Hybrid framework of perfect and dissipative spin hydrodynamics

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Hybrid approach

There are currently several approaches to spin hydrodynamics:

1. Determine spin polarization using only gradients of hydrodynamic fields on the freezeout hypersurface (the main objects are thermal vorticity $\varpi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right)$ and thermal shear $\xi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu\right)$, where $\beta^\mu = \frac{u^\mu}{T}$).
2. Obtain hydrodynamic equations from kinetic theory.
3. Consider mathematically possible forms of the e-m and spin tensors and apply conservation laws and entropy production.
4. Use the spin-extended Lagrangian formalism.

We propose a framework that combines (2) for the perfect fluid description and (3) for the inclusion of dissipation.

Z. D., W. Florkowski, M. Hontarenko, *Hybrid approach to perfect and dissipative spin hydrodynamics*, Phys. Rev. D **110**, 096018 (2024)

Phenomenological version of thermodynamic relations

The fundamental thermodynamic relation, first law of thermodynamics and the Gibbs-Duhem relation are

$$\varepsilon + P = T\sigma + \mu n + \frac{1}{2}\Omega_{\alpha\beta} S^{\alpha\beta}, \quad (1)$$

$$\mathrm{d}\varepsilon = T\mathrm{d}\sigma + \mu \mathrm{d}n + \frac{1}{2}\Omega_{\alpha\beta} \mathrm{d}S^{\alpha\beta}, \quad (2)$$

$$\mathrm{d}P = \sigma \mathrm{d}T + n\mathrm{d}\mu + \frac{1}{2} S^{\alpha\beta}\mathrm{d}\Omega_{\alpha\beta}. \quad (3)$$

Multiplication by u^μ leads to tensor equations

$$S^\mu = P\beta^\mu - \xi N^\mu + \beta_\lambda T^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta} S^{\mu,\alpha\beta}, \quad (4)$$

$$\mathrm{d}S^\mu = -\xi \mathrm{d}N^\mu + \beta_\lambda \mathrm{d}T^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta}\mathrm{d}S^{\mu,\alpha\beta}, \quad (5)$$

$$\mathrm{d}(P\beta^\mu) = N^\mu \mathrm{d}\xi - T^{\lambda\mu}\mathrm{d}\beta_\lambda + \frac{1}{2} S^{\mu,\alpha\beta}\mathrm{d}\omega_{\alpha\beta}, \quad (6)$$

with $\xi = \frac{\mu}{T}$, the spin polarization $\omega_{\alpha\beta}$, spin chemical potential $\Omega_{\alpha\beta} = T\omega_{\alpha\beta}$, spin density $S^{\alpha\beta} = u_\lambda S^{\lambda,\alpha\beta}$, and the spin tensor $S^{\mu,\alpha\beta} = u^\mu S^{\alpha\beta}$.

- $S^{\mu,\alpha\beta} = u^\mu S^{\alpha\beta}$ appears already in a model of spinning fluid by Weyssenhoff and Raabe 1947 and has been used many times since then.

  J. Weyssenhoff, A. Raabe, Relativistic Dynamics of Spin-Fluids and Spin-Particles, Acta Phys. Pol. 9, 7 (1947)
- It is analogous to perfect fluid expressions $N^\mu_{eq} = nu^\mu$, $T^{\lambda\mu}_{\rm eq} = (\varepsilon + P)u^\lambda u^\mu - Pg^{\lambda\mu} = \varepsilon u^\lambda u^\mu - P\Delta^{\lambda\mu}$.
- However, it is not justified by microscopic calculations.

Kinetic theory with a classical description of spin

Internal angular momentum

$$s^{\alpha\beta} = \frac{1}{m}\epsilon^{\alpha\beta\gamma\delta}p_\gamma s_\delta, \tag{7}$$

where the spin 4-vector $s \cdot p = 0$, $s^\alpha = \frac{1}{2m}\epsilon^{\alpha\beta\gamma\delta}p_\beta s_{\gamma\delta}$.
In the particle rest frame, $p^\mu = (m, 0, 0, 0)$, $s^\alpha = (0, \mathbf{s}_*)$, $|\mathbf{s}_*| = \mathfrak{s}$.

$$\mathfrak{s}^2 = 1/2\,(1 + 1/2) = 3/4. \tag{8}$$

Local equilibrium distribution functions for particles $(+)$ and for antiparticles $(-)$ have the Fermi-Dirac form

$$f^{\pm}_{\rm eq}(x, p, s) = \left[\exp\left(\mp\xi(x) + p\cdot\beta(x) - \frac{1}{2}\,\omega(x) : s\right) + 1\right]^{-1} = \frac{1}{e^{y^{\pm}} + 1}, \tag{9}$$

$$y^{\pm} = \mp\xi(x) + p\cdot\beta(x) - \frac{1}{2}\omega(x) : s, \tag{10}$$

with $\omega(x) : s \equiv \omega_{\mu\nu}s^{\mu\nu}$.

Preliminaries

Parametrization of the spin polarization tensor

$$\omega_{\alpha\beta} = k_\alpha u_\beta - k_\beta u_\alpha + t_{\alpha\beta}, \tag{11}$$

with $k \cdot u = \omega \cdot u = 0, t_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} u^\gamma \omega^\delta$, $t^\mu = t^{\mu\nu} k_\nu = \epsilon^{\mu\nu\alpha\beta} k_\nu u_\alpha \omega_\beta$.
Integration measures

$$\mathrm{d}P = \frac{\mathrm{d}^3 p}{(2\pi)^3 E_p}, \tag{12}$$

$$\mathrm{d}S = \frac{m}{\pi \mathfrak{s}} \mathrm{d}s \delta(s \cdot s + \mathfrak{s}^2) \delta(p \cdot s). \tag{13}$$

Currents and tensors

The baryon current, the energy-momentum tensor, and the spin tensor in local equilibrium

$$N^{\mu}_{\rm eq} = \int \mathrm{d}P\, \mathrm{d}S\, p^{\mu} \, \left[f^{+}_{\rm eq}(x,p,s) - f^{-}_{\rm eq}(x,p,s) \right], \tag{14}$$

$$T^{\mu\nu}_{\rm eq} = \int \mathrm{d}P\, \mathrm{d}S\, p^{\mu} p^{\nu} \, \left[f^{+}_{\rm eq}(x,p,s) + f^{-}_{\rm eq}(x,p,s) \right], \tag{15}$$

$$S^{\lambda,\mu\nu}_{\rm eq} = \int \mathrm{d}P\, \mathrm{d}S\; p^{\lambda}\, s^{\mu\nu} \left[f^{+}_{\rm eq}(x,p,s) + f^{-}_{\rm eq}(x,p,s) \right]. \tag{16}$$

In addition, we define the current $\mathcal{N}^{\mu}$ ($= P\beta^{\mu}$ in traditional hydrodynamics)

$$\mathcal{N}^{\mu} = -\int \mathrm{d}P\, \mathrm{d}S\, p^{\mu} \, \left[\ln(1 - f^{+}_{\rm eq}) + \ln(1 - f^{-}_{\rm eq}) \right]. \tag{17}$$

The entropy current

$$S^\mu_{\rm eq} = -\int \mathrm{d}P\,\mathrm{d}S\,p^\mu\,\left[f^+_{\rm eq}\ln f^+_{\rm eq} - f^+_{\rm eq}\ln(1-f^+_{\rm eq}) + \ln(1-f^+_{\rm eq})\right]. \tag{18}$$

We use the formula

$$f_{\rm eq}\ln f_{\rm eq} - f_{\rm eq}\ln(1-f_{\rm eq}) = -y f_{\rm eq} \tag{19}$$

and express $S^\mu_{\rm eq}$ in terms of other tensors and currents

$$S^\mu_{\rm eq} = \int \mathrm{d}P\,\mathrm{d}S\,p^\mu\left[f^+_{\rm eq}\left(-\xi + p\cdot\beta - \tfrac{1}{2}\omega : s\right) + f^-_{\rm eq}\left(\xi + p\cdot\beta - \tfrac{1}{2}\omega : s\right)\right] + \mathcal{N}^\mu, \tag{20}$$

$$S^\mu_{\rm eq} = -N^\mu_{\rm eq}\xi + T^{\mu\alpha}_{\rm eq}\beta_\alpha - \frac{1}{2}S^{\mu,\alpha\beta}_{\rm eq}\omega_{\alpha\beta} + \mathcal{N}^\mu. \tag{21}$$

This is similar to Eq. (4), except $\mathcal{N}^\mu \neq N\beta^\mu$ for particles with spin.

Expansion for small spin

For small magnitudes of ω, we can expand the FD distribution functions around $y_s = 0$

$$\frac{1}{e^{y_0^\pm + y_s} + 1} = \frac{1}{e^{y_0^\pm} + 1} - \frac{e^{y_0^\pm}}{(e^{y_0^\pm} + 1)^2}\, y_s + \frac{e^{y_0^\pm}\,(e^{y_0^\pm} - 1)}{2\,(e^{y_0^\pm} + 1)^3}\, y_s^2 + \cdots, \quad (22)$$

where $y_0^\pm = \mp\xi(x) + p \cdot \beta(x)$, $y_s = -\frac{1}{2}\omega : s$.

$$\begin{aligned} f_{\rm eq}^\pm &= f_0^\pm - f_0^\pm(1 - f_0^\pm) y_s + \frac{1}{2} f_0^\pm (1 - f_0^\pm)(1 - 2 f_0^\pm) y_s^2 + \cdots \\ &\equiv f_0^\pm + \frac{1}{2}\, f_1^\pm\, \omega : s + \frac{1}{8}\, f_2^\pm\, (\omega : s)^2 + \cdots \end{aligned} \quad (23)$$

The baryon current

Expansion up to quadratic order in ω

$$N^\mu_{\rm eq} = \int \mathrm{d}P\,\mathrm{d}S\,p^\mu\ \left[f^+_{\rm eq}(x,p,s) - f^-_{\rm eq}(x,p,s)\right] = 2(Z_0^{+\mu} - Z_0^{-\mu})$$
$$+\,\mathfrak{s}^2\frac{\omega:\omega}{6}(Z_2^{+\mu} - Z_2^{-\mu}) + \frac{\mathfrak{s}^2}{3m^2}(Z_2^{+\mu\alpha\beta} - Z_2^{-\mu\alpha\beta})\omega^\gamma{}_\alpha\omega_{\beta\gamma} + \dots, \quad (24)$$

$$Z_n^{\pm\alpha\beta\dots} \equiv \int \mathrm{d}Pp^\alpha p^\beta \dots f_n^\pm. \quad (25)$$

In the Boltzmann case,
$Z_n^{\pm\alpha\beta\dots} \to e^{\pm\xi}Z^{\alpha\beta\dots} \equiv e^{\pm\xi}\int \mathrm{d}Pp^\alpha p^\beta\cdots e^{-\beta\cdot p}$, $n=0,1,2,$ and so

$$N^\mu_{\rm eq} = (n_0 + n_2^k + n_2^\omega)u^\mu + n_t t^\mu, \quad (26)$$

with coefficients $n_0 = \frac{2\sinh\xi}{\pi^2}z^2T^3K_2(z)$, $n_2^k = -\frac{2\mathfrak{s}^2\sinh\xi}{3\pi^2}zT^3K_3(z)k^2$,
$n_2^\omega = -\frac{\mathfrak{s}^2\sinh\xi}{3\pi^2}zT^3\left[zK_2(z)+2K_3(z)\right]\omega^2$, $n_t = -\frac{2\mathfrak{s}^2\sinh\xi}{3\pi^2}zT^3K_3(z)$,
$K_n(z)$ – Bessel functions of the second kind.

The energy-momentum tensor

Expansion up to quadratic order in ω

$$T^{\mu\nu}_{\rm eq} = \int \mathrm{d}P\,\mathrm{d}S\, p^\mu p^\nu \, \left[f^+_{\rm eq}(x,p,s) + f^-_{\rm eq}(x,p,s)\right] = 2\left(Z_0^{+\mu\nu} + Z_0^{-\mu\nu}\right)$$
$$+ \mathfrak{s}^2 \frac{\omega:\omega}{6}\left(Z_2^{+\mu\nu} + Z_2^{-\mu\nu}\right) + \frac{\mathfrak{s}^2}{3m^2}\left(Z_2^{+\mu\nu\alpha\beta} + Z_2^{-\mu\nu\alpha\beta}\right)\omega^\gamma{}_\alpha \omega_{\beta\gamma} + \dots \tag{27}$$

$$Z_n^{\pm\alpha\beta\dots} = \int \mathrm{d}P p^\alpha p^\beta \dots f_n^\pm.$$

In the Boltzmann case,

$$T^{\mu\nu}_{\rm eq} = (\varepsilon_0 + \varepsilon_2^k + \varepsilon_2^\omega) u^\mu u^\nu - (P_0 + P_2^k + P_2^\omega)\Delta^{\mu\nu}$$
$$+ P_{k\omega}(k^\mu k^\nu + \omega^\mu \omega^\nu) + P_t(t^\mu u^\nu + t^\nu u^\mu), \tag{28}$$

with

$$\varepsilon_0 = \frac{2\cosh\xi}{\pi^2} z^2 T^4 [zK_3(z) - K_2(z)], \quad \varepsilon_2^k = -\frac{2\mathfrak{s}^2\cosh\xi}{3\pi^2} zT^4[zK_2(z) + 5K_3(z)]k^2$$
$$\varepsilon_2^\omega = -\frac{\mathfrak{s}^2\cosh\xi}{3\pi^2} zT^4[zK_2(z) + (z^2+10)K_3(z)\omega^2, \quad P_0 = \frac{2\cosh\xi}{\pi^2} z^2 T^4 K_2(z),$$
$$P_2^k = -\frac{4\mathfrak{s}^2\cosh\xi}{3\pi^2} zT^4 K_3(z) k^2, \quad P_2^\omega = -\frac{\mathfrak{s}^2\cosh\xi}{3\pi^2} zT^4[zK_2(z) + 4K_3(z)]\omega^2,$$
$$P_{k\omega} = -\frac{2\mathfrak{s}^2\cosh\xi}{3\pi^2} zT^4 K_3(z), \quad P_t = \frac{2\mathfrak{s}^2\cosh\xi}{f\pi^2} zT^4[K_3(z) - zK_4(z)].$$

The spin tensor

Expansion up to linear order in ω

$$S_{\rm eq}^{\lambda,\mu\nu} = \int \mathrm{d}P\,\mathrm{d}S\; p^\lambda\, s^{\mu\nu}\left[f_{\rm eq}^{+}(x,p,s) + f_{\rm eq}^{-}(x,p,s)\right] = \frac{2\,\mathfrak{s}^2}{3}\,\omega^{\mu\nu}\left(Z_1^{+\lambda} + Z_1^{-\lambda}\right)$$
$$+ \frac{2\,\mathfrak{s}^2}{3m^2}\left[(Z_1^{+\lambda\alpha\mu} + Z_1^{-\lambda\alpha\mu})\omega^{\nu}{}_{\alpha} - (Z_1^{+\lambda\alpha\nu} + Z_1^{-\lambda\alpha\nu})\omega^{\mu}{}_{\alpha}\right] + \dots \tag{29}$$

In the Boltzmann case,

$$S_{\rm eq}^{\lambda,\mu\nu} = u^\lambda\left[A\,(k^\mu u^\nu - k^\nu u^\mu) + A_1 t^{\mu\nu}\right]$$
$$+ \frac{A}{2}\left(t^{\lambda\mu}u^\nu - t^{\lambda\nu}u^\mu + \Delta^{\lambda\mu}k^\nu - \Delta^{\lambda\nu}k^\mu\right), \tag{30}$$

where

$$A = -\frac{4\,\mathfrak{s}^2\cosh\xi}{3\pi^2} z T^3 K_3(z), \quad A_1 = \frac{2\,\mathfrak{s}^2\cosh\xi}{3\pi^2} z T^3\left[zK_2(z) + 2K_3(z)\right].$$

Not just $S_{\rm eq}^{\mu,\alpha\beta} = u^\mu S^{\alpha\beta}$.

Entropy current for the Boltzmann case

$$S^{\mu}_{\rm eq} = T^{\mu\alpha}_{\rm eq}\beta_{\alpha} - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}_{\rm eq} - \xi N^{\mu}_{\rm eq} + \mathcal{N}^{\mu}. \tag{31}$$

$$S^{\mu}_{\rm eq} = \bar{\sigma} u^{\mu} + \sigma_t t^{\mu}, \tag{32}$$

with $\bar{\sigma} = \frac{\bar{\varepsilon}}{T} + (\coth\xi - \xi)\bar{n} - \bar{s}, \quad \sigma_t = \frac{P_t}{T} + (\coth\xi - \xi)n_t - s_t.$

Note on Fermi-Dirac statistics

Tensors

$$Z_n^{\pm\alpha\beta\cdots} \equiv \int \mathrm{d}P p^\alpha p^\beta \cdots f_n^\pm$$

instead of

$$Z^{\alpha\beta\cdots} \equiv \int \mathrm{d}P p^\alpha p^\beta \cdots e^{-\beta\cdot p}$$

admit the same decomposition in terms of generic tensors of the same symmetry built out of u^μ and $g^{\mu\nu}$, but the expressions contain integrals other than the Bessel functions.
In the limit case $\epsilon \to 0$, we retrieve the Boltzmann case.

Close-to-equilibrium dynamics

- We write general nonequilibrium expressions as the equilibrium terms plus corrections
$$N^\mu = N^\mu_{eq} + \delta N^\mu, \quad T^{\mu\nu} = T^{\mu\nu}_{eq} + \delta T^{\mu\nu}, \quad S^{\mu,\alpha\beta} = S^{\mu,\alpha\beta}_{eq} + \delta S^{\mu,\alpha\beta}. \tag{33}$$
- The spin-orbit interaction is introduced through dissipative terms.
- $T^{\mu\nu}$ contains nonsymmetric parts, and the spin tensor is not seperately conserved
$$\partial_\mu J^{\mu,\alpha\beta} = 0, \quad J^{\mu,\alpha\beta} = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} + S^{\mu,\alpha\beta},$$
$$\partial_\mu S^{\mu,\alpha\beta} = T^{\beta\alpha} - T^{\alpha\beta}, \quad \partial_\mu N^\mu = 0, \quad \partial_\mu T^{\mu\nu} = 0 \tag{34}$$
- From Eqs. (31, 33, 34), we get
$$\partial_\mu S^\mu = \delta N^\mu \partial_\mu \xi + \delta T^{\mu\nu}_s \partial_\mu \beta_\nu + \delta T^{\mu\nu}_a (\partial_\mu \beta_\nu - \omega_{\nu\mu}) - \frac{1}{2} \delta S^{\mu,\alpha\beta} \partial_\mu \omega_{\alpha\beta}. \tag{35}$$

(Compare, e.g., K. Hattori et al., *Fate of spin polarization in a relativistic fluid. An entropy-current analysis*, Phys. Lett. B **795** (2019) 100-106,

F. Becattini, A. Daher, X.-L. Sheng, *Entropy current and entropy production in relativistic spin hydrodynamics*, Phys. Lett. B **850** (2024), 138533.)

General tensor decomposition

To find the form of the deviations δN^μ, $\delta T^{\mu\nu}$, $\delta S^{\mu,\alpha\beta}$, we use a decomposition of general tensors of the given symmetry via projections along u^μ and separation into symmetric and antisymmetric parts

$$\begin{aligned} N^\mu &= a u^\mu + b^\mu, \\ T^{\mu\nu} &= c u^\mu u^\nu + d_s^\mu u^\nu + d_s^\nu u^\mu + d_a^\mu u^\nu - d_a^\nu u^\mu + e_a^{\mu\nu} + e_s^{\mu\nu}, \\ S^{\lambda,\mu\nu} &= u^\lambda \left[(f^\mu u^\nu - f^\nu u^\mu) + \epsilon^{\mu\nu\rho\sigma} u_\rho w_\sigma \right] + i^{\lambda\mu} u^\nu - i^{\lambda\nu} u^\mu + j^{\lambda\mu\nu}, \end{aligned} \tag{36}$$

with the constraints $b^\mu u_\mu = 0$, $d_s^\mu u_\mu = d_a^\mu u_\mu = e_a^{\mu\nu} u_\mu = e_s^{\mu\nu} u_\mu = 0$, $f^\mu u_\mu = 0$, $h^{\mu\nu} = -h^{\nu\mu}$, $h^{\mu\nu} u_\mu = 0$, $i^{\lambda\mu} u_\lambda = i^{\lambda\mu} u_\mu = 0$, $j^{\lambda\mu\nu} = -j^{\lambda\nu\mu}$, $j^{\lambda\mu\nu} u_\lambda = j^{\lambda\mu\nu} u_\mu = j^{\lambda\mu\nu} u_\nu = 0$, $w \cdot u = 0$.

The dissipative corrections

Some parts of the general tensors have the same form as the equilibrium ones. When we enforce the Landau matching conditions

$$
\begin{aligned}
N^{\mu} u_{\mu} &= N^{\mu}_{\rm eq} u_{\mu}, \\
T^{\mu\nu} u_{\mu} u_{\nu} &= T^{\mu\nu}_{\rm eq} u_{\mu} u_{\nu}, \\
S^{\lambda,\mu\nu} u_{\lambda} &= S^{\lambda,\mu\nu}_{\rm eq} u_{\lambda},
\end{aligned}
\tag{37}
$$

we obtain

$$
\begin{aligned}
a = \bar{n}(T, \xi, k^2, \omega^2), &\quad c = \bar{\varepsilon}(T, \xi, k^2, \omega^2), \\
f^{\mu} = A(T, \xi) k^{\mu}, &\quad w^{\mu} = A_1(T, \xi) \omega^{\mu}.
\end{aligned}
\tag{38}
$$

The remaining terms are

$$\begin{aligned} \delta N^\mu &= V^\mu, \\ \delta T_s^{\mu\nu} &= -\Pi\Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu}, \\ \delta T_a^{\mu\nu} &= d_a^\mu u^\nu - d_a^\nu u^\mu + e_a^{\mu\nu}, \\ \delta S^{\lambda,\mu\nu} &= \Sigma^{\lambda\mu} u^\nu - \Sigma^{\lambda\nu} u^\mu + \phi^{\lambda\mu\nu}, \end{aligned} \tag{39}$$

with

$$\begin{aligned} &V^\mu = b^\mu - n_t t^\mu, \quad \Pi = e - \bar{P}_{k\omega}, \quad W^\mu = d_s^\mu - P_t t^\mu, \\ &\pi^{\mu\nu} = e_s^{\langle\mu\nu\rangle} - P_{k\omega}(k^{\langle\mu} k^{\nu\rangle} + \omega^{\langle\mu}\omega^{\nu\rangle}), \quad \Sigma^{\lambda\mu} = i^{\lambda\mu} - \frac{A}{2} t^{\lambda\mu}, \\ &\phi^{\lambda\mu\nu} = j^{\lambda\mu\nu} - \frac{A}{2}(\Delta^{\lambda\mu} k^\nu - \Delta^{\lambda\nu} k^\mu). \end{aligned} \tag{40}$$

Equations (39) have the same form as in (Biswas 2023) and can be expressed in terms of gradients of hydrodynamic variables multiplied by kinetic coefficients.

R. Biswas, A. Daher, A. Das, W. Florkowski, R. Ryblewski, *Relativistic second-order spin hydrodynamics: An entropy-current analysis*, Phys. Rev. D **108**, 014024 (2023).

Thus, we get the form of the tensors

$$
\begin{aligned}
b^\mu &= \lambda \nabla^\mu \xi + n_t t^\mu. \quad c = \bar{\varepsilon}(T, \xi, k^2, \omega^2), \quad d_s^\mu = -\kappa(Du^\mu - \beta \nabla^\mu T) + P_t t^\mu, \\
d_a^\mu &= \lambda_a \beta^{-1}(\beta Du^\mu + \beta^2 \nabla^\mu T - 2k^\mu), \quad e = \bar{P} - \zeta\theta - (1/3)P_{k\omega}(k^2 + \omega^2), \\
e_s^{\langle\mu\nu\rangle} &= 2\eta\sigma^{\mu\nu} + P_{k\omega}(k^{\langle\mu} k^{\nu\rangle} + \omega^{\langle\mu}\omega^{\nu\rangle}), \quad e_a^{\mu\nu} = \gamma\beta\nabla^{[\mu} u^{\nu]}, \\
i^{\lambda\mu} &= -\chi_1 \Delta^{\lambda\mu} u^\beta \nabla^\alpha \omega_{\alpha\beta} - \chi_2 u_\nu \nabla^{\langle\lambda}\omega^{\mu\rangle\nu} - \chi_3 u_\nu \Delta_\rho^{[\mu}\nabla^{\lambda]}\omega^{\rho\nu} + \frac{A}{2} t^{\lambda\mu}, \\
j^{\lambda\mu\nu} &= \frac{\chi_4}{2}\nabla^\lambda \omega^{\langle\mu\rangle\langle\nu\rangle} + \frac{A}{2}(\Delta^{\lambda\mu} k^\nu - \Delta^{\lambda\nu} k^\mu),
\end{aligned}
\tag{41}
$$

with $D = u^\mu \partial_\mu$, $\theta = p_\mu u^\mu$, $\sigma^{\mu\nu} = p^{\langle\mu} u^{\nu\rangle}$, η, ζ – shear and bulk viscosity, κ – thermal conductivity, λ_a, γ – coefficients introduced in (Hattori 2019), $\chi_1, \chi_2, \chi_3, \chi_4$ – coefficients from (Biswas 2023).

- This, together with the conservation laws, forms a framework of dissipative spin hydrodynamics.
- We include second-order terms in ω in a consistent way.
- Including second-order terms in gradients is straightforward.

Summary

- We combine the perfect-fluid results of kinetic theory for particles with spin $1/2$ with the Israel-Stewart approach for including nonequilibrium processes.
- Two-fold expansion: in ω and in gradients of hydrodynamic variables.
- With spin degrees of freedom, the perfect-fluid description contains seemingly dissipative, transverse terms.
- Genuine dissipative terms come from the condition of positive entropy production.
- The results can be implemented in code for numerical simulations.

Thank you for your attention!