)} = -\frac{2n+3}{\tau}\mathcal{R}_L^{(n)} + \frac{1}{\tau}\mathcal{R}_L^{(n+1)}$$

…to test against the exact solutions

10.1103/PhysRevD.108.036015

Hydrodynamics

$$\dot{\mathcal{E}} = -\frac{\mathcal{E}+\mathcal{P}_L}{\tau}$$

$$\dot{\mathcal{P}}_L + \frac{1}{\tau_R}\left(\mathcal{P}_L - \frac{1}{3}\mathcal{E}\right) = -\frac{3}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_L^{(1)}\Big|_{eq}$$

$$\dot{\mathcal{P}}_T + \frac{1}{\tau_R}\left(\mathcal{P}_T - \frac{1}{3}\mathcal{E}\right) = -\frac{1}{\tau}\mathcal{P}_T + \frac{1}{\tau}\mathcal{R}_T^{(1)}\Big|_{eq}$$

What can we say for the isotropic case

$$R_L^{eq.} = \frac{1}{5}\mathcal{E} \qquad R_L^0 = -\frac{1}{5}\mathcal{E}$$

$$R_T^{eq.} = \frac{1}{15}\mathcal{E} \qquad R_T^0 = -\frac{1}{15}\mathcal{E}$$

$$\frac{\delta\dot{\mathcal{P}}_L}{\dot{\mathcal{P}}_L}\Big|_0 = -\frac{1}{3}$$

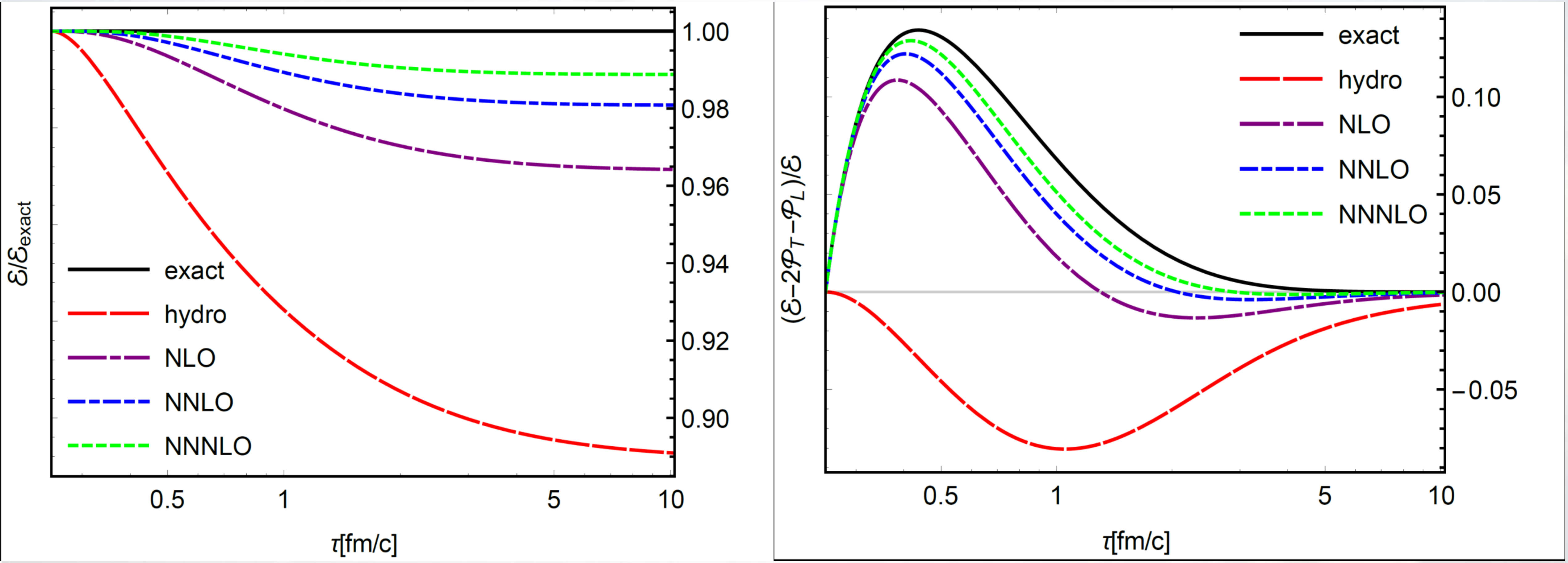
$$\frac{\delta\dot{\mathcal{P}}_T}{\dot{\mathcal{P}}_T}\Big|_0 = -\frac{1}{3}$$

$$\delta\mathcal{P}_L = \int_{\tau_0}^{\tau} ds\, \delta\dot{\mathcal{P}}_L \Rightarrow \frac{\delta\mathcal{P}_L}{\mathcal{P}_L} = \frac{\int \delta\dot{\mathcal{P}}_L}{\mathcal{P}_L} \Rightarrow \text{Maximum if } 0 = \partial_\tau\left(\frac{\delta\mathcal{P}_L}{\mathcal{P}_L}\right) = \frac{\delta\dot{\mathcal{P}}_L}{\mathcal{P}_L} - \frac{\delta\mathcal{P}_L}{\mathcal{P}_L}\frac{\dot{\mathcal{P}}_L}{\mathcal{P}_L} \Rightarrow \frac{\delta\mathcal{P}_L}{\mathcal{P}_L} = \frac{\delta\dot{\mathcal{P}}_L}{\dot{\mathcal{P}}_L}$$

$$\frac{\delta\mathcal{E}}{\mathcal{E}} = \frac{\delta\dot{\mathcal{E}}}{\dot{\mathcal{E}}} = \frac{\delta\mathcal{E}+\delta\mathcal{P}_L}{\mathcal{E}+\mathcal{P}_L} \Rightarrow \frac{\delta\mathcal{E}}{\mathcal{E}} \simeq \frac{\delta\mathcal{P}_L}{\mathcal{P}_L}$$

...but for the trace anomaly $\mathcal{E} - 2\mathcal{P}_T - \mathcal{P}_L = -3\Pi$ $\qquad \frac{\delta\dot{\Pi}}{\dot{\Pi}} = -1$

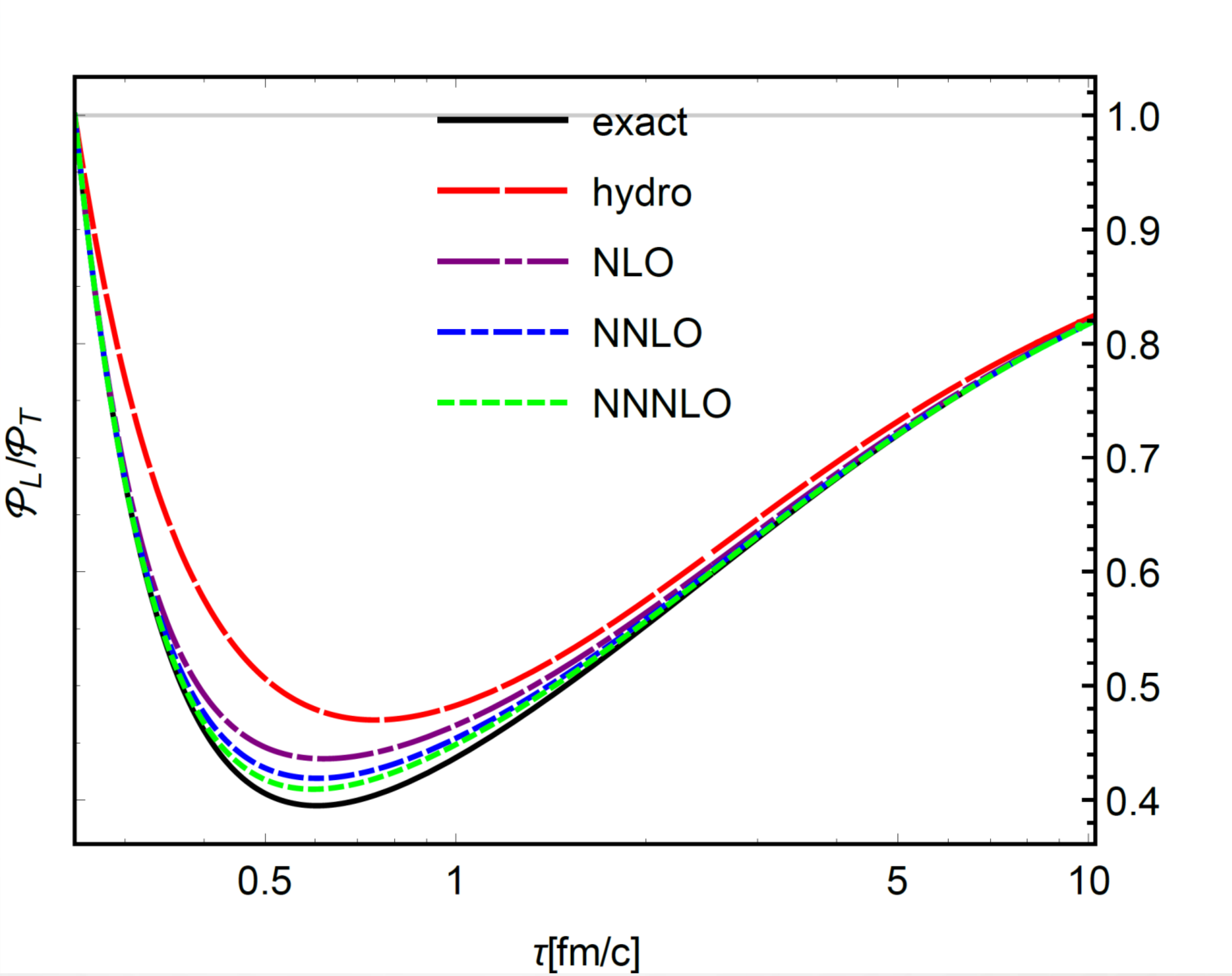
Comparisons with the exact solutions



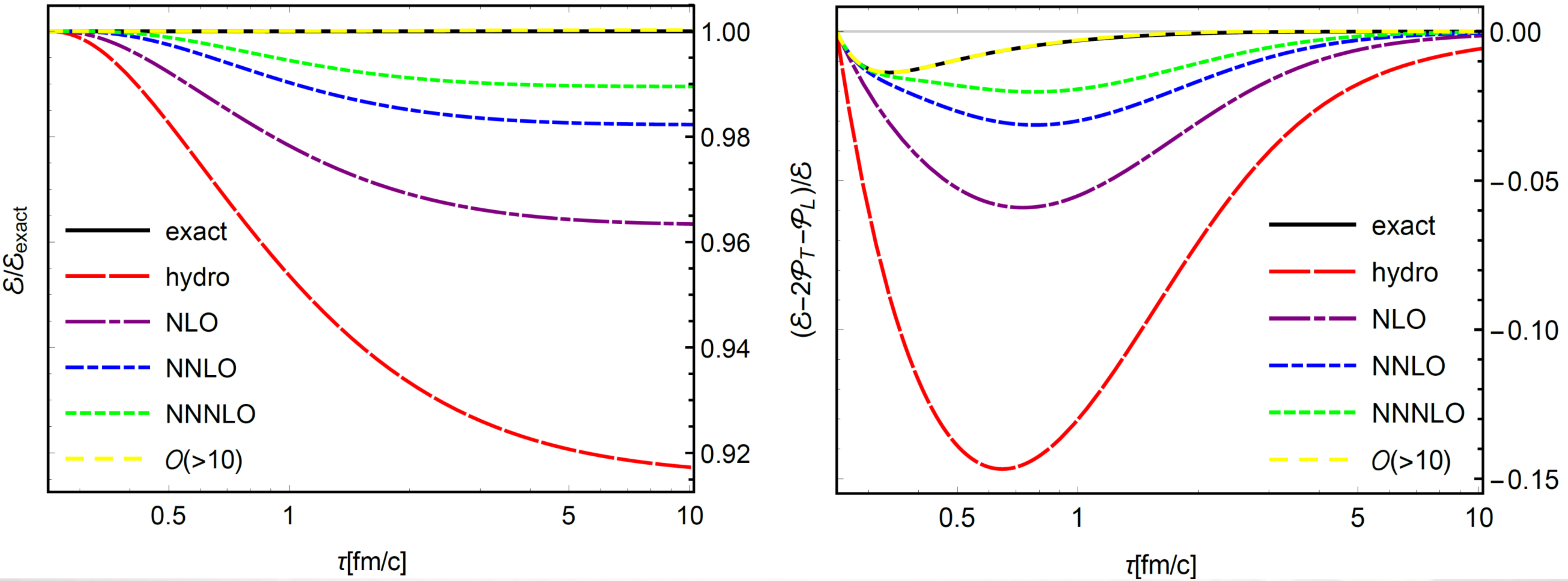
$$(\mathcal{E}-2\mathcal{P}_T-\mathcal{P}_L)/\mathcal{E}=-\frac{3\Pi}{\mathcal{E}}=-\frac{\Pi}{\mathcal{P}}$$

Comparisons with the exact solutions

fast convergence for the pressure anisotropy too



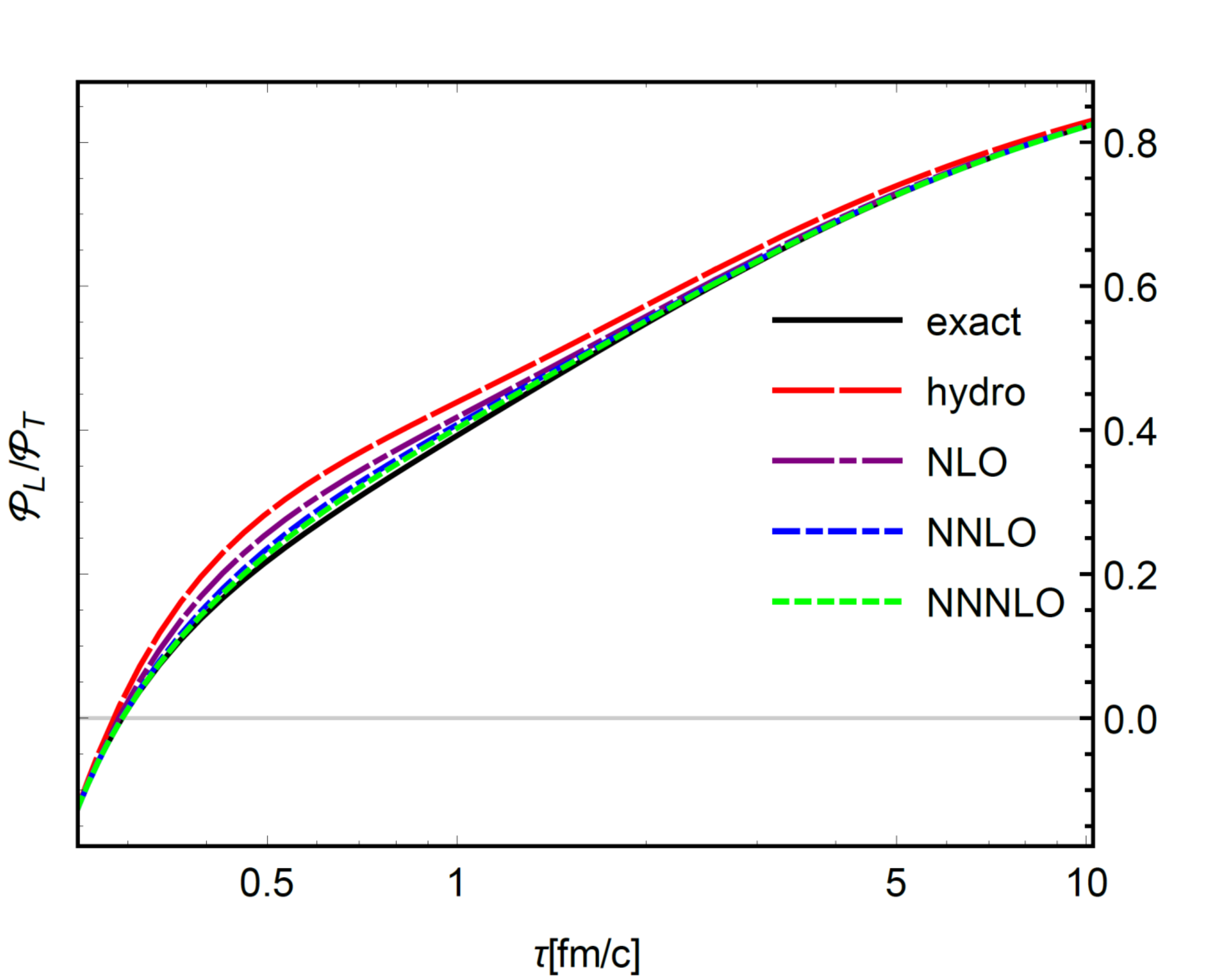
Comparisons for the anisotropic initial conditions



similar conclusions

Comparisons for the anisotropic initial conditions

reasonable approximation for the pressure anisotropy from the start



similar conclusions

$$\int [g(x) + h(x)]\,dx \neq \int g(x)dx + \int h(x)dx$$

$$\int \lim_{\varepsilon\to 0} f(\varepsilon, x)dx \neq \lim_{\varepsilon\to 0} \int f(\varepsilon, x)dx$$

$$\frac{1}{\beta} = \int_0^\infty \left[-\partial_\beta \left(\frac{e^{-\beta x}}{x}\right)\right] dx \neq -\partial_\beta \left(\int_0^\infty \frac{e^{-\beta x}}{x} dx \equiv \infty\right)$$

$$\frac{1}{x} = \int_0^\infty e^{-\alpha x} d\alpha$$

$$\frac{1}{(\alpha+\beta)^2} = \int_0^\infty dx \left[-\partial_\beta \left(e^{-(\alpha+\beta)x}\right)\right] = -\partial_\beta \left(\int_0^\infty dx\, e^{-(\alpha+\beta)x} = \frac{1}{\alpha+\beta}\right),$$

$$\int_0^\infty d\alpha \left[\frac{1}{(\alpha+\beta)^2} = \partial_\alpha \left(-\frac{1}{\alpha+\beta}\right)\right] = \frac{1}{\beta}$$

Particles interacting with external fields

Boltzmann-Vlasov equation

$$p\cdot\partial f + m\partial_\alpha m\,\partial^\alpha_{(p)} f + qF_{\alpha\beta}p^\beta\partial^\alpha_{(p)} f = -\mathcal{C}[f]$$

Immediate (but problematic) generalization

$$\begin{aligned}\dot{\mathcal{F}}_r^{\mu_1\cdots\mu_s} + C_{r-1}^{\mu_1\cdots\mu_s} \;=\;& r\,\dot{u}_\alpha\,\mathcal{F}_{r-1}^{\alpha\mu_1\cdots\mu_s} - \nabla_\alpha\mathcal{F}_{r-1}^{\alpha\mu_1\cdots\mu_s} + (r-1)\,\nabla_\alpha u_\beta\,\mathcal{F}_{r-2}^{\alpha\beta\mu_1\cdots\mu_s}\\ &+m\dot{m}\,(r-1)\,\mathcal{F}_{r-2}^{\mu_1\cdots\mu_s} + s\,m\partial^{(\mu_1}m\,\mathcal{F}_{r-1}^{\mu_2\cdots\mu_s)}\\ &-q(r-1)\,E_\alpha\,\mathcal{F}_{r-2}^{\alpha\mu_1\cdots\mu_s} - q\,s\,g_{\alpha\beta}F^{\alpha(\mu_1}\mathcal{F}_{r-1}^{\mu_2\cdots\mu_s)\beta}\end{aligned}$$

$$F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \varepsilon_{\mu\nu\rho\sigma}u^\rho B^\sigma$$

Moments with large negative r needed, infrared catastrophe!

arXiv:1808.06436

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu}$$

$$f_{r}^{\mu_1\cdots\mu_s} = \mathcal{F}_{r}^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}$$

$$\phi_{r}^{\mu_1\cdots\mu_s} = \Phi_{r}^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}$$

$$\begin{aligned}
\dot{f}_{r}^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + (\mathcal{F}_{\text{coll.}})_{r}^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = & -q\, s\, \varepsilon^{\rho\sigma\alpha(\mu_1} f_{r-1}^{\mu_2\cdots\mu_s)\beta} g_{\alpha\beta} u_\rho B_\sigma - q(r-1)\, E_\alpha\, f_{r-2}^{\alpha\mu_1\cdots\mu_s} - q\, s\, E^{(\mu_1} f_{r}^{\mu_2\cdots\mu_s)} \\
& + m\dot{m}\, (r-1)\, f_{r-2}^{\mu_1\cdots\mu_s} + s\, m\nabla^{(\mu_1} m\, f_{r-1}^{\mu_2\cdots\mu_s)} \\
& + r\, \dot{u}_\alpha f_{r-1}^{\alpha\mu_1\cdots\mu_s} - s\dot{u}^{(\mu_1} f_{r+1}^{\mu_2\cdots\mu_s)} \\
& - \nabla_\alpha f_{r-1}^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle} - \theta\, f_{r}^{\mu_1\cdots\mu_s} - s\, \nabla_\alpha u^{(\mu_1} f_{r}^{\mu_2\cdots\mu_s)\alpha} \\
& + (r-1)\nabla_\alpha u_\beta\, f_{r-2}^{\alpha\beta\mu_1\cdots\mu_s},
\end{aligned}$$

$$\begin{aligned}
\dot{\phi}_1^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + (\Phi_{\text{coll.}})_1^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = & -q\left[s\, E^{(\mu_1}\phi_1^{\mu_2\cdots\mu_s)} - 2\xi^2\left(E_\alpha\, \phi_1^{\alpha\mu_1\cdots\mu_s} + m\dot{m}\, \phi_1^{\mu_1\cdots\mu_s}\right)\right] \\
& + s\frac{1}{\sqrt{\pi}}\int_{\xi^2}^{\infty}\frac{d\upsilon}{\sqrt{\upsilon-\xi^2}}\left[m\nabla^{(\mu_1} m\, \phi_1^{\mu_2\cdots\mu_s)} - q\, \varepsilon^{\rho\sigma\alpha(\mu_1}\phi_1^{\mu_2\cdots\mu_s)\beta} g_{\alpha\beta}u_\rho B_\sigma\right] \\
& + \frac{1}{\sqrt{\pi}}\int_{\xi^2}^{\infty}\frac{d\upsilon}{\sqrt{\upsilon-\xi^2}}\left[\dot{u}_\alpha\phi_1^{\alpha\mu_1\cdots\mu_s} + s\, \dot{u}^{(\mu_1}\partial_\upsilon\phi_1^{\mu_2\cdots\mu_s)} + 2\xi^2\, \dot{u}_\alpha\, \partial_\upsilon\phi_1^{\alpha\mu_1\cdots\mu_s} - \nabla_\alpha\phi_1^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle}\right] \\
& - \theta\, \phi_1^{\mu_1\cdots\mu_s} - s\, \nabla_\alpha u^{(\mu_1}\phi_1^{\mu_2\cdots\mu_s)\alpha} - 2\xi^2\nabla_\alpha u_\beta\, \phi_1^{\alpha\beta\mu_1\cdots\mu_s}.
\end{aligned}$$