

Instantons and $U(1)_A$ restoration in hot QCD

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Symmetries of QCD and their realization

- $m_u \approx m_d \approx 0$
- Symmetries: $SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A$
 - $U(1)_A$ anomalous
 - $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$ spontaneously broken below T_c

- Order parameter of the symmetry breaking (Banks-Casher formula):

$$\langle \bar{\psi}\psi \rangle \propto \frac{1}{V} \sum_i \frac{1}{\lambda_i + m} \propto \int_{-\Lambda}^{\Lambda} d\lambda \frac{m}{\lambda^2 + m^2} \rho(\lambda) \xrightarrow{m \rightarrow 0} \rho(0)$$

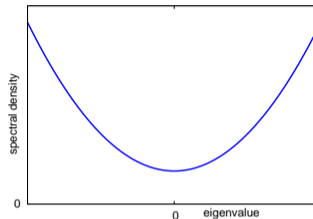
λ_i : eigenvalues of the Dirac operator, $\rho(\lambda)$: its spectral density

The finite temperature transition

Standard picture

Below T_c

- Chiral symmetry broken
- Order parameter:
 $\rho(0) \neq 0$



The finite temperature transition

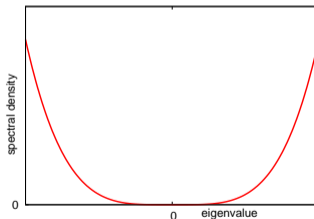
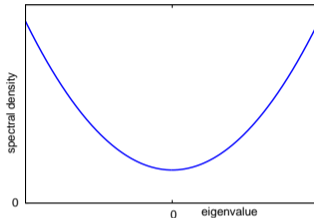
Standard picture

Below T_c

- Chiral symmetry broken
- Order parameter:
 $\rho(0) \neq 0$

Above T_c

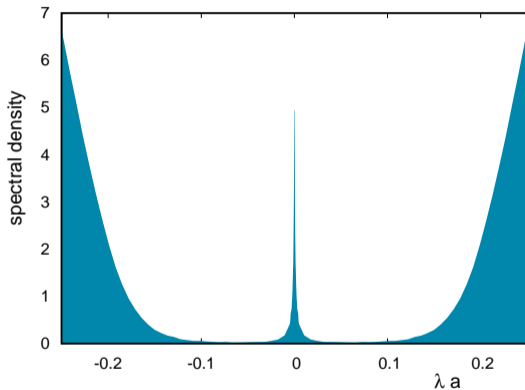
- Chiral symmetry restored
- Order parameter $\rho(0) = 0$
- (Pseudo)gap
lowest Matsubara mode



spectral density at 0 \iff realization of chiral symmetry

Spectral density at $T = 1.1 T_c$ on the lattice

quenched (quark back reaction omitted)



$$Z = \int \mathcal{D}U \prod_f \det(D[U] + m_f) \cdot e^{-S_g[U]}$$

Peak at zero in the spectral density!

Edwards et al. PRD 61 (2000); Alexandru & Horvath, PRD 92 (2015); 2404.12298; Kaczmarek, Mazur, Sharma, PRD 104 (2021) 2021

- Why is there a peak at zero?
- How is it suppressed if the quark determinant is included?
- How does the peak influence chiral symmetry as $m \rightarrow 0$?

Instantons \rightarrow zero eigenvalues of $D(A)$

- (Anti)instanton
 \rightarrow zero eigenvalue of $D(A)$ with $(-)+$ chirality eigenmode
- High T :
large instantons “squeezed out” in the temporal direction
 \rightarrow dilute gas of instantons and antiinstantons
- Zero modes exponentially localized:

$$\psi(r) \propto e^{-\pi Tr}$$

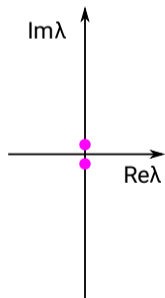
Instanton-antiinstanton pair

The Dirac operator in the subspace of zero modes

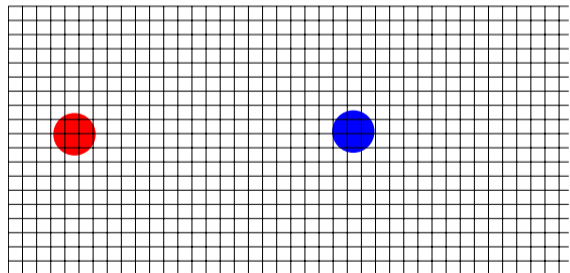
$$D(A) = \begin{pmatrix} 0 & iw \\ iw & 0 \end{pmatrix}$$

$$w \propto e^{-\pi T r}$$

Spectrum of $D(A)$



Instanton and antiinstanton



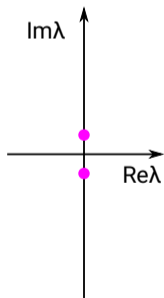
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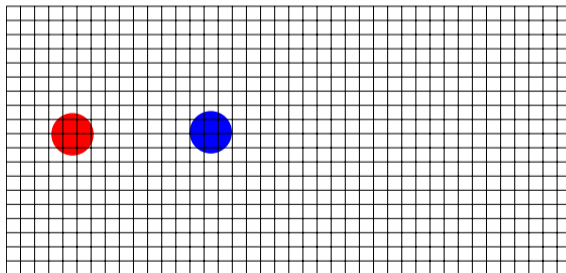
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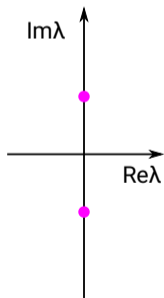
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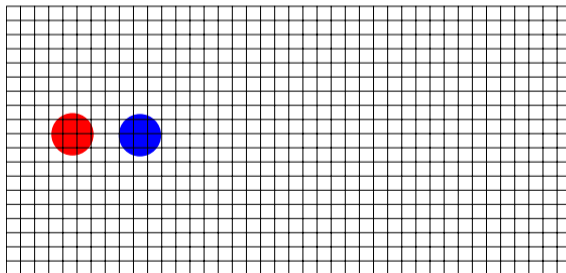
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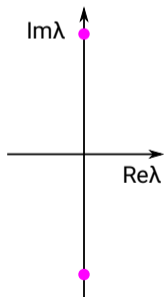
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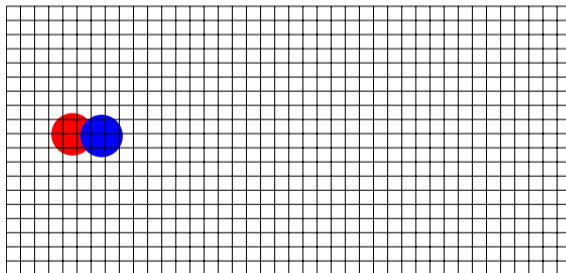
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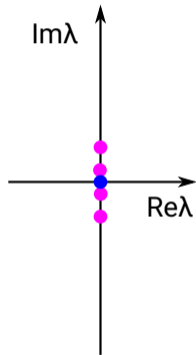
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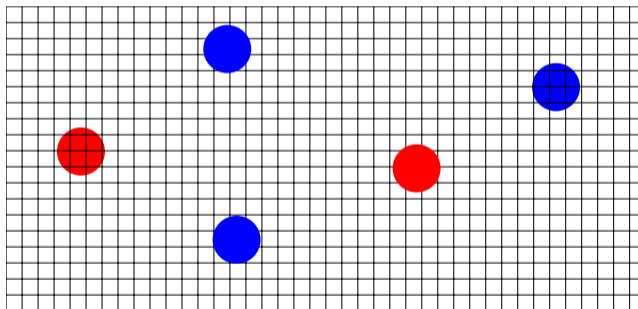
Spectrum of $D(A)$ in dilute gas of instantons

The Dirac operator in the subspace of zero modes

Spectrum of $D(A)$



Instantons and antiinstantons



n_i instantons n_a antiinstantons

→ $|n_i - n_a|$ exact zero modes + mixing near zero modes

Dirac operator in the subspace of zero modes (ZMZ)

Work by E.V. Shuryak, J.J.M. Verbaarschot, T. Schäfer (1990-2000)...

- Given n_i instantons, n_a antiinstantons in 3d box of size L^3
- Construct $(n_i + n_a) \times (n_i + n_a)$ matrix:

$$D = \begin{pmatrix} \overbrace{0}^{n_i} & \overbrace{iW}^{n_a} \\ iW^\dagger & 0 \end{pmatrix}$$

- $w_{ij} = A \cdot \exp(-\pi T \cdot r_{ij})$ r_{ij} is the distance of instanton i and antiinstanton j

Random matrix model of $D(A)$ in the zero mode zone

- How to choose instanton numbers (n_i, n_a) and locations?

- Quenched lattice $T > 1.05 T_c \rightarrow$ free instanton gas

Bonati et al. PRL 110 (2013); Vig R. & TGK, PRD 103 (2021)

- n_i and n_a independent identical Poisson-distributed

$$p(n_i, n_a) = e^{-\chi V} \cdot \frac{(\chi V/2)^{n_i}}{n_i!} \cdot \frac{(\chi V/2)^{n_a}}{n_a!}$$

χ is the topological susceptibility

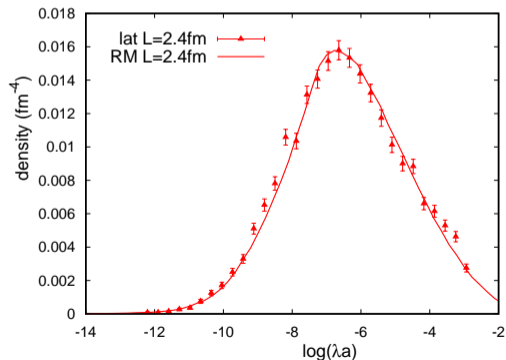
- Locations random (uniform)
- \rightarrow $D(A)$ in quenched QCD: ensemble of random matrices

Fit parameters to quenched lattice Dirac spectrum

$T = 1.1 T_c$ overlap Dirac spectrum

- Two parameters:
 - χ – topological susceptibility: from exact zero modes $\rightarrow \chi = \langle Q^2 \rangle / V$
 - A – prefactor of the exponential mixing between zero modes
- Fit A to distribution of Dirac eigenvalues (lowest eigenvalue)

$L = 2.4\text{fm}$ fit



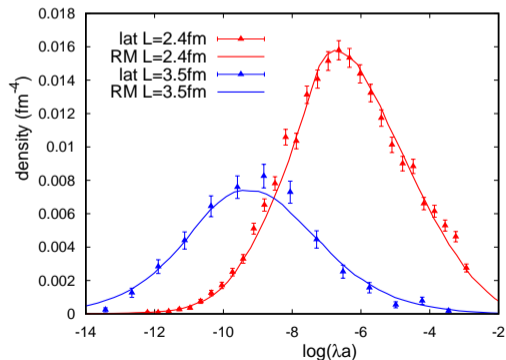
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$L = 2.4\text{fm}$ fit

$L = 3.5\text{fm}$ prediction



Random matrix model of full QCD zero mode zone

- Include $\det(D + m)^{N_f}$ in Boltzmann weight

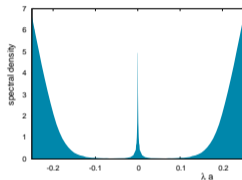
- $\det(D + m) = \prod_{\text{zms}} (\lambda_i + m) \times \prod_{\text{bulk}} (\lambda_i + m)$

- Bulk weakly correlated with zero mode zone

- Approximate det with $\prod_{\text{zms}} (\lambda_i + m)$

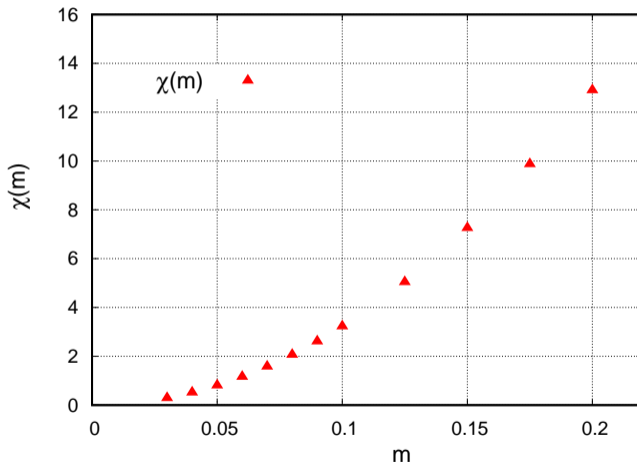
- Consistently included in RM model:

$$P(n_i, n_a) = \underbrace{e^{-\chi_0 V} \frac{1}{n_i!} \frac{1}{n_a!} \left(\frac{\chi_0 V}{2} \right)^{n_i + n_a}}_{\text{free instanton gas with random locations}} \times \det(D + m)^{N_f}$$



Random matrix simulation: results for $N_f = 2$

Topological susceptibility:



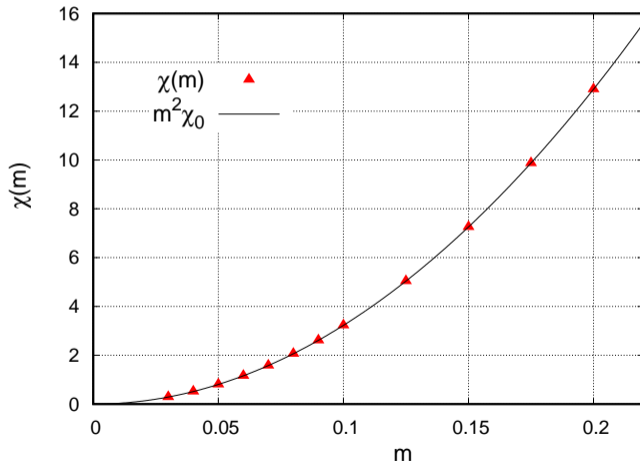
Random matrix simulation: results for $N_f = 2$

Topological susceptibility:

$$\chi(m) = m^2 \chi_0$$

not a fit!

↑ quenched susceptibility



Explanation: free instanton gas

- Quark determinant for n_i instantons and n_a antiinstantons:

$$\det(D + m)^{N_f} = \prod_{n_i, n_a} (\lambda_j + m)^{N_f} \approx m^{N_f(n_i + n_a)}$$

if $|\lambda_j| \ll m$

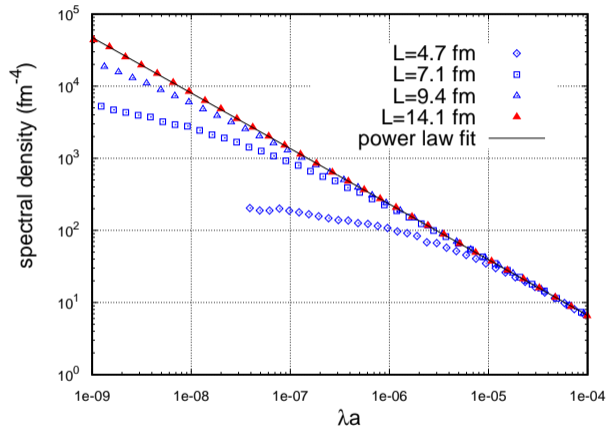
- Reweighting depends on number of topological objects, not on their type or location

$$P(n_i, n_a) \propto \left(\frac{\chi_0 V}{2}\right)^{n_i + n_a} \times \det(D + m)^{N_f} \approx \left(\frac{m^{N_f} \chi_0 V}{2}\right)^{n_i + n_a}$$

- Free gas, but susceptibility suppressed as $\chi_0 \rightarrow m^{N_f} \chi_0$
- As $m \rightarrow 0$ instanton gas more dilute $\Rightarrow |\lambda_j|$ smaller
- Even in the chiral limit $|\lambda_j| \ll m \Rightarrow$ free instanton gas

Spectral density singular at the origin for $V \rightarrow \infty$

RM model simulation, parameters from quenched $T = 1.1 T_c$ overlap spectrum



$$\rho(\lambda) \propto \lambda^\alpha$$

fit: $\alpha = -0.770(5)$

Singular spectral density from similar instanton model:

Sharan and Teper, hep-ph/9910216

Banks-Casher for a singular spectral density?

“Banks-Casher” for singular spectral density

$$\langle \bar{\psi} \psi \rangle \propto \left\langle \sum_i \frac{m}{m^2 + \lambda_i^2} \right\rangle \approx \underbrace{\left(\text{avg. number of in-stantons in free gas} \right)}_{m^{N_f} \chi_0 V} \cdot \frac{1}{m} = m^{N_f-1} \chi_0 V$$

$|\lambda_i| \ll m$

“Banks-Casher” for singular spectral density

$$\langle \bar{\psi} \psi \rangle \propto \left\langle \sum_i \frac{m}{m^2 + \lambda_i^2} \right\rangle \approx \underbrace{\left(\text{avg. number of in-stantons in free gas} \right)}_{m^{N_f} \chi_0 V} \cdot \frac{1}{m} = m^{N_f-1} \chi_0 V$$

$|\lambda_i| \ll m$

$U(1)_A$ breaking susceptibility $\chi_\pi - \chi_\delta$

$$\left\langle \sum_i \frac{m^2}{(m^2 + \lambda_i^2)^2} \right\rangle \approx \underbrace{\left(\text{avg. number of in-stantons in free gas} \right)}_{m^{N_f} \chi_0 V} \cdot \frac{1}{m^2} = m^{N_f-2} \chi_0 V$$

$$\rightarrow \lim_{m \rightarrow 0} (\chi_\pi - \chi_\delta) \neq 0 \quad \text{for } N_f = 2$$

- At high T non-interacting degrees of freedom: free instantons
- Dirac spectral density has singular peak at zero
at any finite T , for any nonzero quark mass
- Chiral symmetry restoration nontrivial $\rightarrow N_f = 2$: $U(1)_A$ anomaly remains
- Spontaneously broken $SU(2)$ restored,
but order of the $m \rightarrow 0$ and $V \rightarrow \infty$ limit still important

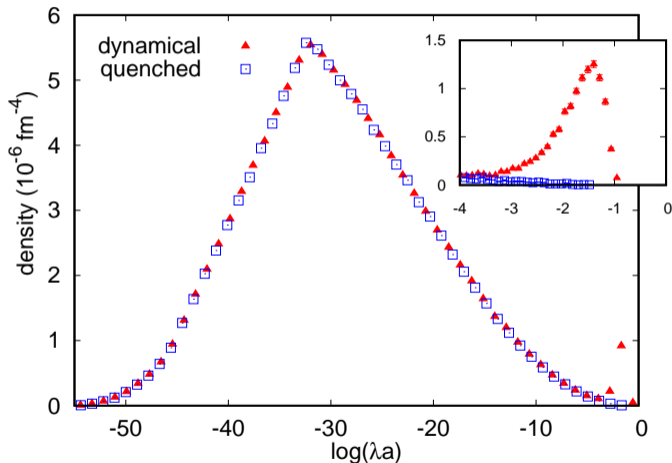
BACKUP SLIDES

Related developments & outlook

- RM model @ small $m \rightarrow$ instanton-antiinstanton molecules
do not contribute to $\langle \bar{\psi}\psi \rangle$ and $\chi_\pi - \chi_\delta$ in the chiral limit
- Constraints on the Dirac spectrum from chiral symmetry restoration
 \rightarrow consistent with free instanton gas [M. Giordano, 2404.03546 \(2024\)](#)
- Localization properties of eigenmodes in ZMZ
[M. Giordano and TGK, Universe 7 \(2021\);](#) [A. Alexandru and I Horvath, PRL 127 \(2021\), PLB 833 \(2022\)](#)
- What is the lowest temperature where the instanton gas is ideal?
 \rightarrow dynamical simulation with chiral quark action
- Anderson model with off-diagonal disorder on bipartite lattice
Chiral symmetry, power-law singularity in spectral density at zero, delocalization of modes toward zero.
[S.N. Evangelou and D.E.Katsanos, J. Phys. A 36 \(2003\)](#)

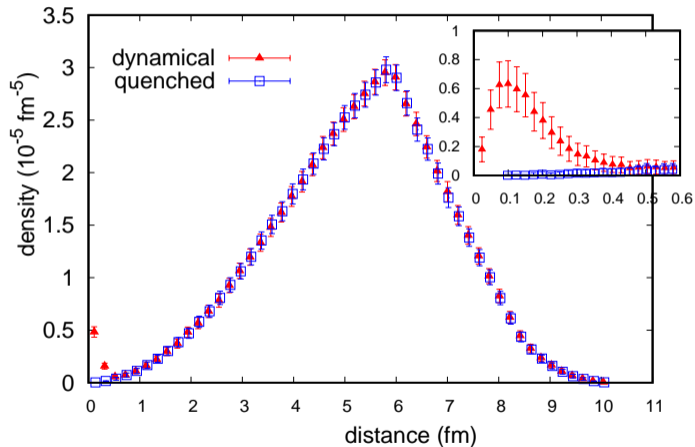
Spectral density – full QCD vs. ideal instanton gas

random matrix model, same topological susceptibility



Instanton-antiinstanton molecules

density of closest opposite charge pairs at given distance



Direct lattice simulations?

- Important to resolve small Dirac eigenvalues
→ chiral action needed [JLQCD, PRD 103 \(2021\)](#)
- To see spectral peak: large volume, close to T_c needed
- $\frac{\chi_\pi - \chi_\delta}{\chi_{\text{top}}} \propto m^{-2}$ instanton contribution independent of T
- Explore how far down in T free instanton gas persists
 - Compare eigenvalue statistics to prediction of free instanton gas
 - Can be done in each topological sector separately