Thermodynamic relations for perfect spin hydrodynamics

JAGIELLONIAN UNIVERSITY IN KRAKÓW

Mykhailo Hontarenko, Wojciech Florkowski

02-06.12.2024, 24th Zimányi School, Budapest

NATIONAL SCIENCE CENTRE POLAND

Generalized thermodynamic relations are introduced into the framework of a relativistic perfect spin hydrodynamics. They allow for consistent treatment of spin degrees of freedom, including the use of spin tensors whose structure follows from microscopic calculations. The obtained results are important for establishing consistency between different formulations of spin hydrodynamics and form the basis for introducing dissipative corrections.

Introduction

Recent measurements of non-zero spin polarization of hyperons and vector mesons produced in relativistic heavy-ion collisions have triggered broad interest in the spin polarization phenomena in strongly interacting matter. On the theory side, there exist several approaches (classical and quantum) to incorporate spin degrees of freedom into the framework of relativistic hydrodynamics as [5], [3]. The later has become the main theoretical tool used to describe the spacetime evolution of strongly interacting matter produced in heavy-ion collisions, hence, the inclusion of spin dynamics in the hydrodynamics formalism seems to be an inevitable necessity. The differences appears already at the basic level of relativistic thermodynamic relations and definitions of the fundamental macroscopic quantities such as the spin tensor.

In this work we critically reexamine thermodynamic relations used in perfect spin hydrodynamics of particles with spin 1/2 and propose to introduce their generalized (tensor) forms that can be used for large values of the spin polarization tensor, $\omega_{\mu\nu}$, and with kinetic-theory motivated forms of the GLW spin tensor $S^{\lambda,\mu\nu}$. In this way, we remove a gap between the works that use kinetic-theory concepts as the starting point and the works that use phenomenological expressions for the spin tensor and construct dissipative corrections using the positivity of the entropy production as the main physical ansatz.

Israel-Stewert approach to spin hydrodynamics

To describe relativistic systems, one must generalize the scalar thermodynamic relation to tensorial forms. This generalization is achieved by multiplying the scalar relation by the flow four-vector u^μ. To incorporate spin, [4] begins by postulating phenomenological thermodynamic relations, introducing the spin chemical potential $\Omega_{\alpha\beta}$ and the spin density tensor $S^{\alpha\beta}_{\rm eq}$, and combining these into the following equation

$$\varepsilon + P = T\sigma + \mu n + \frac{1}{2}\Omega_{\alpha\beta}S^{\alpha\beta}_{\rm eq}. \tag{1}$$

Furthermore, by multiplying the scalar equation by u^μ, as prescribed by the Israel-Stewart (IS) method, the spin current is defined as

$$S^{\lambda,\mu\nu}_{\rm eq} = u^\lambda S^{\mu\nu}_{\rm eq}.$$

This approach extends the thermodynamic relations accordingly

$$S^\mu_{\rm eq} = P\beta^\mu - \xi N^\mu_{\rm eq} + \beta_\lambda T^{\lambda\mu}_{\rm eq} - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}_{\rm eq}, \tag{2}$$

$$dS^\mu_{\rm eq} = -\xi\, dN^\mu_{\rm eq} + \beta_\lambda\, dT^{\lambda\mu}_{\rm eq} - \frac{1}{2}\omega_{\alpha\beta}dS^{\mu,\alpha\beta}_{\rm eq}, \tag{3}$$

$$d(P\beta^\mu) = N^\mu_{\rm eq}\, d\xi - T^{\lambda\mu}_{\rm eq}\, d\beta_\lambda + \frac{1}{2}S^{\mu,\alpha\beta}_{\rm eq}d\omega_{\alpha\beta}. \tag{4}$$

The equilibrium energy-momentum tensor and baryon number current are given in their standard forms as $T^{\mu\nu}_{\rm eq} = \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu}$, $\quad N^\mu_{\rm eq} = nu^\mu$. To study the system's behavior, one must adopt a counting scheme. A frequently used scheme assumes $\omega_{\mu\nu} \sim \mathcal{O}(\partial^1)$ and $S^{\mu\nu} \sim \mathcal{O}(\partial^0)$. However, this approach is inconsistent, as $S^{\mu\nu} \sim \omega_{\mu\nu}$. Similarly, considering $\omega_{\mu\nu} \sim S^{\mu\nu} \sim \mathcal{O}(\partial^1)$ fails because the term $\omega_{\mu\nu}S^{\mu\nu}$, which is a second-order contribution, is neglected. One solution to this issue is to describe the local equilibrium with spin while excluding gradients. This approach focuses on expanding and counting quantities in terms of the spin polarization tensor $\mathcal{O}(\omega_{\mu\nu})$, rather than relying on gradients. Another challenge arises from the discrepancy between the assumed form of the spin tensor, $S^{\lambda,\mu\nu}_{\rm eq} = u^\lambda S^{\mu\nu}_{\rm eq}$, and the forms derived in certain calculations, such as those in kinetic theory.

Insights from Kinetic Theory

The basic object used in the kinetic theory is the phase-space distribution function $f(x,\boldsymbol{p})$. For particles with spin, $f(x,\boldsymbol{p})$ is generalized to a spin dependent distribution $f(x,\boldsymbol{p},s)$. In local equilibrium, the spin dependent distribution functions for particles (+) and antiparticles (−) have the form

$$f^\pm(x,p,s) = \exp\left(-p_\mu\beta^\mu \pm \xi + \tfrac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right). \tag{5}$$

where β^μ, ξ and $\omega_{\alpha\beta}$ are functions of space and time coordinates x and play the same role as β^μ, ξ and $\omega_{\alpha\beta}$ defined in thermodynamic relations. By integrating the equilibrium distribution functions over momentum and spin degrees of freedom, one obtains the macroscopic currents and tensors

$$N^\mu_{eq} = \int dP\, dS\, p^\mu\, \left[f^+_{eq}(x,p,s) - f^-_{eq}(x,p,s)\right], \tag{6}$$

$$T^{\mu\nu}_{eq} = \int dP\, dS\, p^\mu p^\nu\, \left[f^+_{eq}(x,p,s) + f^-_{eq}(x,p,s)\right], \tag{7}$$

$$S^{\lambda,\mu\nu}_{eq} = \int dP\, dS\, p^\lambda\, s^{\mu\nu}\, \left[f^+_{eq}(x,p,s) + f^-_{eq}(x,p,s)\right]. \tag{8}$$

The internal angular momentum tensor $s^{\alpha\beta}$ presented in distribution function is defined in terms of the particle four-momentum p_α and spin four-vector s_α

$$s^{\alpha\beta} = \frac{1}{m}\varepsilon^{\alpha\beta\gamma\delta}p_\gamma s_\delta, \;\; p^\alpha s_\alpha = 0, \;\; s^\alpha = \frac{1}{2m}\varepsilon^{\alpha\beta\gamma\delta}p_\beta s_{\gamma\delta}. \tag{9}$$

The Boltzmann entropy for the system in kinetic theory is defined as

$$S^\mu_{\rm eq} = -\int dP\, dS\, p^\mu\left[f^+\left(\ln f^+ - 1\right) + f^-\left(\ln f^- - 1\right)\right]. \tag{10}$$

From this definition, we gain additional insights from kinetic theory, particularly regarding thermodynamics, the entropy current, and the equilibrium forms of $T^{\mu\nu}$ and N^μ. As shown in [1], these relations are expressed as follows:

$$S^\mu = \mathcal{N}^\mu - \xi N^\mu + \beta_\lambda T^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}, \quad \mathcal{N}^\mu = {\rm ctgh}(\xi)\, N^\mu_{\rm eq} \neq P\beta^\mu, \tag{11}$$

$$dS^\mu = -\xi\, dN^\mu + \beta_\lambda\, dT^{\lambda\mu} - \frac{1}{2}\omega_{\alpha\beta}\, dS^{\mu,\alpha\beta}, \tag{12}$$

$$d\mathcal{N}^\mu = N^\mu\, d\xi - T^{\lambda\mu}\, d\beta_\lambda + \frac{1}{2}S^{\mu,\alpha\beta}\, d\omega_{\alpha\beta}. \tag{13}$$

We note that these forms represent a slight generalization of those obtained in the Israel-Stewart (IS) approach. However, to treat the system consistently, we must compute the equilibrium forms of the currents.

Spin Polarization Tensor Expansion

The main idea is to expand the distribution function for each current as follows:

$$f^\pm_{\rm eq} = \exp(-p\cdot\beta(x) \pm \xi(x))\left[1 - \frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta} + \frac{1}{8}(\omega_{\alpha\beta}s^{\alpha\beta})^2 + \dots\right]. \tag{14}$$

We calculate all currents up to second order in $\omega_{\mu\nu}$, which is dimensionless in natural units. An important technical step in understanding this expansion is the following integral property:

$$\int dS\, s_\alpha = 0. \tag{15}$$

This property explains why the second-order expansion is crucial to revealing the spin's presence in the system. Equations (6)–(7), after expansion, account only for the zero-spin and quadratic contributions, whereas Equation (8) is already linear in spin. The linear term in the expansion treats $S^{\lambda,\mu\nu}$ as quadratic in spin. This approach is consistent with equation (11), where quadratic terms such as $\omega_{\alpha\beta}S^{\alpha\beta}$ appear explicitly.

Tensor Structure of Currents

After performing the second-order expansion as outlined in [1, 2], we observe that, in the equilibrium setup, the currents take the following form:

$$N^\mu_{eq} = \bar{n}u^\mu + n_t t^\mu, \tag{16}$$

$$T^{\mu\nu}_{eq} = \bar{\varepsilon}u^\mu u^\nu - \bar{P}\Delta^{\mu\nu} + P_k\, k^\mu k^\nu + P_\omega\, \omega^\mu\omega^\nu + P_t\,(t^\mu u^\nu + t^\nu u^\mu), \tag{17}$$

$$S^{\lambda,\mu\nu}_{eq} = u^\lambda\left[A\,(k^\mu u^\nu - k^\nu u^\mu) + A_1 t^{\mu\nu}\right] + \frac{A_3}{2}\left(t^{\lambda\mu}u^\nu - t^{\lambda\nu}u^\mu + \Delta^{\lambda\mu}k^\nu - \Delta^{\lambda\nu}k^\mu\right). \tag{18}$$

All present coefficients depends on (T, μ, k^2, ω^2) [1, 2].

Outlook and discussion

In this work we have introduced generalized thermodynamic relations into the framework of a relativistic perfect spin hydrodynamics. They allow for a consistent treatment of spin degrees of freedom, including the use of spin tensors whose structure follows from microscopic calculations. Our main observation that a commonly used scalar version of thermodynamic relations should be replaced by the tensor form is very general — the spin hydrodynamics introduces a new hydrodynamic variable that has a tensor structure, hence, in local equilibrium all currents and tensors may a priori have a richer structure than that used in spinless hydrodynamics.

Acknowledgments

This work was supported in part by the Polish National Science Centre Grant No. 2022/47/B/ST2/01372.

References

[1] W. Florkowski and M. Hontarenko, [arXiv:2405.03263 [hep-ph]].

[2] Z. Drogosz, W. Florkowski and M. Hontarenko, Phys. Rev. D **110** (2024) no.9, 096018 doi:10.1103/PhysRevD.110.096018 [arXiv:2408.03106 [hep-ph]].

[3] F. Becattini, A. Daher and X. L. Sheng, Phys. Lett. B **850** (2024), 138533 doi:10.1016/j.physletb.2024.138533

[4] K. Hattori, M. Hongo, X. G. Huang, M. Matsuo and H. Taya, Phys. Lett. B **795** (2019), 100-106 doi:10.1016/j.physletb.2019.05.040 [arXiv:1901.06615 [hep-th]].

[5] W. Florkowski, A. Kumar and R. Ryblewski, Prog. Part. Nucl. Phys. **108** (2019), 103709 doi:10.1016/j.ppnp.2019.07.001 [arXiv:1811.04409 [nucl-th]].