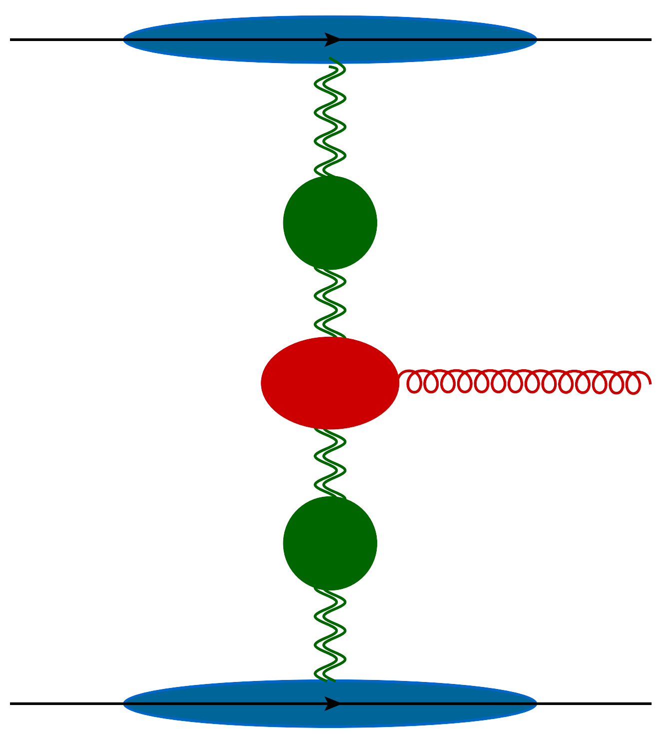
CERN Theory Colloquium

QCD scattering in the Regge limit

Einan Gardi
Higgs Centre for Theoretical Physics
University of Edinburgh



QCD Scattering in the Regge Limit

Abstract

Fixed-order computations of QCD amplitudes in general kinematics are limited to either one, two or three loops, depending on the number of particles produced. This strongly motivates our theoretical research programme aimed at understanding the behaviour of quark and gluon scattering amplitudes in special kinematic limits, in which new factorization and exponentiation properties arise. A particularly interesting limit is the Regge limit, where major simplifications take place. A remarkable and well-known property of this limit is the exponentiation of energy logarithms, a phenomenon known as *gluon Reggeization*, leading to power-like dependence on the energy (a Regge pole). This phenomenon and its breaking can be investigated using non-linear rapidity evolution equations. In the planar limit the evolution is consistent with a Regge pole to any logarithmic accuracy. However in full non-planar QCD multi-Reggeon interactions give rise to Regge cuts, in addition to the pole. Over the past decade an effective theory of multi-Reggeon interactions was developed, leading to a clear interpretation of state-of-the-art $2 \to 2$ and $2 \to 3$ QCD amplitude computations. Specifically, we are now able to fix all parameters associated with the Regge pole to the next-to-next-to-leading order (NNLO): the 3-loop trajectory, the 2-loop impact factors, and since recently, the two-loop Reggeon-gluon-Reggeon vertex. These, along with the effective multi-Reggeon theory, can be used to determine or resum higher-loop corrections in $2 \to n$ scattering in the multi-Regge limit, and push BFKL theory to NNLO.

QCD Scattering in the Regge Limit

Outline of the talk

- Inspiration: the rising total cross section
- Partonic amplitudes in the Regge limit:
  - ✓ Reggeization; factorization and its violation
  - ✓ Towards an effective *multi-Reggeon* two-dimensional theory from rapidity evolution (will not discuss alternative approaches, such as Glauber-SCET).
- The Loops and Legs frontier:
  - ✓ Disentangling pole from cut at NNLL in signature-odd $2 \to 2$ amplitudes
  - ✓ Determining the Lipatov Vertex at 2 loops

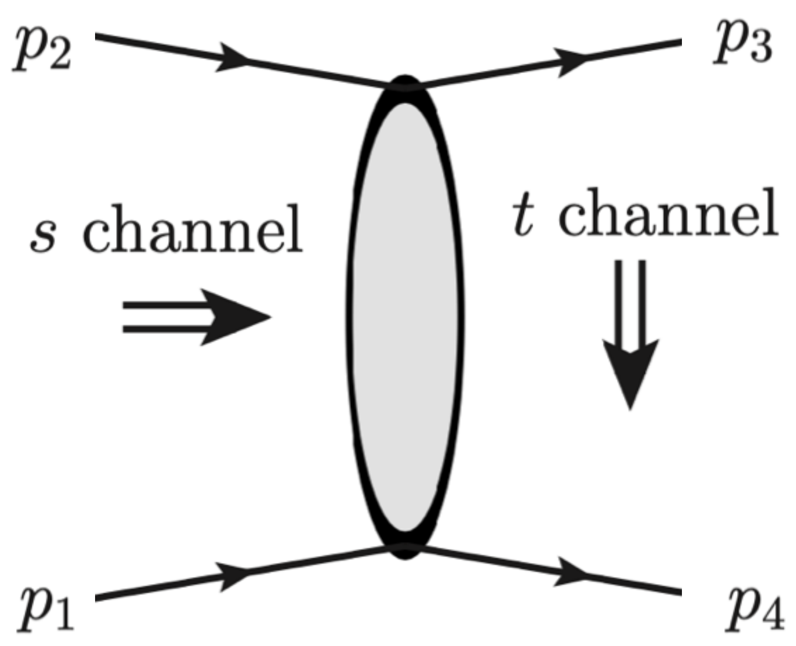
Inspiration: cross section in proton-proton scattering

- The total cross section rises with energy s

- $\sigma_{\text{tot}} \simeq \dfrac{\text{Im}\mathcal{M}}{s}$ seems to be growing like s^{α}

  while the Froissart bound: $\sigma_{\text{tot}} < \ln^2(s)$

- σ_{tot} receives contributions of small t.



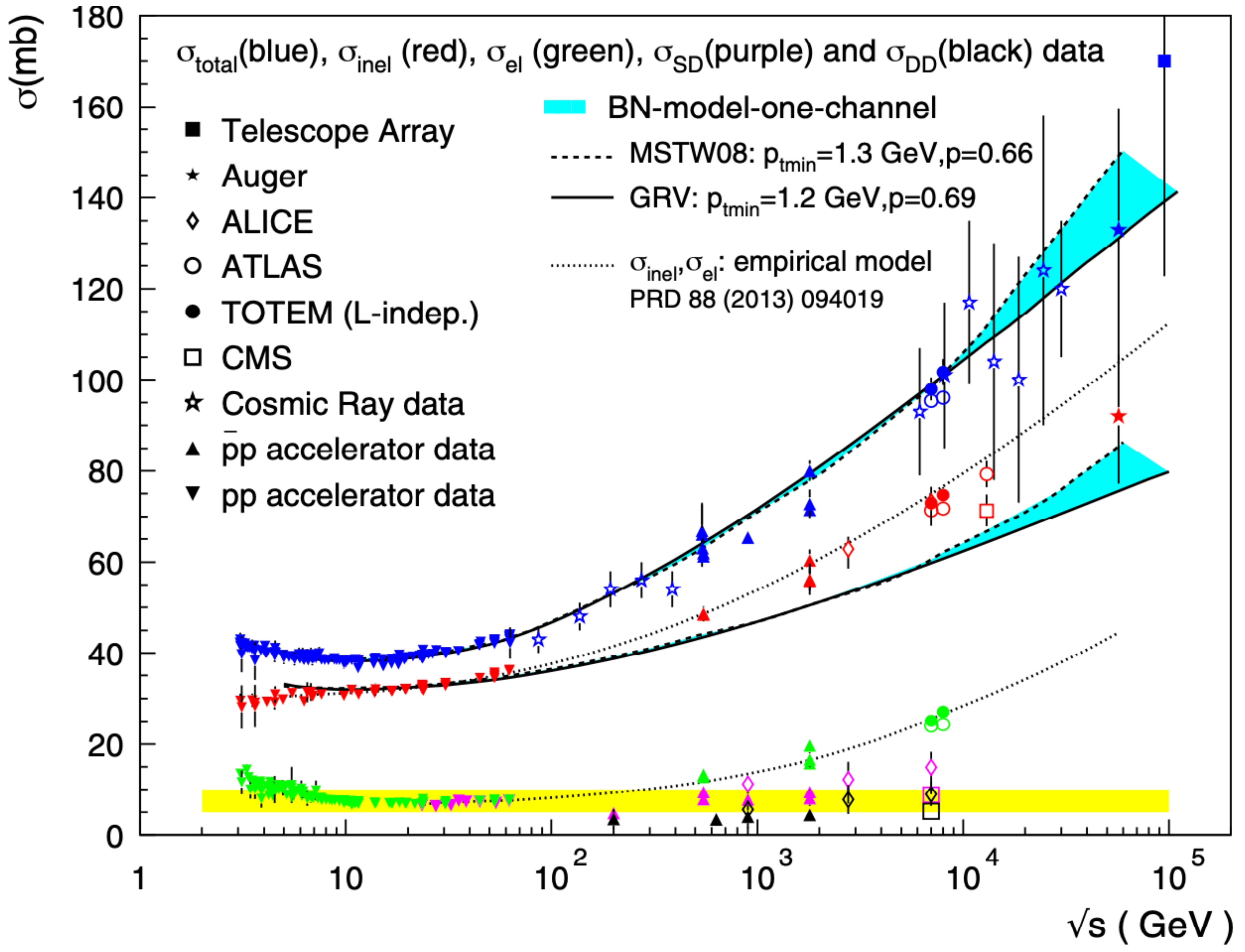


Figure from review by Pancheri and Srivastava.
Data and model compilation
by D. Fagundes, A. Grau and O. Shekhovtsova

- Cross sections can be computed perturbatively if $-t \gg \Lambda^2_{\text{QCD}}$.

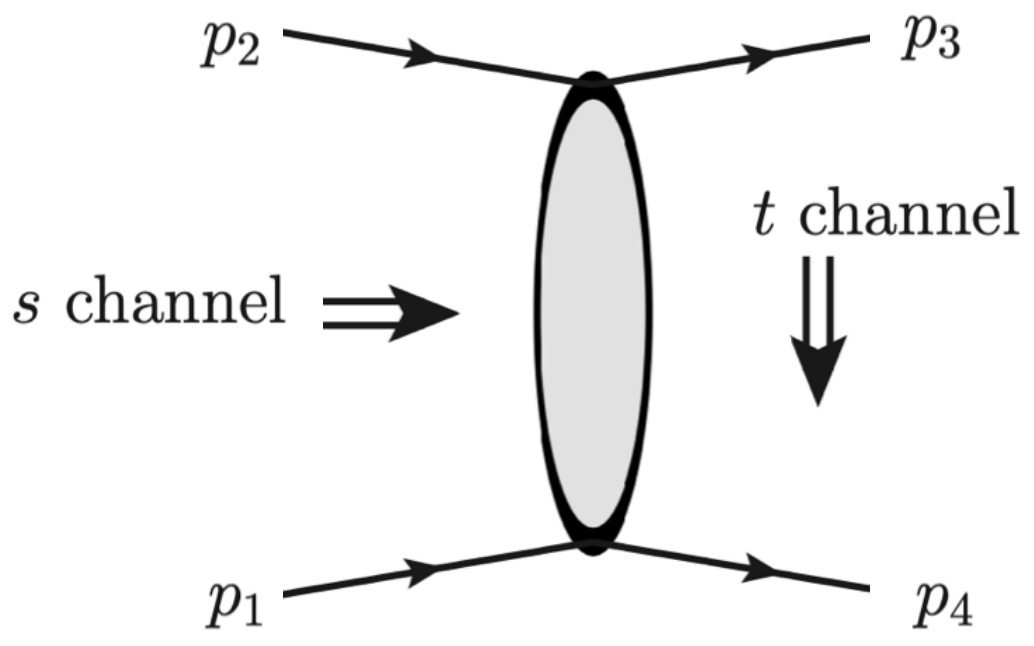
- We consider $s \gg -t \gg \Lambda^2_{\text{QCD}}$ and study $\mathcal{M}$ in this limit *perturatively*

QCD scattering in the Regge limit

Motivation and objectives

- Understand the high-energy behaviour of **quark and gluon** scattering amplitudes in **full colour**
- Study the **exponentiation** of high-energy logarithms
- Connect **rapidity evolution equations** to properties of scattering amplitudes
- Establish connection with **Regge poles and cuts** in the complex angular momentum plane
- Understand the interplay between the Regge limit and **soft gluon exponentiation**

$2 \to 2$ amplitudes: signature and reality properties



- Regge theory is based on expressing the t-channel amplitude as a sum over states with a given angular momentum ℓ, and analytically continuing to the s channel.

  This requires separating between even and odd values of ℓ, leading to **even/odd signature**.

- Defining **signature** *even* and *odd* amplitudes under $s \leftrightarrow u$

$$\mathcal{M}^{(\pm)}(s,t) = \tfrac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big)$$

- The spectral representation of the amplitude implies:

$$\mathcal{M}^{(+)}(s,t) = i\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{2\sin(\pi j/2)}\, a_j^{(+)}(t)\, e^{jL}\,,$$

$$\mathcal{M}^{(-)}(s,t) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{2\cos(\pi j/2)}\, a_j^{(-)}(t)\, e^{jL}\,,$$

with $\left(a^{\pm}_{j^*}(t)\right)^* = a^{\pm}_j(t)$

$$L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2} = \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right)$$

- Expanding the amplitude in the signature-symmetric log, L, the coefficients in $\mathcal{M}^{(+)}$ are imaginary, while in $\mathcal{M}^{(-)}$ real.

[See 1701.05241 Caron-Huot, EG, Vernazza]

The singularity structure of 2 → 2 amplitudes in the complex angular momentum plane: pole vs. cut

- The signature-odd amplitude admits

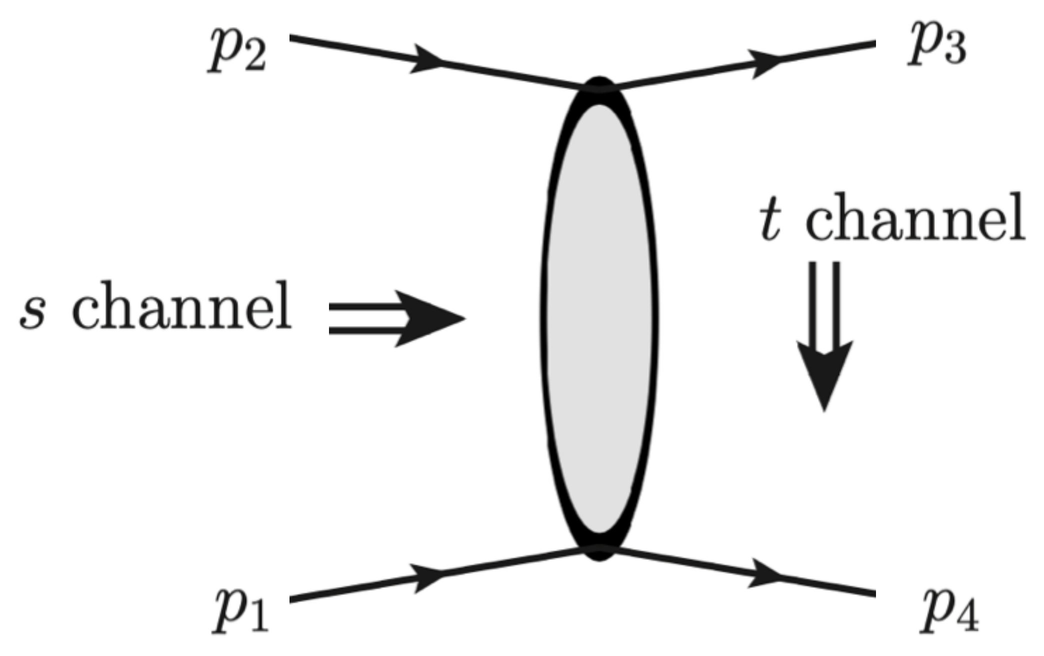
$$\mathcal{M}^{(-)}(s,t) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{2\cos(\pi j/2)}\, a_j^{(-)}(t)\, e^{jL}\,,$$

	singularity	amplitude asymptotics
pole	$a_j^{(-)}(t) \simeq \frac{1}{j-1-\alpha(t)}$	$\mathcal{M}^{(-)}(s,t)\vert_{\text{Regge pole}} \simeq \frac{\pi}{\sin\frac{\pi\,\alpha(t)}{2}}\,\frac{s}{t}\, e^{L\,\alpha(t)} + \ldots,$
cut	$a_j^{(-)}(t) \simeq \frac{1}{(j-1-\alpha(t))^{1+\beta(t)}}$	$\mathcal{M}^{(-)}(s,t)\vert_{\text{Regge cut}} \simeq \frac{\pi}{\sin\frac{\pi\,\alpha(t)}{2}}\,\frac{s}{t}\,\frac{1}{\Gamma\left(1+\beta(t)\right)} L^{\beta(t)}\, e^{L\,\alpha(t)} + \text{subleading logs}$

- Reggeization of the **signature-odd** amplitude (NLL): a manifestation of a pure **Regge pole**.

The Regge limit of 2 → 2 gauge-theory amplitudes

- **Regge theory:** the amplitude should be dominated by the t-channel exchange of the particle with the highest spin, $\mathcal{M} \sim s^{\ell}$.

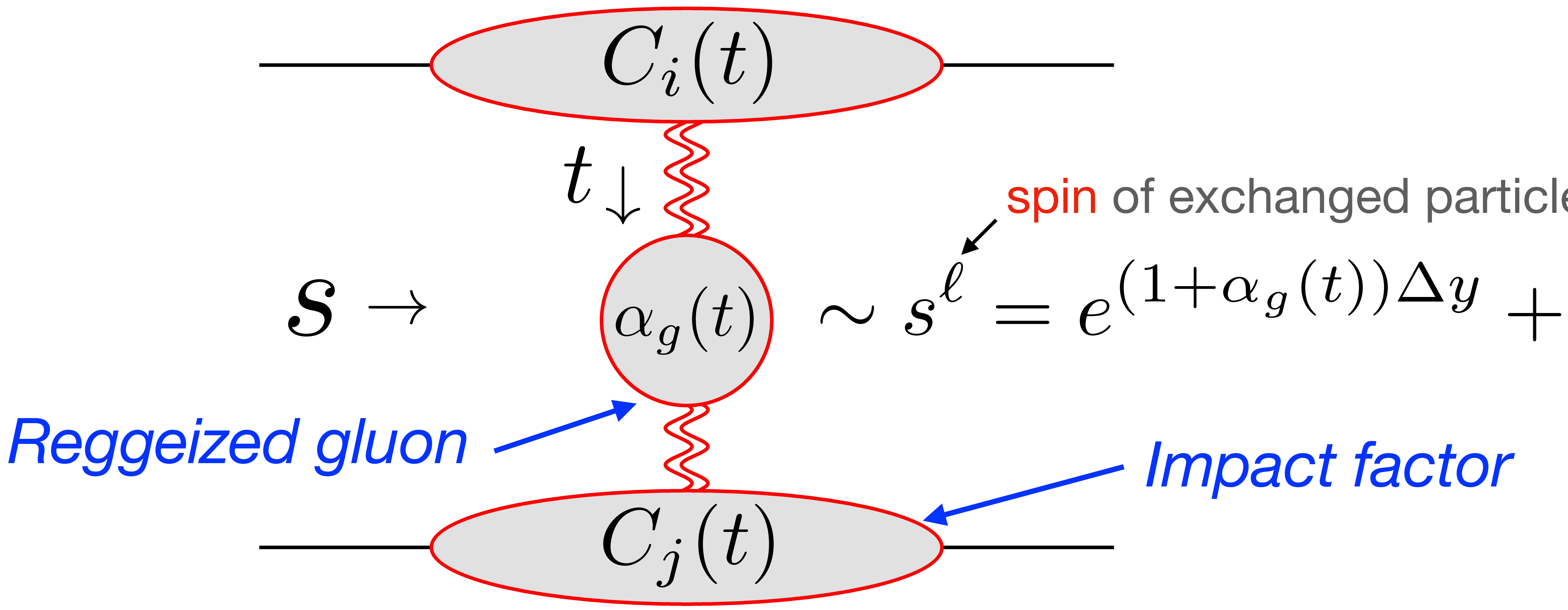


- **QCD**: Simplification at leading power in t/s: helicity is conserved, and indeed, t-channel **gluon** exchange is dominant.

- **Reggeization** (Regge-pole): $$\frac{s}{t} \longrightarrow \frac{s}{t}\left(\frac{s}{-t}\right)^{\alpha_g(t)}$$

  resumming all leading logarithms $\left[\alpha_s(-t)\,\ln\left(\frac{s}{-t}\right)\right]^n$

- **Factorization:**



spin of exchanged particle

$$\sim s^{\ell} = e^{(1+\alpha_g(t))\Delta y} + \ldots$$

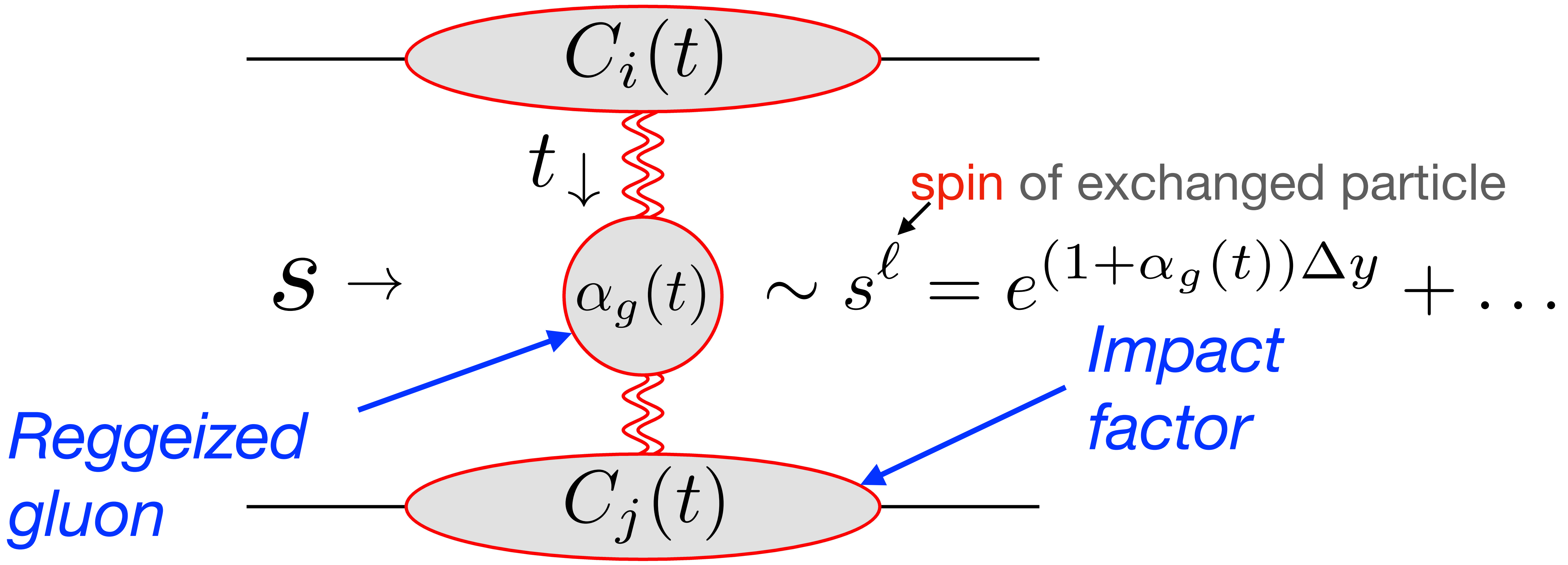
The high-energy limit of 2 → 2 gauge-theory amplitudes

- The gluon Regge trajectory can be computed in perturbation theory. At one loop:

$$\alpha_g(t) = -\alpha_s \mathbf{T}_t^2 (\mu^2)^\epsilon \int \frac{d^{2-2\epsilon}k_\perp}{(2\pi)^{2-2\epsilon}} \frac{q_\perp^2}{k_\perp^2 (q_\perp - k_\perp)^2} + \mathcal{O}(\alpha_s^2)$$
$$= \frac{\alpha_s}{\pi} \mathbf{T}_t^2 \left(\frac{-t}{\mu^2}\right)^{-\epsilon} \frac{B_0(\epsilon)}{2\epsilon} + \mathcal{O}(\alpha_s^2)$$

$$B_0(\epsilon) = e^{\epsilon\gamma_E} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} = 1 - \frac{\zeta_2}{2}\epsilon^2 - \frac{7\zeta_3}{3}\epsilon^3 + \ldots$$

- Regge-pole factorization amounts to a **relation** between gg→gg, qg→qg, qq→qq



- This holds for the **real part** of the amplitude through NLL.
Beyond that it is violated by **non-planar** corrections associated with **multi-Reggeon** exchange forming **Regge cuts.** These effects are now better understood.

Signature, number of Reggeons and t-channel colour flow

- The signature odd and even sectors decouple

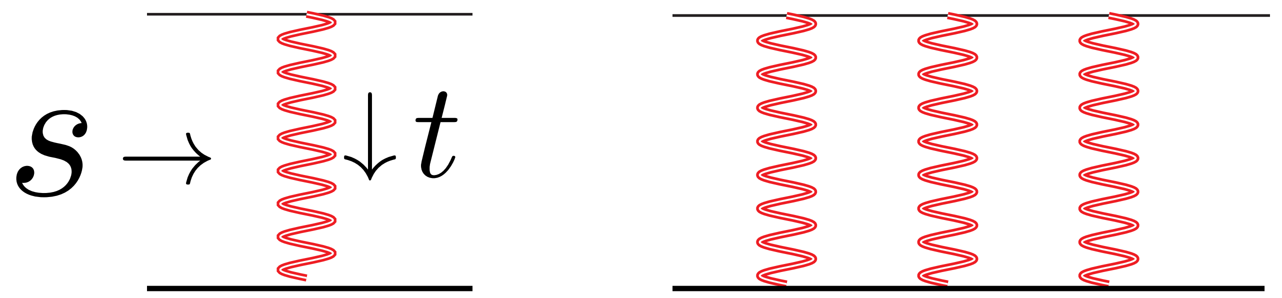
$$\mathcal{M}_{ij\to ij} \xrightarrow{\text{Regge}} \mathcal{M}^{(-)}_{ij\to ij} + \mathcal{M}^{(+)}_{ij\to ij}$$

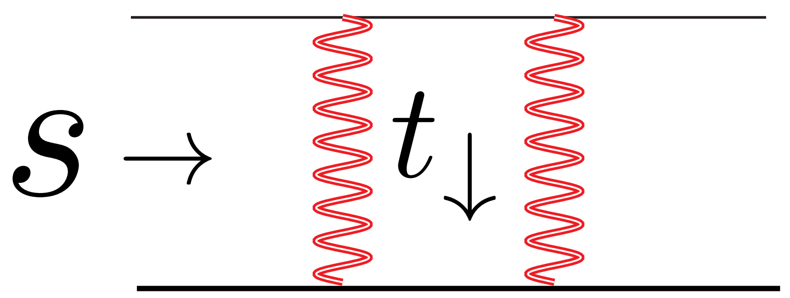
- odd/even signature amplitude is governed by the exchange of an odd/even number of Reggeons.

- Bose symmetry in *gg* → *gg* correlates odd/even signature with odd/even colour representations in the *t* channel.

$(\mathbf{8}\otimes\mathbf{8})_{gg}$

odd	even
$8_a \oplus (10 \oplus \overline{10})$	$0 \oplus 1 \oplus 8_s \oplus 27$





More generally we use channel colour operators: $\mathbf{T}_t^2$ is even, $\mathbf{T}^2_{s-u} \equiv \dfrac{\mathbf{T}_s^2 - \mathbf{T}_u^2}{2}$ is odd

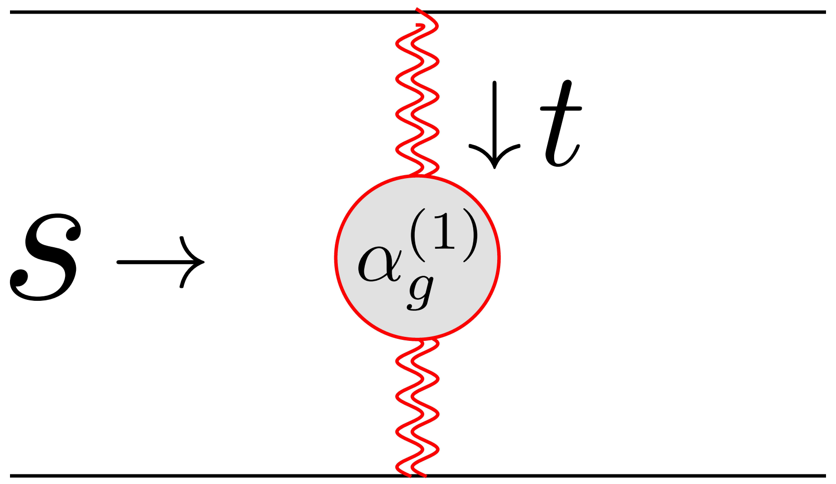
Signature-odd amplitudes: Regge-pole factorisation and its breaking

Regge factorization and violation:

$$\mathcal{M}^{(-)}_{ij\to ij} = C_i(t)\, e^{\alpha_g(t)\, C_A\, L}\, C_j(t)\mathcal{M}^{\text{tree}}_{ij\to ij} + \text{MR}$$

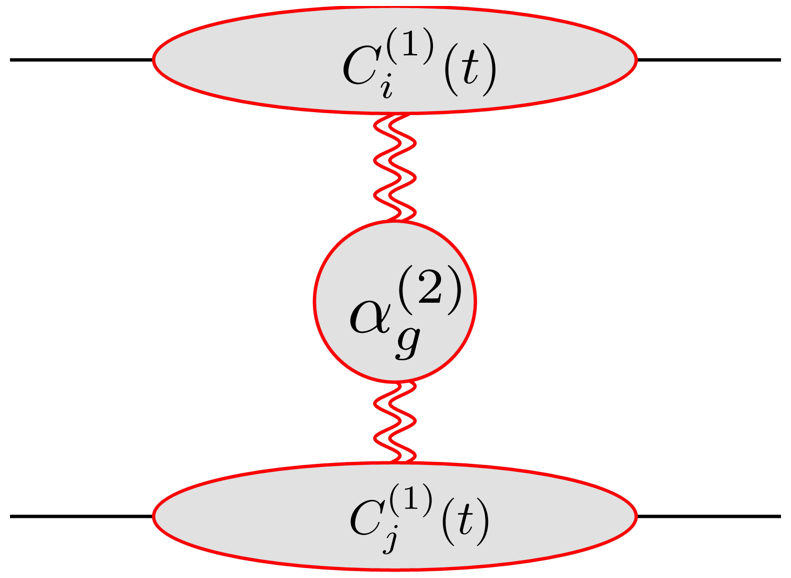
Colour **octet** exchange in the t channel: single Reggeon

LL $\alpha_s^n \log^n\left(\frac{s}{-t}\right)$



one-loop Regge trajectory

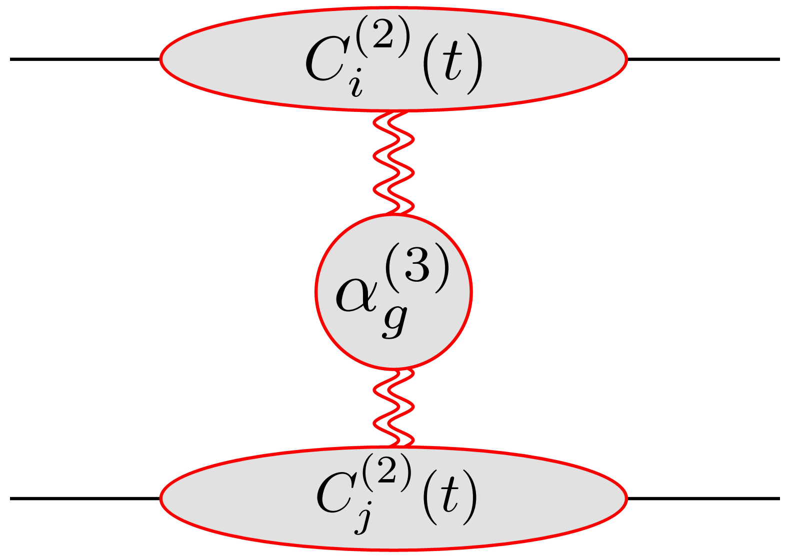
NLL $\alpha_s^n \log^{n-1}\left(\frac{s}{-t}\right)$



two-loop Regge trajectory

one-loop impact factors

NNLL $\alpha_s^n \log^{n-2}\left(\frac{s}{-t}\right)$



three-loop Regge trajectory

two-loop impact factors

Regge factorisation breaking (starting at 2 loops) can be inferred from comparing

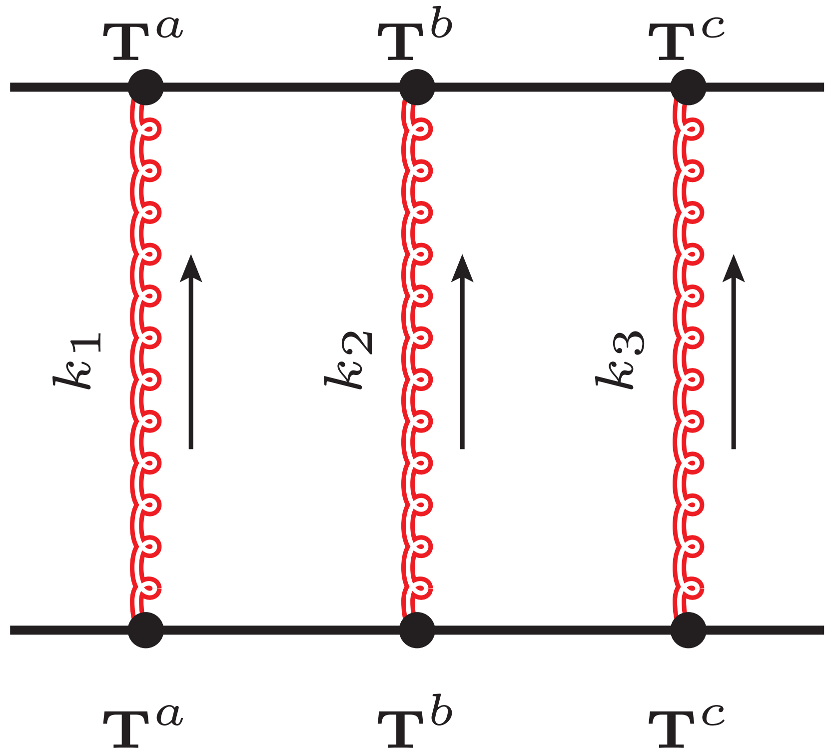
gg→gg, qg→qg, qq→qq

amplitudes

[Del Duca, Glover '01]

[Del Duca, Falcioni, Magnea, Vernazza '14]

But until recently unknown how to account for it



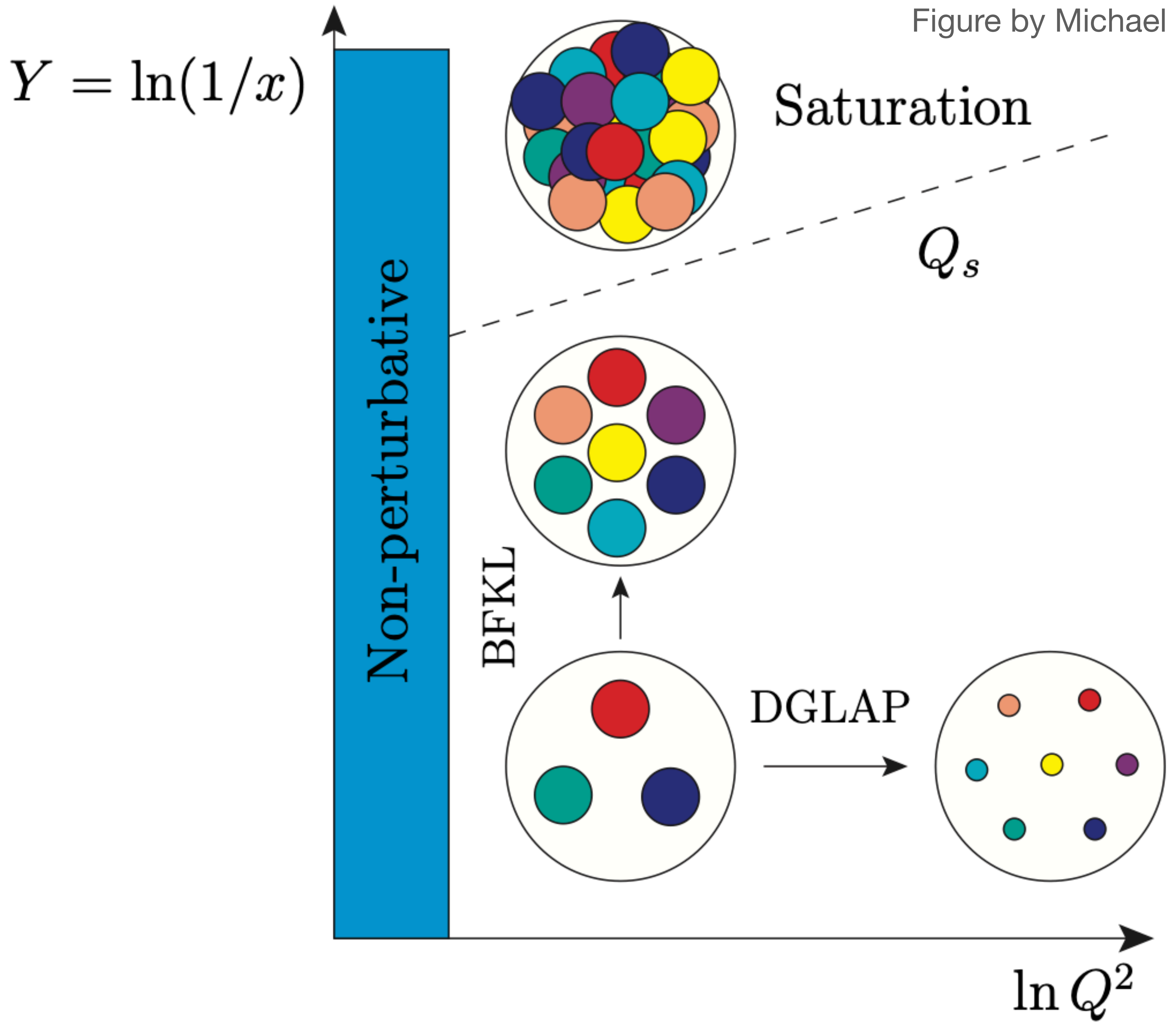
parton density evolution in Q^2 and in rapidity

- DGLAP (Dokshitzer-Gribov-Lipatov-Altarelli-Parisi) evolution resums logarithms of Q^2/μ_F^2

- BFKL (Balitsky-Fadin-Kuraev-Lipatov) evolution resums energy logarithms (= rapidity Y)

  The high-gluon-density saturation regime requires a generalisation: Balitsky-JIMWLK (Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner) non-linear evolution in rapidity

Figure by Michael Fucilla

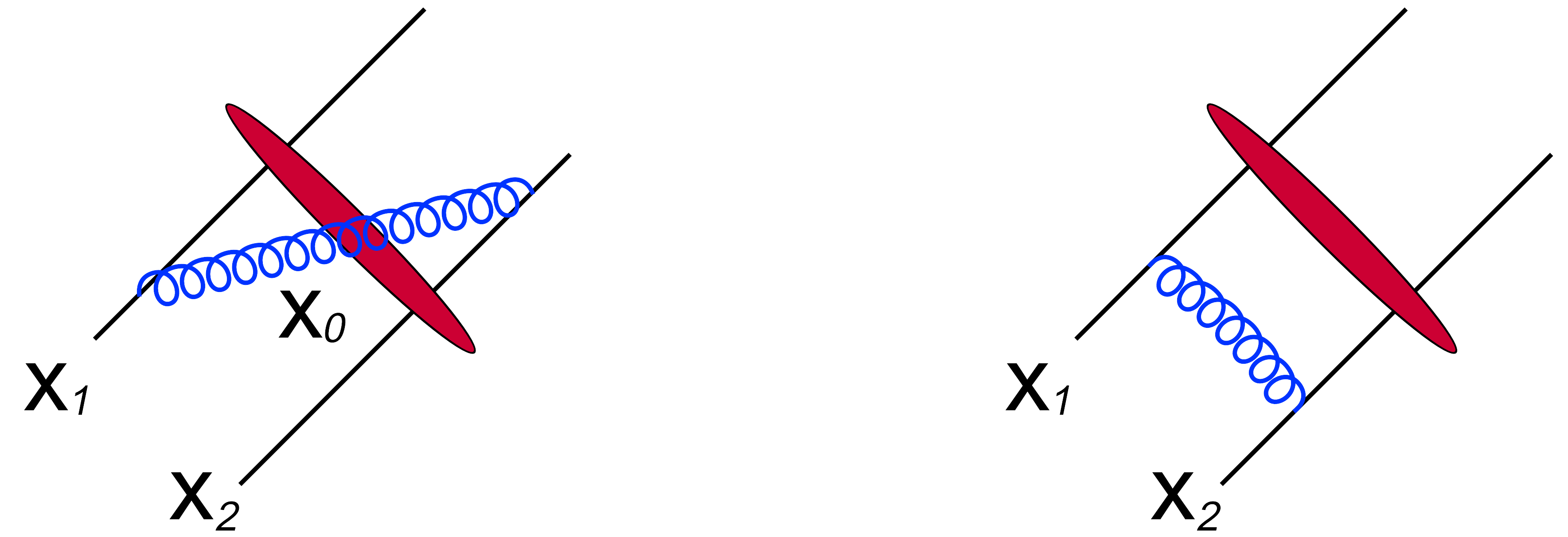


We shall study the consequences of this non-linear evolution on the perturbative amplitude.

The shock-wave formalism and non-linear rapidity evolution

- The colliding particles are replaced by (sets of) infinite lightlike Wilson lines

$$U(\mathbf{x}) = \mathcal{P}\exp\left\{ig_s\int_{-\infty}^{\infty} dx^+ A_+^a(x^+, x^- = 0; \mathbf{x})T^a\right\}$$



- Rapidity evolution equation [Balitsky-JIMWLK]

$$-\frac{d}{d\eta}\left[U(\mathbf{x}_1)\ldots U(\mathbf{x}_n)\right] = H\left[U(\mathbf{x}_1)\ldots U(\mathbf{x}_n)\right]$$

$$H = \frac{\alpha_s}{2\pi^2}\int d\mathbf{x}_i d\mathbf{x}_j d\mathbf{x}_0 \frac{\mathbf{x}_{0i}\cdot\mathbf{x}_{0j}}{\mathbf{x}_{0i}^2\mathbf{x}_{0j}^2}\left(T^a_{i,L}T^a_{j,L} + T^a_{i,R}T^a_{j,R} - U^{ab}_{\text{adj}}(\mathbf{x}_0)(T^a_{i,L}T^b_{j,R} + T^a_{j,L}T^b_{i,R})\right)$$

$$T^a_{i,L} \equiv T^a U(\mathbf{x}_i)\frac{\delta}{\delta U(\mathbf{x}_i)}, \qquad T^a_{i,R} \equiv U(\mathbf{x}_i)T^a\frac{\delta}{\delta U(\mathbf{x}_i)}$$

Provides complete separation between the light-cone directions
and the transverse plane: **2-dimensional dynamics**

Towards an effective theory: Defining the Reggeon

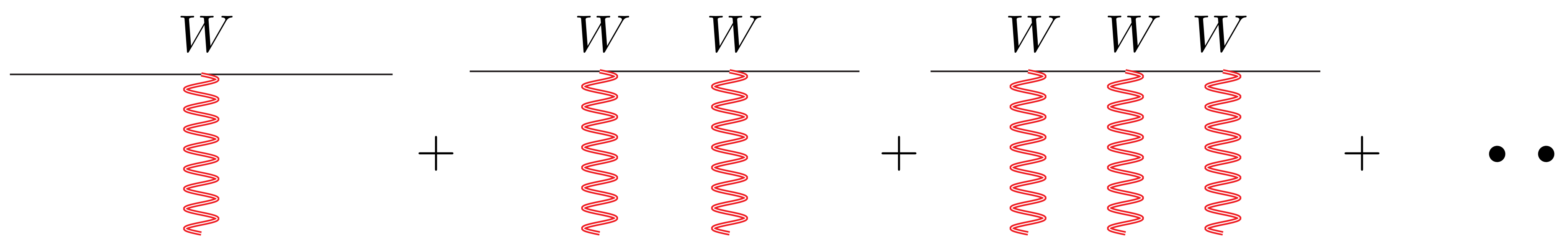
- In the perturbative regime $U(\mathbf{x}) \simeq 1$ it is natural to expand in terms of W Simon Caron-Huot (2013)

$$U(\mathbf{x}) = \mathcal{P}\exp\left\{ig_s\int_{-\infty}^{\infty} dx^+ A_+^a(x^+, x^- = 0; \mathbf{x})T^a\right\} = e^{ig_s\, T^a W^a(\mathbf{x})}.$$

W sources a Reggeon

- Scattered particles are expanded in states of a definite number of Reggeons

$$|\psi_i\rangle \equiv \frac{Z_i^{-1}}{2p_1^+} a_i(p_4)a_i^\dagger(p_1)|0\rangle \sim g_s\,|W\rangle + g_s^2\,|WW\rangle + g_s^3\,|WWW\rangle + \ldots =$$

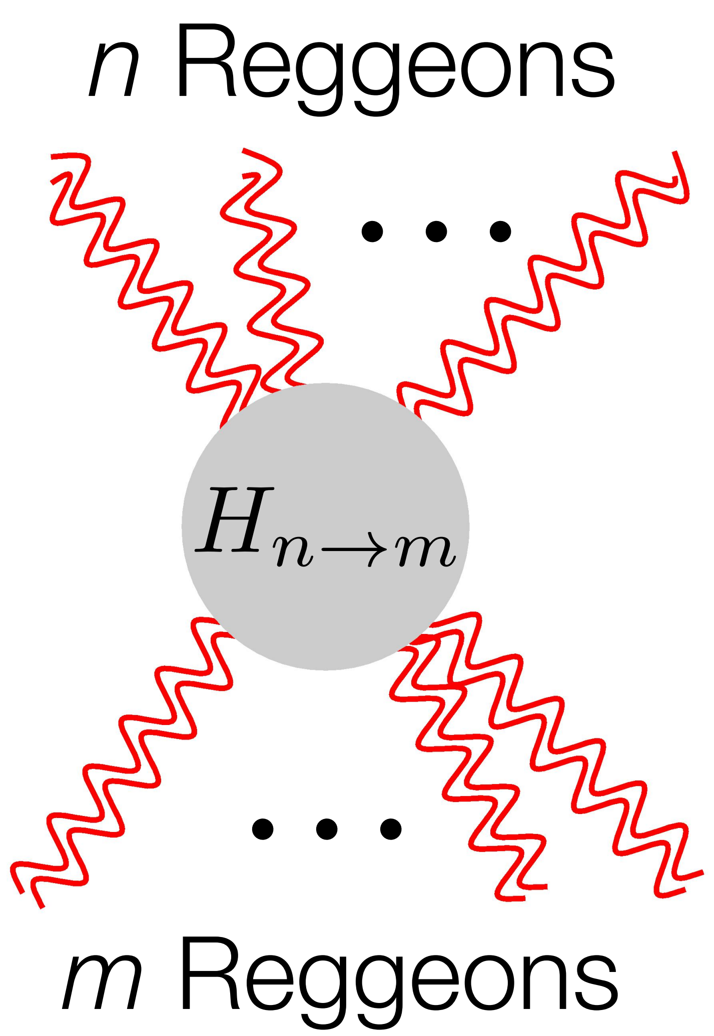


- Amplitudes are governed by rapidity evolution between the target and projectile:

$$\frac{i(Z_iZ_j)^{-1}}{2s}\mathcal{M}_{ij\to ij} = \langle\psi_j|e^{-HL}|\psi_i\rangle$$

$$-\frac{d}{d\eta}\,|\psi_i\rangle = H\,|\psi_i\rangle$$

$$H\begin{pmatrix} W \\ WW \\ WWW \\ \ldots \end{pmatrix} \equiv \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & \ldots \\ 0 & H_{2\to2} & 0 & \ldots \\ H_{1\to3} & 0 & H_{3\to3} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}\begin{pmatrix} W \\ WW \\ WWW \\ \ldots \end{pmatrix}$$

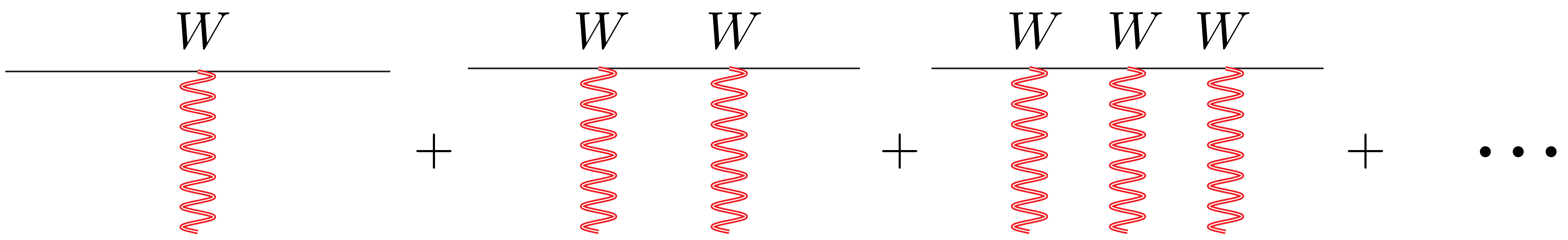


- Each action of the Hamiltonian generates an extra power of the high-energy log L

Computing multi-Regge exchanges using non-linear rapidity evolution

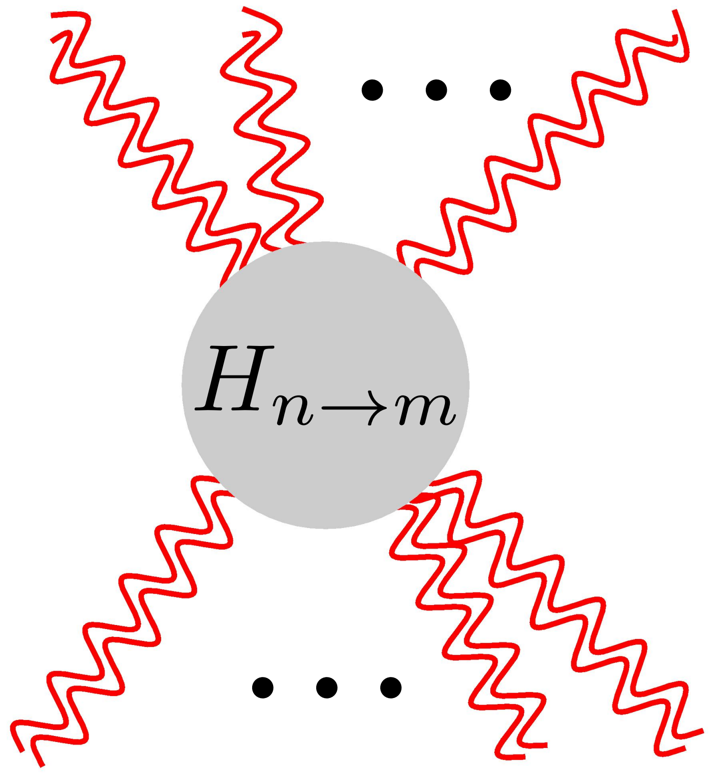
1701.05241 Caron-Huot, EG, Vernazza

Projectile $|\psi_i\rangle =$



n Reggeons to *m* Reggeons transition Hamiltonian [1701.05241]

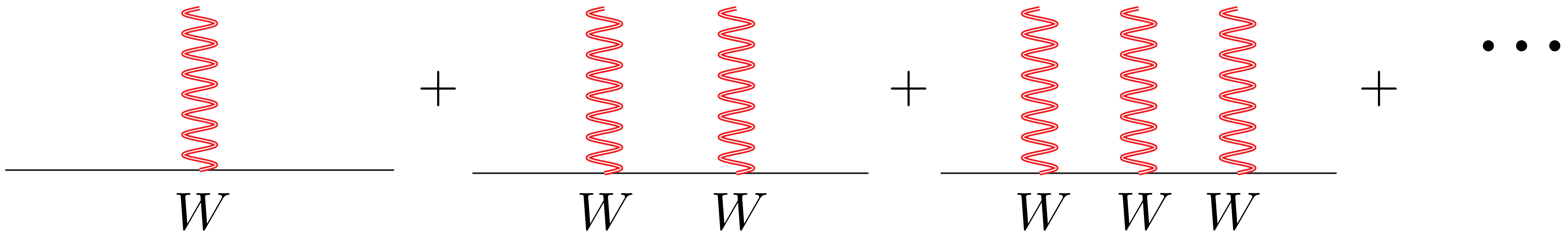
$$\sum_{n,m}$$



$$H\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix} \equiv \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & \cdots \\ 0 & H_{2\to2} & 0 & \cdots \\ H_{1\to3} & 0 & H_{3\to3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix}$$

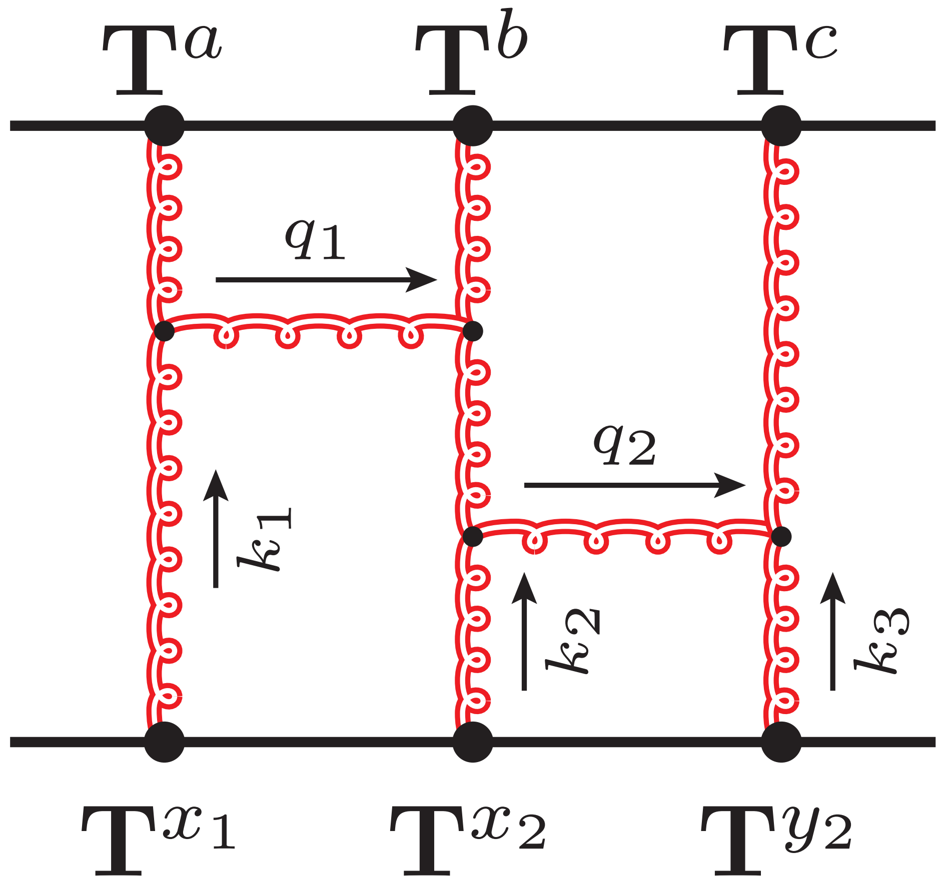
$$\sim \begin{pmatrix} g_s^2 & 0 & g_s^4 & \cdots \\ 0 & g_s^2 & 0 & \cdots \\ g_s^4 & 0 & g_s^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix}$$

Target $\langle\psi_j| =$



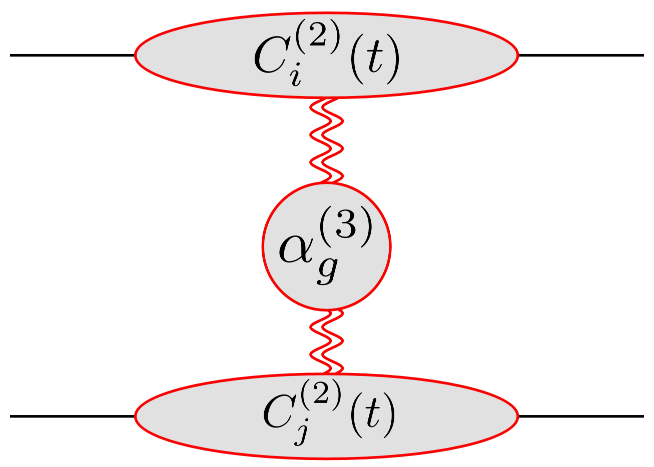
Signature-odd 2 → 2 amplitudes: understanding the NNLL tower

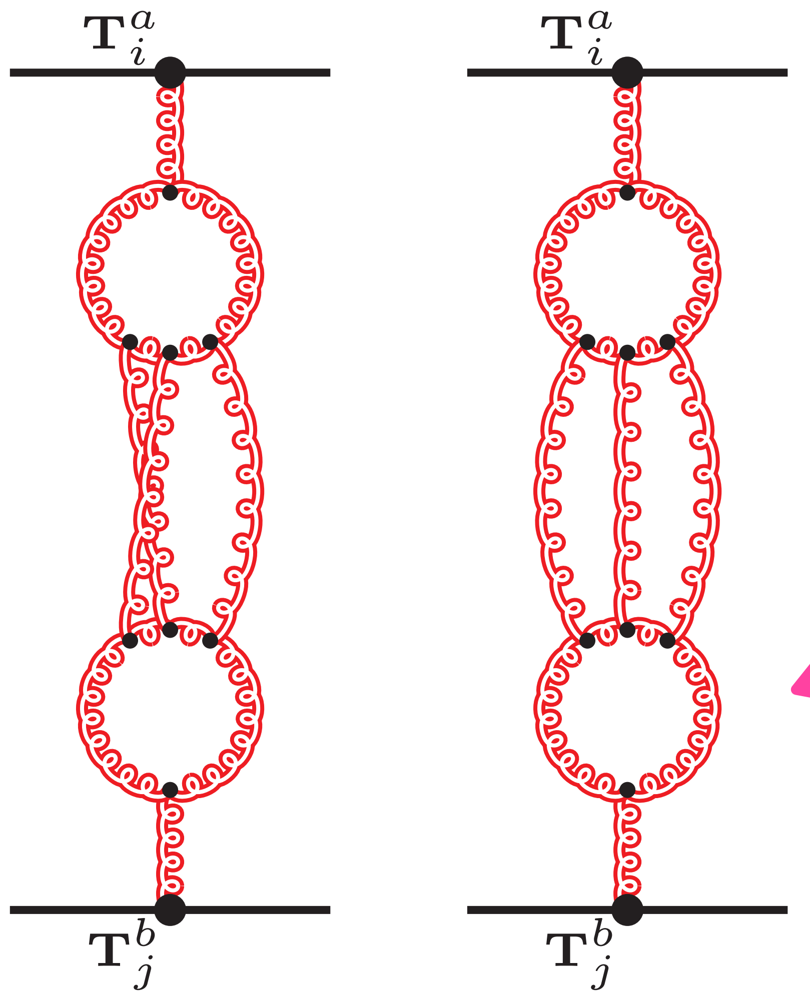
- Using **non-linear** rapidity evolution, the NNLL tower is determined to all orders in terms of **one** and **three** Reggeon exchanges

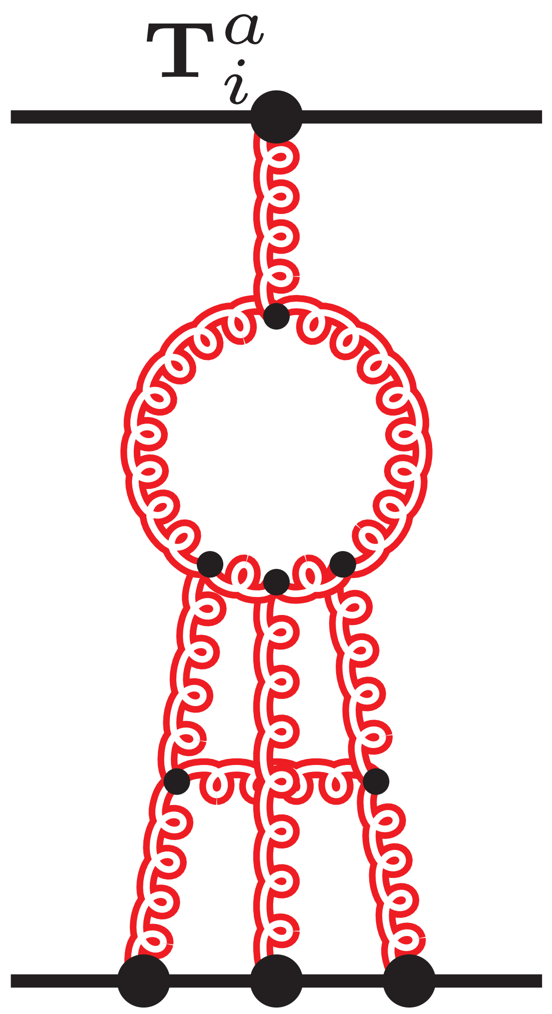


- Expanding in $X \equiv \frac{\alpha_s}{\pi} r_\Gamma L$

$$\frac{i}{2s}\hat{\mathcal{M}}^{(-),\mathrm{NNLL}}_{ij\to ij} = \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg\{ \langle j_1|i_1\rangle^{\mathrm{NNLO}} + r_\Gamma^2\pi^2 \sum_{k=0}^{\infty} \frac{(-X)^k}{k!} \Bigg[\langle j_3|\hat H^k_{3\to3}|i_3\rangle + \Theta(k\ge 1)\Big[\langle j_1|\hat H_{3\to1}\hat H^{k-1}_{3\to3}|i_3\rangle + \langle j_3|\hat H^{k-1}_{3\to3}\hat H_{1\to3}|i_1\rangle\Big] + \Theta(k\ge 2)\langle j_1|\hat H_{3\to1}\hat H^{k-2}_{3\to3}\hat H_{1\to3}|i_1\rangle \Bigg]^{\mathrm{LO}} \Bigg\}$$







- All diagrams computed to four loops

Caron-Huot, EG, Vernazza
JHEP 06 (2017) 016
Falcioni, EG, Milloy, Vernazza
Phys. Rev. D 103 (2021) L111501

Signature odd 2 → 2 amplitude at NNLL: Regge pole and cut

Requiring that the Regge cut is strictly non-planar fixes the separation between Regge pole vs. Regge cut

Falcioni, EG, Maher, Milloy, Vernazza
Phys.Rev.Lett. 128 (2022) 13, 13;
JHEP 03 (2022) 053

$$\mathcal{M}^{(-)}_{ij\to ij} = \underbrace{\mathcal{M}^{(-)\,\mathrm{SR}}_{ij\to ij} + \mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{planar}}} + \mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{nonplanar}}$$

$$= \qquad \mathcal{M}^{(-)\,\mathrm{pole}}_{ij\to ij} \qquad + \mathcal{M}^{(-)\,\mathrm{cut}}_{ij\to ij}$$

$$\mathcal{M}^{(-)\,\mathrm{pole}}_{ij\to ij} = C_i(t)\, e^{\alpha_g(t)\, C_A\, L}\, C_j(t) \mathcal{M}^{\mathrm{tree}}_{ij\to ij}$$

$\mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{planar}}$ must be **universal** (gg, gq, qq) to be absorbed in the factorizing pole term.

$\mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{planar}}$ ***cannot*** contribute beyond 3 loops: the NNLL Regge pole term has **no** free parameters!

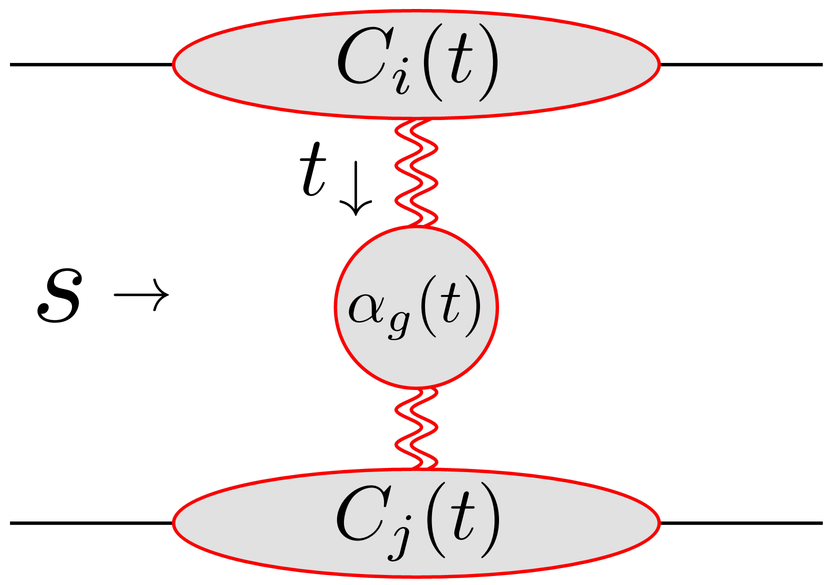
Indeed, at 4 loops **planar Multi Regge contributions conspire to cancel!**

Signature odd amplitude at NNLL: Regge pole and cut properties

All-order structure through NNLL for any gauge theory, any representation:

$$\mathcal{M}^{(-)}_{ij\to ij} = Z_i(t)\,\bar{D}_i(t)\,Z_j(t)\,\bar{D}_j(t)\left[\left(\frac{-s}{-t}\right)^{C_A\alpha_g(t)} + \left(\frac{-u}{-t}\right)^{C_A\alpha_g(t)}\right]\mathcal{M}^{\text{tree}}_{ij\to ij} + \sum_{n=2}^{\infty} a^n L^{n-2}\mathcal{M}^{(\pm,n,n-2)\,\text{cut}}$$

Regge pole-factorized



✓ single Reggeon; colour octet
✓ dominant in planar limit

✓ Trajectory and impact factors at NNLL are **fixed** by matching to (qq, gg, qg) scattering amplitudes*

Regge cut: breaks factorization

✓ multiple Reggeons; various colour reps.
✓ suppressed in planar limit
✓ proportional to $(i\pi)^2$

✓ no dependence on the matter content: the same for any gauge theory!

✓ Sensitive to soft singularities beyond the dipole formula.

* 3-loop Amplitudes: Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi, JHEP 10 (2021) 206]
Caola et al. Phys.Rev.Lett. 128 (2022) 21, 21

Falcioni, EG, Maher, Milloy, Vernazza, Phys.Rev.Lett. 128 (2022) 13, 13; *JHEP* 03 (2022) 053

Regge-pole factorisation for multi-leg amplitudes in MRK

Multi-Regge Kinematics (MRK)

4-momentum $p = (p^+, p^-; \mathbf{p})$

target $p_1 = (0, p_1^-; \mathbf{0})$

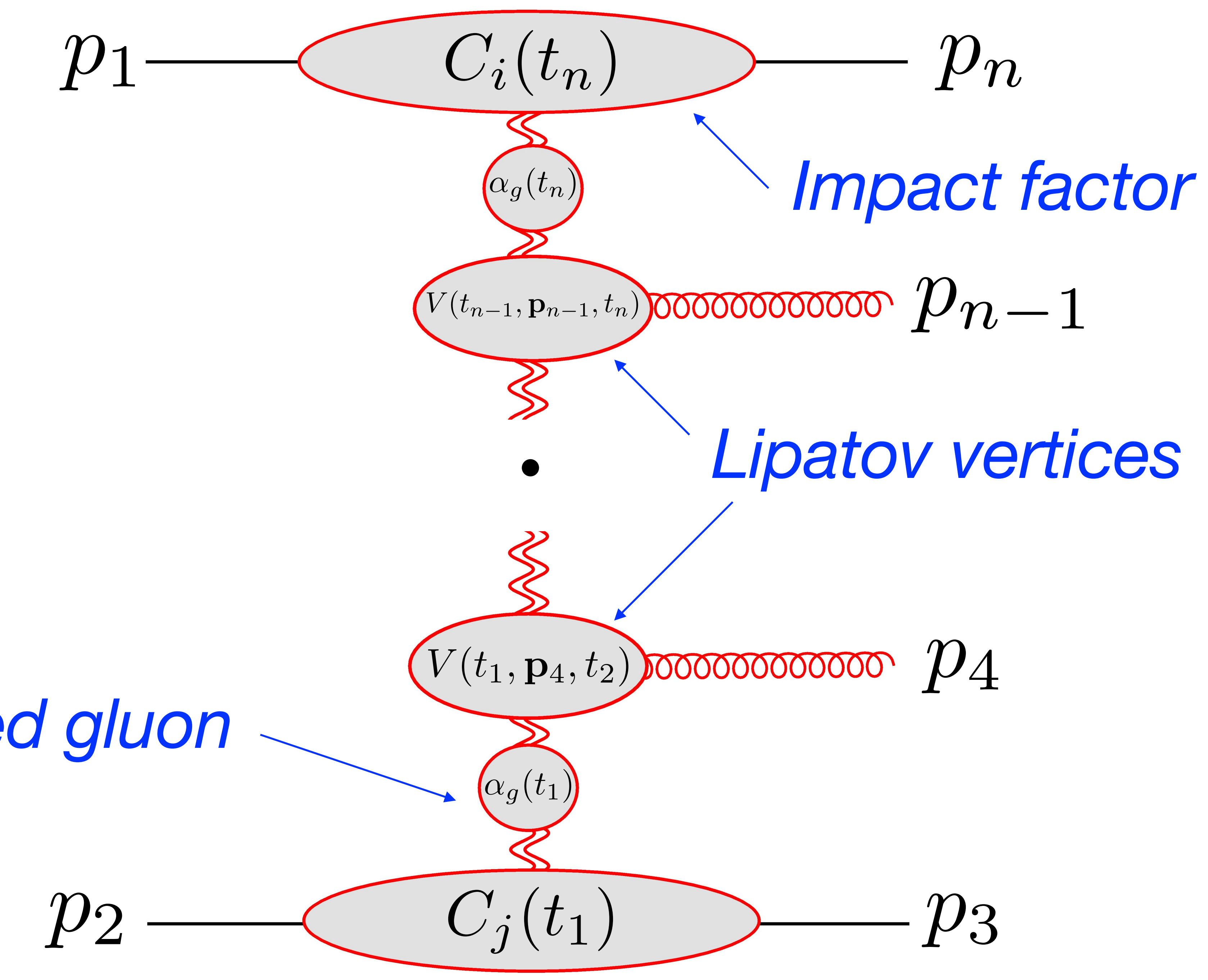
projectile $p_2 = (p_2^+, 0; \mathbf{0})$

strong hierarchy of light-cone components $p_3^+ \gg p_4^+ \gg ... \gg p_n^+$

$p_3^- \ll p_4^- \ll ... \ll p_n^-$

no ordering of transverse components $|\mathbf{p}_3| \sim |\mathbf{p}_4| \sim ... \sim |\mathbf{p}_n|$

Regge (pole) factorization in MRK



Regge (pole) factorization holds in MRK for the dispersive (real part) of the amplitudes through NLL; established using unitarity [Fadin et al. 2006]

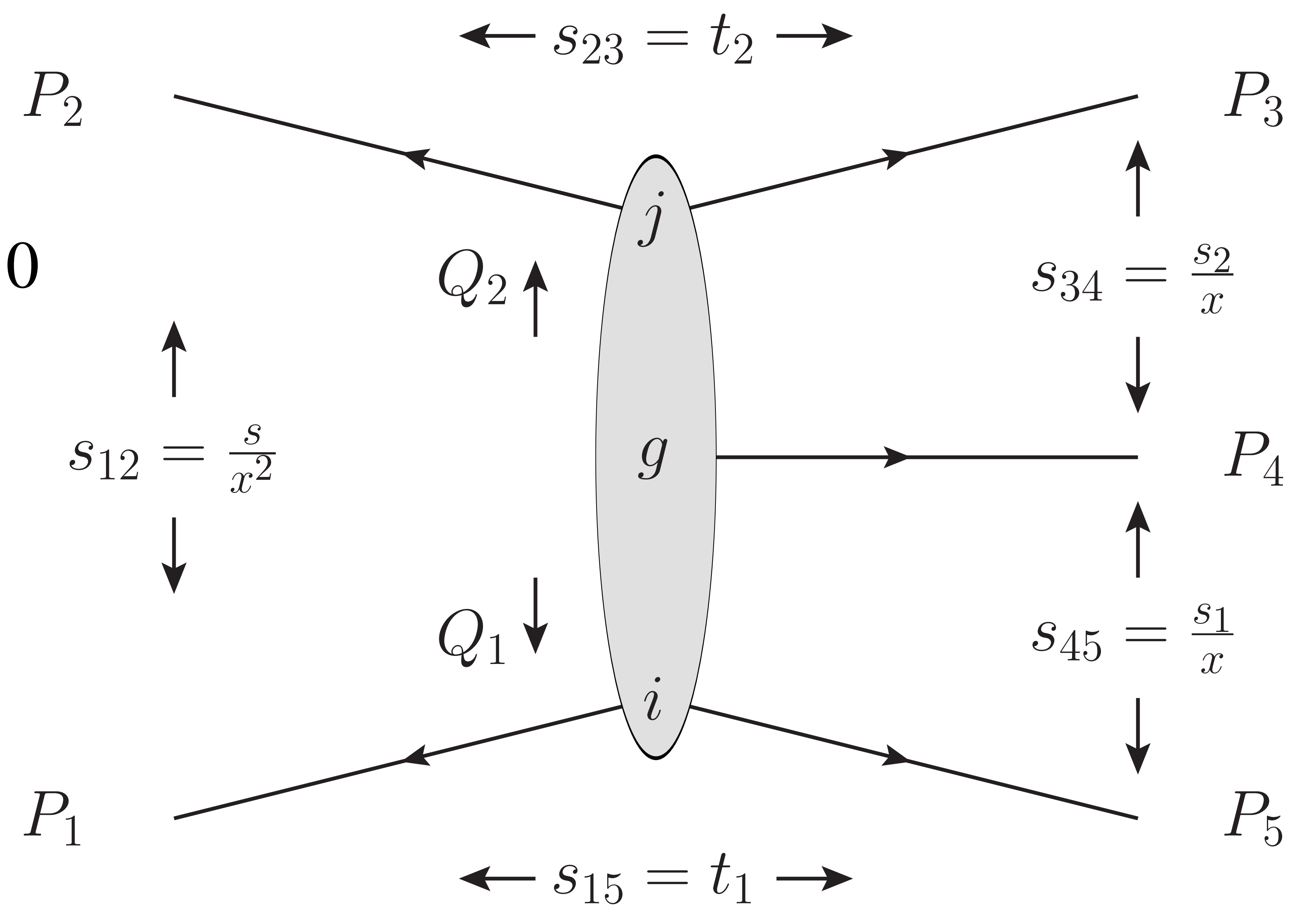
Planar limit:

- Four- and five-point planar amplitudes have only Regge poles. Essential for the BDS ansatz in SYM.
- Six and higher-point planar amplitudes have also Regge cuts in some special kinematic regions [Bartels, Lipatov, Sabio Vera (2008)]. All multiplicity planar results are available [Del Duca *et al.* (2019)]

2 → 3 amplitudes in multi-Regge kinematics

- Multi-Regge kinematics:

$$s_{12} \to \frac{s_{12}}{x^2} \qquad s_{45} \to \frac{s_1}{x} \qquad s_{34} \to \frac{s_2}{x} \qquad s_{15} \to t_1 \qquad s_{23} \to t_2 \qquad \text{for} \quad x \to 0$$



- Signature symmetry operations:

$$\begin{aligned} (1 \leftrightarrow 5) \quad &\to \quad \{s \to -s, \quad s_{45} \to -s_{45}\}, \\ (2 \leftrightarrow 3) \quad &\to \quad \{s \to -s, \quad s_{34} \to -s_{34}\}. \end{aligned}$$

- t-channel colour basis: diagonal operators:

$$\mathbf{T}_{t_1}^2 \equiv (\mathbf{T}_1 + \mathbf{T}_5)^2$$
$$\mathbf{T}_{t_2}^2 \equiv (\mathbf{T}_2 + \mathbf{T}_3)^2$$

Signature-preserving operator on line i, j: $\quad \mathbf{T}_{(++)} = (\mathbf{T}_1^a + \mathbf{T}_5^a) \cdot (\mathbf{T}_2^a + \mathbf{T}_3^a)$,

Signature-preserving on line i, inverting on j: $\quad \mathbf{T}_{(+-)} = (\mathbf{T}_1^a + \mathbf{T}_5^a) \cdot (\mathbf{T}_2^a - \mathbf{T}_3^a)$,

Signature-preserving on line j, inverting on i: $\quad \mathbf{T}_{(-+)} = (\mathbf{T}_1^a - \mathbf{T}_5^a) \cdot (\mathbf{T}_2^a + \mathbf{T}_3^a)$,

Signature-inverting operator on lines i, j: $\quad \mathbf{T}_{(--)} = (\mathbf{T}_1^a - \mathbf{T}_5^a) \cdot (\mathbf{T}_2^a - \mathbf{T}_3^a)$.

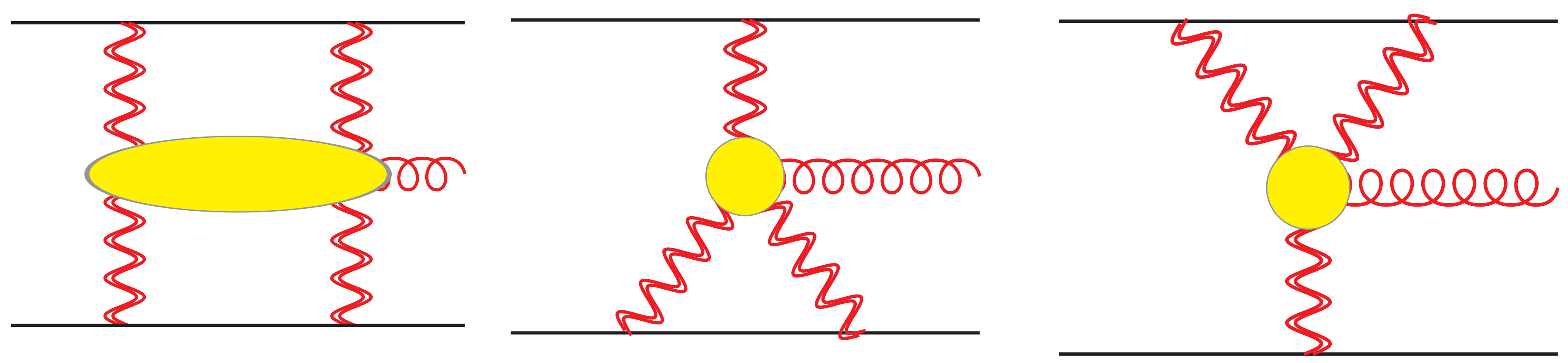
$2 \to 3$ amplitudes at one loop: multi-Reggeon contributions

- A new feature compared to $2 \to 2$ scattering: even and odd signature mix

Caron-Huot, Chicherin, Henn, Zhang, Zoia, JHEP 10 (2020) 188

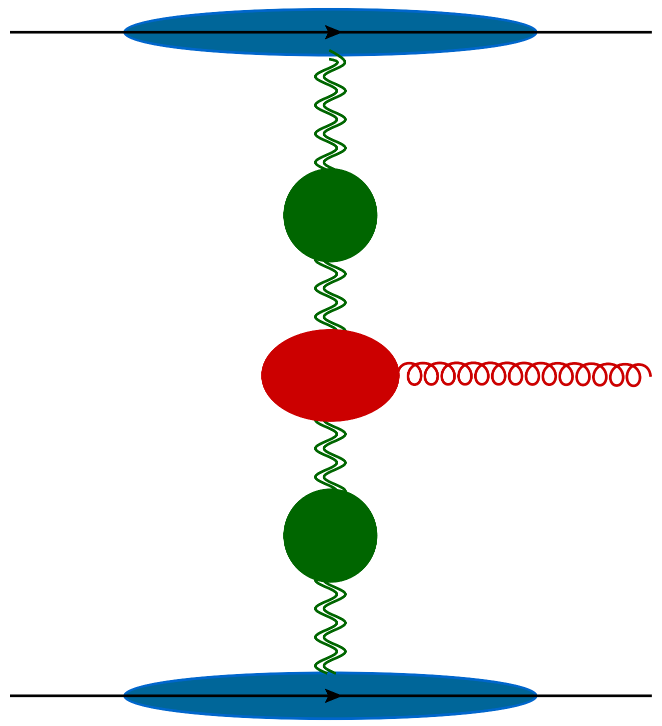
These have now been eventuated in the effective multi-Reggeon framework:

$$\mathcal{M}^{\mathrm{MR}\,(1)}_{ij\to i'gj'} = \mathcal{M}^{(1)}_{\mathcal{R}^2 g\mathcal{R}^2} + \mathcal{M}^{(1)}_{\mathcal{R}g\mathcal{R}^2} + \mathcal{M}^{(1)}_{\mathcal{R}^2 g\mathcal{R}}$$
$$= \frac{i\pi}{4}\left\{\frac{1}{\epsilon}\left(\mathbf{T}_{(--)} + \mathbf{T}_{(+-)} + \mathbf{T}_{(-+)}\right) + \log\frac{p_4^2}{p_3^2 p_5^2}\mathbf{T}_{(--)} + \log\frac{p_3^2}{p_4^2 p_5^2}\mathbf{T}_{(-+)} + \log\frac{p_5^2}{p_3^2 p_4^2}\mathbf{T}_{(+-)} + \mathcal{O}(\epsilon)\right\}\mathcal{M}^{\mathrm{tree}}_{ij\to i'gj'}$$



- As in $2 \to 2$ scattering, at one-loop multi-Reggeon exchanges **do not affect** the odd-odd signature part of the amplitude, hence factorization (for the [8,8] component) holds just as at tree level:

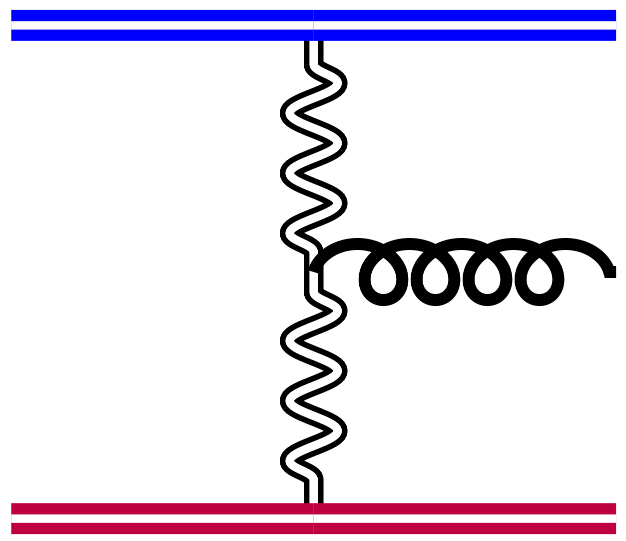
$$\frac{\mathcal{M}^{(-,-)}_{ij\to i'gj'}\Big|^{\text{1-Reggeon}}}{\mathcal{M}^{\mathrm{tree}}_{ij\to i'gj'}} = c_i(t_1,\tau)\, e^{\omega_1\,\eta_1}\, v(t_1,t_2,\mathbf{p}_4^2,\tau)\, e^{\omega_2\,\eta_2}\, c_j(t_2,\tau)$$



Extracting the Lipatov vertex from one-loop amplitudes

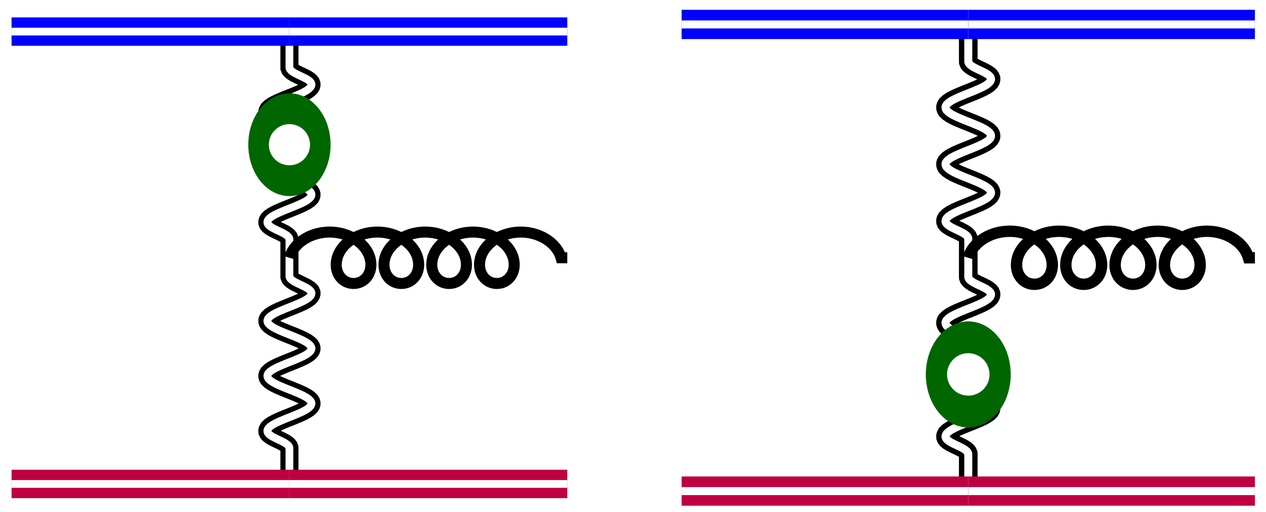
1-loop vertex extracted/computed in Fadin & Lipatov (1993); Del Duca and Schmidt (1998); Del Duca, Duhr, Glover (2009); Fadin, Fucilla, Papa (2023), Buccioni, Caola, Devoto, Gambuti — 2411.14050 [hep-ph] (Figures below), Abreu, De Laurentis, Falcioni, EG, Milloy and Vernazza — 2412.20578 [hep-ph]

Tree



One-loop: $(-,-)$ signature

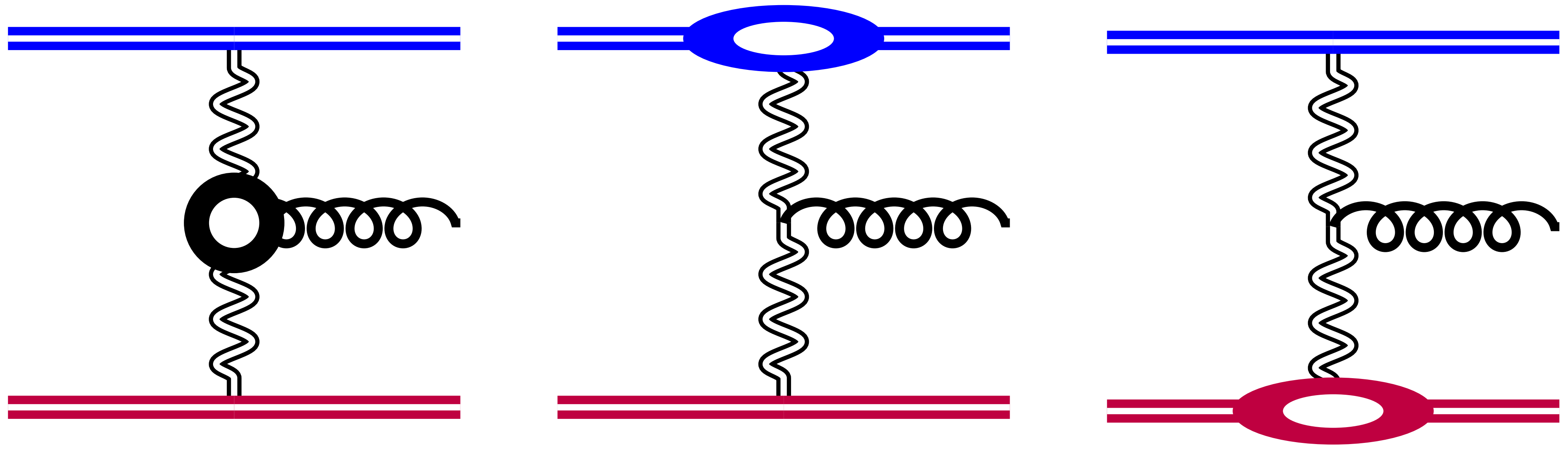
Leading Logarithms



One-loop

$(-,-)$ signature

No Logarithms ("Next to Leading")

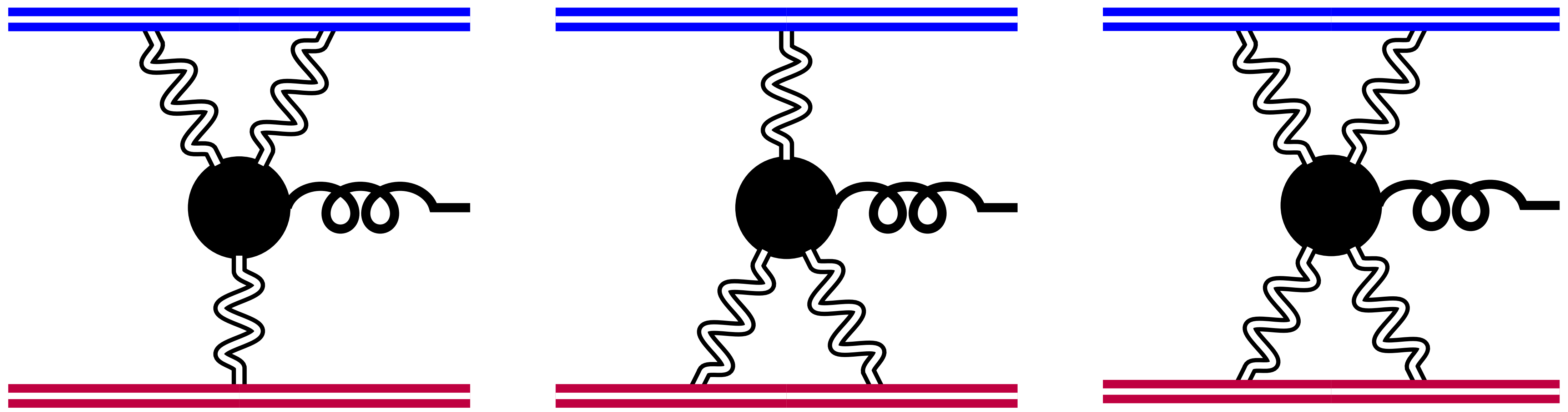


One-loop

$(-,+), (+-), (+,+)$ signature

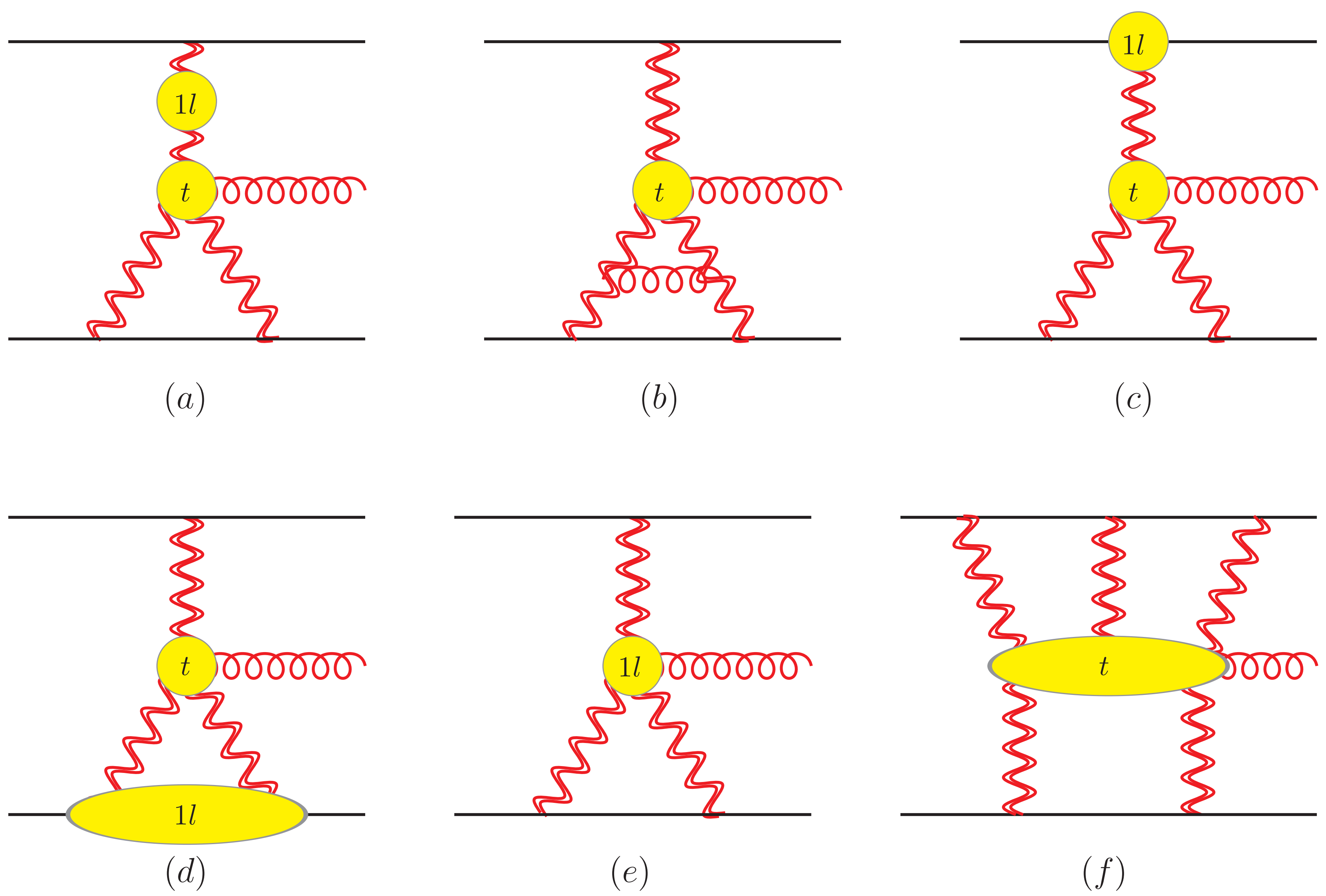
No Logarithms

No [8,8] colour component

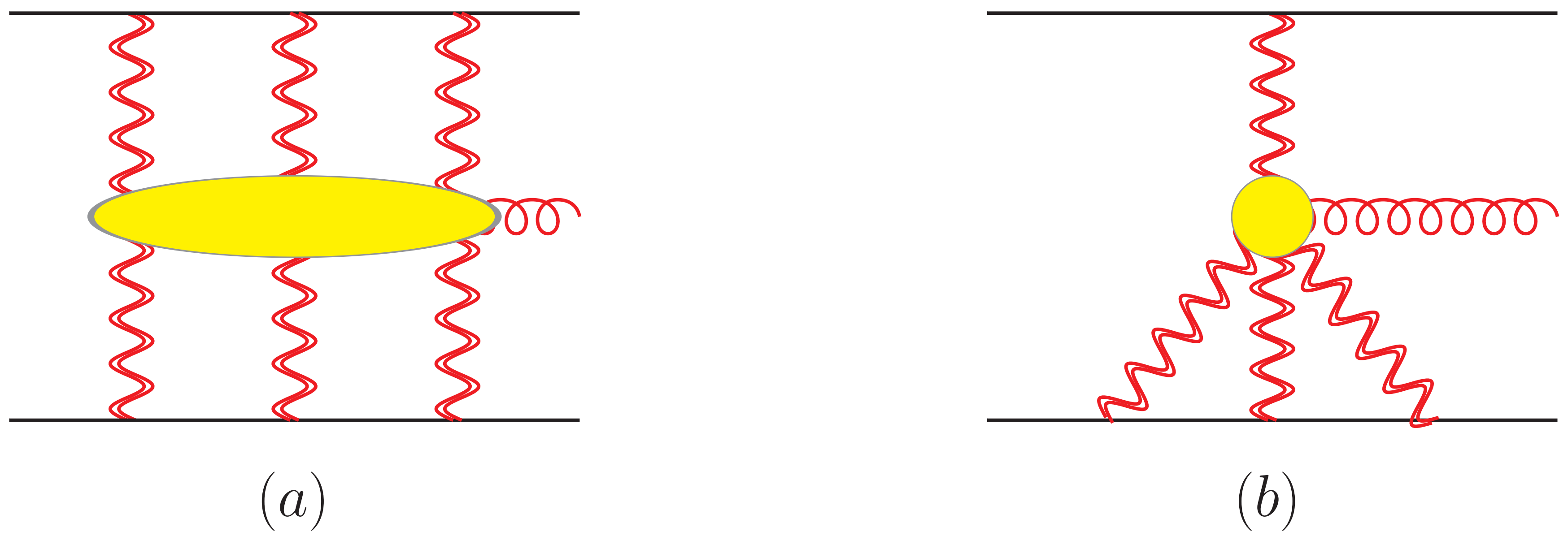


Multiple-Reggeon effect in 2 → 3 scattering: two loops

- At two loops there are many contributions of mixed odd-even signature



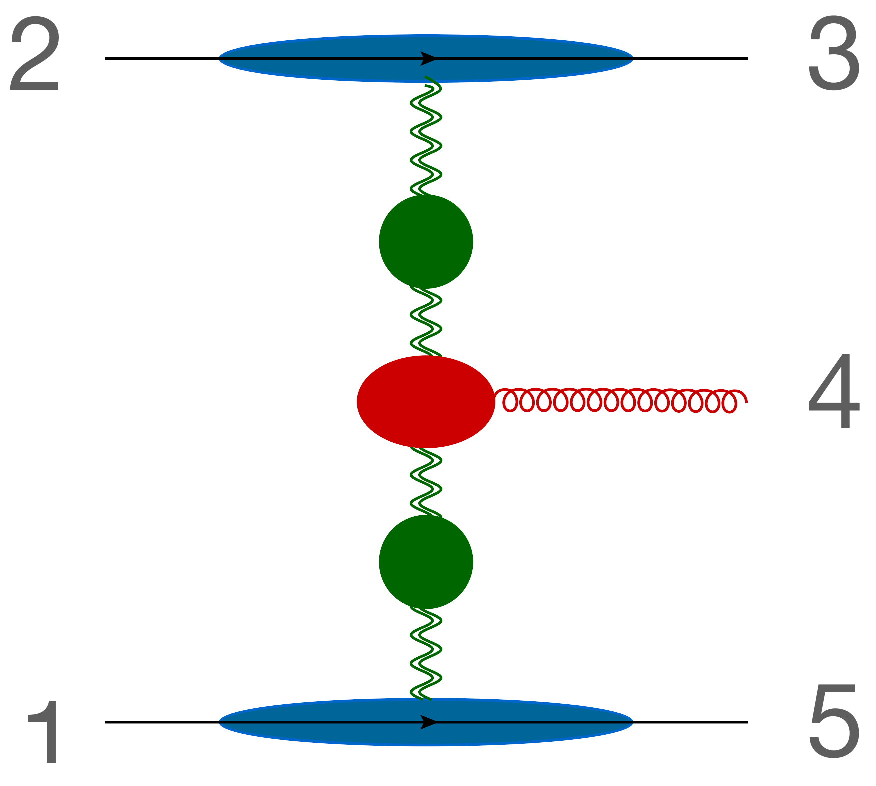
- But importantly, **there are also odd-odd contributions from multiple Reggeons**

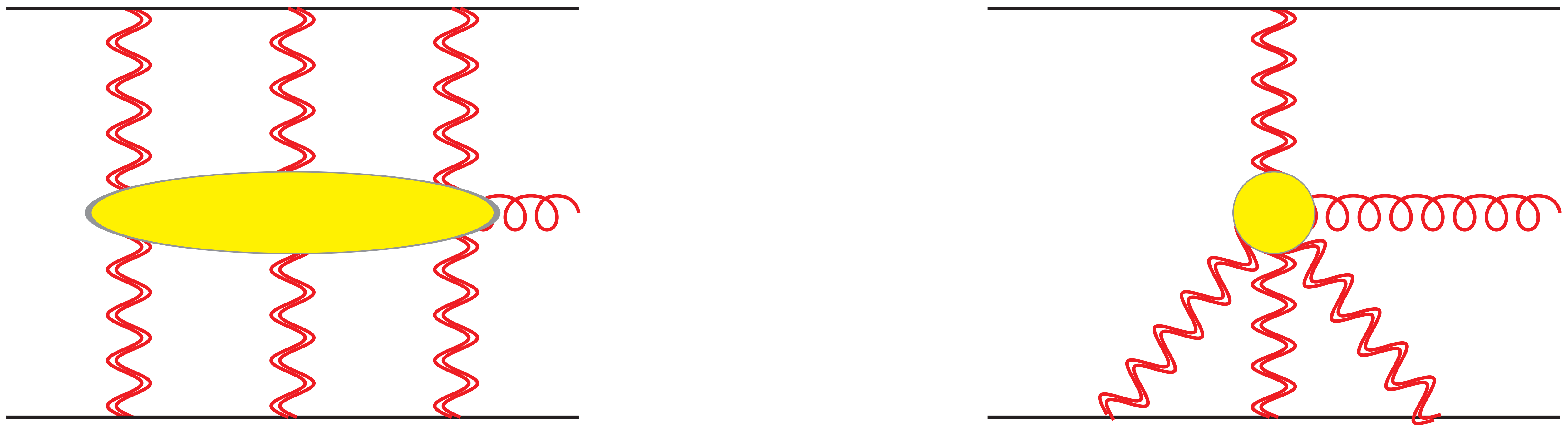


These break factorization!

Signature odd-odd 2 → 3 amplitude at two loops

- Two-loop contributions of odd-odd signature: $\mathcal{M}^{(-,-)\,(2)}_{ij\to i'gj'} = \mathcal{M}^{(2)}_{\mathcal{R}g\mathcal{R}} + \mathcal{M}^{(2)}_{\mathcal{R}^3g\mathcal{R}^3} + \mathcal{M}^{(2)}_{\mathcal{R}g\mathcal{R}^3} + \mathcal{M}^{(2)}_{\mathcal{R}^3g\mathcal{R}}$





(a) (b)

$$
\begin{aligned}
C_{\mathcal{R}^3g\mathcal{R}^3} &= \mathbf{T}_i^{\{a,b,c\}}\, i f^{ca_4d}\, \mathbf{T}_j^{\{a,b,d\}} \\
&= \frac{1}{144}\left\{9\mathbf{T}^2_{(--)} + \mathbf{T}^2_{(++)} + 4N_c\mathbf{T}_{(++)} + 3\left(\mathbf{T}^2_{(-+)} + \mathbf{T}^2_{(+-)}\right)\right\}\mathcal{C}^{(0)}_{ij} \\
&= \begin{cases} \frac{1}{72}\left(N_c^2 - 6 + \frac{18}{N_c^2}\right) c^{[8,8]_a} & \text{for } qq \\ \frac{1}{72}\left(N_c^2+6\right) c^{[8,8_a]_a} & \text{for } qg \\ \frac{1}{72}\left(N_c^2+36\right) c^{[8_a,8_a]} - \frac{1}{4}\sqrt{N_c^2-4}\, c^{[10,\overline{10}]_1} & \text{for } gg \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
C_{\mathcal{R}g\mathcal{R}^3} &= \mathbf{T}_i^b f^{bck} f^{kge} f^{eda_4}\mathbf{T}_j^{\{c,d,g\}} \\
&= \frac{1}{24}\left(2N_c\,\mathbf{T}_{(++)} + 2(\mathbf{T}_{(++)})^2 + 6(\mathbf{T}_{(-+)})^2\right)\mathcal{C}^{(0)}_{ij} \\
&= \begin{cases} \left(\frac{N_c^2}{24} + \frac{3}{2}\right) c^{[8_a,8_a]} - \frac{3\sqrt{N_c^2-4}}{4\sqrt{2}} c^{[10+10,8_a]} & \text{for } gg \\ \left(\frac{N_c^2}{24} + \frac{1}{4}\right) c^{[8,8]_a} & \text{for } qq \\ \left(\frac{N_c^2}{24} + \frac{1}{4}\right) c^{[8,8_a]_a} & \text{for } qg \\ \left(\frac{N_c^2}{24} + \frac{3}{2}\right) c^{[8,8_a]_a} - \frac{3\sqrt{N_c^2-4}}{4\sqrt{2}} c^{[8,10+10]} & \text{for } gq \end{cases}
\end{aligned}
$$

- These affects the [8,8] colour channel
- Their large-Nc limit is universal

Factorizable and non-factorizable contributions in 2 → 3 amplitudes

- The [8,8] odd-odd component of the Multi-Reggeon (MR) amplitude, splits into Regge-factorizable (planar) terms and non-factorizable terms

$$\mathcal{M}_{\rm MR}^{(2),\,[8,8]} = \frac{(i\pi)^2}{72}\left(\frac{\mu^2}{|\mathbf{p}_4|^2}\right)^{2\epsilon}\mathcal{M}^{(0),\,[8,8]} \times \begin{cases} (N_c^2+36)F_{\rm fact}(z,\bar z) & \text{for } gg \\ N_c^2 F_{\rm fact}(z,\bar z) + F^{qq}_{\text{non-fact}}(z,\bar z) & \text{for } qq \\ N_c^2 F_{\rm fact}(z,\bar z) + F^{qg}_{\text{non-fact}}(z,\bar z) & \text{for } qg \end{cases}$$

$$F_{\rm fact}(z,\bar z) = \frac{1}{\epsilon^2} - \frac{1}{2\epsilon}\log|z|^2|1-z|^2 + 3\,D_2(z,\bar z) - \zeta_2 + \frac{5}{4}\log^2|z|^2 + \frac{5}{4}\log^2|1-z|^2 - \frac{1}{2}\log|z|^2\log|1-z|^2$$

$D_2(z,\bar z)$: Block-Wigner Dilogarithm

$$F^{qq}_{\text{non-fact}} = \frac{9}{\epsilon}\log|z|^2|1-z|^2 + \frac{9}{2}\Big(12D_2(z,\bar z) - \log^2|z|^2 - 2\log|z|^2\log|1-z|^2 - \log^2|1-z|^2\Big)$$
$$+\frac{3}{N_c^2}\left(\frac{3}{\epsilon^2} - \frac{6}{\epsilon}\log|z|^2|1-z|^2 - 18D_2(z,\bar z) + 6\log^2|z|^2 + 3\log|z|^2\log|1-z|^2 + 6\log^2|1-z|^2 - \frac{\pi^2}{2}\right)$$

$$F^{qg}_{\text{non-fact}} = \frac{27}{2\epsilon^2} - \frac{9}{\epsilon}\Big(2\log|z|^2 - 3\log|1-z|^2\Big) + \frac{9}{4}\Big(48D_2(z,\bar z) + 10\log^2|z|^2 - 8\log|z|^2\log|1-z|^2 - \pi^2\Big)$$

Abreu, Falcioni, EG, Milloy and Vernazza — *PoS* LL2024 (2024) 085

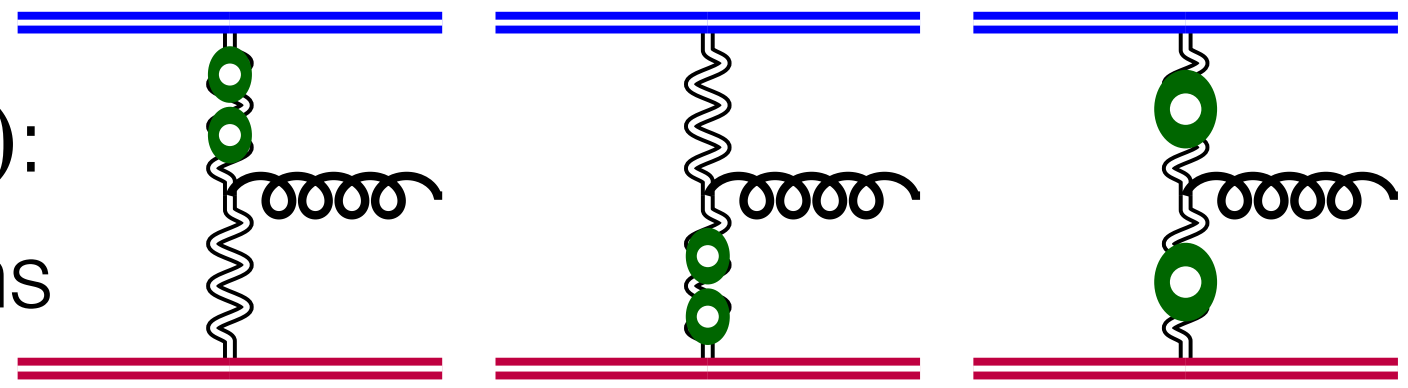
Buccioni, Caola, Devoto, Gambuti — 2411.14050 [hep-ph]

Abreu, De Laurentis, Falcioni, EG, Milloy and Vernazza — 2412.20578 [hep-ph]

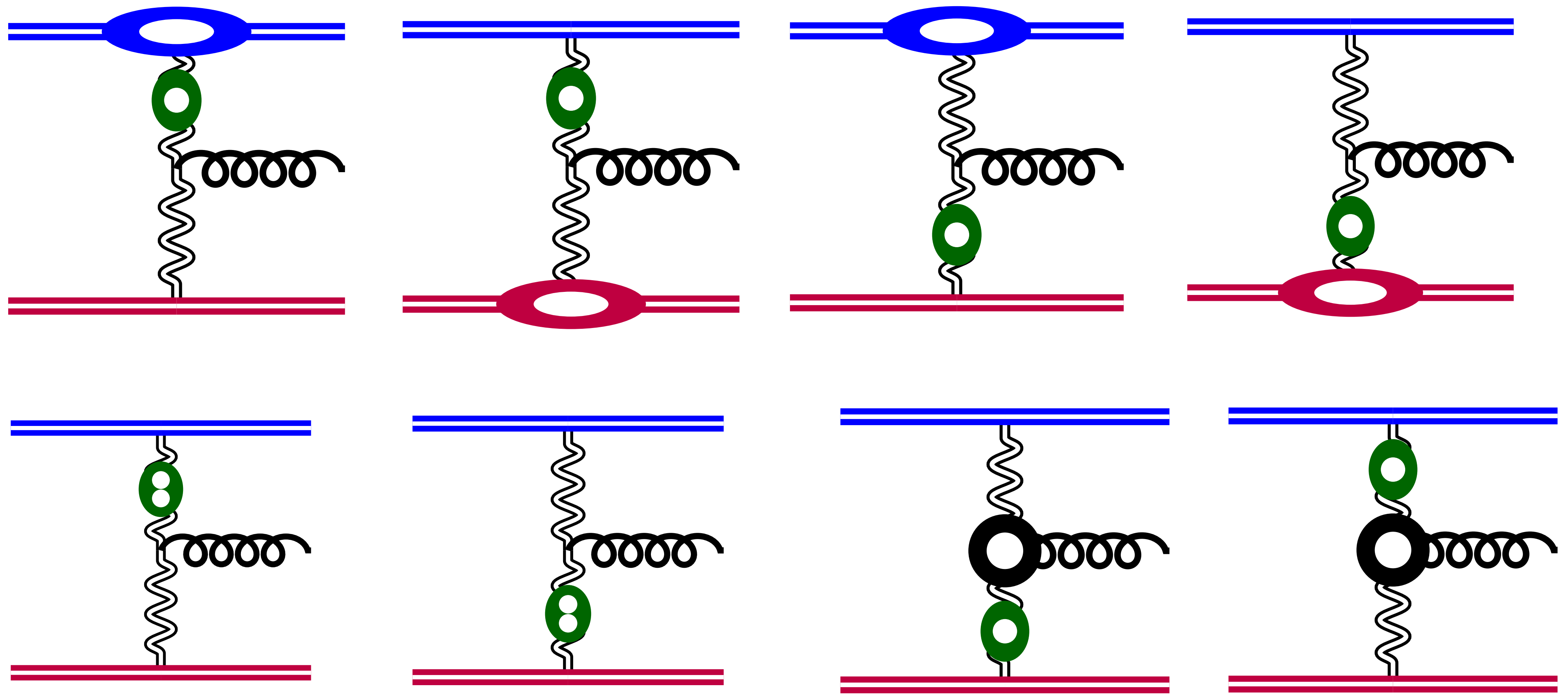
Extracting the Lipatov vertex from 2 → 3 (− , −) signature amplitudes

Two-loop amplitudes are available since last year: G. De Laurentis, H. Ita , M. Klinkert, V. Sotnikov 2311.10086, 2311.18752,
B. Agarwal, F. Buccioni, F. Devoto, G. Gambuti, A. von Manteuffel, L. Tancredi, 2311.09870

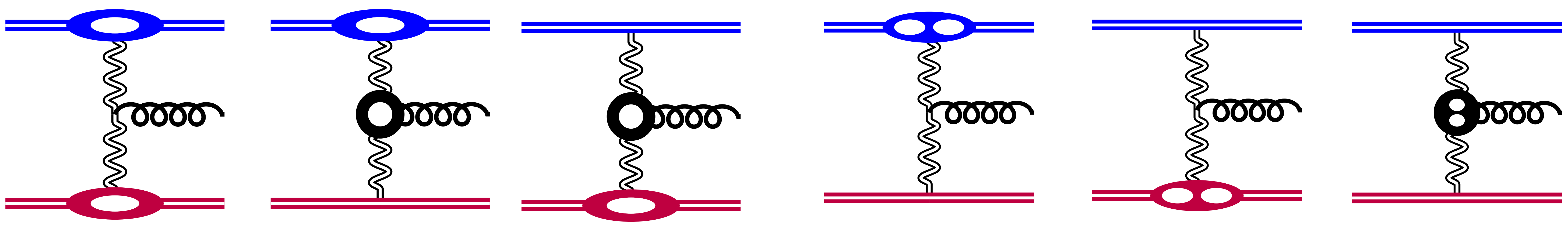
Two-loop (− , −):
Leading Logarithms

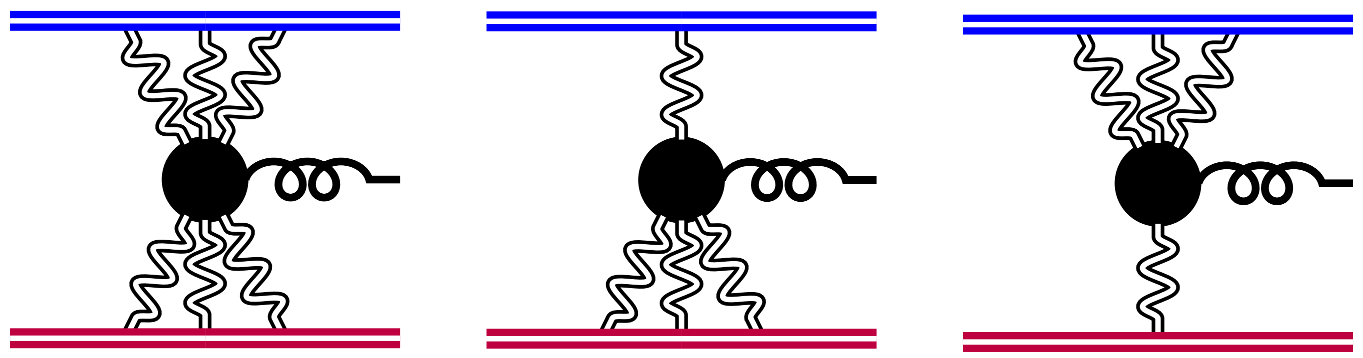


Two-loop
(− , −):
NLL



Two-loop (− , −): Next-to-next-to Leading Logarithms





Multi-Reggeon [8,8] contributions

Robust check: we obtained the (same) expression for the 2-loop QCD Lipatov Vertex from all three partonic channels!

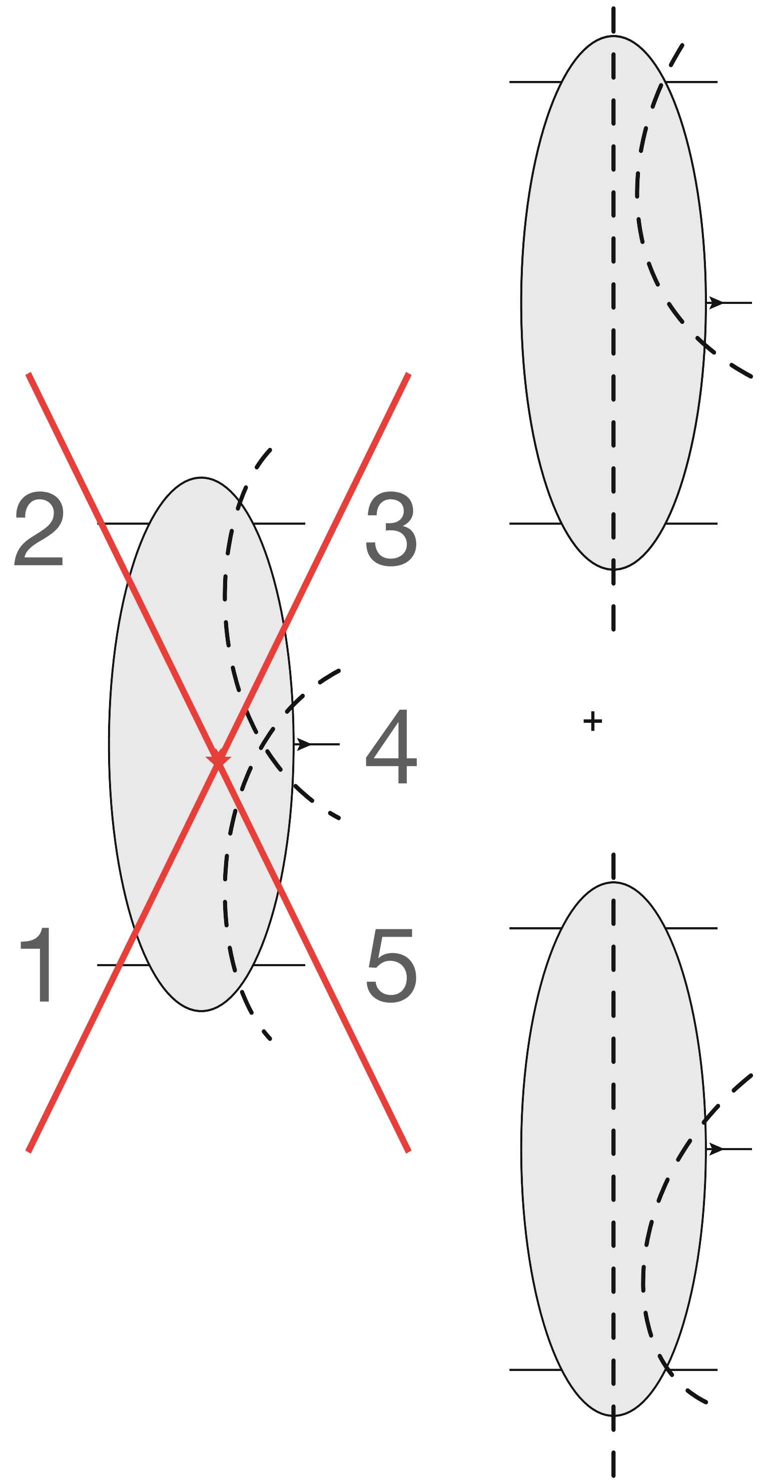
Abreu, Falcioni, EG, Milloy and Vernazza — *PoS* LL2024 (2024) 085

Buccioni, Caola, Devoto, Gambuti 2411.14050 [hep-ph]; Abreu, De Laurentis, Falcioni, EG, Milloy and Vernazza: 2412.20578

Odd-Odd 2 → 3 amplitude: discontinuity structure

- Steinmann relations forbid unitarity cuts in partially overlapping channels.

- Allowed iterated discontinuities: s_{12} and s_{45} or s_{12} and s_{34}
  compatible with the signature



- All-order factorization formula for 2 → 3 amplitudes in Multi-Regge kinematics in terms of two real-valued vertex functions

$$\frac{\mathcal{M}^{(-,-)}_{ij\to i'gj'}\Big|^{\text{1-Reggeon}}}{\mathcal{M}^{\text{tree}}_{ij\to i'gj'}} = c_i(t_1,\tau)\frac{1}{4}\Bigg\{\left[\left(\frac{s_{34}}{\tau}\right)^{\omega_2-\omega_1}+\left(\frac{-s_{34}}{\tau}\right)^{\omega_2-\omega_1}\right]\left[\left(\frac{s}{\tau}\right)^{\omega_1}+\left(\frac{-s}{\tau}\right)^{\omega_1}\right]v_R(t_1,t_2,|\mathbf{p}_4|^2,\tau)+$$
$$\left[\left(\frac{s_{45}}{\tau}\right)^{\omega_1-\omega_2}+\left(\frac{-s_{45}}{\tau}\right)^{\omega_1-\omega_2}\right]\left[\left(\frac{s}{\tau}\right)^{\omega_2}+\left(\frac{-s}{\tau}\right)^{\omega_2}\right]v_L(t_1,t_2,|\mathbf{p}_4|^2,\tau)\Bigg\}c_j(t_2,\tau)$$

[Bartels (1980), Fadin and Lipatov (1993),…, Fadin, Fucilla, Papa (2023)]

Odd-Odd 2 → 3 amplitude

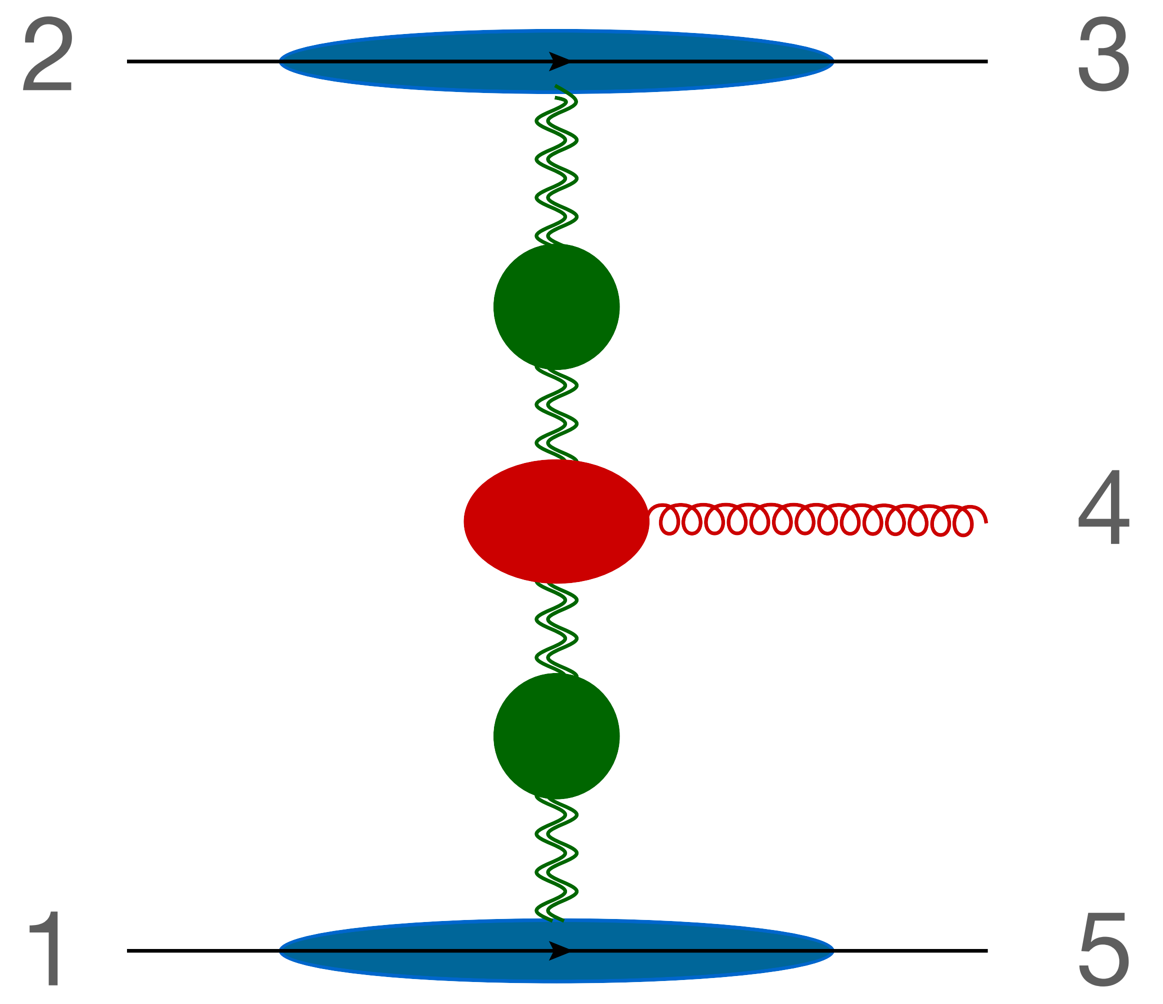
- All-order factorization formula for 2 → 3 amplitudes in Multi-Regge kinematics in terms of two real-valued vertex functions v_R, v_L

$$\omega_1 = C_A \alpha_g(t_1), \qquad \omega_2 = C_A \alpha_g(t_2)$$

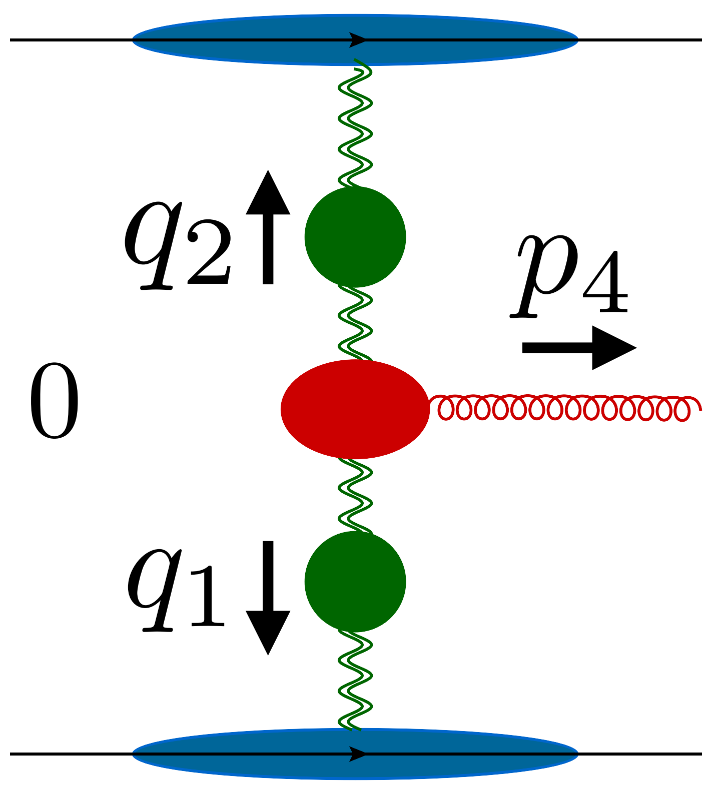
$$\frac{\mathcal{M}^{(-,-)}_{ij\to i'gj'}\Big|^{\text{1-Reggeon}}}{\mathcal{M}^{\text{tree}}_{ij\to i'gj'}} = c_i(t_1,\tau)\frac{1}{4}\Bigg\{\left[\left(\frac{s_{34}}{\tau}\right)^{\omega_2-\omega_1} + \left(\frac{-s_{34}}{\tau}\right)^{\omega_2-\omega_1}\right]\left[\left(\frac{s}{\tau}\right)^{\omega_1} + \left(\frac{-s}{\tau}\right)^{\omega_1}\right] v_R(t_1,t_2,|\mathbf{p}_4|^2,\tau) + \left[\left(\frac{s_{45}}{\tau}\right)^{\omega_1-\omega_2} + \left(\frac{-s_{45}}{\tau}\right)^{\omega_1-\omega_2}\right]\left[\left(\frac{s}{\tau}\right)^{\omega_2} + \left(\frac{-s}{\tau}\right)^{\omega_2}\right] v_L(t_1,t_2,|\mathbf{p}_4|^2,\tau)\Bigg\} c_j(t_2,\tau)$$

- **Equivalently**: a single complex-valued vertex rapidity variables absorb a phase: $\eta_1 = \log\frac{s_{45}}{\tau} - \frac{i\pi}{4}, \qquad \eta_2 = \log\frac{s_{34}}{\tau} - \frac{i\pi}{4}$

$$\frac{\mathcal{M}^{(-,-)}_{ij\to i'gj'}\Big|^{\text{1-Reggeon}}}{\mathcal{M}^{\text{tree}}_{ij\to i'gj'}} = c_i(t_1,\tau)\, e^{\omega_1\,\eta_1}\, v(t_1,t_2,\mathbf{p}_4^2,\tau)\, e^{\omega_2\,\eta_2}\, c_j(t_2,\tau)$$



Complex-valued vertex: properties

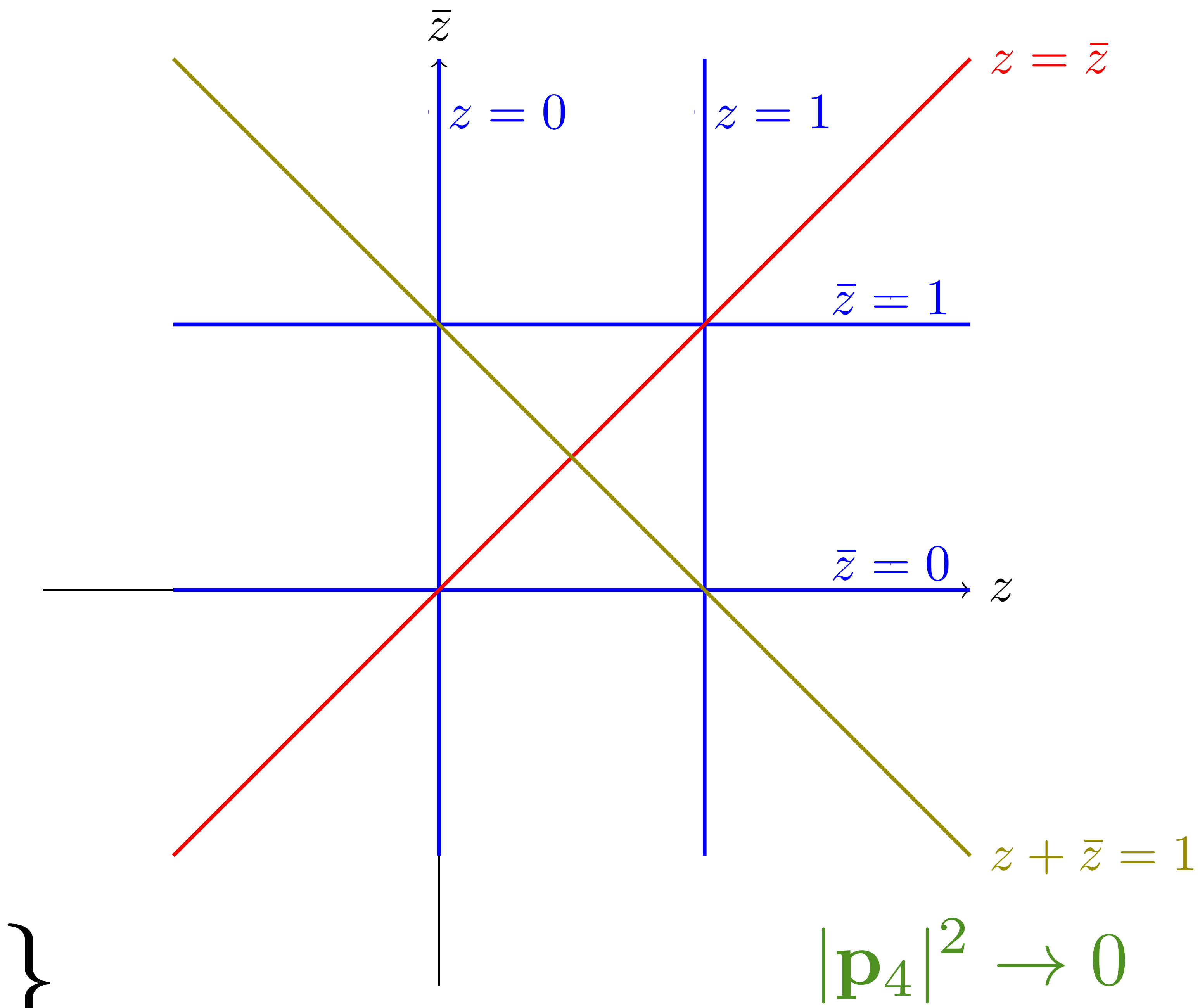


$$v(t_1, t_2, |\mathbf{p}_4|^2, \tau) = \frac{\mathcal{M}^{(-,-)}_{ij\to i'gj'}\Big|^{\text{1-Reggeon}}}{c_i(t_1,\tau)\, e^{\omega_1\,\eta_1}\, e^{\omega_2\,\eta_2}\, c_j(t_2,\tau)\mathcal{M}^{\text{tree}}_{ij\to i'gj'}}$$

- 2-dim momenta: $$\frac{-t_1}{|\mathbf{p}_4|^2} = (1-z)(1-\bar{z}), \qquad \frac{-t_2}{|\mathbf{p}_4|^2} = z\bar{z}$$

$$v(t_1, t_2, |\mathbf{p}_4|^2, \tau) = v(z, \bar{z}) \left(\frac{\tau}{|\mathbf{p}_4|^2}\right)^{\frac{1}{2}(\omega_1+\omega_2)}$$

$$\frac{(\mathbf{q}_1\cdot\mathbf{q}_2)^2 - |\mathbf{q}_1|^2|\mathbf{q}_2|^2}{|\mathbf{p}_4|^2/4} = (z-\bar{z})^2 \to 0$$

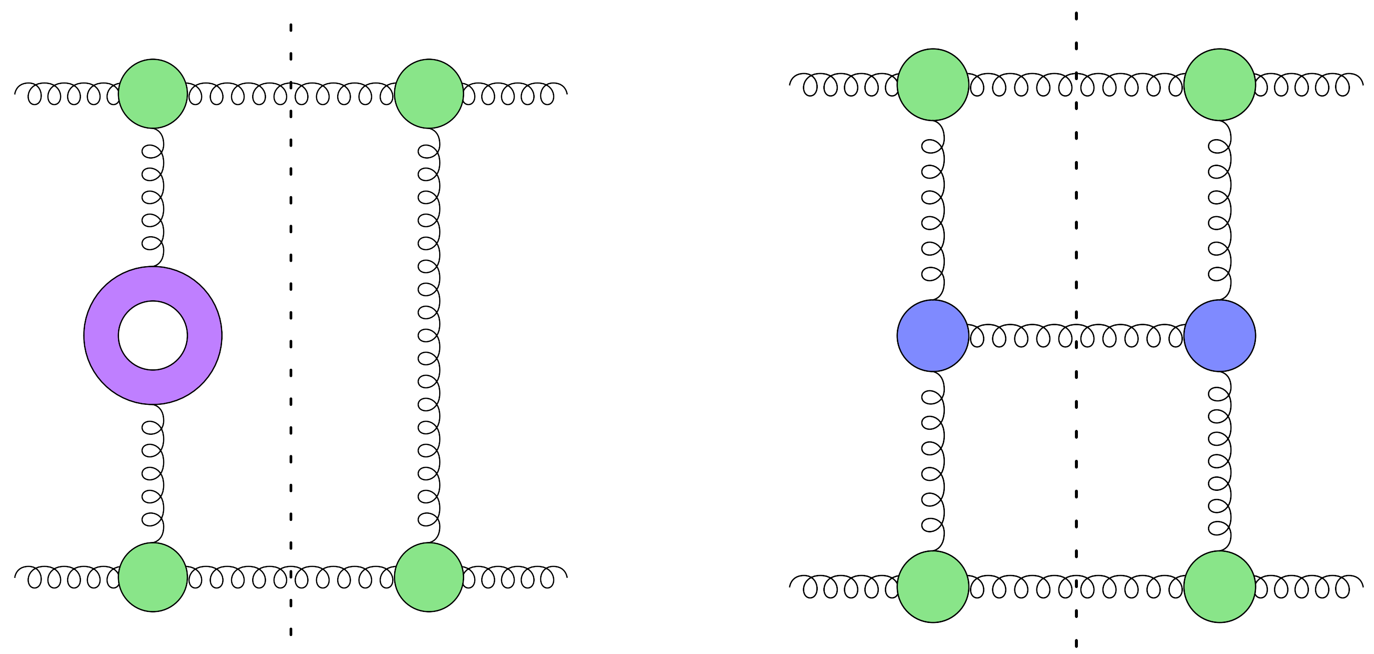


$|\mathbf{p}_4|^2 \to 0$

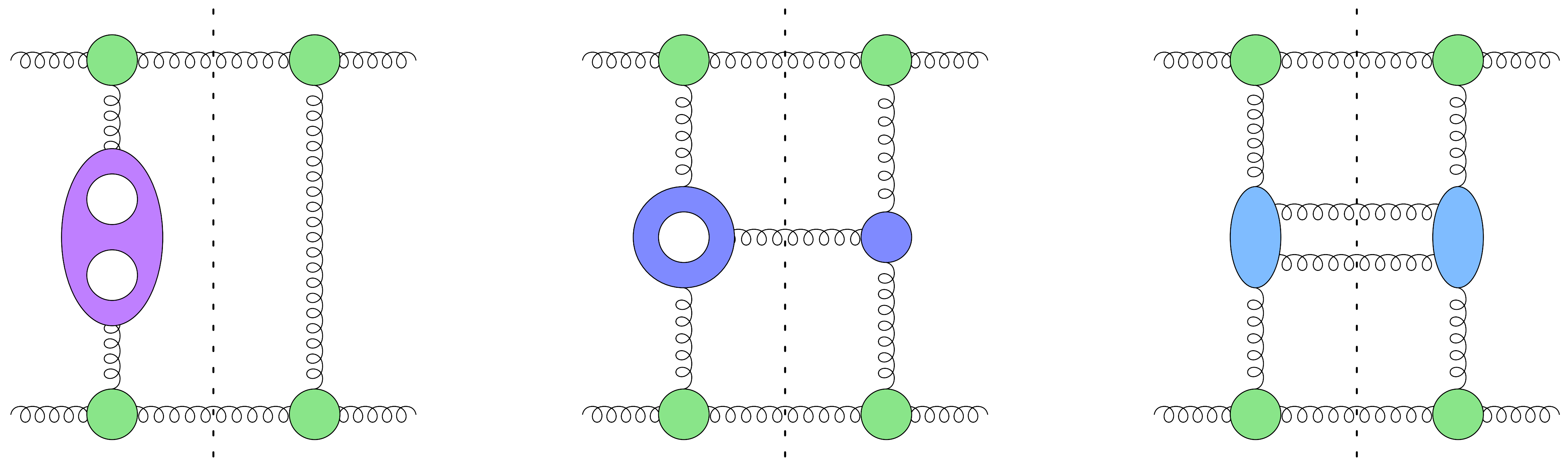
- Absence of discontinuities in physical kinematics ($z = \bar{z}^*$) implies that the transcendental functions $f(z, \bar{z})$ in the complex vertex should be **Single-Valued GPLs**
- The **reality** of v_L and v_R and **target-projectile symmetry** imply definite $z \leftrightarrow \bar{z}$ and $z \leftrightarrow 1 - z$ symmetry properties
- Symbol alphabet: $\{z,\ \bar{z},\ 1-z,\ 1-\bar{z},\ z-\bar{z},\ 1-z-\bar{z}\}$
- Rational factors have <u>spurious</u> singularities on the line $z + \bar{z} = 1$

Contributions to BFKL kernel at increasing logarithmic accuracy

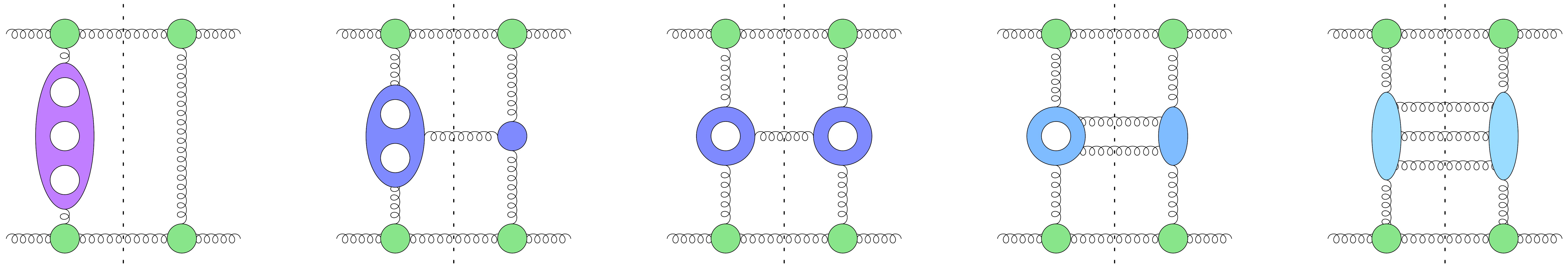
Leading
Order



Next to
Leading Order



Next to next to
Leading Order



Multiple parton Central Emission Vertices at : 2-gluon CeV in sYM: Byrne, Del Duca, Dixon, EG, Smillie 2204.12459
and on-going work with De Laurentis, Byrne, Del Duca, Mo, Smillie

Conclusions

Sufficient evidence to say that the Reggon field has been identified:

(1) The shock-wave formalism and rapidity evolution equations

facilitate efficient computation of multiple-Reggeon interactions in the (multi) Regge limit:

NLL for signature even $2 \to 2$ amplitudes (all orders)

NNLL for signature odd $2 \to 2$ amplitudes (so far to four loops)

NNLL for signature odd-odd $2 \to 3$ amplitudes (so far to two loops)

(2) **Regge-pole factorization violations in $2 \to 2$ and $2 \to 3$ amplitudes - Regge cut contributions - are non-planar**

(3) Based on (1), (2) and 3-loop 4-point calculations **we now know all Regge-pole parameters to NNLO.**

(4) Based on (1), (2) and (3) and recent 2-loop 5-point calculations **we determined the 2-loop Lipatov vertex in QCD.**

This vertex is one of the building blocks of **NNLO BFKL Kernel**. The remaining ones will be available soon.

Great prospects to further exploiting the synergy between effective computations in the Regge limit and the calculation and study of multi-leg amplitudes