# The chiral transition in a 20 fm$^3$ box


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Wuppertal-Budapest collaboration.
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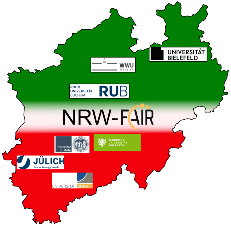
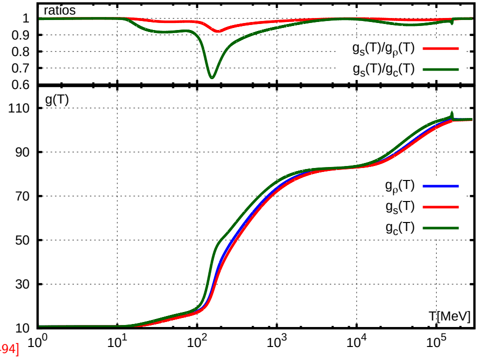



Zimanyi School 2024

# The full equation of state

Adding QCD to the free light particles and the electroweak theory:

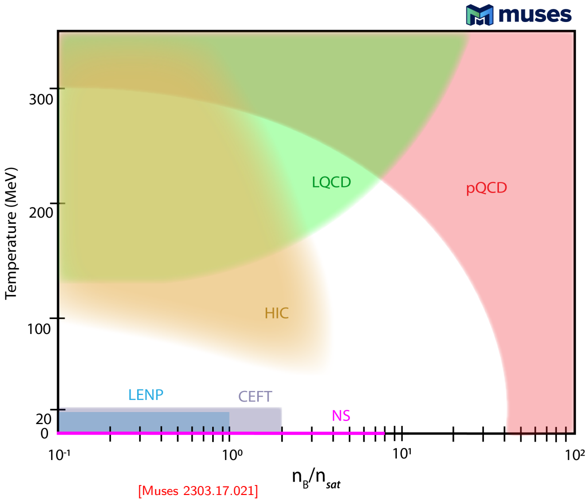
number of effective degrees of freedom:



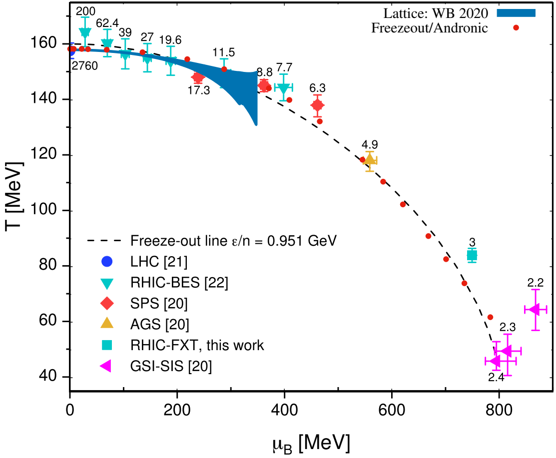

[Wuppertal-Budapest 1606.07494]

energy dens. $\rho = g_\rho \frac{\pi^2}{30} T^4$ entropy dens. $s = g_s \frac{2\pi^2}{45} T^3$ heat cap. $c = g_c \frac{2\pi^2}{15} T^3$

cooling rate in early universe $\frac{dT}{dt} = -\frac{T^3}{M_{pl}} \frac{2\pi^{3/2}}{3\sqrt{5}} \frac{\sqrt{g_\rho} g_s}{g_c}$



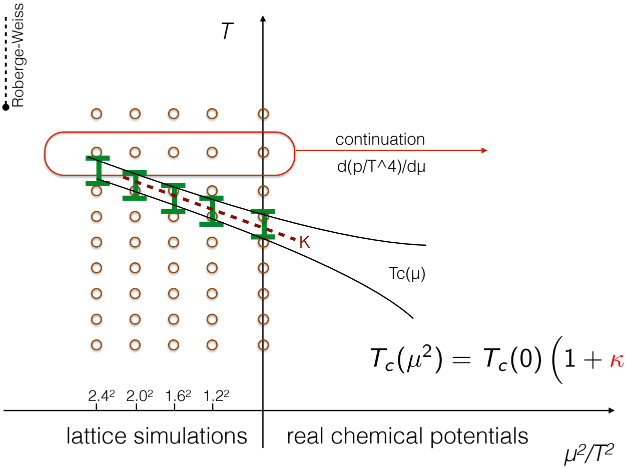

[Muses 2303.17.021]




Compiled by [Vovchenko et al 2408.06473]

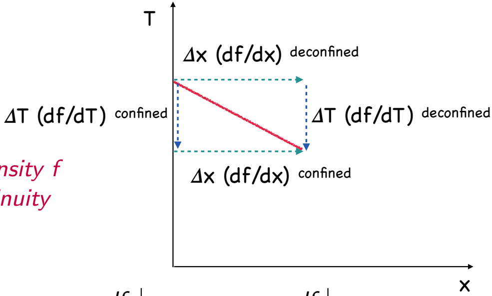
Lattice result on the chiral transition line:[Wuppertal-Budapest PRL 125 (2020) 052001]

*What can Lattice QCD add to this phase diagram?*




$$T_c(\mu^2) = T_c(0)\left(1 + \kappa \frac{\mu^2}{T^2} + \dots\right)$$

# Clausius Clapeyron equation




*Free energy density f has no discontinuity*

$$\Delta x \left.\frac{df}{dx}\right|_{\text{confined}} + \Delta T \left.\frac{df}{dT}\right|_{\text{confined}} = \Delta x \left.\frac{df}{dx}\right|_{\text{deconfined}} + \Delta T \left.\frac{df}{dT}\right|_{\text{deconfined}}$$

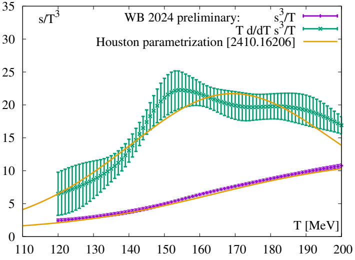
$$-\Delta s = \left.\frac{df}{dT}\right|_{\text{deconfined}} - \left.\frac{df}{dT}\right|_{\text{confined}} \quad ; \quad -\frac{1}{2}\Delta\chi = \left.\frac{df}{dx}\right|_{\text{deconfined}} - \left.\frac{df}{dx}\right|_{\text{confined}}$$

$$\Delta T \Delta s = -\Delta x \frac{1}{2}\Delta\chi$$

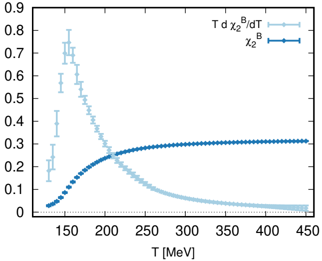
$$\kappa := \frac{1}{T}\cdot\frac{\Delta T}{\Delta x} = -\frac{\Delta\chi/T^4}{2\Delta s/T^3}$$

# Entropy and baryon susceptibility

$$\frac{s}{T^3} \qquad T\frac{d}{dT}\frac{s}{T^3} \qquad \chi_2^B \qquad T\frac{d}{dT}\chi_2^B$$




[Wuppertal Budapest [preliminary]]




[Wuppertal Budapest [2102.06660]]

$$\kappa \;=\; -\frac{1}{2}\cdot\frac{\Delta\text{susceptibility}}{\Delta\text{entropy}} \rightarrow -\frac{1}{2}\cdot\frac{\frac{d\text{susceptibility}}{dT}}{\frac{d\text{entropy}}{dT}} \approx -\frac{1}{2}\cdot\frac{0.75}{22.3} \approx -0.017$$

- *Taylor coefficients* of the pressure

$$\frac{p(\mu_B)}{T^4} = \frac{p(0)}{T^4} + \frac{1}{2!}\frac{\mu_B^2}{T^2}\chi_2^B(T) + \frac{1}{4!}\frac{\mu_B^4}{T^4}\chi_4^B(T) + \frac{1}{6!}\frac{\mu_B^6}{T^6}\chi_6^B(T) + \dots$$

- These Taylor coefficients are equal to the Grand Canonical fluctuations

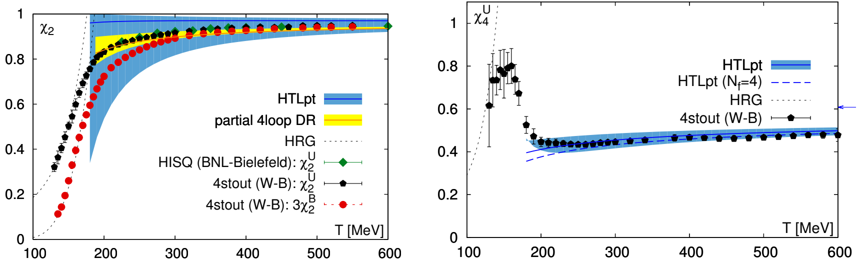
$$\chi_2^B(T) = \left\langle B^2 \right\rangle - \left\langle B \right\rangle^2 = \frac{1}{VT}\frac{\partial^2 \log Z(V,T,\mu_B,\mu_Q,\mu_S)}{\partial {\mu_B}^2}$$

- Higher fluctuations are the Taylor coefficients of lower fluctuations

$$\chi_2^B(\mu_B) = \chi_2^B(\mu_B=0) + \frac{1}{2!}\frac{\mu_B^2}{T^2}\chi_4^B(\mu_B=0) + \frac{1}{4!}\frac{\mu_B^4}{T^4}\chi_6^B(\mu_B=0) + \dots$$

- Taylor coefficients can be used to reveal analytic structure of the thermodynamic potential
  - Repulsive interactions beyond ideal HRG
  - Searching the critical end point
- Hints for chiral O(4) universality

# Coefficients below and above the transition region




$$\frac{\partial^2(p/T^2)}{\partial\mu_u^2} \qquad \frac{\partial^4(p/T^4)}{\partial\mu_u^4}$$

- Low temperature coefficients:
  *Hadron Resonance Gas model estimates lattice data well*
- High temperature coefficients:
  *Improved perturbation theory describes lattice data well*

HTL results: [Haque et al 1309.3968,1402.6907]

Lattice results: [Wuppertal-Budapest: 1507.04627] [BNL-Bielefeld: 1507.06637 ]

$$\chi_{200}^{uds} = +\langle B_u\rangle + \langle A_u A_u\rangle$$

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$$\begin{aligned}\chi_{400}^{uds} = \; & +\langle D_u\rangle + 3\langle B_u B_u\rangle - 3\langle B_u\rangle\langle B_u\rangle + 4\langle A_u C_u\rangle \\ & +\langle A_u A_u A_u A_u\rangle - 3\langle A_u A_u\rangle\langle A_u A_u\rangle + 6\langle A_u A_u B_u\rangle - 6\langle B_u\rangle\langle A_u A_u\rangle\end{aligned}$$

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$$\begin{aligned}\chi_{600}^{uds} = \; & +\langle F_u\rangle + 10\langle C_u C_u\rangle + 15\langle B_u D_u\rangle + 15\langle B_u B_u B_u\rangle + 6\langle A_u E_u\rangle + 60\langle A_u B_u \\ & +45\langle A_u A_u B_u B_u\rangle + 20\langle A_u A_u A_u C_u\rangle + 15\langle A_u A_u A_u A_u B_u\rangle + \langle A_u A_u A_u A \\ & -15\langle D_u\rangle\langle A_u A_u\rangle - 45\langle B_u\rangle\langle B_u B_u\rangle - 60\langle B_u\rangle\langle A_u C_u\rangle - 90\langle B_u\rangle\langle A_u A_u B_u \\ & -15\langle B_u\rangle\langle A_u A_u A_u A_u\rangle - 45\langle B_u B_u\rangle\langle A_u A_u\rangle - 60\langle A_u C_u\rangle\langle A_u A_u\rangle - 90\langle A \\ & -15\langle A_u A_u\rangle\langle A_u A_u A_u A_u\rangle + 30\langle B_u\rangle\langle B_u\rangle\langle B_u\rangle + 90\langle B_u\rangle\langle B_u\rangle\langle A_u A_u\rangle \\ & +90\langle B_u\rangle\langle A_u A_u\rangle\langle A_u A_u\rangle + 30\langle A_u A_u\rangle\langle A_u A_u\rangle\langle A_u A_u\rangle\end{aligned}$$

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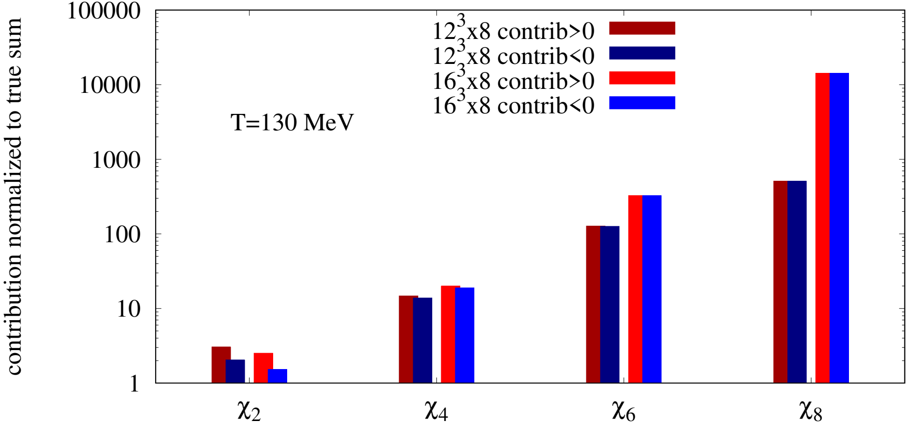
$$\chi_{800}^{uds} = 79 \text{ terms}\ldots$$

*$A, B, C, \ldots$ are defined as*

$$[\det M(\mu_u)]^{1/4} = [\det M(0)]^{1/4} \exp\left(1 + A_u\mu_u + \frac{B_u}{2!}\mu_u^2 + \frac{C_u}{3!}\mu_u^3 + \ldots\right)$$

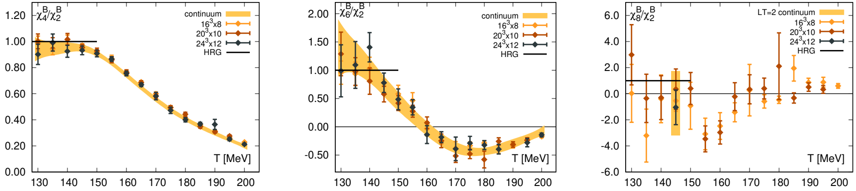
*Data analysis uses computer generated code.*

# Sign problem in the Taylor coefficients

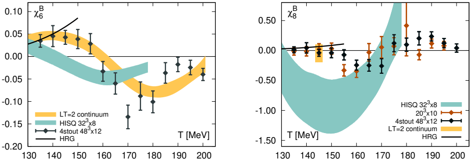


# High order coefficients in a $LT = 2$ box

4Hex continuum result [Phys.Rev.D 110 (2024) 1, L011501]




## Comparison with literature

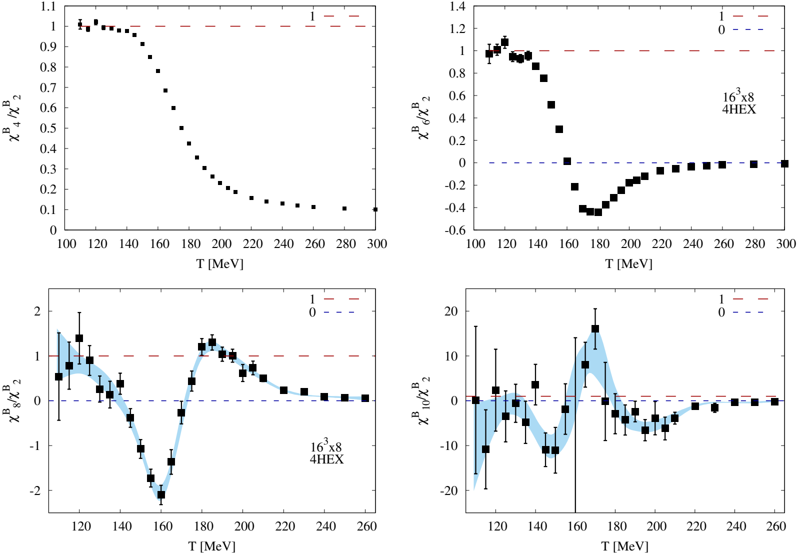



4stout data (imaginary $\mu_B$, $48^3 \times 12$ lattice) : [Wuppertal-Budapest, 1805.04445]

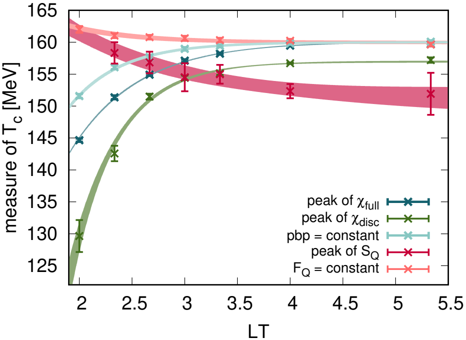
HISQ data: ($\mu_B = 0$, $32^3 \times 8$ lattice, inexact charge conservation) [BNL-Bielefeld, 2202.09184,2212.09043]

# Status of $16^3 \times 8$ simulations

*Baryon cumulant ratios*



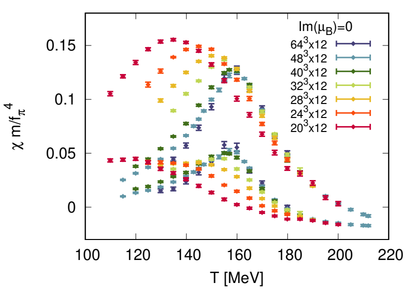
# How is physics distorted in a smaller volume?

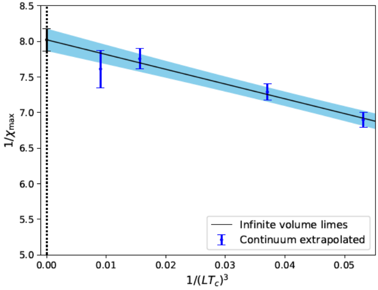



*Chiral observables suffer from the reduced volume, but heavy quark observables (e.g. Polyakov loop) are much less affected.* [Wuppertal-Budapest 2405.1232]

Upper curves: *Full chiral susceptibility*
Lower curves: *Disconnected part of chiral susceptibility*

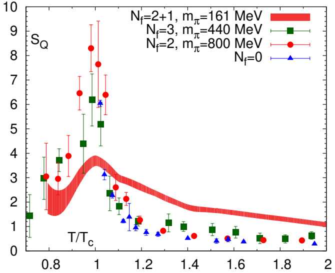
$$\chi_{\bar{\psi}\psi} \sim \frac{\partial^2 \log Z}{\partial m^2}$$



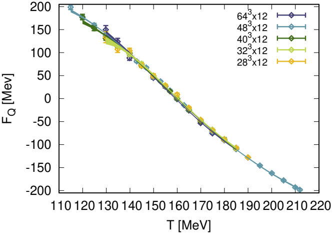



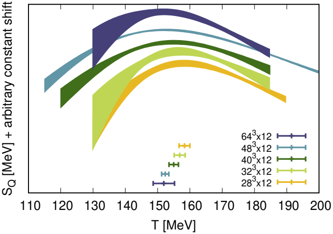

[Wuppertal-Budapest 2405.12320]

$$
\begin{aligned}
P &= \frac{1}{V}\left\langle \mathrm{Tr} \prod_i U_i \right\rangle \\
F_Q &= -T \log P \\
S_Q &= -\frac{\partial F_Q(T,\mu_B)}{\partial T} \\
T_c &: \quad \textit{peak of } S_Q \textit{ in } T
\end{aligned}
$$



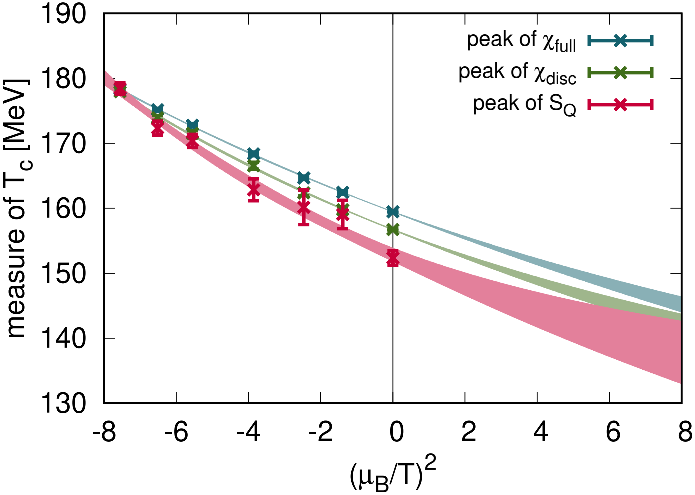

[Left: TUM QCD, Bazavov et al 1603.06637]




[Right: Wuppertal-Budapest 2405.12320]

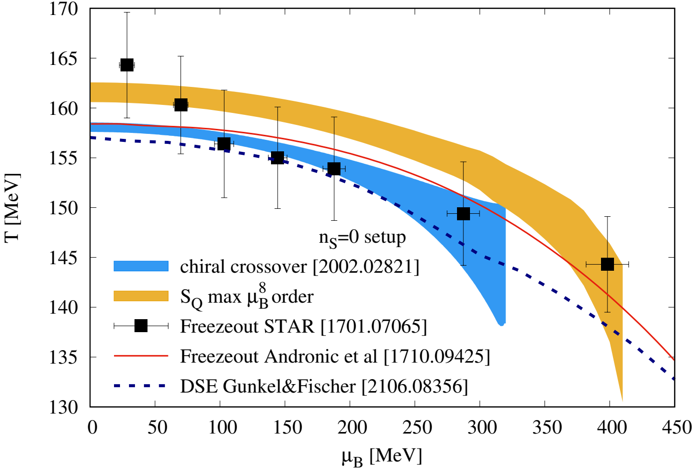
*We repeat the $T_c$ calculation at many imaginary chemical potentials.*




*This plot:* $48^3 \times 12$ *lattice, 4stout*

[Wuppertal-Budapest 2405.12320]

$S_Q$ itself can be extrapolated *($16^3 \times 8$ lattices)*.
The temperature of the $S_Q$-peak is calculated for each $\mu_B$.




[Wuppertal-Budapest 2410.06216]

Besides light baryons hyperons are also generated with $\mu_B > 0$: $\Longrightarrow \langle S \rangle < 0$

In experiment (at chemical freeze-out) $\langle S \rangle = 0$.

*We achieve this by adding $\mu_S > 0$, this is $T$ and $\mu_B$ dependent.*

$$\mu_S^*(\mu_B, T) = s_1(T)\mu_B + s_3(T)\mu_B^3 + s_5(T)\mu_B^5 + \dots$$

One obtains $s_1(T)$, $s_3(T)$ and $s_5(T)$ from the standard Taylor coefficients

[HotQCD 1208.1220; 1701.04325]

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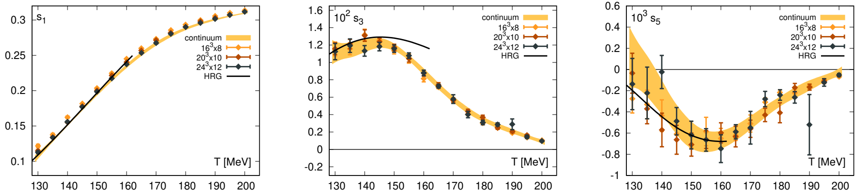
Our recent continuum results [Wuppertal-Budapest 2312.07528]



# Strangeness neutrality in a crosscheck

Besides ligth baryons hyperons are also generated with $\mu_B > 0$: $\Longrightarrow$ $\langle S \rangle < 0$

In experiment (at chemical freeze-out) $\langle S \rangle = 0$.

*We achieve this by adding $\mu_S > 0$, this is $T$ and $\mu_B$ dependent.*

$$\mu_S^*(\mu_B, T) = s_1(T)\mu_B + s_3(T)\mu_B^3 + s_5(T)\mu_B^5 + s_7(T)\mu_B^7 + s_9(T)\mu_B^9 + \dots$$

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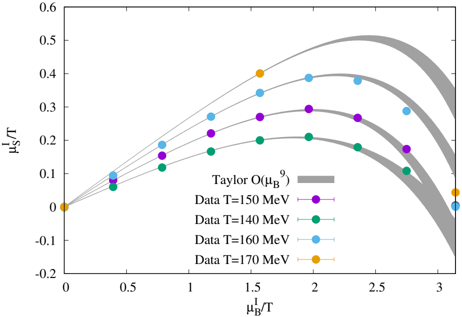
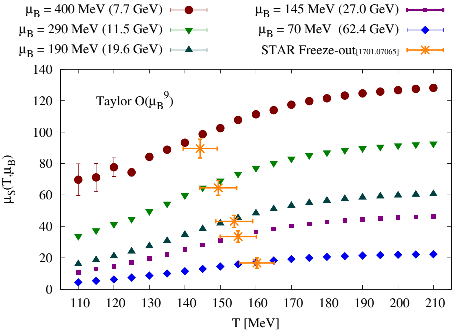
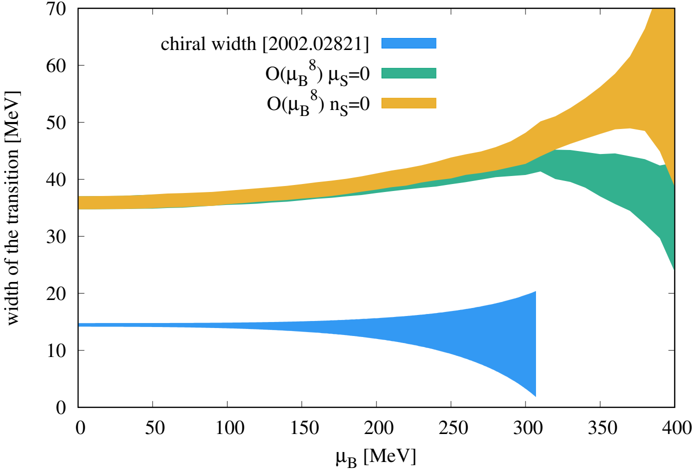



*Wuppertal-Budapest preliminary data, $16^3 \times 8$ lattice.*

# Width of the transition extrapolated to eight order

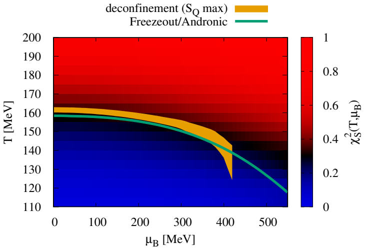
$S_Q$ itself can be extrapolated *($16^3 \times 8$ lattices)*.
The **width** of the $S_Q$-peak is calculated for each $\mu_B$.

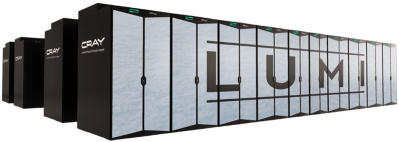



[Wuppertal-Budapest 2410.06216]

It is still very expensive to access high density physics from the lattice.

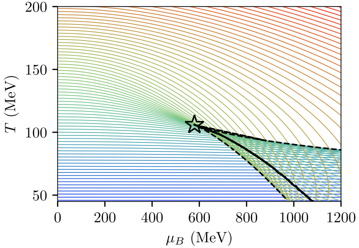
- Continuum extrapolated high order coefficients
- Extreme statistics on $16^3 \times 8$ lattices
  *( $\frac{1}{2}$ years in Jülich
  $+5$ months on LUMI)*
  one can attempt to extrapolate to so far unattainable parts of the phase diagram.
  (today only on coarse lattices)

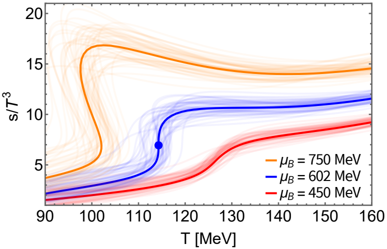



# backup

Idea: look at contours of constant entorpy/$T^3$, find spinodal regions.




[Black-Hole-Engineering model Hippert et al[2309.00579]] [QCD: Shah et al [2410.16206]]

Leading order: Entropy contours are exact parabolas: $T' = A + B\mu_B^2$

Optimistic assumption on error propagation.

Critical endpoint estimate: $T_c = 114.3 \pm 6.9$ MeV, $\mu_{B,c} = 602.1 \pm 62.1$ MeV

*Is this a first principles result on a CEP?*

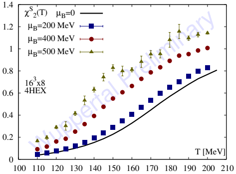
No, an expansion is defined, where

each order can be computed from first principles,

this expansion does not automatically break down near the CEP.

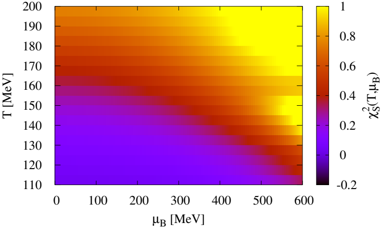
# Strangeness susceptibility extrapolations

*NNLO expansion on a $16^3 \times 8$ lattice*

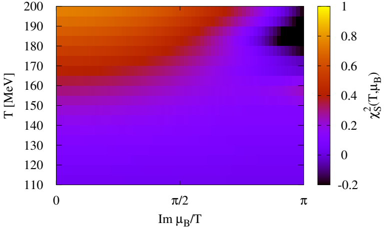



$$\chi_2^S(T,\mu_B) \approx \chi_2^S(T,0) + \frac{\mu_B^2}{2T^2}\chi_{22}^{BS}(T,0) + \frac{\mu_B^4}{24T^4}\chi_{42}^{BS}(T,0) + \frac{\mu_B^6}{720T^6}\chi_{62}^{BS}(T,0)$$

*Real $\mu_B$*

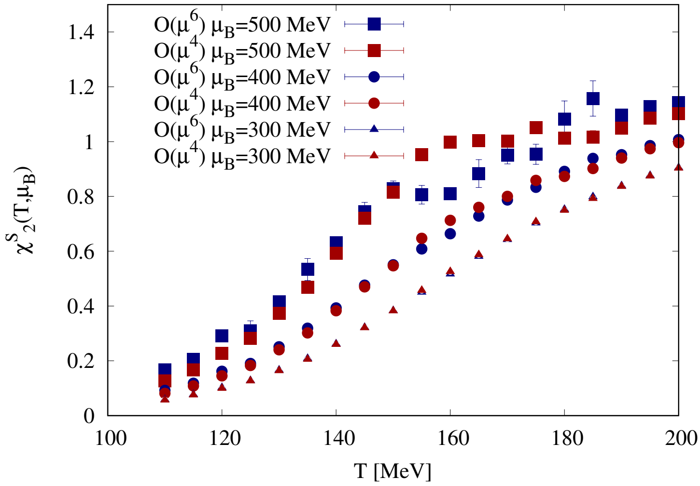



*Imaginary $\mu_B$*



# $\chi_2^S(\mu_B)$ extrapolation

$$\chi_2^S(T,\mu_B) \approx \chi_2^S(T,0) + \frac{\mu_B^2}{2T^2}\chi_{22}^{BS}(T,0) + \frac{\mu_B^4}{24T^4}\chi_{42}^{BS}(T,0) + \frac{\mu_B^6}{720T^6}\chi_{62}^{BS}(T,$$

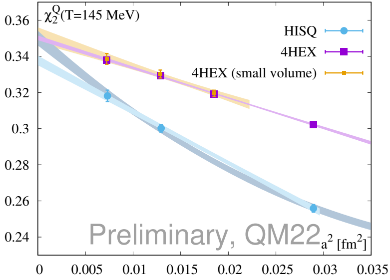


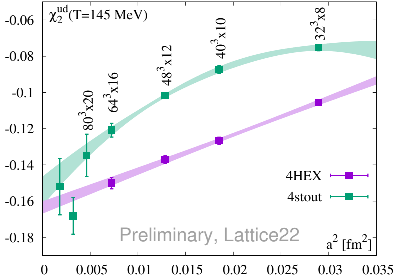
# Testing continuum limits with the 4HEX action

*4HEX staggered action with strongly reduced taste breaking.*

*Let's look at those fluctuations e.g. chanrge, that is sensitive to it.*

*Continuum extrapolation $T = 145$ MeV with large volume up to $N_t = 16$*




**4STOUT:** Wuppertal-Budapest (2013–2023[Wuppertal-Budapest [1507.04627]]

**HISQ:** BNL-Bielefeld (2011–. . . )[HotQCD [2107.10011]]

**4HEX:** Wuppertal-Budapest (2022–. . . ) [Wuppertal-Budapest **QM2022**]