Existence theorem on the UV limit of Euclidean Wilsonian RG flows

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Outline

- Mathematics of Euclidean Feynman functional integral.
- Mathematics of Wilsonian regularization.
- Mathematics of Wilsonian renormalization.

Recap on distribution theory

Will consider only scalar and bosonic fields for simplicity.

Will consider only flat (affine) spacetime manifold for simplicity.

- $\mathcal{E}$: space of all smooth fields over spacetime.
  They form a vector space with a topology: collection of "open" sets
  $\varphi_i \in \mathcal{E}$ $(i \in \mathbb{N}_0) \to 0$ iff all derivatives locally uniformly converge to zero.

- $\mathcal{S}$: space of rapidly decreasing smooth fields (Schwartz fields) over spacetime.
  They form a vector space with a topology:
  $\varphi_i \in \mathcal{S}$ $(i \in \mathbb{N}_0) \to 0$ iff all derivatives uniformly converge to zero faster than polynomial.

- $\mathcal{D}$: space of compactly supported smooth fields (test fields) over spacetime.
  They form a vector space with a topology:
  $\varphi_i \in \mathcal{D}$ $(i \in \mathbb{N}_0) \to 0$ iff they stay within a compact set and $\to 0$ in $\mathcal{E}$ sense.

only meaningful on flat spacetime

$$\underbrace{\mathcal{E}}_{\text{smooth fields}} \supset \underbrace{\mathcal{S}}_{\text{Schwartz fields}} \supset \underbrace{\mathcal{D}}_{\text{test fields}}$$

Distributions are continuous duals of $\mathcal{E}$, $\mathcal{S}$, $\mathcal{D}$.

- $\mathcal{E}'$: continuous $\mathcal{E} \to \mathbb{R}$ linear functionals.
  They are the compactly supported distributions.
- $\mathcal{S}'$: continuous $\mathcal{S} \to \mathbb{R}$ linear functionals.
  They are the tempered or Schwartz distributions.
- $\mathcal{D}'$: continuous $\mathcal{D} \to \mathbb{R}$ linear functionals.
  They are the space of all distributions.

They carry a corresponding natural topology (notion of "open" sets).

only meaningful on flat spacetime

$$\underbrace{\mathcal{E}'}_{\text{compactly supp distributions}} \subset \underbrace{\mathcal{S}'}_{\text{Schwartz distributions}} \subset \underbrace{\mathcal{D}'}_{\text{all distributions}}$$

[Of course, functions are also distributions, e.g. $\mathcal{D} \subset \mathcal{E}'$ and $\mathcal{E} \subset \mathcal{D}'$ etc.]

Recap on measure / integration / probability theory

- Let X be a set (their elements called elementary events).
- Let Σ be a collection of subsets of X such that:
  - X is in Σ,
  - for all A in Σ, their complement is in Σ.
  - for all countably infinite system $A_i \in \Sigma$ $(i \in \mathbb{N}_0)$, their union $\bigcup_{i\in\mathbb{N}_0} A_i$ is in Σ.

  Then, Σ is called a sigma-algebra (their elements called composite events).
  Typically, if X carries open sets (topology), the sigma-alg generated by them is used.
- Let $\mu : \Sigma \to \mathbb{R}_0^+$ be a set-function, such that:
  - $\mu(\emptyset) = 0$,
  - for all countably infinite disjoint system $A_i \in \Sigma$ $(i \in \mathbb{N}_0)$: $\mu\big(\bigcup_{i\in\mathbb{N}_0} A_i\big) = \sum_{i\in\mathbb{N}_0} \mu(A_i)$,
  - $\exists$ some countably infinite system $A_i \in \Sigma$ $(i \in \mathbb{N}_0)$ with $\mu(A_i) < \infty$: $X = \bigcup_{i\in\mathbb{N}_0} A_i$.

  Then, μ is called a measure.

(X, Σ, μ) is called a measure space. [E.g. probability measure space if $\mu(X) =$ finite.]
Will study probability measures on $X = \mathcal{E}, \mathcal{S}, \mathcal{D}, \mathcal{E}', \mathcal{S}', \mathcal{D}'$.

- A function $f : X \to \mathbb{C}$ is called measurable iff in good terms with mesure theory.
  Theorem: f is measurable iff approximable pointwise by “histograms” with bins from Σ.

- The integral $\int\limits_{\phi\in X} f(\phi)\,\mathrm{d}\mu(\phi)$ is defined via the histogram “area” approximations.
  Theorem: this is well-defined.

- Let (X,Σ,μ) be a measure space and (Y,Δ) an other one, with unspecified measure.
  Let $C : X \to Y$ be a measurable mapping.
  Then, one can define the pushforward (or marginal) measure $C_*\mu$ on Y.
  [For all $B\in\Delta$ one defines $(C_*\mu)(B) := \mu\big(\overset{-1}{C}(B)\big)$.]

- Pushforward (marginal) measure means simply transformation of integration variable.
  If forgetful transformation, the “forgotten” d.o.f. are “integrated out”.

- If μ is a probability measure, $Z(j) := \int\limits_{\phi\in X} \mathrm{e}^{\,\mathrm{i}(j|\phi)}\,\mathrm{d}\mu(\phi)$ is its Fourier transform.

Mathematics of Euclidean Feynman functional integral

- Take an Euclidean action $S = T + V$, with kinetic + potential term splitting.
  Say, $T(\varphi) = \int \varphi\,(-\Delta + m^2)\varphi$, and $V(\varphi) = g \int \varphi^4$.

- Then T, i.e. $(-\Delta + m^2)$ has a propagator $K(\cdot,\cdot)$ which is positive definite:
  - $(-\Delta + m^2)_x\, K(x, y) = \delta_y(x)$,
  - for all $j \in \mathcal{S}$ sources: $(K|j \otimes j) \geq 0$.

- Due to above, the function $Z_T(j) := \mathrm{e}^{-(K|j\otimes j)}$ $(j \in \mathcal{S})$ has “quite nice” properties.

- Bochner-Minlos theorem: because of
  - “quite nice” properties of Z_T,
  - “quite nice” properties of the space $\mathcal{S}$,

  $\exists|$ probability measure γ_T on $\mathcal{S}'$, whose Fourier transform is Z_T.
  It is the Feynman measure for free theory: $\int\limits_{\phi\in\mathcal{S}'} (\ldots)\,\mathrm{d}\gamma_T(\phi) = \int\limits_{\phi\in\mathcal{S}'} (\ldots)\,\mathrm{e}^{-T(\phi)}$ “$\mathrm{d}\phi$”.

- Tempting definition for Feynman measure of interacting theory:

$$\int\limits_{\phi\in\mathcal{S}'} (\ldots)\,\mathrm{e}^{-V(\phi)}\,\mathrm{d}\gamma_T(\phi) \qquad \left[= \int\limits_{\phi\in\mathcal{S}'} (\ldots)\,\underbrace{\mathrm{e}^{-(T(\phi)+V(\phi))}}_{=\mathrm{e}^{-S(\phi)}}\,\text{“}\mathrm{d}\phi\text{”}\right]$$

Mathematics of Wilsonian regularization

- Problem, the interacting Feynman measure $\mu := \mathrm{e}^{-V} \cdot \gamma_T$ is undefined:

$$\int\limits_{\phi\in\mathcal{S}'} (\ldots) \underbrace{\mathrm{d}\mu(\phi)}_{\text{wannabe Feynman measure}} := \int\limits_{\phi\in\mathcal{S}'} (\ldots) \underbrace{\mathrm{e}^{-V(\phi)}}_{\text{lives on function sense fields}} \underbrace{\mathrm{d}\gamma_T(\phi)}_{\text{lives on distribution sense fields}}$$

  Because V is spacetime integral of pointwise product of fields, e.g. $V(\varphi) = g\int\varphi^4$.
  How to bring e^{-V} and γ_T to common grounds?

- Physicist solution: in desperation, Wilsonian regularization.
  Take a continuous linear mapping $C:$ (distributional fields) $\to$ (function sense fields).
  Take the pushforward Gaussian measure $C_*\gamma_T$, which lives on $\mathrm{Ran}(C)$.
  Those are functions, so safe to integrate e^{-V} there:

$$\int\limits_{\varphi\in\mathrm{Ran}(C)} (\ldots)\ \mathrm{e}^{-V(\varphi)}\ \mathrm{d}\big(C_*\gamma_T\big)(\varphi) \qquad \left[= \int\limits_{\varphi\in\mathrm{Ran}(C)} (\ldots)\ \mathrm{e}^{-(T_C(\varphi)+V(\varphi))}\ \text{``}\mathrm{d}\varphi\text{''}\right]$$

a space of UV regularized fields

[Schwartz kernel theorem: C is convolution by a test function, if translationally invariant. I.e., it is a momentum space damping, or coarse-graining of fields.]

Mathematics of Wilsonian renormalization

- Issue, the so defined $\mathrm{e}^{-V} \cdot C_{*}\gamma_T$ is C-dependent. How to get rid of C-dependence?

- Take a family V_C ($C \in$ {coarse-grainings}) of interaction terms. $\leftrightarrow \mu_C := \mathrm{e}^{-V_C} \cdot C_{*}\gamma_T$
  We say that it is a Wilsonian renormalization group (RG) flow iff:
  $\exists$ some continuous functional z : {coarse-grainings} $\to \mathbb{R}$, such that
  $\forall$ coarse-grainings C, C', C'' with $C'' = C'\,C$:
  $$z(C'')_{*}\,\mu_{C''} \quad = \quad z(C)_{*}\,C'_{*}\,\mu_C.$$
  [z is called the running wave function renormalization factor.]

- Physics idea behind:
  expectation value of a smoothed observable can be calculated at least C-consistently.

- Expressed on moments: let $\mathcal{G}_C = (\mathcal{G}_C^{(0)}, \mathcal{G}_C^{(1)}, \mathcal{G}_C^{(2)}, \dots)$ be the moments of μ_C, then
  $\exists$ some continuous functional z : {coarse-grainings} $\to \mathbb{R}$, such that
  $\forall$ coarse-grainings C, C', C'' with $C'' = C'\,C$:
  $$z(C'')^n\,\mathcal{G}_{C''}^{(n)} \quad = \quad z(C)^n \otimes^n C'\,\mathcal{G}_C^{(n)} \qquad \text{for all } n = 0, 1, 2, \dots$$
  [Valid also in Lorentz signature and on manifolds, for formal moments (correlators).]

[We can always set $z(C) = 1$, by rescaling fields: $\tilde{\mu}_C := z(C)_{*}\,\mu_C$ or $\tilde{\mathcal{G}}_C^{(n)} := z(C)^n\,\mathcal{G}_C^{(n)}$.]

For bosonic fields over flat spacetime, we have some theorems.

[A.László, Z.Tarcsay, *Class.Quant.Grav.* **41**(2024)125009 + *new manuscript in prep.*]

- Existence of UV limit of correlators (any signature):
  $\exists|$ C-independent distributional correlator $G=(G^{(0)},G^{(1)},G^{(2)},\dots)$, such that $\mathcal{G}_C^{(n)}=\otimes^n C\,G^{(n)}$ holds.

- Existence of UV limit of Feynman measure (Euclidean signature):
  $\exists|$ C-independent probability measure μ on the distribution sense fields, such that $\mu_C=C_*\,\mu$ holds.

- Existence of UV limit of interaction potential (Euclidean signature):
  $\exists|$ C-independent interaction potential V on the distribution sense fields ϕ, such that $V_C(C\,\phi)=V(\phi)$ holds γ_T-a.e., for positive Fourier spectrum coarse-grainings C.

- Preservation of lower bound (Euclidean signature):
  if V_C were bounded from below for a positive Fourier spectrum C, then V and all V_C are bounded from below, and with the same bound.
  $\rightarrow$ This leads to a new kind of renormalizability test.

Summary

Under mild conditions:

- Wilsonian RG flow of correlators have a UV limit.
  (any signature)
- Wilsonian RG flow of Feynman measures have UV limit.
  (Euclidean signature)
- Wilsonian RG flow of interaction potentials have UV limit.
  (Euclidean signature)
- The flow preserves lower bound of interaction potentials.
  (Euclidean signature)
  $\rightarrow$ Can be used for a renormalizability test.

Backup slides

Relation to usual RG theory:

Fix some $\eta \in \mathcal{S}$ such that $\int \eta = 1$ and $F(\eta) > 0$.
Introduce scaled η, that is $\eta_\Lambda(x) := \Lambda^N \eta(\Lambda\, x)$ (for all $x \in \mathbb{R}^N$ and scaling $1 \leq \Lambda < \infty$).
One has $\eta_\Lambda \xrightarrow{\mathcal{S}'} \delta$ as $\Lambda \longrightarrow \infty$.

By our theorem, for all Λ, one has $V_{C_{\eta_\Lambda}}(C_{\eta_\Lambda}\, \phi) = V(\phi)$ for γ_T-a.e. $\phi \in \mathcal{S}'$.

$\Downarrow$

Informally: ODE for $V_{C_{\eta_\Lambda}}$, namely $\frac{\mathrm{d}}{\mathrm{d}\Lambda} V_{C_{\eta_\Lambda}}(C_{\eta_\Lambda}\, \phi) = 0$ for $1 \leq \Lambda < \infty$.

QFT people try to solve such flow equation, given initial data $V_{C_\Lambda}\big|_{\Lambda=1}$.

But why bother? By our theorem, all RG flows of such kind has some V at the UV end.
Look directly for V?

What really the game is about?

Original problem:

- We had $\mathcal{V}:\ \{\text{function sense fields}\}\to\mathbb{R}\cup\{\pm\infty\}$, say $\mathcal{V}(\varphi)=g\int\varphi^4$.
- We would need to integrate it against γ_T, but that lives on $\mathcal{S}'$ fields.
- γ_T known to be supported "sparsely", i.e. not on function fields, but really on $\mathcal{S}'$.
- So, we really need to extend $\mathcal{V}$ at least γ_T-a.e. to make sense of $\mu:=\mathrm{e}^{-V}\,\gamma_T$.

Concern of physicists: this may be impossible.

- We are afraid that V on S' might not exist.
- Instead, let us push γ_T to smooth fields by C, do there $\mu_C:=\mathrm{e}^{-V_C}\,C_*\,\gamma_T$.
- Then, get rid of C-dependence of μ_C by concept of Wilsonian RG flow.
  Maybe even $\mu_C\to\mu$ could exist as $C\to\delta$ if we are lucky...

Our result: we are back to the start.

- The UV limit Feynman measure μ then indeed exists.
- But we just proved that then there must exist some extension V of $\mathcal{V}$ to $\mathcal{S}'$, γ_T-a.e.
- So, we'd better look for that ominous extension V.
- For bounded from below $\mathcal{V}$, bounded from below measurable V needed.
  If we find one, $\mu:=\mathrm{e}^{-V}\,\gamma_T$ is then finite measure automatically.
  Only pathology: overlap integral of e^{-V} and γ_T expected small, maybe zero.
  We only need to make sure that $\int_{\phi\in\mathcal{S}'}\mathrm{e}^{-V(\phi)}\,\mathrm{d}\gamma_T(\phi)>0$!

We had $\mathcal{V}: \{\text{function sense fields}\} \to \mathbb{R} \cup \{\pm\infty\}$, say $\mathcal{V}(\varphi) = g \int \varphi^4$.
We need an extenson V on the distribution sense fields with the same lower bound.
If V bounded from below, only possible pathology: $\mu := \mathrm{e}^{-V} \cdot \gamma_T = 0$.
There is an optimal extension, the "greedy" extension.

$$V(\cdot) \;:=\; \underset{\{\eta_n \to \delta\}}{(\gamma_T)\inf} \; \liminf_{n\to\infty} \; \mathcal{V}(\eta_n \star \,\cdot\,)$$

This is the lower bound of extensions, i.e. overlap of e^{-V} and γ_T largest.
But is V measurable at all? Not evident.

Theorem[A.László, Z.Tarcsay *manuscript in prep.*]:
1. The "greedy extension" is measurable.
2. The interacting Feynman measure $\mu := \mathrm{e}^{-V} \cdot \gamma_T$ by greedy extension is nonzero iff

$$\exists\, \eta_n \to \delta \;:\; \qquad \int\limits_{\phi\in\mathcal{S}'} \limsup_{n\to\infty}\, \mathrm{e}^{-\mathcal{V}(\eta_n\star\phi)}\, \mathrm{d}\gamma_T(\phi) \quad > \quad 0.$$

E.g. a sufficient condition:

$$\exists\, \eta_n \to \delta \;:\; \qquad \lim_{n\to\infty} \int\limits_{\phi\in\mathcal{S}'} \mathrm{e}^{-\mathcal{V}(\eta_n\star\phi)}\, \mathrm{d}\gamma_T(\phi) \quad > \quad 0.$$