Beyond mean-field: Gaussian approximation in a quark-meson model

Győző Kovács
University of Wrocław | Wigner RCP
gyozo.kovacs@uwr.edu.pl | kovacs.gyozo@wigner.hu

In collaboration with:

Péter Kovács (HUN-REN Wigner RCP)
György Wolf (HUN-REN Wigner RCP)
Zsolt Szép (NKFIH-ELTE)

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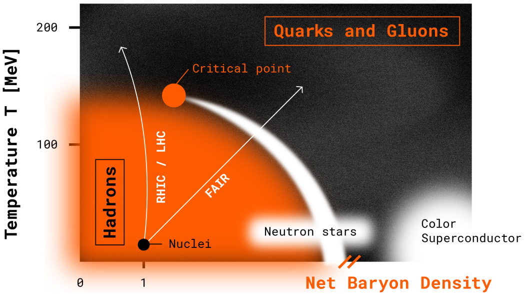
QCD phase diagram

Finite T and μ:
Effective models



Mean-field approximation

pros:

- Clear and simple
- It works well! (mostly)

cons:

- It can miss something important
- It can generate nonphysical effects

How to go beyond?

- Functional methods
- Add further correction, e.g. in a gaussian approximation [**PRD 104 (2021) no.5, 056013**]

What will be shown is one way to go beyond mean-field

Simple model

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V_{cl}(\phi) + \bar{\psi}(i\partial\!\!\!/ - g\phi)\psi$$

Recipe

$$Z = \int \mathcal{D}\phi\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS} \qquad \Omega = \frac{i}{\mathcal{V}_4}\log(Z)$$

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Mean-field approximation: Set $\varphi \to 0 \quad \Rightarrow \quad Z = e^{i\left(S_{\text{mes}}(\bar{\phi}) - i\log\text{Det}(i\mathcal{S}^{-1}(\bar{\phi}))\right)}$

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To go beyond: $S(\bar{\phi} + \varphi) = S(\bar{\phi}) + \varphi \left.\frac{\delta S(\phi)}{\delta \phi}\right|_{\phi=\bar{\phi}} + \varphi \left.\frac{\delta S(\phi)}{\delta\phi\delta\phi}\right|_{\phi=\bar{\phi}} \varphi + \ldots$

Gaussian approximation

$$\int \mathcal{D}\varphi \; e^{\frac{i}{2}\int_x \varphi (i\mathcal{G}^{-1})\varphi} = Det(i\mathcal{G}^{-1})^{-\frac{1}{2}}$$

$$\Omega = V_{cl}(\bar{\phi}) + i \, \text{tr} \int_k \log(i\mathcal{S}^{-1}) - \frac{i}{2}\text{tr} \int_k \log(i\mathcal{G}^{-1})$$

Simple models with:

- **Mesons** as bosons: main degrees of freedom
- **Chiral symmetry** as basic structure
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Lagrangian:

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$$+ \mathrm{Tr}(H\phi) + c_1 \left(\mathrm{Det}(\phi) + \mathrm{Det}(\phi^\dagger)\right) + \bar{\psi}\left(i\partial\!\!\!/ - g\mathcal{M})\right)\psi$$

with $\phi = S + i\,P$, containing pion, (kaon,) eta, sigma, etc..., while $\mathcal{M} = S + i\gamma_5 P$

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+ Vector and axial-vector mesons (ELSM)
+ Polyakov loop to mimic the confinement
+ Further hadronic fields to describe more dofs

Thermodynamics: **Mean-field level** effective potential

- Classical potential.
- Fermionic one-loop correction (vacuum part properly renormalized)

$$\Omega_{MF}(T,\mu_q) = V_{cl} + i\mathrm{tr}\int_K \log\left(iS_0^{-1}(T,\mu_q)\right)$$

Field equations (FE):

$$\frac{\partial\Omega_{MF}}{\partial\phi_N} = \frac{\partial\Omega_{MF}}{\partial\phi_S} = \frac{\partial\Omega_{MF}}{\partial\bar{\Phi}} = \frac{\partial\Omega_{MF}}{\partial\Phi} = 0$$

Curvature meson masses:

$$M^2_{(MF)ab} = \left.\frac{\partial^2\Omega_{MF}}{\partial\varphi_a\partial\varphi_b}\right|_{\{\varphi_i\}=0}$$

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$$\Omega_G(T,\mu_q) = \Omega_{MF}(T,\mu_q) - \frac{i}{2}\mathrm{tr}\int_K \log\left(iG^{-1}(T,\mu_q,M^2_{(MF)})\right)$$

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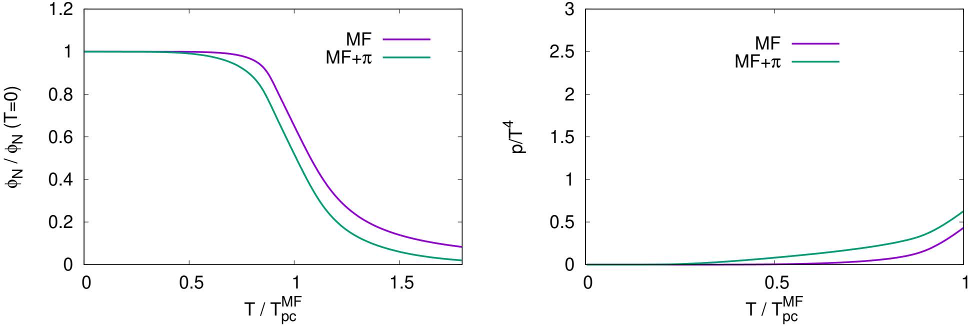
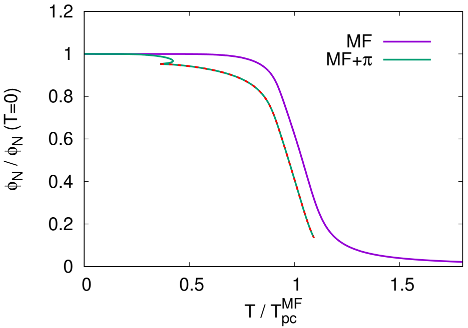
The transition temperature decreases

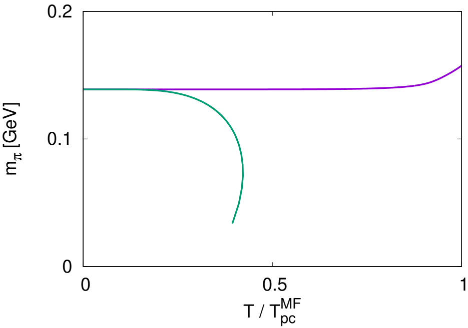
The mesonic pressure corrects the $T < T_{pc}$ behavior (tuned parameters)



What can go wrong?
The pion mass!

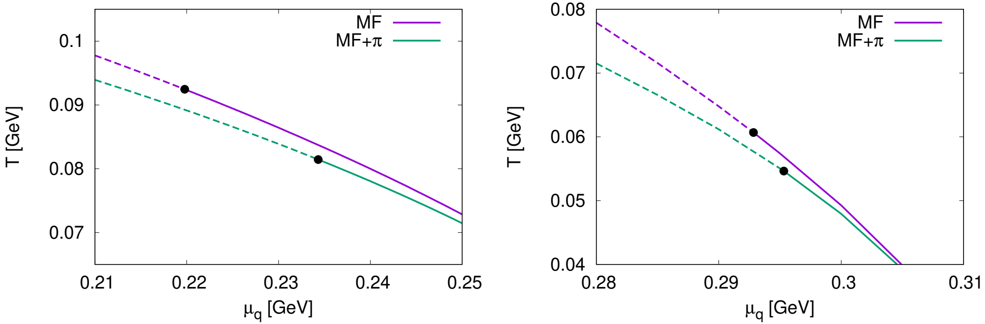
This is non-physical





A slight shift in the critical endpoint

Note: the meson fluctuations have no μ_B dependence ($\mu_B = 3\mu_q$ in this case)

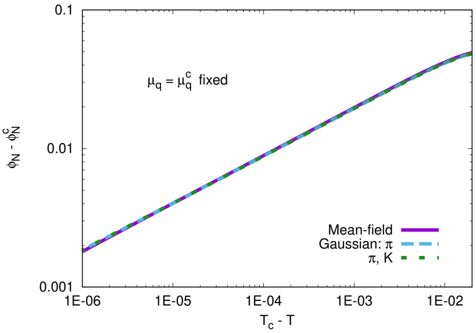


Is there a modification in the critical scaling?

Only a slight difference in β

Look at $\tilde{\phi}_N \propto |T_c - T|^\beta$

Seems to be almost inseparable,
but there is a visible difference



- Effective models are simple and useful tools to investigate the phase diagram.
- The usual mean-field approximation works surprisingly well.
- **We implemented a beyond mean-field, Gaussian approximation.**
- The Gaussian approximation slightly modifies the phase diagram but also qualitatively justifies the mean-field results.

- The problem with the pion mass at $N_f = 2 + 1$ (at $N_f = 2$, this is not there).
- How to handle properly the mesonic vacuum fluctuations?
- Meson fluctuations in the meson masses?

Thank you!

Backup: Some clarification

Questions might arise naively:

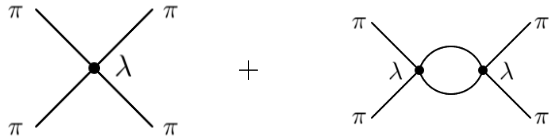
- What is the small parameter?
- How we get closer to the physical case?

Backup: Some clarification

Questions might arise naively:

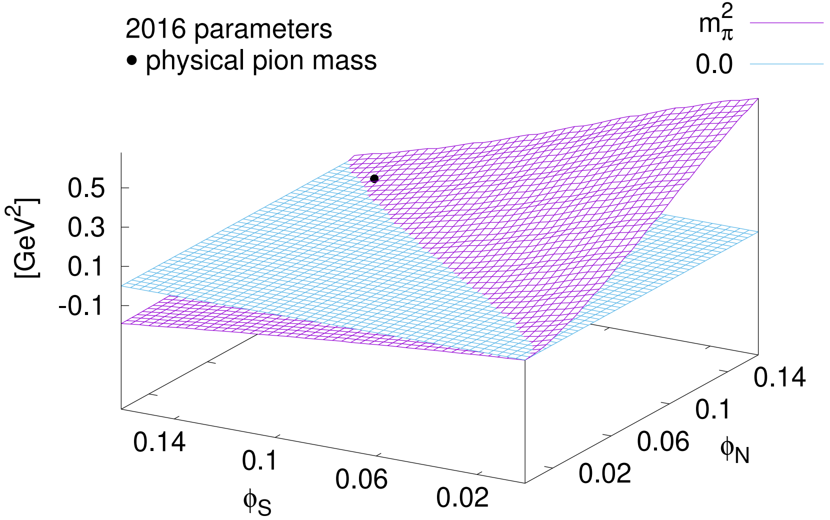
- What is the small parameter? — **No small parameter**
- How we get closer to the physical case? — **We are always "there"**

Physics
all loop and
everything =



Even in the simplest case: **Physics contained** in the parameters
Keeps the **structure of the underlying symmetry**

Going beyond: **Extra structure** can be important


2016 parameters
• physical pion mass
m_π^2
0.0
[GeV^2]
0.5
0.3
0.1
-0.1
0.14
0.1
0.06
0.02
ϕ_S
0.14
0.1
0.06
0.02
ϕ_N