Turbulence in financial markets

Chapter 11 of Mantegna and Stanley's book

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R. N. Mantegna and H. E. Stanley:
An introduction to econophysics
Correlations and complexity in finance

Chapter 11:
Intro: physics w/o Hamiltonian
11.1 Turbulence
11.2 Parallels: fluids and markets
11.3 Scalings in fluids and markets
11.4 Discussion: similarities and differences

Introduction to Chapter 11

Basic problem: can physics discuss systems w/o a Hamiltonian?
w/o a known dynamical equation?

Examples: granular material
modelling friction
Viscous fluid dynamics – not yet mathematically solvable
(One of the unsolved Millennium Problems)

On a qualitative level, turbulence and financial markets are attractively similar. For example, in turbulence, one injects energy at a large scale by, e.g., stirring a bucket of water, and then one observes the manner in which the energy is transferred to successively smaller scales. In financial systems 'information' can be injected into the system on a large scale and the reaction to this information is transferred to smaller scales – down to individual investors. Indeed, the word 'turbulent' has come into common parlance since price fluctuations in finance qualitatively resemble velocity fluctuations in turbulence. Is this qualitative parallel useful on a quantitative level, such that our understanding of turbulence might be relevant to understanding price fluctuations?

Opportunity: cross-fertilization

In this chapter, we will discuss fully developed turbulence in parallel with the stochastic modeling of stock prices. Our aim is to show that cross-fertilization between the two disciplines might be useful, not that the turbulence analogy is quantitatively correct. We shall find that the formal correspondence between turbulence and financial systems is not supported by quantitative calculations.

11.1 Turbulencia

Navier-Stokes egyenletek

Millennium Prize Problems

Birch and Swinnerton-Dyer conjecture
Hodge conjecture
Navier–Stokes existence and smoothness
P versus NP problem
Poincaré conjecture (solved)
Riemann hypothesis
Yang–Mills existence and mass gap

V · T · E

Navier–Stokes existence and smoothness [edit]

Main article: Navier–Stokes existence and smoothness

$$\overbrace{\underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{\text{Variation}} + \underbrace{(\mathbf{u}\cdot\nabla)\mathbf{u}}_{\text{Convection}}}^{\text{Inertia (per volume)}} \overbrace{-\underbrace{\nu\,\nabla^2\mathbf{u}}_{\text{Diffusion}} = \underbrace{-\nabla w}_{\text{Internal source}}}^{\text{Divergence of stress}} + \underbrace{\mathbf{g}}_{\text{External source}}.$$

The Navier–Stokes equations describe the motion of fluids, and are one of the pillars of fluid mechanics. However, theoretical understanding of their solutions is incomplete, despite its importance in science and engineering. For the three-dimensional system of equations, and given some initial conditions, mathematicians have not yet proven that smooth solutions always exist. This is called the *Navier–Stokes existence and smoothness* problem.

The problem, restricted to the case of an incompressible flow, is to prove either that smooth, globally defined solutions exist that meet certain conditions, or that they do not always exist and the equations break down. The official statement of the problem was given by Charles Fefferman.[13]

11.1 Turbulencia

Navier-Stokes egyenletek 2.

Navier–Stokes existence and smoothness [edit]

Main article: Navier–Stokes existence and smoothness

$$\overbrace{\underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{\text{Variation}} + \underbrace{(\mathbf{u}\cdot\nabla)\mathbf{u}}_{\text{Convection}}}^{\text{Inertia (per volume)}} \overbrace{- \underbrace{\nu\nabla^2\mathbf{u}}_{\text{Diffusion}} = \underbrace{-\nabla w}_{\text{Internal source}}}^{\text{Divergence of stress}} + \underbrace{\mathbf{g}}_{\text{External source}}.$$

Consider a simple system that exhibits turbulence, a fluid of kinematic viscosity ν flowing with velocity V in a pipe of diameter L. The control parameter whose value determines the 'complexity' of this flowing fluid is the Reynolds number,

$$\mathrm{Re} \equiv \frac{LV}{\nu}. \tag{11.1}$$

When Re reaches a particular threshold value, the 'complexities of the fluid explode' as it suddenly becomes turbulent.

The equations describing the time evolution of an incompressible fluid have been known since Navier's work was published in 1823 [128], which led to what are now called the Navier–Stokes equations,

$$\frac{\partial}{\partial t}\mathbf{V}(\mathbf{r},t) + (\mathbf{V}(\mathbf{r},t)\cdot\nabla)\mathbf{V}(\mathbf{r},t) = -\nabla P + \nu\nabla^2\mathbf{V}(\mathbf{r},t), \tag{11.2}$$

and

$$\nabla\cdot\mathbf{V}(\mathbf{r},t) = 0. \tag{11.3}$$

Here $\mathbf{V}(\mathbf{r},t)$ is the velocity vector at position $\mathbf{r}$ and time t, and P is the pressure. The Navier–Stokes equations characterize completely 'fully developed turbulence', a technical term indicating turbulence at a high Reynolds number. The analytical solution of (11.2) and (11.3) has proved impossible, and even numerical solutions are impossible for very large values of Re.

11.1 Turbulencia

Navier-Stokes 3: Kolmogorov's scaling

$$\frac{\partial}{\partial t}\mathbf{V}(\mathbf{r},t) + (\mathbf{V}(\mathbf{r},t)\cdot\nabla)\mathbf{V}(\mathbf{r},t) = -\nabla P + \nu\nabla^2\mathbf{V}(\mathbf{r},t), \tag{11.2}$$

and

$$\nabla\cdot\mathbf{V}(\mathbf{r},t) = 0. \tag{11.3}$$

In 1941, a breakthrough in the description of fully developed turbulence was achieved by Kolmogorov [82–84]. He showed that in the limit of infinite Reynolds numbers, the mean square velocity increment

$$\langle[\Delta V(\ell)]^2\rangle = \langle[V(r+\ell) - V(r)]^2\rangle \tag{11.4}$$

behaves approximately as

$$\langle[\Delta V(\ell)]^2\rangle \sim \ell^{2/3} \tag{11.5}$$

in the inertial range, where the dimensions are smaller than the overall dimension within which the fluid's turbulent behavior occurs and larger than the typical length below which kinetic energy is dissipated into heat.

Kolmogorov's theory describes well the second-order $\langle[\Delta V(\ell)]^2\rangle$ and provides the exact relation for the third-order $\langle[\Delta V(\ell)]^3\rangle$ moments observed in experiments, but fails to describe higher moments.

11.1 Turbulencia

Navier-Stokes 4: Kolmogorov's scaling and dimensional analysis

$$\frac{\partial}{\partial t}\mathbf{V}(\mathbf{r},t) + (\mathbf{V}(\mathbf{r},t)\cdot\nabla)\mathbf{V}(\mathbf{r},t) = -\nabla P + \nu\nabla^2\mathbf{V}(\mathbf{r},t),$$

and

$$\nabla\cdot\mathbf{V}(\mathbf{r},t) = 0.$$

Navier-Stokes 5: dimensional analysis from 11.3

Next we show that dimensional consistency requires that the mean square velocity increment assumes the form

$$\langle[\Delta V(\ell)]^2\rangle = C\epsilon^{2/3}\ell^{2/3}, \qquad (11.7)$$

where C is a dimensionless constant. This equation is the only one possible because the energy dissipation rate per unit mass has the dimensions $[L]^2[T]^{-3}$. In fact, if we define a to be the exponent of ϵ, and b to be the exponent of ℓ in Eq. (11.7), then dimensional consistency requires that

$$\frac{[L]^2}{[T]^2} = \frac{[L]^{2a}}{[T]^{3a}}[L]^b, \qquad (11.8)$$

where the equality indicates that both sides of the equation have the same dimension. This condition is satisfied by equating powers of L and T,

$$\begin{cases} 2 = 2a + b \\ 2 = 3a. \end{cases} \qquad (11.9)$$

Hence $a = 2/3$ and $b = 2/3$.

11.2 Parallels between prizes and flows

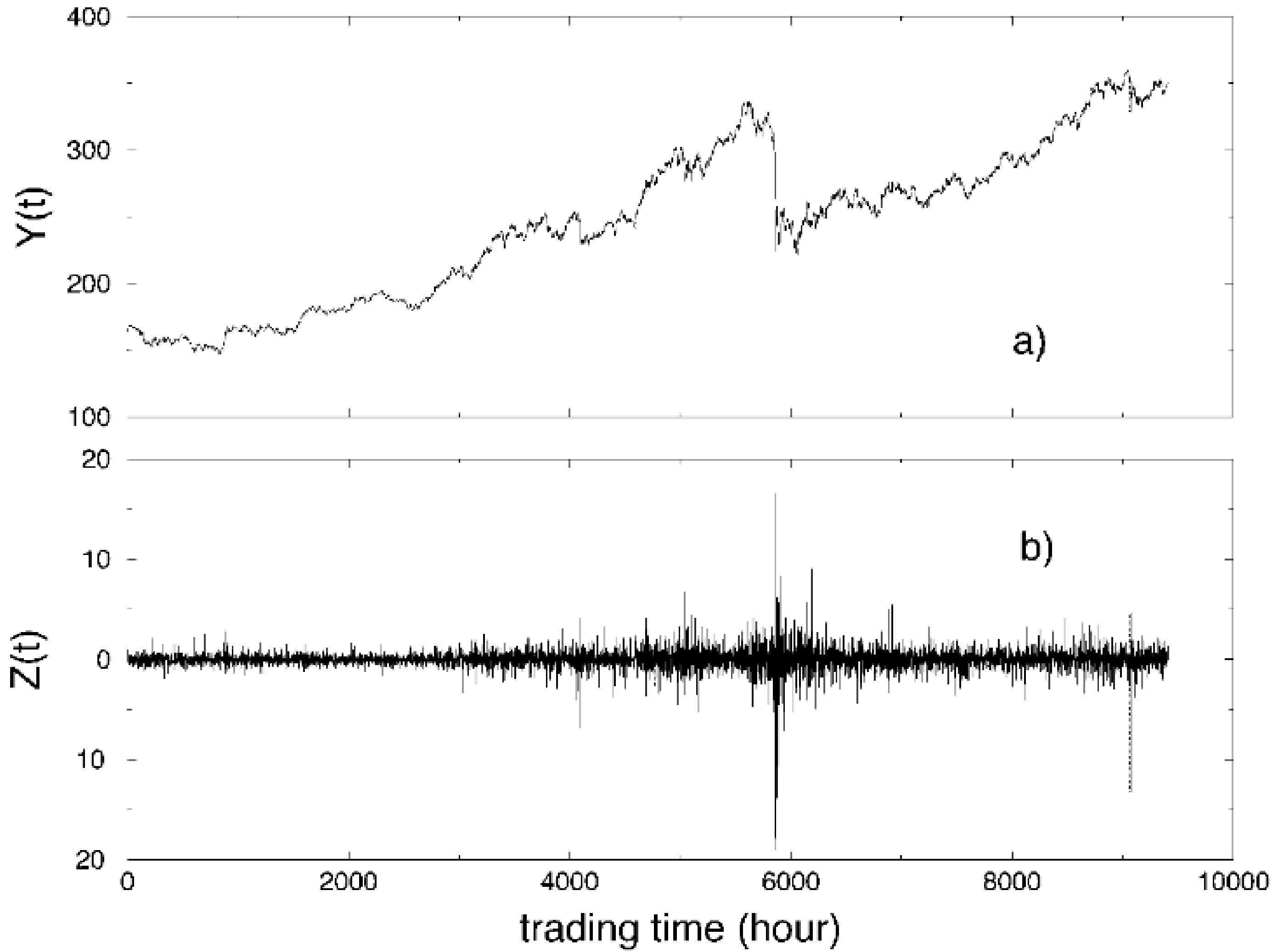

Fig. 11.1. (a) Time evolution of the S&P 500, sampled with a time resolution $\Delta t = 1$ h, over the period January 1984 to December 1989. (b) Hourly variations of the S&P 500 index in the 6-year period January 1984 to December 1989.

11.2 Parallels between prizes and flows

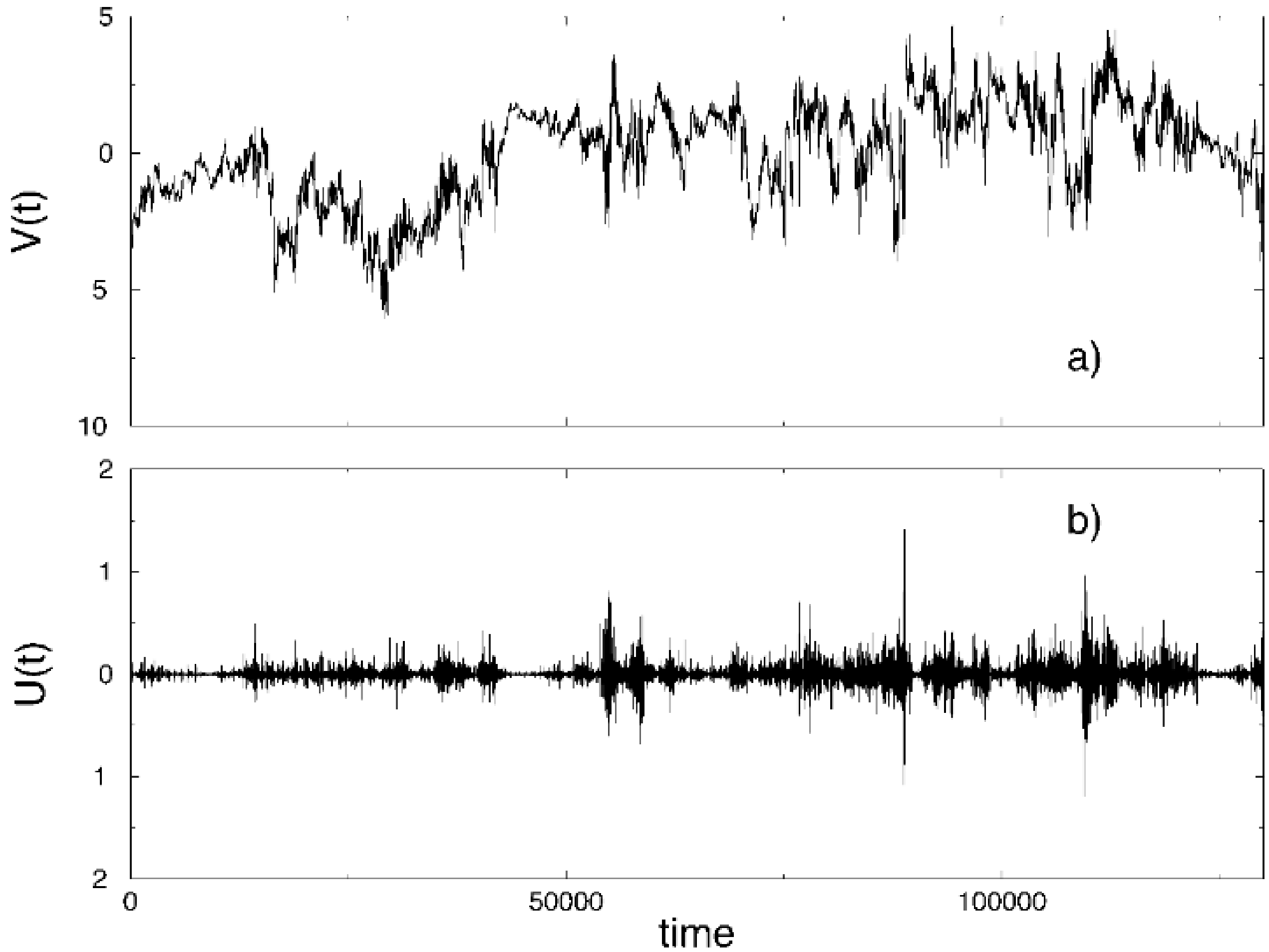

Fig. 11.2. Time evolution of the fluid velocity in fully developed turbulence. (a) Time evolution of the wind velocity recorded in the atmosphere at extremely high Reynolds number; the Taylor microscale Reynolds number is of the order of 1,500. The time units are given in arbitrary units. (b) Velocity differences of the time series given in (a). Adapted from [113].

11.2 Parallels between prizes and flows

Similar scaling laws, different exponents!

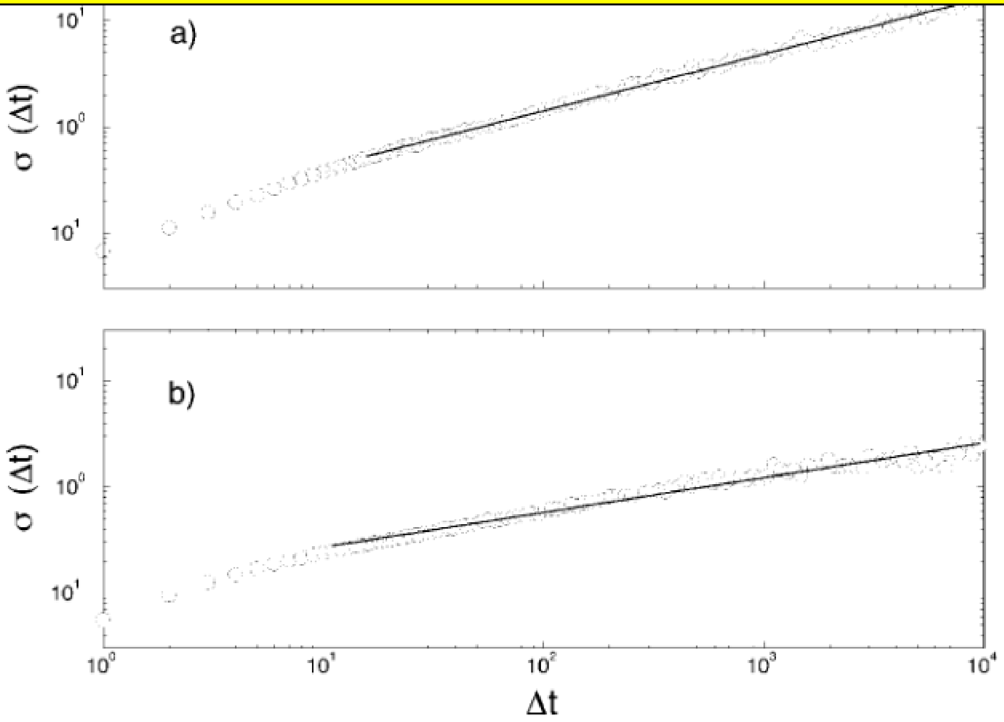

Fig. 11.3. (a) Standard deviation $\sigma(\Delta t)$ of the probability distribution $P(Z)$ characterizing the increments $Z_{\Delta t}(t)$ plotted double logarithmically as a function of Δt for the S&P 500 time series. After a time interval of superdiffusive behavior ($0 < \Delta t \leq 15$ minutes), a diffusive behavior close to the one expected for a random process with uncorrelated increments is observed; the measured diffusion exponent 0.53 (the slope of the solid line) is close to the theoretical value 1/2. (b) Standard deviation $\sigma(\Delta t)$ of the probability distribution $P(U)$ characterizing the velocity increments $U_{\Delta t}(t) \equiv V(t + \Delta t) - V(t)$ plotted double logarithmically as a function of Δt for the velocity difference time series in turbulence. After a time interval of superdiffusive behavior ($0 < \Delta t \leq 10$), a subdiffusive behavior close to the one expected for a fluid in the inertial range is observed. In fact, the measured diffusion exponent 0.33 (the slope of the solid line) is close to the theoretical value 1/3. Adapted from [112].

11.4. Discussion

- *similarities:* intermittency, non-Gaussian pdf, and gradual convergence to a Gaussian attractor in probability, and
- *differences:* the pdfs have different shapes in the two systems, and the probability of return to the origin shows different behavior – for turbulence we do not observe a scaling regime whereas for index changes we observe a scaling regime spanning a time interval of more than three orders of magnitude. Moreover, velocity fluctuations are anticorrelated whereas index (or exchange rate) fluctuations are essentially uncorrelated.

A closer inspection of Kolmogorov's theory explains why the observation of this difference is not surprising. The $2/3$ law for the evolution of the variance of velocity fluctuations, Eq. (11.5), is valid only for a system in which the dynamical evolution is essentially controlled by the energy dissipation rate per unit mass. We do not see any rational reason supporting the idea that assets in a financial market should have a dynamical evolution controlled by a similar variable. Indeed no analog of the $2/3$ law appears to hold for price dynamics.