Finanțat de Uniunea Europeană
NextGenerationEU

Guvernul României

Planul Național de Redresare și Reziliență

Firewall Boundaries for Rotating Quark Matter in LSM_q

Zimányi School 2024.
Winter Workshop on Heavy Ion Physics

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3rd December 2024

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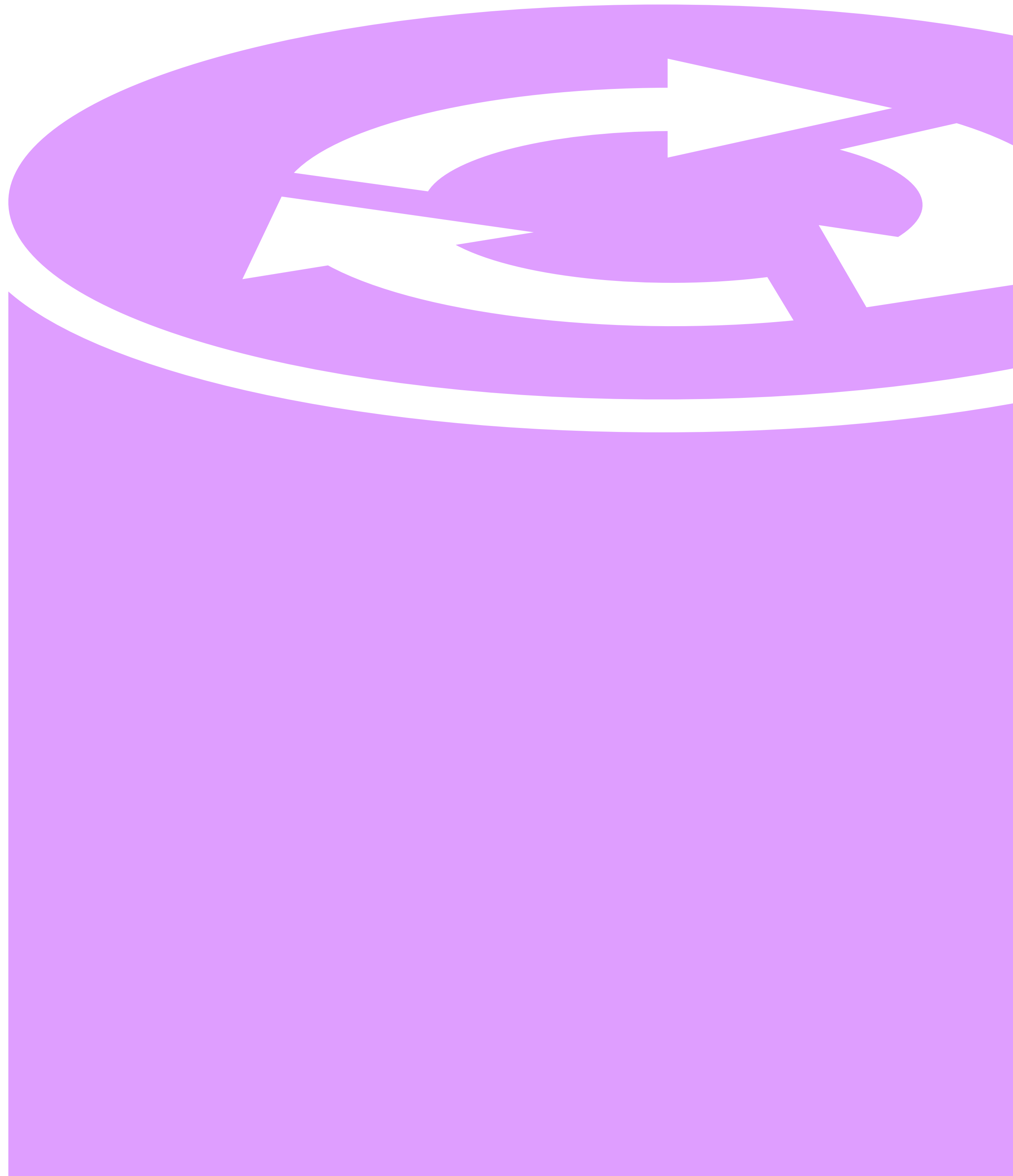

Motivation & Goals

- Interest in rotating QCD matter. (Rotating QGP in heavy-ion collisions).
- Systems under rigid rotation are known to be problematic. Specially if unbounded! They can lead to artifacts coming from superluminal region. [Davies, Dray, Manogue 1996]
- Solution: Place the system in a cylinder and cut the spacetime before the speed-of-light surface. [Ambrus, Winstanley 2016]
- Works in Effective models:
  - ➡ Formally bounded. Use boundary conditions: MIT, spectral… [Chernodub, Gonyo 2016] [Singha, Ambrus, Chernodub 2024]
  - ➡ Unbounded [Chen, Fukushima, Huang, Mameda 2016] [Jiang, Liao 2016]

Both approaches seem to give similar results!

- Typically assume constant value of singlet meson σ or local value without gradients.

Financial support by The European Union NextGenerationEU trough the grant No. 760079/23.05.2023, funded by the Romanian ministry of research, innovation and digitalization through Romania's National Recovery and Resilience Plan, call no. PNRR-III-C9-2022-I8, is gratefully acknowledged.

Rotating states: **Rigid Rotation** vs. **Tolman-Ehrenfest**

- The equilibrium **rigid rotating state** is defined through the following density operator

$$\hat{\rho}_{RR} = e^{-\beta_0(\widehat{H} - \Omega \widehat{J}^z - \mu_0 \widehat{Q})} \, .$$

- Eventually, the evaluation of expectation values require to perform the infinite sum over the angular momentum eigenvalues, which is computationally demanding.

- It has been shown that the sum can be rearranged into the effect of **classical rotation** (captured by the Tolman-Ehrenfest law) **plus quantum corrections.**

- The **Tolman-Ehrenfest** contribution can be efficiently accounted for with the following density operator:

$$\hat{\rho}_{TE} = e^{-\beta \cdot \widehat{P} + \beta_0 \mu_0 \widehat{Q}}$$

where $\beta = \beta_0 / \Gamma$ ⟶ $T = \Gamma T_0 = \dfrac{T_0}{\sqrt{1 - \Omega^2 \rho^2}}$ and $\mu = \Gamma \mu_0 = \dfrac{\mu_0}{\sqrt{1 - \Omega^2 \rho^2}}$

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Linear Sigma Model coupled to **Quarks**

- Model proposed to describe the low-energy properties of quark and mesonic degrees of freedom in a unified way.
- The model is defined through the Lagrangian: $\mathcal{L} = \mathcal{L}_M + \mathcal{L}_q$, where

$$\mathcal{L}_M = \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}) - U(\sigma,\vec{\pi}) \qquad \mathcal{L}_q = \bar{\psi}\left[\frac{i}{2}\overleftrightarrow{\partial\!\!\!/} - g(\sigma + i\gamma^5\vec{\tau}\cdot\vec{\pi})\right]\psi$$

$$U(\sigma,\vec{\pi}) = \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2 - v^2)^2 - h\sigma$$

- ➡ $\vec{\pi}$: Pseudoscalar triplet of the pion fields
- ➡ σ : Scalar singlet meson fields.
- ➡ ψ : Up and down quarks combined into a SU(2) fermion doublet.

Parameters matched to zero-temperature results:

- Pion decay constant: 0.093 GeV
- Pion mass: 0.138 GeV
- Constituent quark mass: 0.307 GeV
- Mass of sigma meson: 0.600 GeV

Linear Sigma Model coupled to **Quarks**

◉ The thermodynamics of the system is encoded in the grand canonical potential $\Phi = -\frac{1}{\beta} \ln \mathcal{Z}_E$ where β is the inverse temperate and $\mathcal{Z}_E$ the Euclidean partition function:

$$\mathcal{Z}_E = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}\sigma \mathcal{D}\vec{\pi} e^{-\int_0^\beta d\tau \int_\mathcal{M} d^3x \mathcal{L}_E}$$

◉ We take the **mean-field approximation for the mesonic fields**, i.e. we study the expectation value of σ and $\vec{\pi}$ while neglecting their quantum fluctuations ⟶ Saddle point approximation for mesonic fields.

$$\mathcal{Z}_E^{\text{m.f.}} = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} e^{-\int_0^\beta d\tau \int_\mathcal{M} d^3x \mathcal{L}_E}$$

From now on, σ and $\vec{\pi}$ denote the expectation values of the fields in a given (rotating) state (to be defined later).

Linear Sigma Model coupled to **Quarks**

- The thermodynamics of the system is encoded in the grand canonical potential $\Phi = -\frac{1}{\beta}\ln \mathcal{Z}_E$ in the mean-field approximation we have

$$\Phi^{\text{m.f.}}(\sigma, \vec{\pi}) = \int_{\mathcal{M}} d^3x \mathcal{L}_M - \frac{1}{\beta}\ln \mathcal{Z}_D$$

- The fermionic contribution to Φ contains only the quark Lagrangian, which is "free", and can be evaluated. At finite temperature and chemical potential, the contribution reads:

$$-\frac{1}{\beta}\ln \mathcal{Z}_D = -2N_f N_c \sum_{\varsigma=\pm}\int_{\mathcal{M}} d^3x \int \frac{d^3p}{(2\pi)^3\beta}\ln\left(1+e^{-\beta(E+\varsigma\mu)}\right)$$

where $E = \sqrt{p^2 + (g\sigma)^2}$.

Evaluating the **Grand Potential** and the **Fermion Condensate** $\langle\bar{\psi}\psi\rangle_{\Omega,\mu,T}$ in the Tolman-Ehrenfest state

- The expectation values in the Tolman-Ehrenfest state are effectively given by the result in the absence of rotation under the replacements:

$$T = \Gamma T_0 = \frac{T_0}{\sqrt{1-\Omega^2\rho^2}} \quad \text{and} \quad \mu = \Gamma\mu_0 = \frac{\mu_0}{\sqrt{1-\Omega^2\rho^2}}$$

- Therefore, the grand canonical potential and the fermion condensate become

$$\Phi^{\text{m.f.}} = \int_{\mathcal{M}} d^3x \left[\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{\lambda}{4}(\sigma^2 - v^2)^2 + h\sigma \right.$$
$$\left. -2N_f N_c \sum_{\varsigma=\pm} \int \frac{\Gamma d^3p}{(2\pi)^3\beta_0} \ln\left(1 + e^{-\beta_0\Gamma^{-1}(E+\varsigma\mu_0)}\right)\right]$$
$$= -P_L V$$

$$\langle\bar{\psi}\psi\rangle_{TE} = 2N_f N_c g\sigma \times \sum_{\varsigma=\pm 1} \int \frac{d^3p}{(2\pi)^3 E} \frac{1}{e^{\beta_0\Gamma^{-1}(E-\varsigma\mu_0)} + 1}$$

The fermionic contribution to Φ as well as the fermion condensate are obtained for constant σ. Understand as **proxy.**

Mean Field Approximation

- We shall solve the σ eom under three approximations of increasing rigor:

(I) $\sigma(\rho) \to \bar{\sigma}$ Takes a **global (constant) averaged value** over the size of the system. Under this assumption, the extremization of the grand canonical potential results into:

$$\lambda(\bar{\sigma}^2 - v^2)\bar{\sigma} = h - g\frac{2}{R^2}\int d\rho\rho\langle\overline{\psi}\psi\rangle(\rho)$$

(II) $\Box\sigma(\rho) \simeq 0$ The singlet meson is **slowly varying** along the transverse direction.

$$\lambda(\sigma^2 + \vec{\pi}^2 - v^2)\ \sigma = h - g\langle\overline{\psi}\psi\rangle(\rho)$$

(III) $\Box\sigma(\rho) \neq 0$ The **general case** where the gradients in the transverse directions are taken into account.

$$\left[\Box + \lambda(\sigma^2 + \vec{\pi}^2 - v^2)\right]\sigma = h - g\langle\overline{\psi}\psi\rangle(\rho)$$

$$\Gamma = \frac{1}{\sqrt{1-\rho^2\Omega^2}}$$

CASE III: **Fully-varying** local ($\Box\sigma \neq 0$) σ

$$-\Omega^2\Gamma^4(\Gamma^2-1)\partial_\Gamma^2\sigma - \Omega^2\Gamma^3(3\Gamma^2-1)\partial_\Gamma\sigma$$
$$+\lambda\sigma(\sigma^2-v^2) = h - g\langle\overline{\psi}\psi\rangle\,.$$

$$\langle\bar{\psi}\psi\rangle_{TE} = 2N_f N_c\, g\sigma \times \sum_{\varsigma=\pm 1}\int\frac{d^3p}{(2\pi)^3 E}\,\frac{1}{e^{\beta_0\Gamma^{-1}(E-\varsigma\mu_0)}+1}$$

- The solution of the differential equation is subject to initial/boundary condition. Any choice of boundary conditions satisfy the variational principle $\delta\Phi = 0$. Space of solutions $\simeq \mathbb{R}^2$. Is there a **sensible choice of boundary conditions??**

- **σ obeys a differential equation and ought to be smooth in ρ!** Can we have a first order phase transition at all??

$$\Gamma = \frac{1}{\sqrt{1-\rho^2\Omega^2}}$$

CASE III: **Fully-varying** local ($\Box\sigma \neq 0$) σ

$$-\Omega^2\Gamma^4(\Gamma^2-1)\partial_\Gamma^2\sigma - \Omega^2\Gamma^3(3\Gamma^2-1)\partial_\Gamma\sigma + \lambda\sigma(\sigma^2-v^2) = h - g\langle\overline{\psi}\psi\rangle\,.$$

$$\langle\bar{\psi}\psi\rangle_{TE} = 2N_f N_c\, g\sigma \times \sum_{\varsigma=\pm1}\int \frac{d^3p}{(2\pi)^3 E}\frac{1}{e^{\beta_0\Gamma^{-1}(E-\varsigma\mu_0)}+1}$$

- **Sensible choice of boundary conditions?** **Step 1:** Study the differential equation around the critical points: the axis of rotation $\Gamma = 1$ ($\rho = 0$) and the firewall $\Gamma \to \infty$ ($\rho = \Omega^{-1}$).

$$\sigma(\Gamma\to 1) = \sigma_{axis} + \overline{C_1}\log(1-\Gamma) + O(1-\Gamma)$$

$$\sigma(\Gamma \gg 1) = C_1\left(1 + \frac{h}{2}\sigma_0^{-1}\frac{\log\Gamma}{\Gamma^2}\right) + \frac{C_2}{\Gamma^2} + O\left(\frac{\log\Gamma}{\Gamma^4}\right)$$

- The solution diverges at the light cylinder unless $\overline{C}_1 = 0$.

First boundary condition $\sigma'_{axis} = 0$. Now the space of solutions $\simeq \mathbb{R}$.

$$\sigma_0 = \frac{h}{g^2}\left(\frac{T_0^2}{6} + \frac{\mu_0^2}{2\pi^2}\right)^{-1}$$

$$\Gamma = \frac{1}{\sqrt{1-\rho^2\Omega^2}}$$

CASE III: **Fully-varying** local ($\Box\sigma \neq 0$) σ

$$\Phi^{\text{m.f.}} = \int_{\mathcal{M}} d^3x \left[\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{\lambda}{4}(\sigma^2 - v^2)^2 + h\sigma \right.$$
$$\left. -2N_f N_c \sum_{\varsigma=\pm} \int \frac{\Gamma d^3p}{(2\pi)^3\beta_0} \ln\left(1 + e^{-\beta_0\Gamma^{-1}(E+\varsigma\mu_0)}\right) \right]$$

- **Sensible choice of boundary conditions? Step 2:** Evaluate the pressure (or grand canonical potential) near the firewall

$$2N_f N_c \sum_{\varsigma=\pm} \int \frac{\Gamma d^3p}{(2\pi)^3\beta_0} \ln\left(1 + e^{-\beta_0\Gamma^{-1}(E+\varsigma\mu_0)}\right) = \frac{N_f N_c}{180\pi^2}(7\pi^4T^4 + 30\pi^2T^2\mu^2 + 15\mu^4)\Gamma^4$$
$$- \frac{N_f N_c}{12\pi^2} g^2\sigma^2\Gamma^2(\pi^2T^2 + 3\mu^2) + O(g^4\sigma^4\Gamma^0)$$

$$\Gamma = \frac{1}{\sqrt{1-\rho^2\Omega^2}}$$

CASE III: **Fully-varying** local ($\Box\sigma \neq 0$) σ

- **Sensible choice of boundary conditions?** Step 2: Evaluate the pressure (or grand canonical potential) near the firewall

$$2N_fN_c\sum_{\varsigma=\pm}\int\frac{\Gamma d^3p}{(2\pi)^3\beta_0}\ln\left(1+e^{-\beta_0\Gamma^{-1}(E+\varsigma\mu_0)}\right)=\frac{N_fN_c}{180\pi^2}(7\pi^4T^4+30\pi^2T^2\mu^2+15\mu^4)\Gamma^4$$

$$-\frac{N_fN_c}{12\pi^2}g^2\sigma^2\Gamma^2(\pi^2T^2+3\mu^2)+O(g^4\sigma^4\Gamma^0)$$

- The first term is infinite, but independent of boundary conditions. It contributes the same given a thermodynamic state (T,μ,Ω) .

- The second term is negative semi-definite! Any solution with a non-zero value of σ at the light cylinder is thermodynamically infinitely penalised.

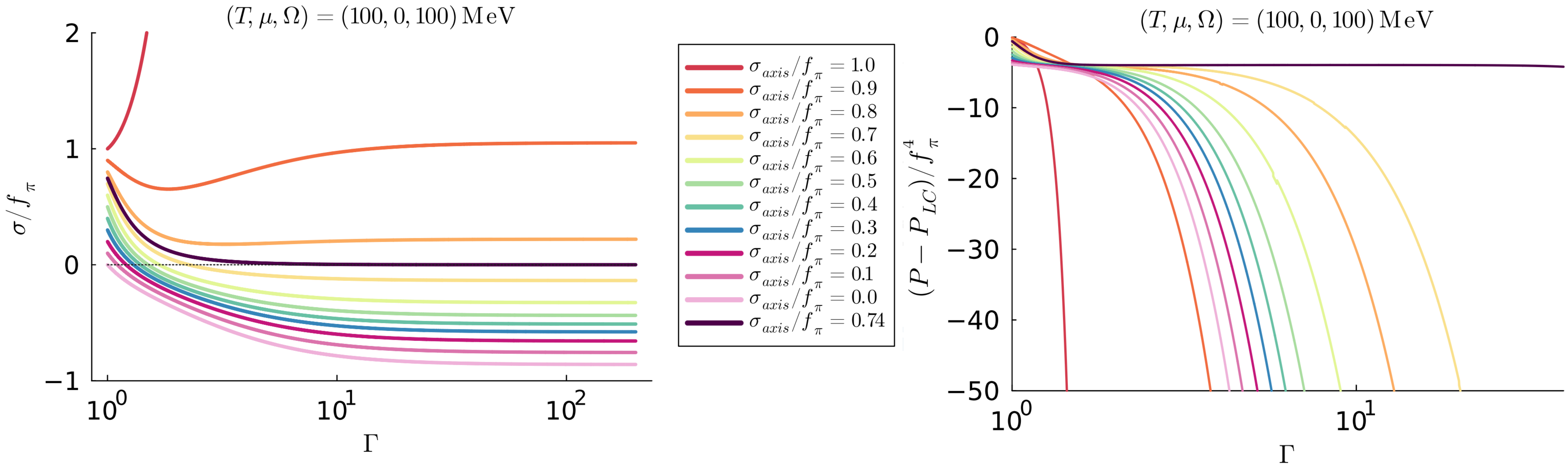
- **Second boundary condition:** $\sigma(\Gamma\to\infty)=0$

(In a finite size system that does not reach the light cylinder, a different boundary condition may be imposed)

$$\Gamma = \frac{1}{\sqrt{1-\rho^2\Omega^2}}$$

CASE III: **Fully-varying** local ($\Box\sigma \neq 0$) σ

◉ **Sensible choice of boundary conditions?** Step 2: Evaluate the pressure (or grand canonical potential) near the firewall



penalised.

$$\Gamma = \frac{1}{\sqrt{1-\rho^2\Omega^2}}$$

CASE III: **Fully-varying** local ($\Box\sigma \neq 0$) σ

- To sum up: Given a thermodynamic state (T, μ, Ω) solve the differential equation

$$\left[-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} + \lambda(\sigma^2 - v^2)\right]\sigma = h - g\langle\bar\psi\psi\rangle$$

$$\langle\bar\psi\psi\rangle_{TE} = 2N_f N_c g\sigma \times \sum_{\varsigma=\pm1}\int \frac{d^3p}{(2\pi)^3 E}\frac{1}{e^{\beta_0\Gamma^{-1}(E-\varsigma\mu_0)}+1}$$

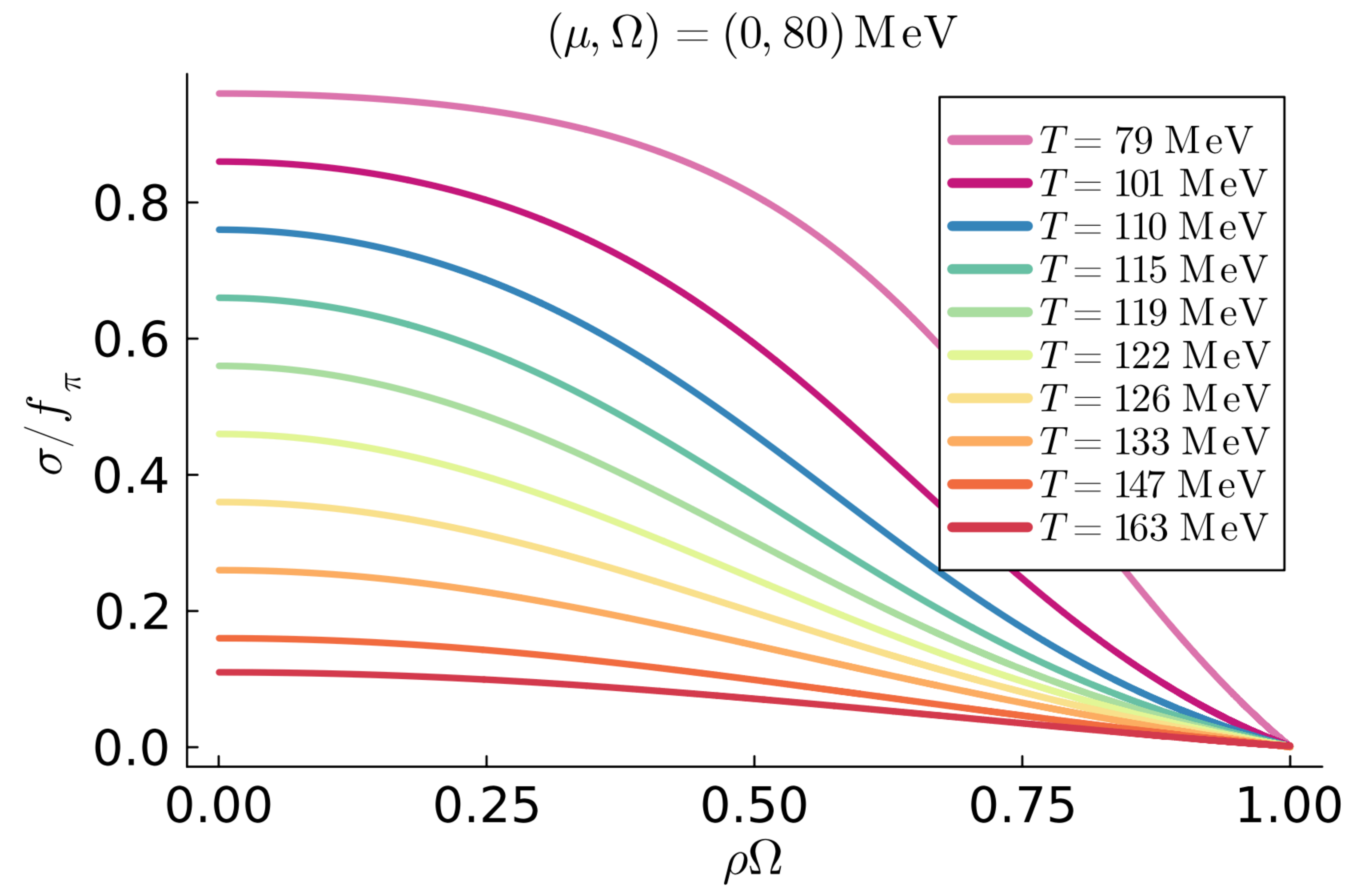
subject to the boundary conditions $\sigma(\rho\Omega \to 1) = 0$ and $\sigma'(\rho \to 0) = 0$. We obtain $\sigma(\rho)$ and in particular $\sigma_{axis} = \sigma(\rho \to 0)$.

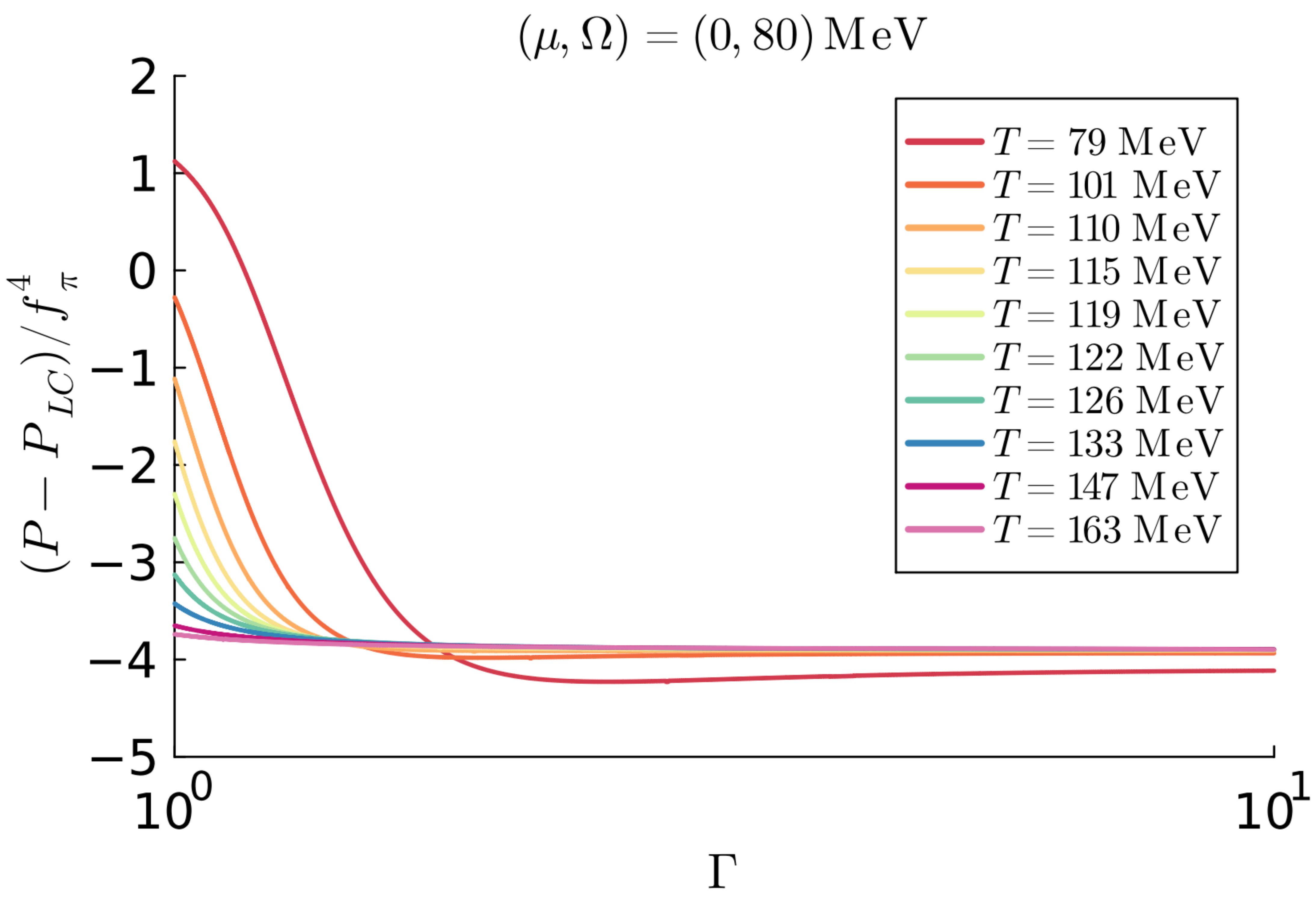
- **The solution to the boundary value problem may not be unique!** In such case, we use the free energy to discriminate the thermodynamically favoured solution. This is how the first order first transition is realised in this setup.

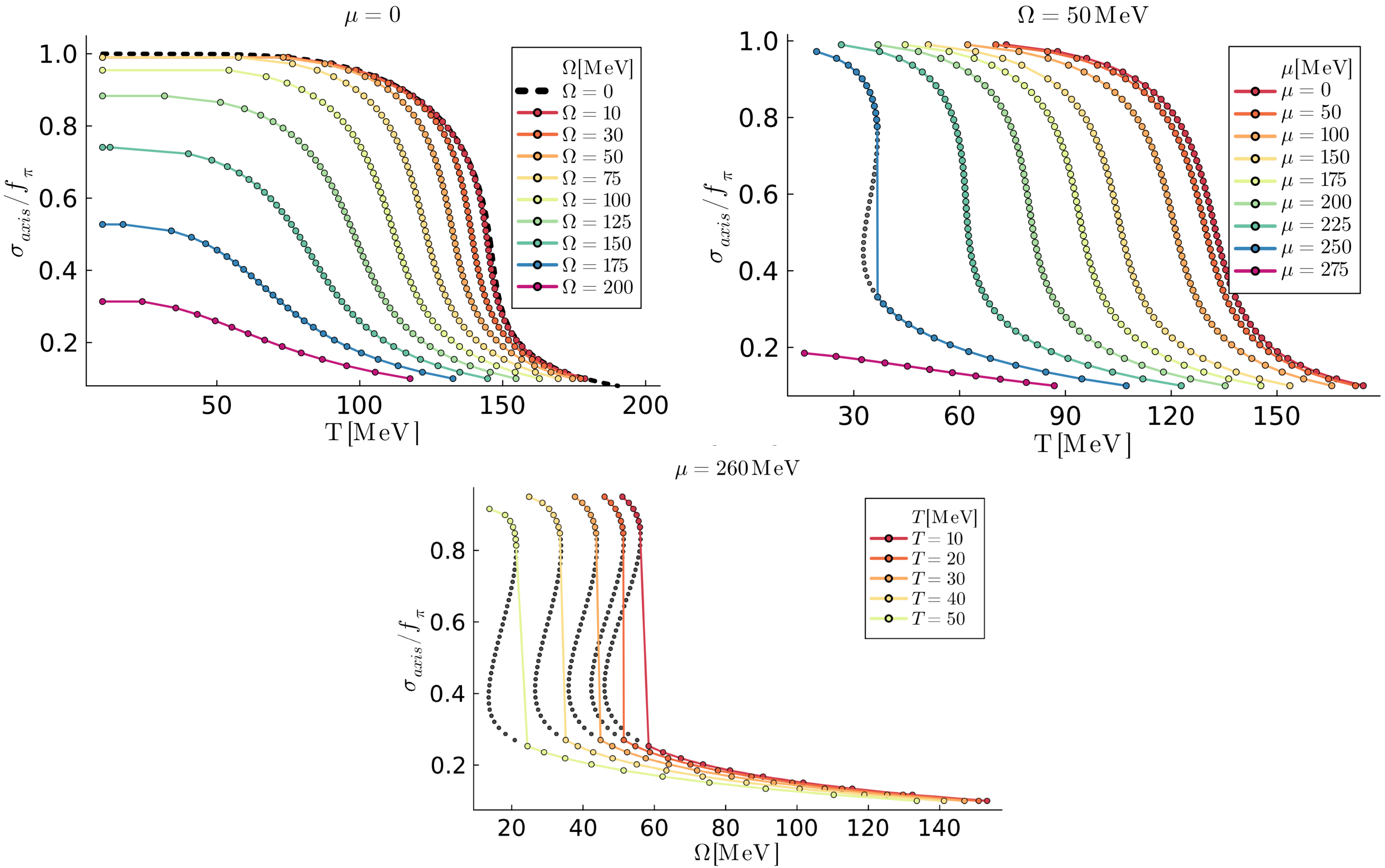
$$\Gamma = \frac{1}{\sqrt{1-\rho^2\Omega^2}}$$

CASE III: **Fully-varying** local ($\Box\sigma \neq 0$) σ

- The system is either in a **chirally restored phase**, where σ is small everywhere, or in a **mixed inhomogeneous phase**, where it is chirally broken in the region close to the rotation axis and chirally restored close to the light-cylinder. Each **phase identified by** σ_{axis}



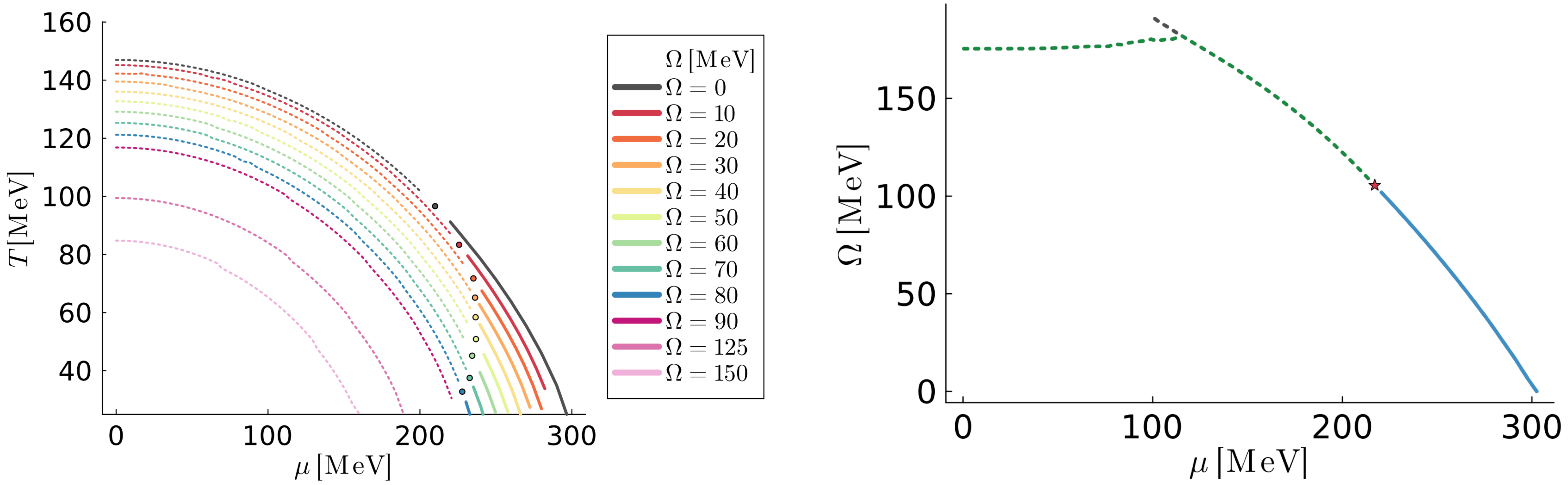




$\mu = 0$
Ω[MeV]
$\Omega = 0$
$\Omega = 10$
$\Omega = 30$
$\Omega = 50$
$\Omega = 75$
$\Omega = 100$
$\Omega = 125$
$\Omega = 150$
$\Omega = 175$
$\Omega = 200$
σ_{axis}/f_π
T [MeV]
$\Omega = 50\,\mathrm{MeV}$
μ[MeV]
$\mu = 0$
$\mu = 50$
$\mu = 100$
$\mu = 150$
$\mu = 175$
$\mu = 200$
$\mu = 225$
$\mu = 250$
$\mu = 275$
σ_{axis}/f_π
T [MeV]
$\mu = 260\,\mathrm{MeV}$
T[MeV]
$T = 10$
$T = 20$
$T = 30$
$T = 40$
$T = 50$
σ_{axis}/f_π
Ω[MeV]

CASE III: Phase Diagram

- The inner region corresponds to the **mixed inhomogeneous phase** while in the outer region the system is in the **chirally restored phase**.
- The critical point is follows a non-monotonic trajectory in phase space as a function of μ.
- There is a critical point at zero temperature.
- At small T and small μ, phase transition driven by angular velocity only. Quantum corrections will be important.



Discussion & Outlook

- In the unbounded system, a natural boundary appears, along with natural boundary conditions for the radial dependent gap equation. They always enforce that the singlet meson vanish on the firewall. The system shields against the superluminal region.
- The approximations that σ is constant or gradients are neglected are only valid for small rotation.
- The phase of the system can be labelled by the value of σ on axis.
- The critical endpoint follows a non-monotonic trajectory in phase space as a function of the chemical potential μ.

(I) Work with the rigid rotating state, i.e. include quantum corrections to Tolman-Ehrenfest

(II) Obtain free energy and fermion condensate for ρ-dependent σ.

(III) Extend these results to the $PLSM_q$.

Thank you!

CASE I: **Global value of σ**

$$\lambda(\overline{\sigma}^2 - v^2)\overline{\sigma} = h - g\frac{2}{R^2}\int d\rho\rho\langle\overline{\psi}\psi\rangle(\rho)$$

CASE II: **Local σ** ($\Box\sigma \simeq 0$)

$$\lambda[\sigma^2(\rho) - v^2]\sigma(\rho) = h - g\langle\bar{\psi}\psi\rangle$$

CASE I: **Global value of σ**

$$\lambda(\overline{\sigma}^2 - v^2)\overline{\sigma} = h - g\frac{2}{R^2}\int d\rho\rho\langle\overline{\psi}\psi\rangle(\rho)$$

CASE II: **Local σ ($\Box\sigma \simeq 0$)**

$$\lambda[\sigma^2(\rho) - v^2]\sigma(\rho) = h - g\langle\bar{\psi}\psi\rangle$$

- As we approach the light cylinder, the **FC diverges** if the singlet attains a finite value there:

$$\frac{2}{R^2}\int_0^R d\rho\,\rho\langle\bar{\psi}\psi\rangle \simeq \frac{2g\bar{\sigma}}{R^2\Omega^2}\left(\frac{T_0^2}{6} + \frac{\mu_0^2}{2\pi^2}\right)\ln\Gamma_R$$

$$\langle\bar{\psi}\psi\rangle \simeq g\bar{\sigma}\Gamma_R^2\left(\frac{T_0^2}{6} + \frac{\mu_0^2}{2\pi^2}\right)$$

- σ (respectively $\overline{\sigma}$) need vanish as it reaches the light cylinder to solve the gap equation:

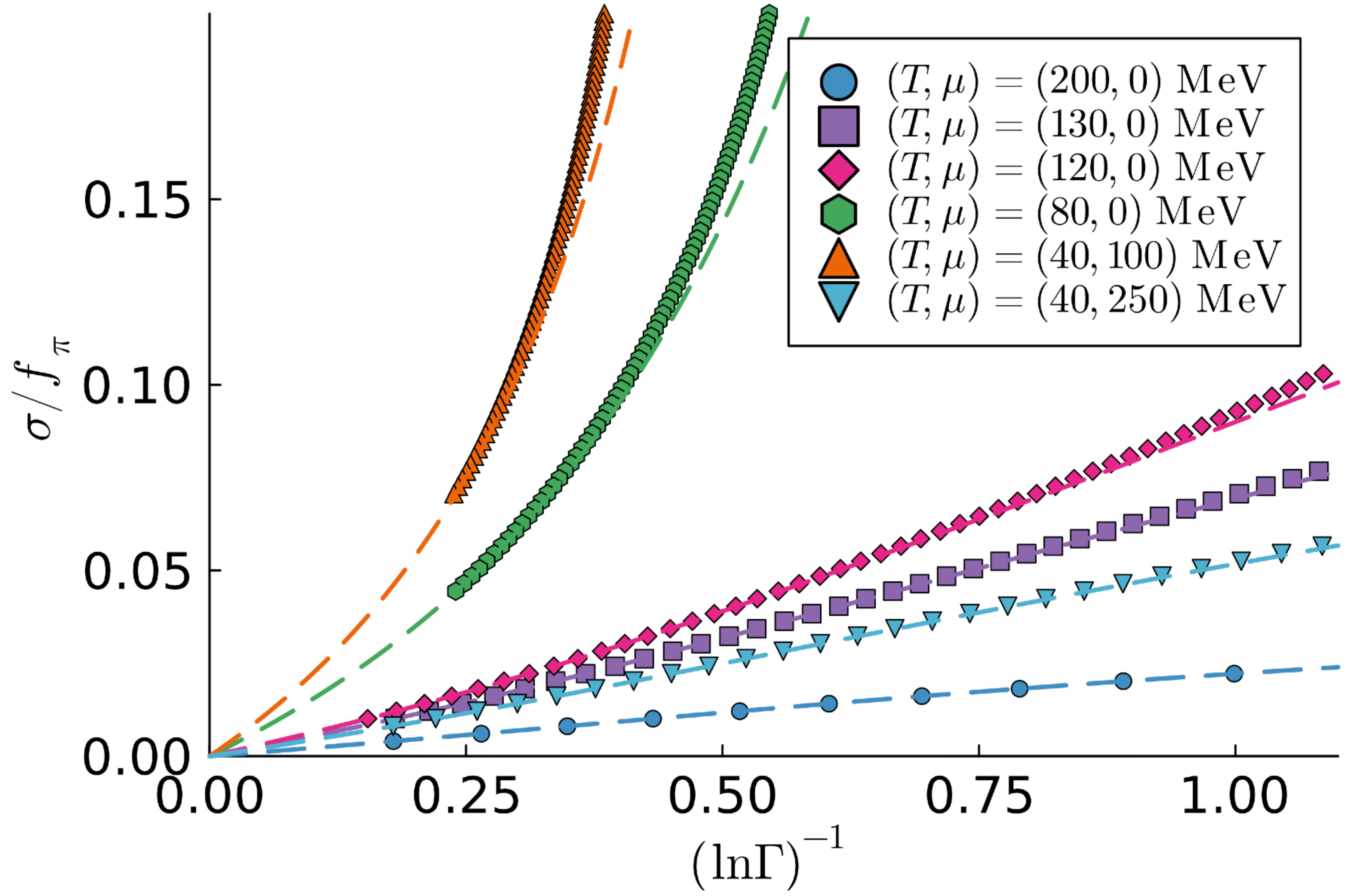
$$\bar{\sigma} \simeq \frac{h}{\frac{2g^2}{R^2\Omega^2}\left(\frac{T_0^2}{6} + \frac{\mu_0^2}{2\pi^2}\right)\ln\Gamma_R - v^2\lambda}.$$

$$\sigma(\rho) \simeq \frac{\sigma_0}{\Gamma^2(\rho) - \sigma_0 h^{-1}\lambda v^2}$$

$$\sigma_0 = \frac{h}{g^2}\left(\frac{T_0^2}{6} + \frac{\mu_0^2}{2\pi^2}\right)^{-1}$$

CASE I: **Global value of σ**

$$\lambda(\overline{\sigma}^2 - v^2)\overline{\sigma} = h - g\frac{2}{R^2}\int d\rho\rho\langle\overline{\psi}\psi\rangle(\rho)$$



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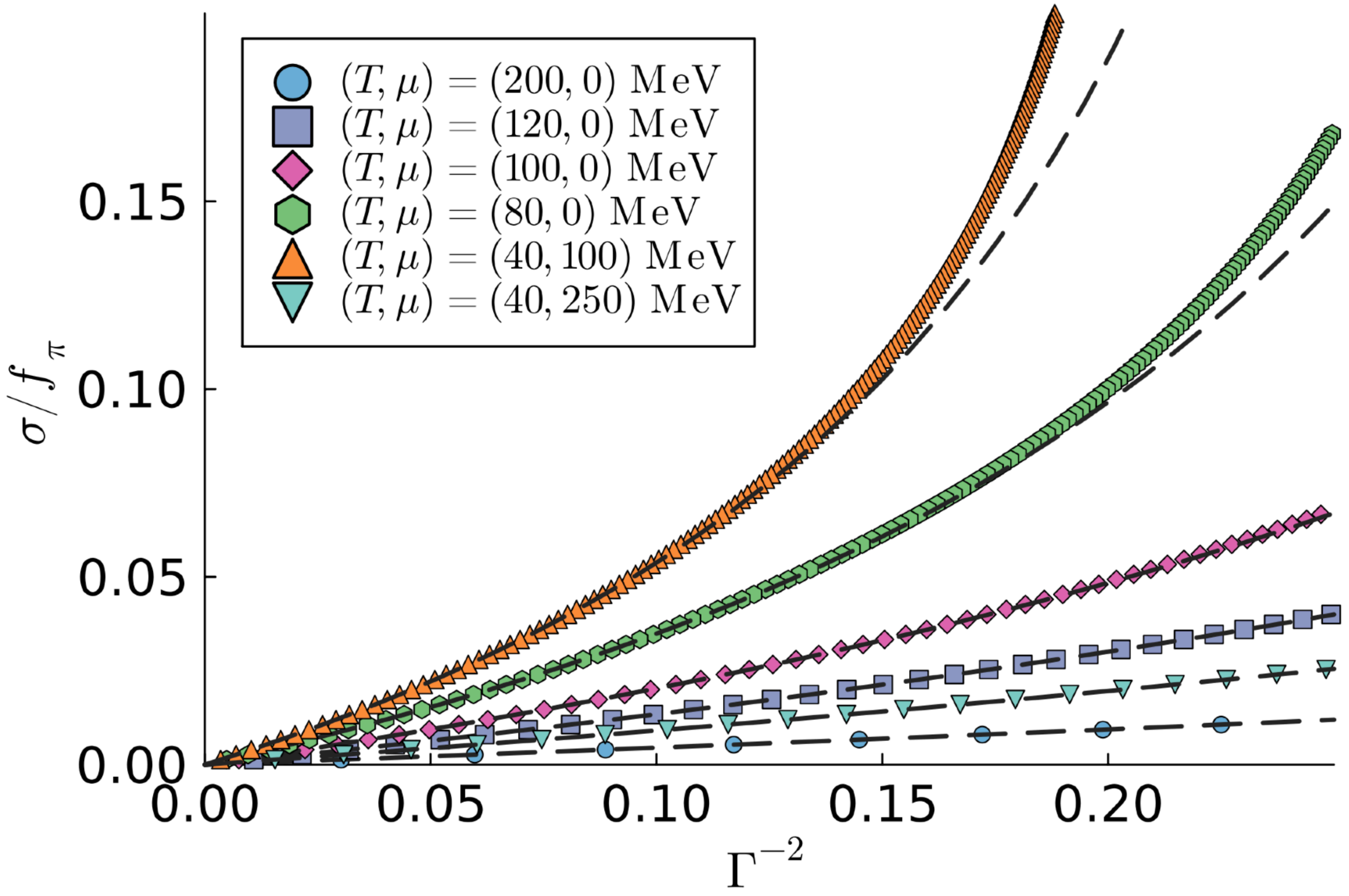
CASE II: **Local σ** ($\Box\sigma \simeq 0$)

$$\lambda[\sigma^2(\rho) - v^2]\sigma(\rho) = h - g\langle\bar{\psi}\psi\rangle$$



$$\sigma(\rho) \simeq \frac{\sigma_0}{\Gamma^2(\rho) - \sigma_0 h^{-1}\lambda v^2}$$

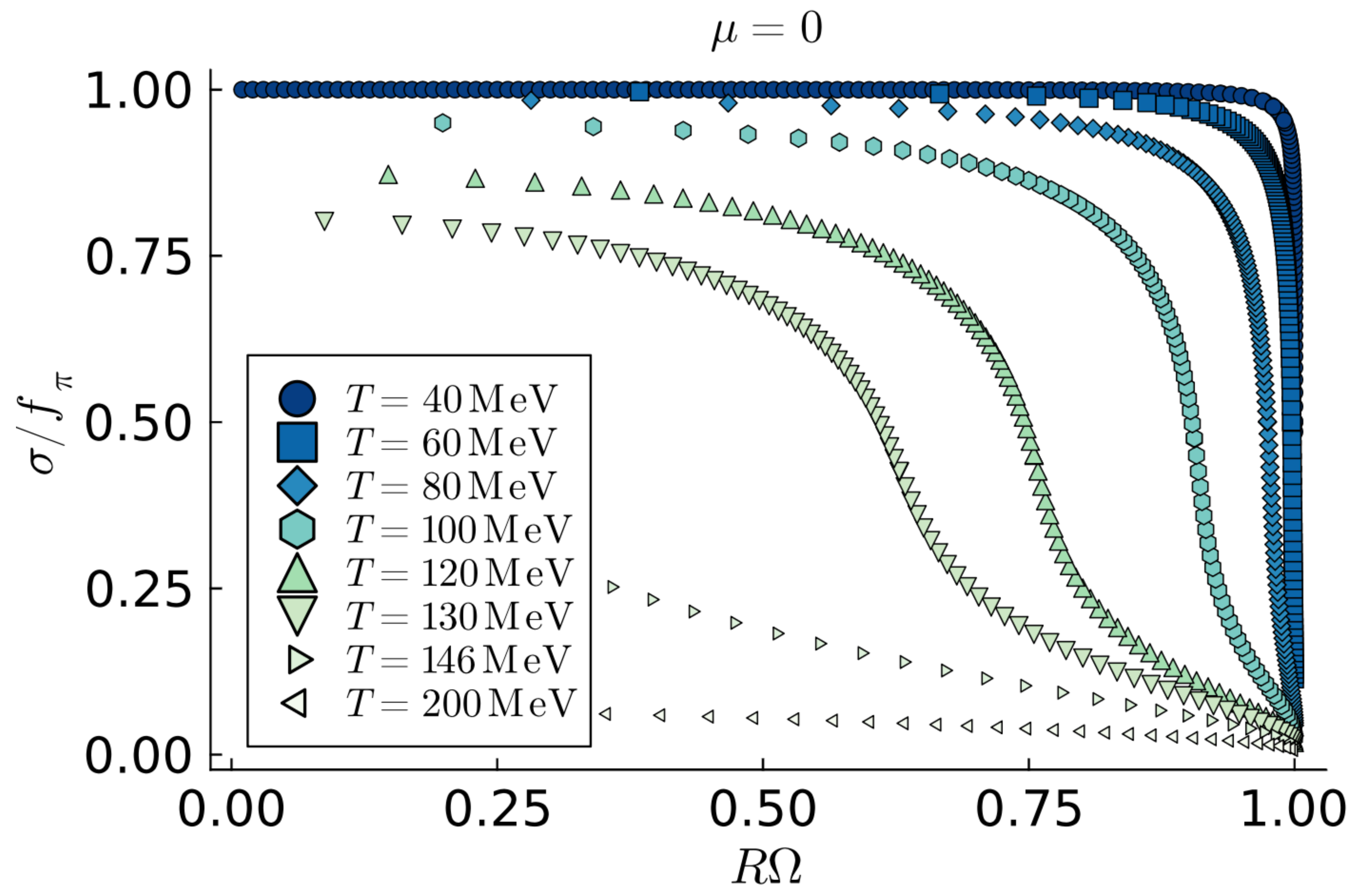
CASE I: **Global value of σ**

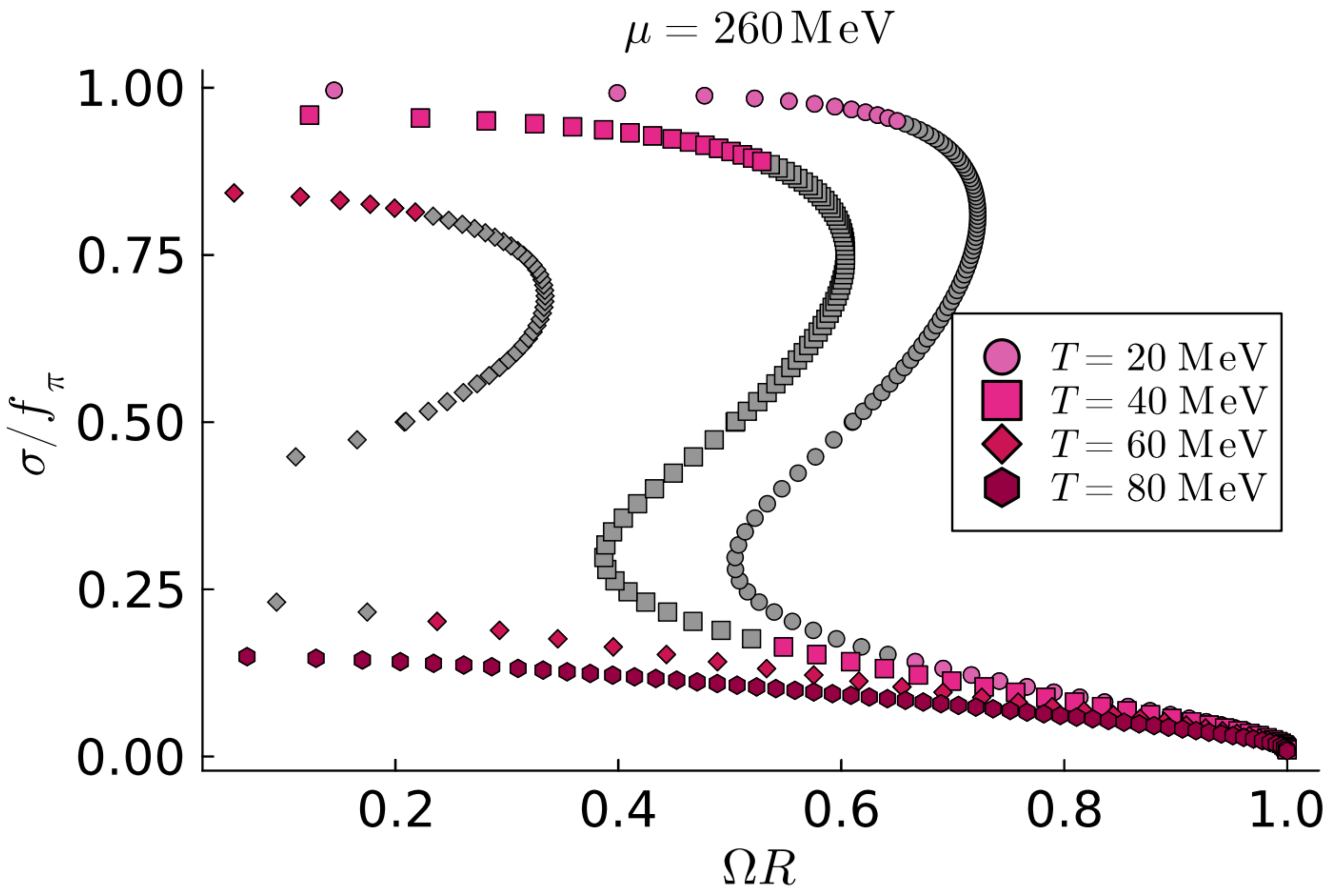
CASE II: **Local σ** ($\Box\sigma \simeq 0$)

$$\lambda(\overline{\sigma}^2 - v^2)\overline{\sigma} = h - g\frac{2}{R^2}\int d\rho\rho\langle\overline{\psi}\psi\rangle(\rho)$$

$$\lambda[\sigma^2(\rho) - v^2]\sigma(\rho) = h - g\langle\bar{\psi}\psi\rangle$$

- The global value of σ is a function of the combination ΩR (only true for the Tolman-Ehrenfest state). Predicts $\sigma(\rho) = 0$ if the system extends to the light-cylinder (i.e. $R = \Omega^{-1}$).





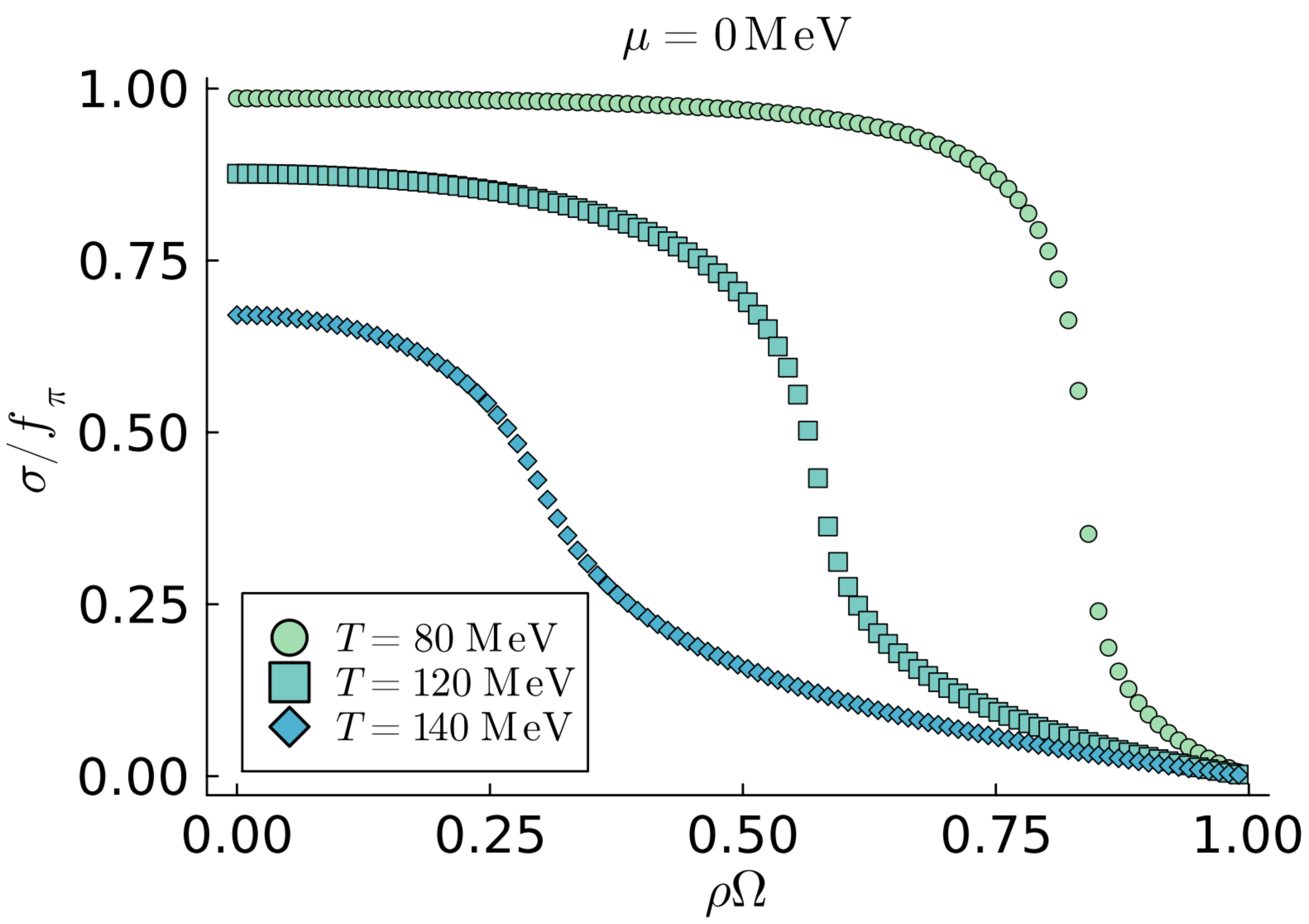
CASE I: **Global value of σ**

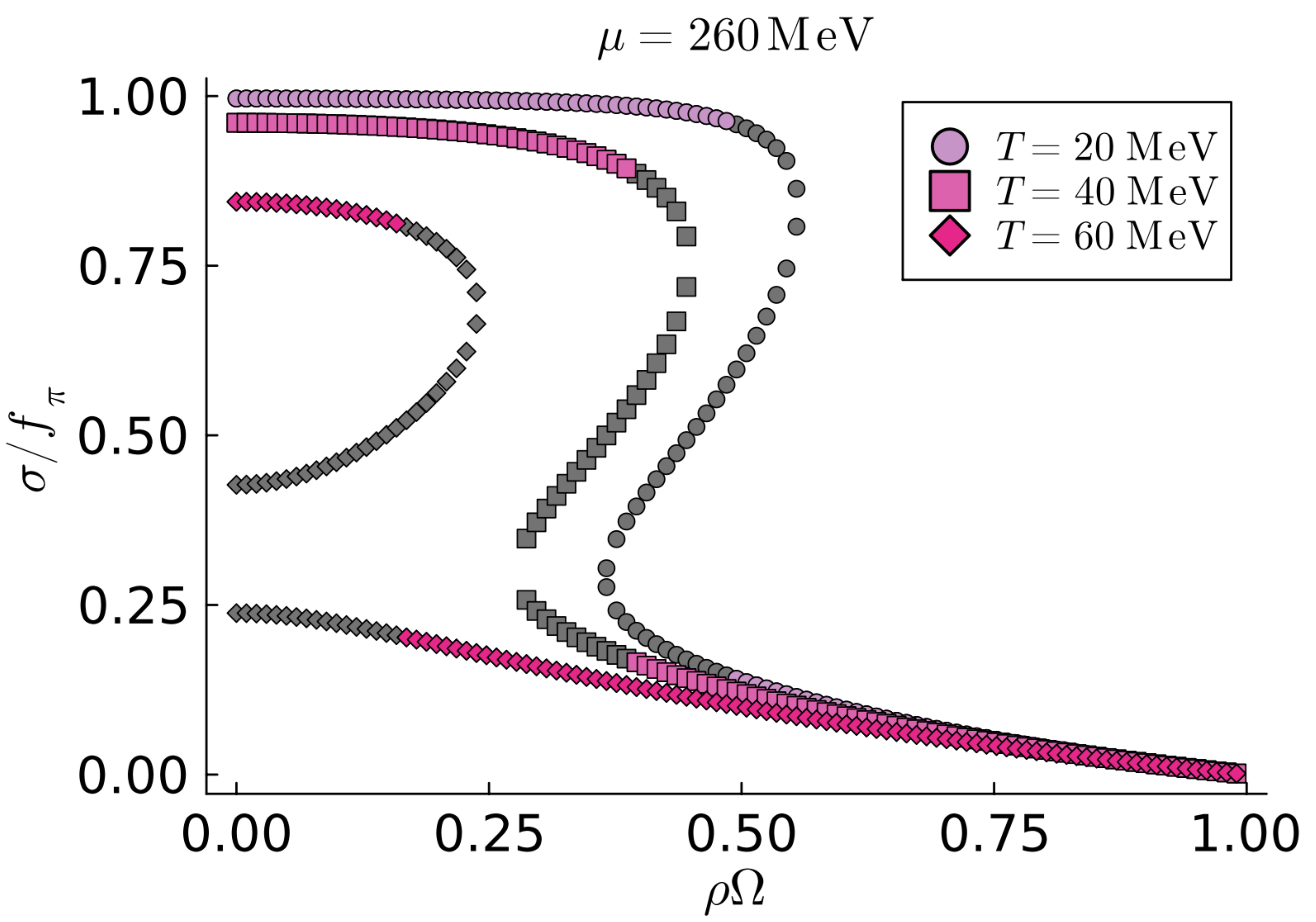
$$\lambda(\overline{\sigma}^2 - v^2)\overline{\sigma} = h - g\frac{2}{R^2}\int d\rho\rho\langle\overline{\psi}\psi\rangle(\rho)$$

CASE II: **Local σ** ($\Box\sigma \simeq 0$)

$$\lambda[\sigma^2(\rho) - v^2]\sigma(\rho) = h - g\langle\bar{\psi}\psi\rangle$$

- The global value of σ is a function of the combination $\rho\Omega$ (only true for the Tolman-Ehrenfest state). Allows for **inhomogeneous phases**. Predicts first order phase transitions at a finite distance from the rotation axis.





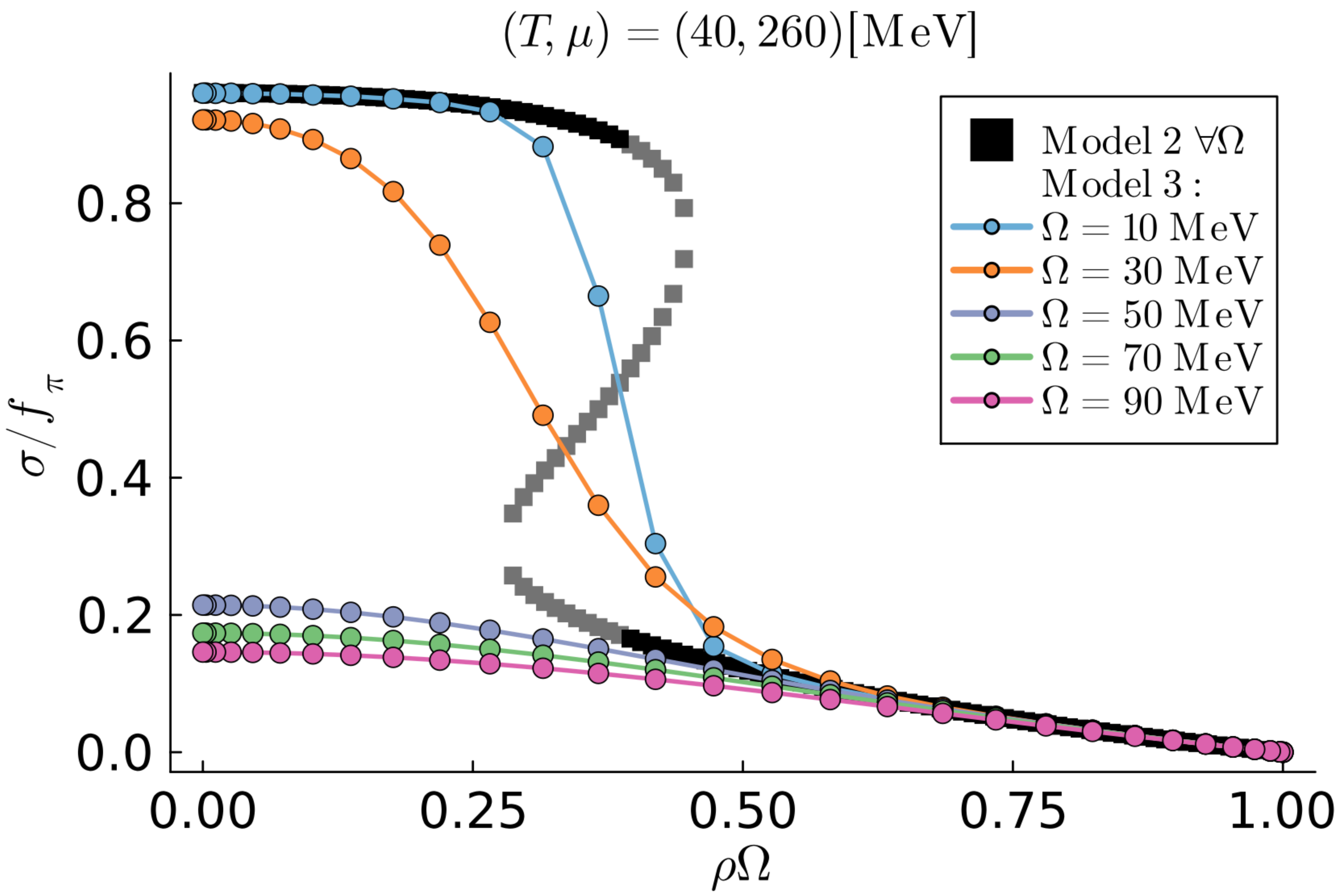
CASE I vs. CASE II

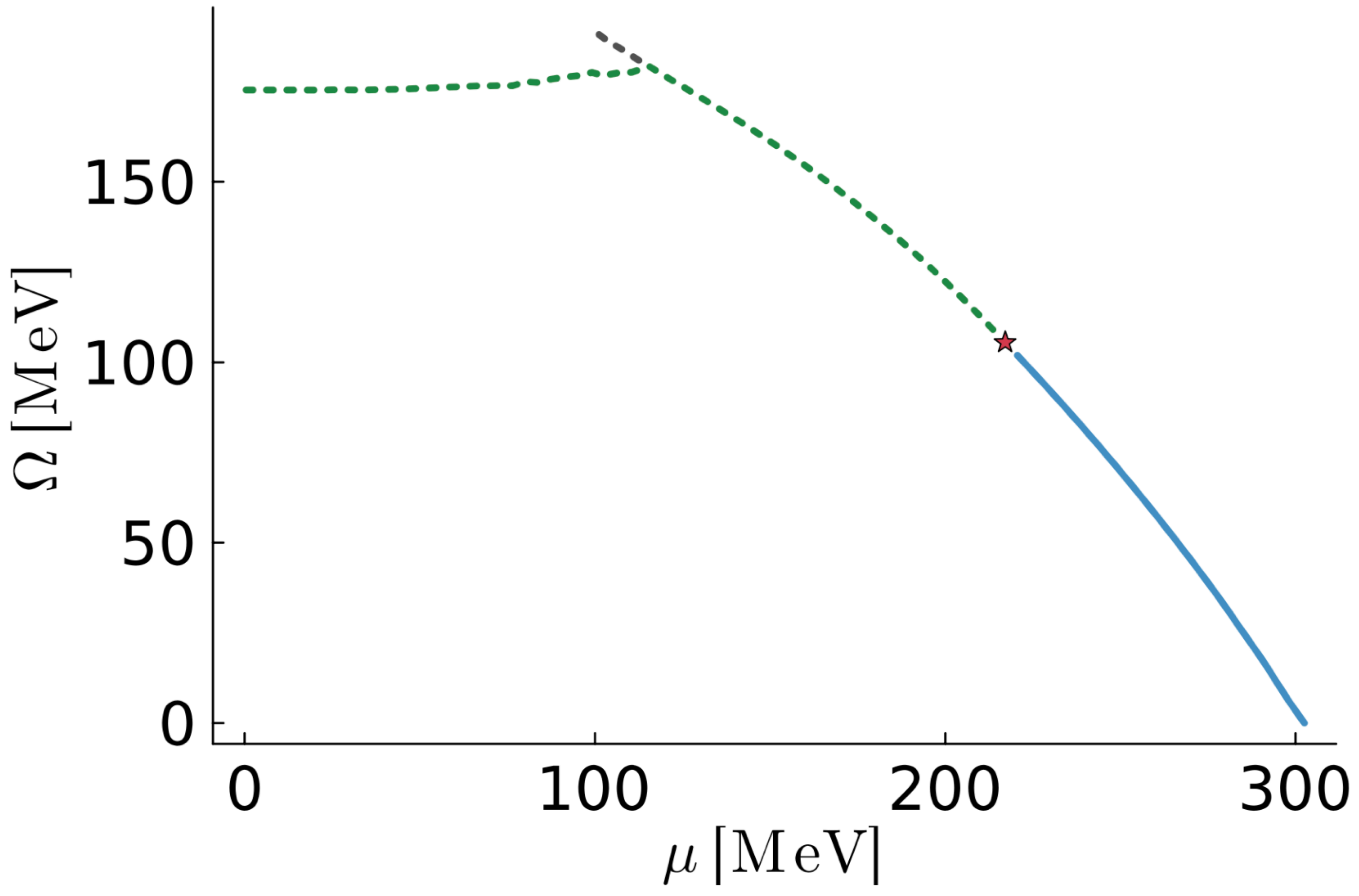
- We compare the global solution $\overline{\sigma}$ (Model 1) with the value of the local σ (Model 2) averaged over the size of the system R (*Model 2*).
- Agreement only before the phase transition. Transition happens "earlier" in Model 2.

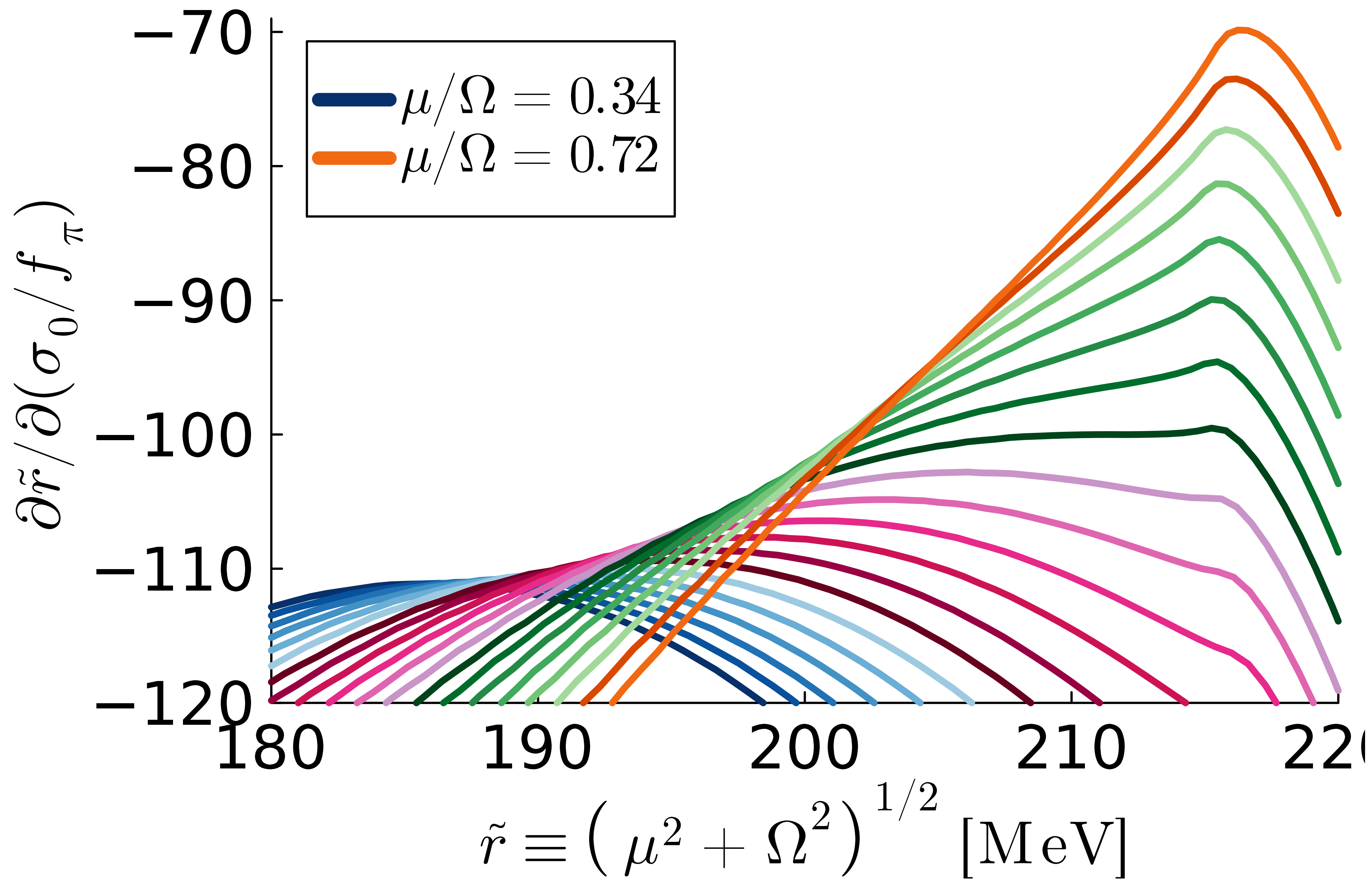
CASE II vs. CASE III

- It is clear from the differential equation that the magnitude of the gradient terms is controlled by the angular velocity $\mathbf{\Omega}$. Therefore, cases II and III should agree only for small angular velocity.

$$-\Omega^2\Gamma^4(\Gamma^2-1)\partial_\Gamma^2\sigma-\Omega^2\Gamma^3(3\Gamma^2-1)\partial_\Gamma\sigma$$
$$+\lambda\sigma(\sigma^2-v^2)=h-g\langle\overline{\psi}\psi\rangle\,.$$







$$\left[-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}+\lambda(\sigma^2-v^2)\right]\sigma = h - g\langle\bar{\psi}\psi\rangle$$

$$\langle\overline{\psi}\psi\rangle_{T=0} \propto g\sigma(\rho)\theta\left(\frac{\mu}{\sqrt{1-\rho^2\Omega^2}} - g\sigma(\rho)\right)$$