Theoretical minimum emittance in storage rings

Reaching ultra-low emittance with variable bending magnets

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Advanced Photon Source

ICFA Beam Dynamics Mini Workshop on Low Emittance Rings 2011, Heraklion, Greece, Oct. 2011

Natural horizontal emittance

$$\epsilon = \frac{C_q \gamma^2}{J_x} \mathcal{F}$$

betatron emittance for uncoupled lattices

$$\mathcal{F} = \frac{\langle \mathcal{H}/|\rho|^3 \rangle}{\langle 1/\rho^2 \rangle} \equiv \langle\!\langle \mathcal{H} \rangle\!\rangle$$

depending on linear lattice design

Dispersion action $\mathcal{H} = \gamma \eta^2 + 2\alpha \eta \eta' + \beta \eta'^2$

For isomagnetic rings with conventional dipoles of bending angle θ:

$$\mathcal{F}^{\min} = \theta^3 / 12\sqrt{15}$$

a remarkable fact known since 1981

"Theoretical minimum emittance in storage rings", C.-x. Wang, presented at ICFA Beam Dynamics Mini Workshop on Low Emittance Rings, Heraklion, Greece, 3-5 Oct. 2011

Lower theoretical minimum emittance

with non-uniform (variable) bending magnets

$$\mathcal{F} = \frac{\langle \mathcal{H}/|\rho|^3 \rangle}{\langle 1/\rho^2 \rangle} \equiv \langle\!\langle \mathcal{H} \rangle\!\rangle$$

constant ρ is not optimal => optimizing $\rho(s)$

A. Wrulich, (PAC99)
Future directions in the storage ring development for light sources.

J. Guo and T. Raubenheimer, (EPAC02)
Low emittance E-/E+ storage ring design using bending magnets with longitudinal gradient.

Y. Papaphilippou, P. Elleaume, (PAC05)
Analytical considerations for reducing the effective emittance with variable dipole field strengths.

R. Nagaoka, A.F. Wrulich, (NIM A575, 2007)
Emittance minimisation with longitudinal dipole field variation.

A. Streun, (2007)
Minimum emittance superbend lattices? PSI Internal Report SLS-TME-TA-2006-0297.

C.-x. Wang et. al., (PR ST-AB 2009, 2011)

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Lower theoretical minimum emittance
with non-uniform (variable) bending magnets

$$\mathcal{F} = \frac{\langle \mathcal{H}/|\rho|^3 \rangle}{\langle 1/\rho^2 \rangle} \equiv \langle\!\langle \mathcal{H} \rangle\!\rangle$$

constant ρ is not optimal => optimizing ρ(s)

Progress made:

1. Minimize *F* for a given ρ(s), i.e., optimizing lattices => elegant minimum emittance theory
2. Numerical search for the optimal bending profile ρ(s) that minimizes F_{min}
3. Analytical calculation of the theoretical minimum emittance with optimal ρ(s)

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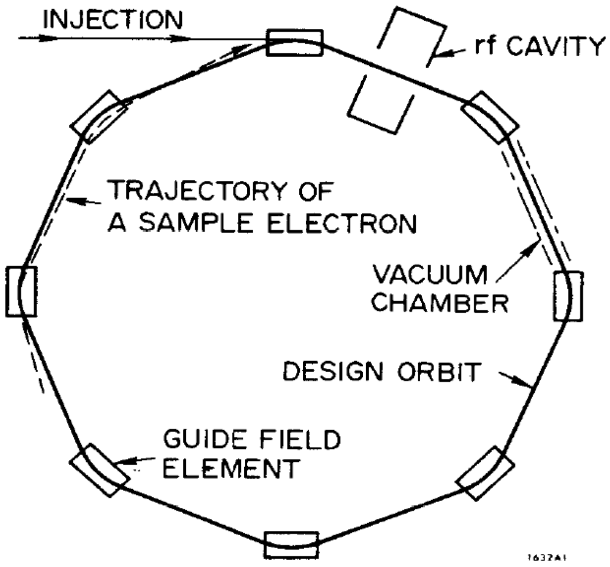
Common lattice types



1. **TME** --- Theoretical Minimum Emittance lattices

   no lattice constraints, figure of merit for damping rings

2. **AME** --- Achromatic Minimum Emittance lattices

   require achromatic arcs for dispersion-free straight section, injection, rf, etc.

3. **EME** --- Effective Minimum Emittance lattices

   no lattice constraints, but minimize the effective emittance for light sources

$$\epsilon_{\text{eff}}^2 \equiv \langle (x+\eta\delta)^2\rangle\langle(x'+\eta'\delta)^2\rangle - \langle (x+\eta\delta)(x'+\eta'\delta)\rangle^2$$
$$= \epsilon_x(\epsilon_x + \mathcal{H}_{\text{ID}}\,\sigma_\delta^2)$$

$$\boxed{\sigma_\delta^2 = C_q\gamma^2\frac{\langle 1/|\rho|^3\rangle}{J_E\langle 1/\rho^2\rangle}}$$

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Minimum emittance theory (rewriting $\mathcal{H}$)

$$\begin{aligned}
\mathcal{H}(s) &= \gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2 \\
&= [\eta, \eta'] \begin{bmatrix} \gamma & \alpha \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} \eta \\ \eta' \end{bmatrix} \\
&= \boldsymbol{\eta}^T \sigma^+ \boldsymbol{\eta} = \mathrm{Tr}\left(\boldsymbol{\eta}\boldsymbol{\eta}^T \sigma^+\right) \\
&= \mathrm{Tr}\left\{[\boldsymbol{\eta}_0 + \hat{\boldsymbol{\xi}}_0(s)][\boldsymbol{\eta}_0 + \hat{\boldsymbol{\xi}}_0(s)]^T \sigma_0^+\right\} \\
&= \mathrm{Tr}\left\{\left(\boldsymbol{\eta}_0\boldsymbol{\eta}_0^T + \boldsymbol{\eta}_0\hat{\boldsymbol{\xi}}_0^T + \hat{\boldsymbol{\xi}}_0\boldsymbol{\eta}_0^T + \hat{\boldsymbol{\xi}}_0\hat{\boldsymbol{\xi}}_0^T\right)\sigma_0^+\right\}
\end{aligned}$$

- a quadratic form
- in matrix form
- using trace
- using an arbitrary reference

allow integration passing through

factor out initial lattice parameters

all dipole effects are contained in the projected dispersion generating vector $\hat{\boldsymbol{\xi}}_0(s)$

$$\boldsymbol{\eta}(s) = M(s)\left[\boldsymbol{\eta}_0 + \hat{\boldsymbol{\xi}}_0(s)\right] = M(s)\boldsymbol{\eta}_0 + \boldsymbol{\xi}(s)$$

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Minimum emittance theory (emittance formula)

$$\mathcal{F} = \frac{\langle \mathcal{H}/|\rho|^3 \rangle}{\langle 1/\rho^2 \rangle} \equiv \langle\!\langle \mathcal{H} \rangle\!\rangle = \mathrm{Tr}\left(G_0\, \sigma_0^+\right)$$

$$\begin{aligned} G_0 &= \check{\rho}\, \boldsymbol{\eta}_0 \boldsymbol{\eta}_0^T + \boldsymbol{\eta}_0 \langle\!\langle \hat{\boldsymbol{\xi}}_0 \rangle\!\rangle^T + \langle\!\langle \hat{\boldsymbol{\xi}}_0 \rangle\!\rangle \boldsymbol{\eta}_0^T + \langle\!\langle \hat{\boldsymbol{\xi}}_0 \hat{\boldsymbol{\xi}}_0^T \rangle\!\rangle \\ &= E + \check{\rho}(\boldsymbol{\eta} + \langle\!\langle \hat{\boldsymbol{\xi}} \rangle\!\rangle / \check{\rho})(\boldsymbol{\eta} + \langle\!\langle \hat{\boldsymbol{\xi}} \rangle\!\rangle / \check{\rho})^T \\ &= E + (q+1)^2 \boldsymbol{\zeta}\boldsymbol{\zeta}^T \end{aligned}$$

assuming $\boldsymbol{\eta}_0 = q\, \langle\!\langle \hat{\boldsymbol{\xi}}_0 \rangle\!\rangle / \check{\rho}$

referenced to the dipole entrance

$$E \equiv \langle\!\langle \hat{\boldsymbol{\xi}} \hat{\boldsymbol{\xi}}^T \rangle\!\rangle - \langle\!\langle \hat{\boldsymbol{\xi}} \rangle\!\rangle \langle\!\langle \hat{\boldsymbol{\xi}} \rangle\!\rangle^T / \check{\rho}\,, \qquad \boldsymbol{\zeta} \equiv \frac{\langle\!\langle \hat{\boldsymbol{\xi}} \rangle\!\rangle - \check{\rho}\, \hat{\boldsymbol{\xi}}(s_0)}{\sqrt{\check{\rho}}}$$

$$\check{\rho} \equiv \langle\!\langle 1 \rangle\!\rangle = \langle 1/|\rho|^3 \rangle / \langle 1/\rho^2 \rangle = I_3 / I_2$$

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Minimum emittance theory (minimization)

$$\mathcal{F} = \mathrm{Tr}\left(G_0\,\sigma_0^{+}\right) \geq 2\sqrt{|G_0|}, \quad \text{equality holds iff } \sigma_0 = \frac{G_0}{\sqrt{|G_0|}}$$

- Schur decomposition: $\mathcal{A}_0 G_0 \mathcal{A}_0^T = V \mathrm{diag}(\lambda_1, \lambda_2) V^{-1}$
- Inequality for positive reals: $\lambda_1 + \lambda_2 \geq 2\sqrt{\lambda_1 \lambda_2}$

$$\mathcal{A}^T \mathcal{A} = \sigma^{+}$$

$$\mathcal{A} = \begin{bmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{bmatrix}$$

$$G = E + (q+1)^2 \zeta\zeta^T$$

$$|G| = |E|[1 + (q+1)^2 a] \qquad a = \mathrm{Tr}(E^{-1}\zeta\zeta^T) > 0$$

AME: $q = 0$, TME: $q = -1$

$$\eta_0 = q \left\langle\!\left\langle \hat{\xi}_0 \right\rangle\!\right\rangle / \check{\rho}$$

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Minimum emittance theory (summary)

$$\mathcal{F}^{\min} = 2\sqrt{|E|}\begin{cases} 1 & \text{TME} \\ \sqrt{1+a} & \text{AME} \\ \sqrt{\dfrac{[1+(q+3)\,qc/2]\,[1+((1+\tau)q+3)\,qc/2]}{(1-c)(1+qc)}} & \text{EME} \end{cases}$$

$$a = \mathrm{Tr}(E^{-1}\zeta\zeta^T)\,, \qquad c = a/(a+1)\,, \quad \tau = J_x/J_E$$

|E|, a, and c are completely determined by the dipole

For uniform dipoles: $2\sqrt{|E|} = \theta^3/12\sqrt{15},\ a = 8,\ \text{and}\ c = 8/9$

$$(1+\tau)q^3 + 2(2+\tau)q^2 + [3 + (2+\tau)/c]q + 2/c = 0$$

This cubic equation determines the optimal dispersion for EME

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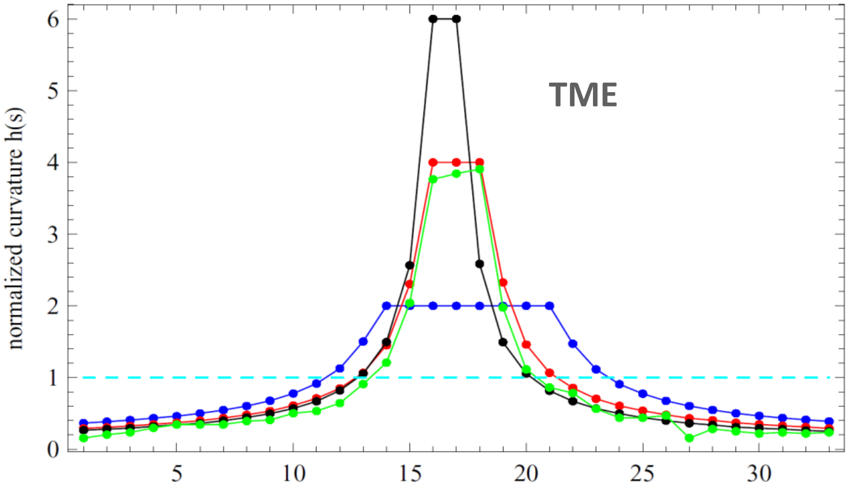
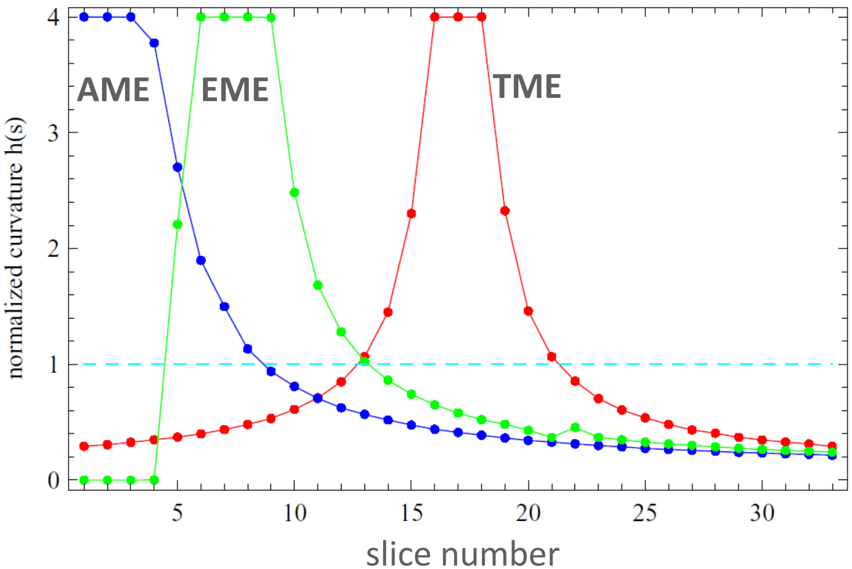
Optimal bending profile

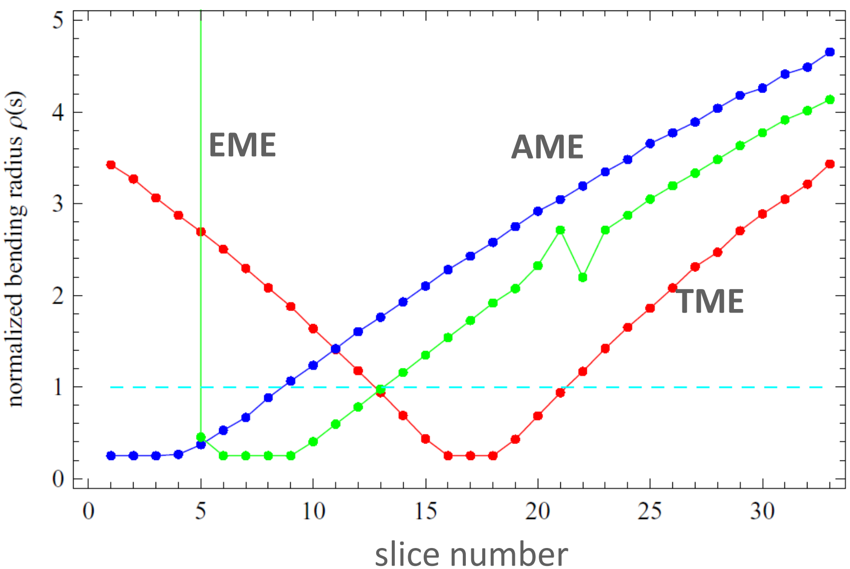


bending curvature (B field)

- dipole of a given length and bending angle is sliced into many slices
- each slice has arbitrary strength (up to a maximum, no polarity change)
- two different optimizers are used to optimize the emittance, etc.
- optimization was done for 3 different peak field (2, 4, and 6 times stronger)



bending radius profile



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Linearly-ramped bending profile

A model sufficiently close to the optimal, yet can be solved analytically

$$\rho(s) = \rho_0 \times \begin{cases} 1 & |s| \le L_0 \\ 1 + g\left(|s|/L_0 - 1\right) & L_0 \le |s| \le L \end{cases}$$

$$\text{where } g = (r-1)/(L/L_0 - 1) \text{ and } r \equiv \rho_{\max}/\rho_0$$

$$\kappa \equiv \frac{B_{\max}}{B_{\mathrm{ref}}} = \frac{\rho_{\mathrm{ref}}}{\rho_{\min}} = \frac{L}{L_0}\frac{\theta_0}{\theta_{\max}} = \frac{g+r-1}{g+\ln r}$$ is chosen as a given parameter

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Theoretical minimum emittance (TME)

f1 and f2 are functions of (g,r)

$$\hat{\mathcal{F}}^{\mathrm{TME}} = \frac{1}{r(g+\ln r)^3[(g+1)r-1]}\sqrt{\frac{f_1 f_2}{32[(2g+1)r^2-1]}}$$

F is normalized by the value of reference uniform dipole, i.e., $\theta^3/12\sqrt{15}$

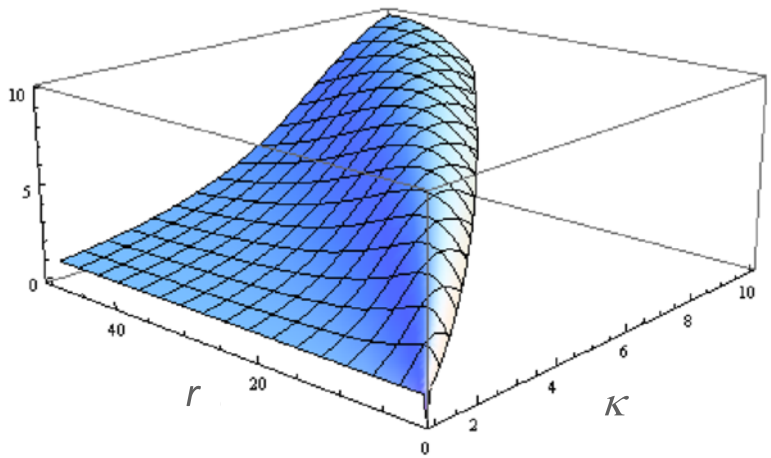
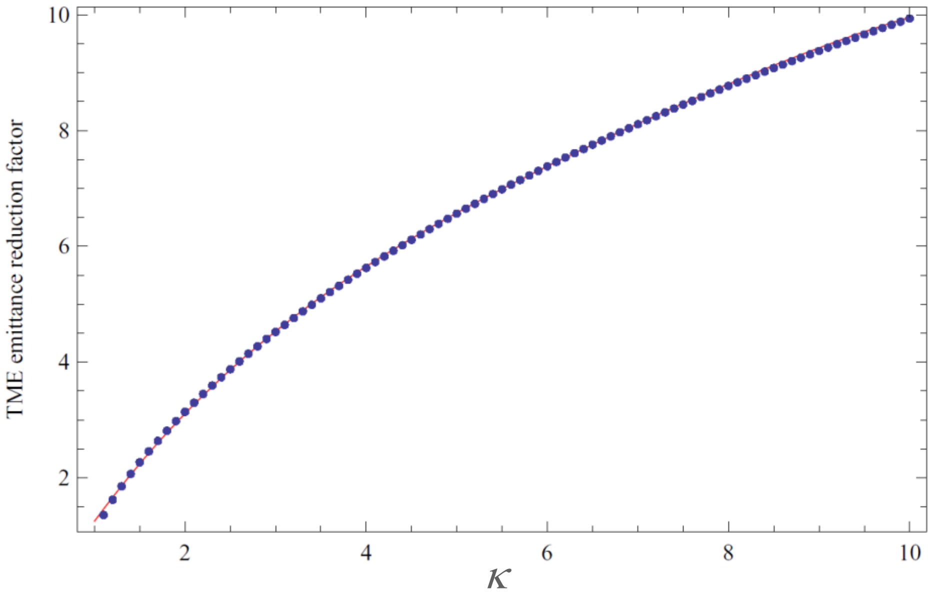


FIG. 1. TME emittance improvement factor $1/\hat{\mathcal{F}}^{\mathrm{TME}}$ as a function of $\kappa = B_{\max}/B_{\mathrm{ref}}$ and $r = \rho_{\max}/\rho_0$. The obvious ridge indicates the maximum emittance reduction.

improvement in TME emittance



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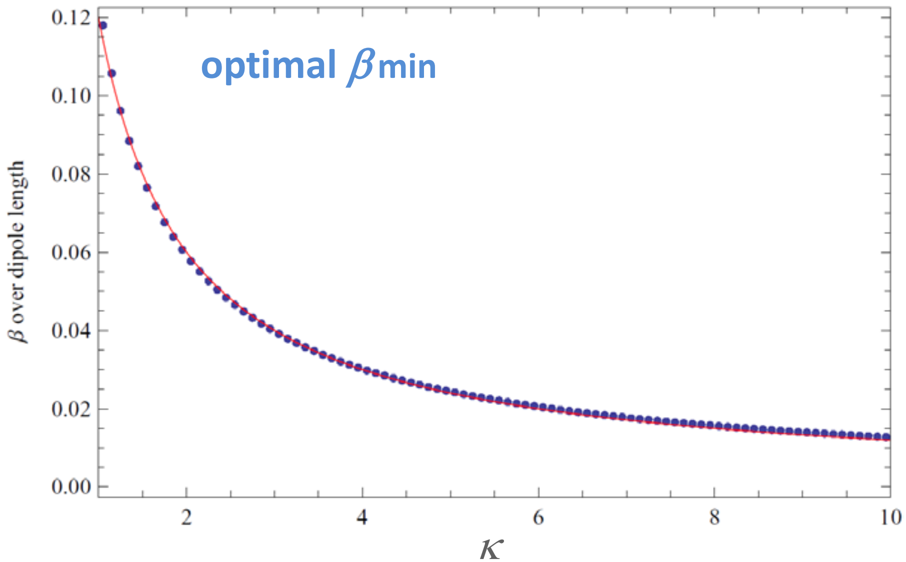
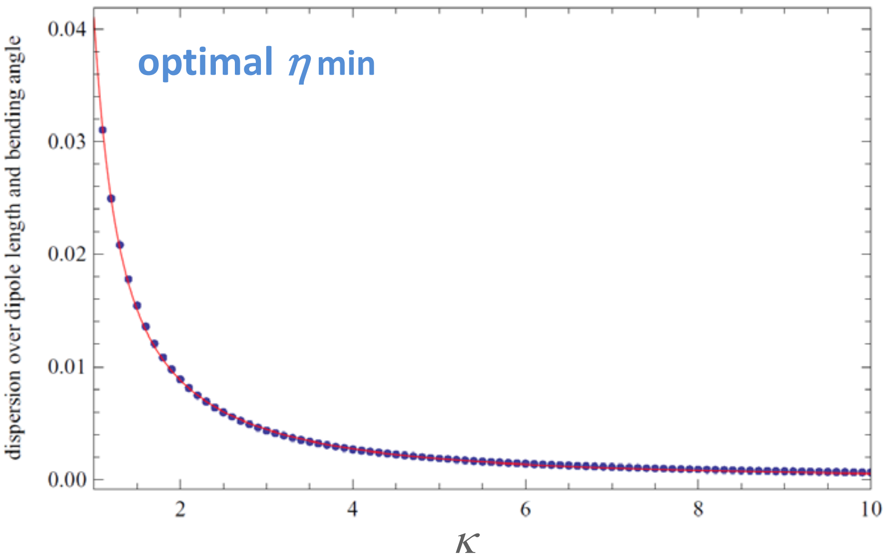
Optimal lattice parameters (TME)

$$\frac{\beta_c}{L_{\text{dipole}}} = \frac{1}{g+r-1}\sqrt{\frac{f_1/f_2}{120[(2g+1)r^2-1]}}$$

$$\frac{\eta_c}{L_{\text{dipole}}\theta} = \frac{2g^3+3[1+g+g^2-\frac{4}{r}+\frac{3-g-g^2+2(1-g)\ln r}{r^2}]}{24(g+\ln r)(g+r-1)(2g+1-1/r^2)}$$





small β challenges implementation of theoretical minimum emittance lattices

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Effects of non-optimal lattice

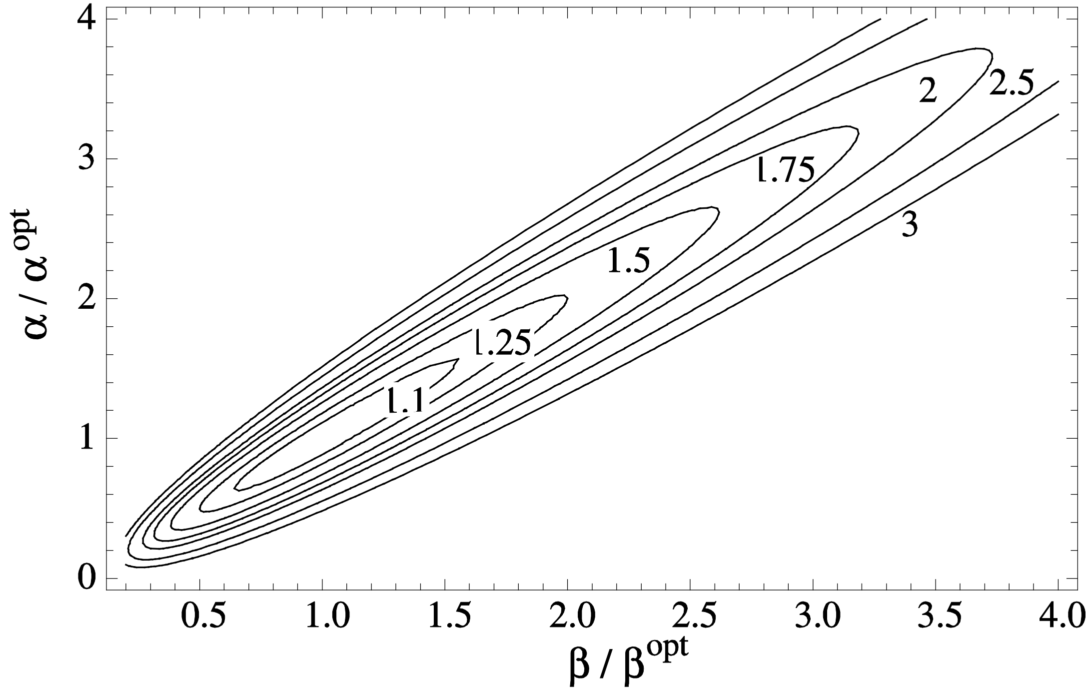
For non-optimal lattice:

$$\tilde{\beta} = \beta_0/\beta_0^{\mathrm{opt}}$$

$$\tilde{\alpha} = \alpha_0/\alpha_0^{\mathrm{opt}}$$

$$\frac{\langle\mathcal{H}\rangle}{\langle\mathcal{H}\rangle_{\min}} = \frac{1}{2}\mathrm{Tr}\left\{\sigma_0^{\mathrm{opt}}\sigma_0^{+}\right\} = \frac{1}{2}\mathrm{Tr}\left\{\left(\sigma_0^{\mathrm{opt}}\right)^{+}\sigma_0\right\}$$

$$= \boxed{\frac{1}{2}\left(\tilde{\beta} + \frac{1}{\tilde{\beta}}\right)} + \frac{1}{2}\left(\sqrt{\tilde{\beta}} - \frac{\tilde{\alpha}}{\sqrt{\tilde{\beta}}}\right)^2 (\alpha_0^{\mathrm{opt}})^2$$



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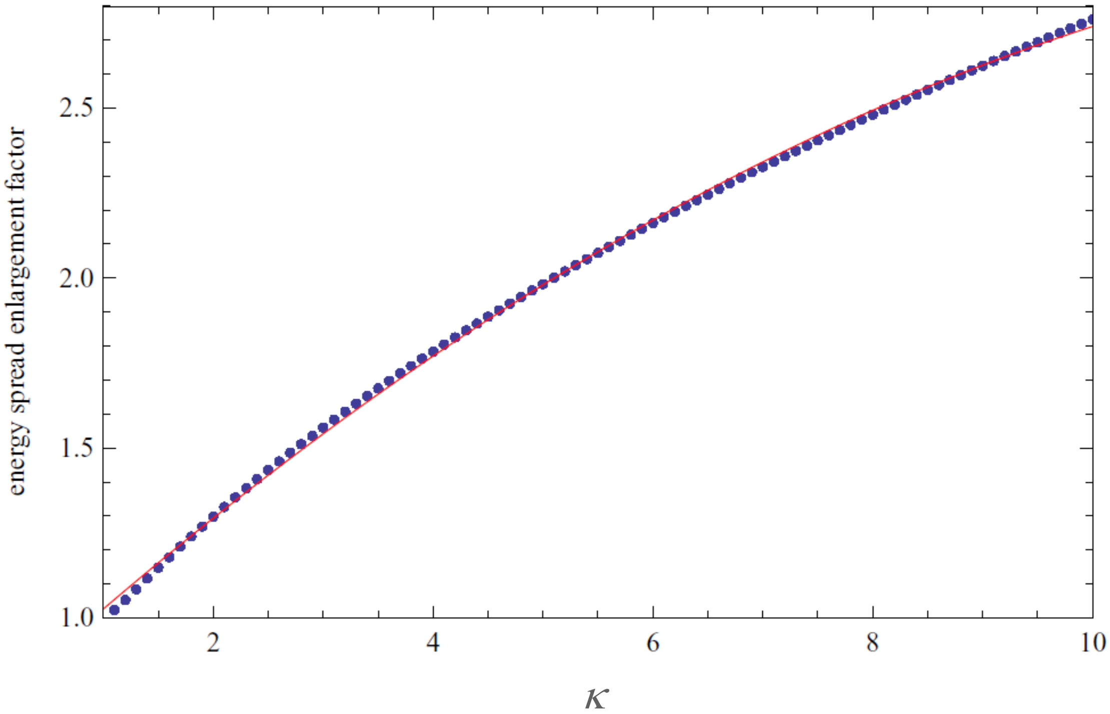
beam energy spread increase (TME)

$$\sqrt{\frac{\check{\rho}}{1/\rho_{\mathrm{ref}}}} = \sqrt{\frac{g+r-1}{g+\ln r}\,\frac{g+(1-1/r^2)/2}{g+1-1/r}}$$



beam energy spread gets worse, similar for TME, AME, EME

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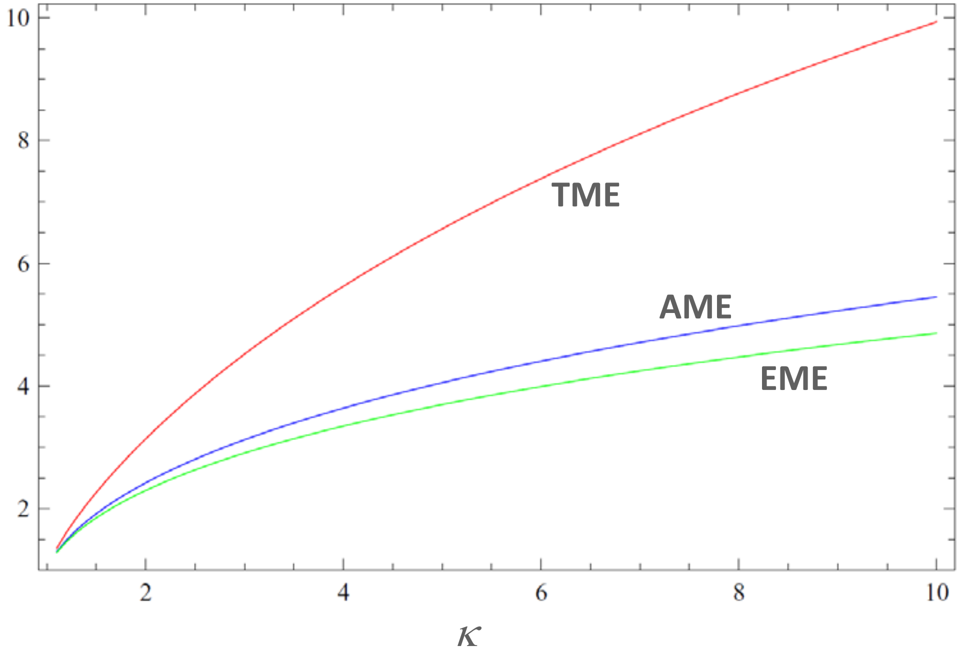
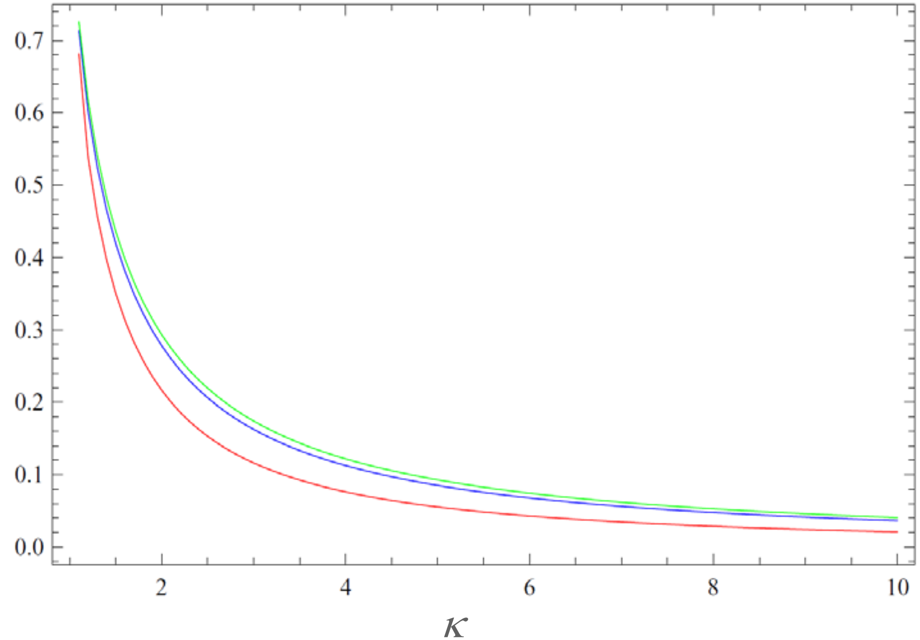
Theoretical minimum emittance (summary)

emittance improvement factor



length of constant field, *L0/L*



- significant emittance reduction is possible with variable bending
- more lattice designs to be explored, moderate gains won't be hard
- more effective for damping rings
- for light sources, multi-bend could benefit more (inner dipoles using TME)

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Transfer matrix for linear-ramp dipole

Solve the dispersion Eq: $$\frac{d^2 D}{ds^2} + \frac{1}{\rho(s)^2} D = \frac{1}{\rho(s)} \quad \text{with } \rho(s) = \rho_0 + \rho' s$$

$$M = \begin{bmatrix} \sqrt{\frac{\rho}{\rho_0}}(\cos u - \frac{1}{2\varpi}\sin u) & \frac{\sqrt{\rho_0 \rho}}{\varpi \rho'} \sin u \\ -\frac{\varpi \rho'}{\sqrt{\rho_0 \rho}}(1 + \frac{1}{4\varpi^2}) \sin u & \sqrt{\frac{\rho_0}{\rho}}(\cos u + \frac{1}{2\varpi}\sin u) \end{bmatrix}$$

the same matrix for rf cavity

$$\xi = \begin{bmatrix} \rho - \sqrt{\rho_0 \rho}(\cos u + \frac{1}{2\varpi}\sin u) \\ \rho' \left\{ 1 - \sqrt{\frac{\rho_0}{\rho}} \left[\cos u + (\frac{1}{4\varpi} - \varpi) \sin u \right] \right\} \end{bmatrix}$$

$$\text{where } u \equiv \varpi \ln \frac{\rho}{\rho_0} \text{ and } \varpi \equiv \sqrt{\frac{1}{\rho'^2} - \frac{1}{4}}$$

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Computation of projected dispersion-generating vector

$$\boldsymbol{\eta}' = JH\boldsymbol{\eta} + \begin{bmatrix} 0 \\ h \end{bmatrix}$$

$$M' = JHM$$

$$\boldsymbol{\eta} = M\left(\boldsymbol{\eta}_0 + \hat{\boldsymbol{\xi}}\right)$$

$$h(s) = 1/\rho(s)$$

$$H = \begin{bmatrix} h^2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{\eta}' = M'(\boldsymbol{\eta}_0 + \hat{\boldsymbol{\xi}}) + M\hat{\boldsymbol{\xi}}' = M'M^{-1}\boldsymbol{\eta} + M\hat{\boldsymbol{\xi}}'$$

$$\boxed{\hat{\boldsymbol{\xi}}' = M^{-1}\begin{bmatrix} 0 \\ h \end{bmatrix}, \qquad \boldsymbol{\xi}_0 = \mathbf{0}}$$

For weak focusing or small angle approximation, M is simply the drift matrix

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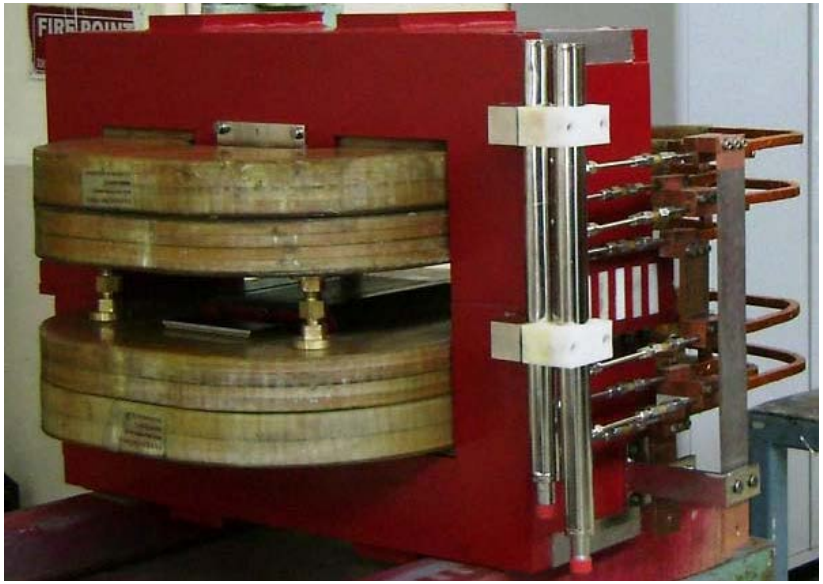
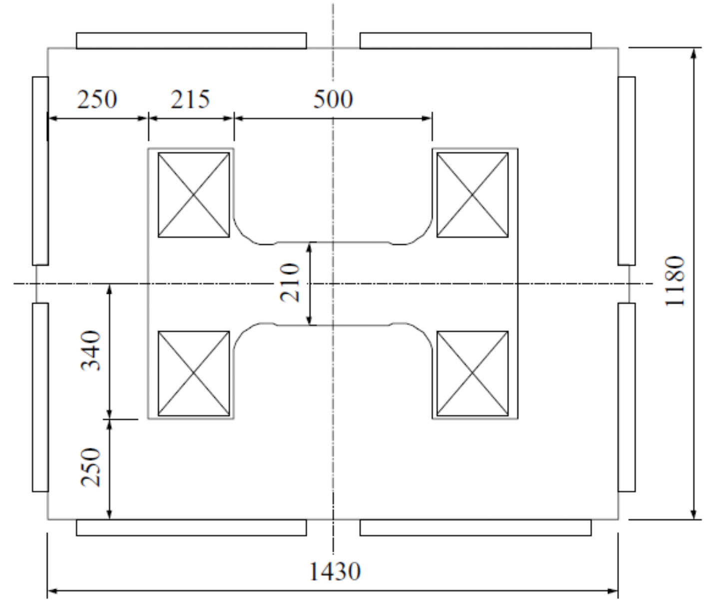
Example of dipole magent

By shaping the laminated core, it is relative easy to make a nonuniform dipole.





$$B_x = \frac{1}{6}(2\kappa b_0'' + \kappa' b_0')y^3$$

$$B_y = b_0 - \frac{1}{2}b_0'' y^2 + \frac{1}{2}(2\kappa b_0'' + \kappa' b_0')xy^2$$

$$B_s = b_0' y - \kappa b_0' xy + \kappa^2 b_0' x^2 y - \frac{1}{6}b_0''' y^3$$

orbit curvature $\kappa(s) = 1/\rho(s) = eb_0(s)/p_s$

dipole field strength

octupole-order nonlinearity strength $\int (2\kappa b_0'' + \kappa' b_0')ds = -(p_s/e)\int (1/\rho)'^2 ds$

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Brief history

❑ Early days

[M. Sands, SLAC-121, 1970]
[R.H. Helm, M.J. Lee, P.L. Morton, PAC73]

- M. Sommer, (Internal Report DCI/NI/20/81, 1981)
- D. Potaux, (Internal Report DCI/NI/30/81, 1981)
- Y. Kamiya, M. Kihara, (KEK report)
- H. Wiedemann, (ESRP report ESRP-IRM-71/84, 1984)

❑ Theory establishment and further development

- L. Teng, Minimum Emittance Lattice … (ANL LS-17, 1985; FNAL/TM-1269, 1984)
- D. Trbojevic, E. Courant, *Low emitance lattices for electron rings revisited (EPAC94)*
- S.Y. Lee, *Emittance optimization in three- and multiple-bend achromats* (PRE, 1996)
- H. Tanaka, A. Ando, *Minimum effective emittance …* (NIM, 1996)
- Y. Shoji, A. Ando, *Minimum emittance of isochronus rings …* (NIM, 1999)
- T.Y. Lee, J. Choi, *Minimum electron beam size of a triple bend lattice* (NIM, 2004)

❑ Recent interests in nonuniform dipoles

- A. Wrulich, *Future directions in the storage ring development for light sources* (PAC99
- R. Nagaoka, A.F. Wrulich, *Emittance minimisation with longitudinal dipole field variation* (NIM, 2007)
- J. Guo, T. Raubenheimer,
  Low emittance E-/E+ storage ring design using dending magnets with longitudinal gradient (EPAC'02)
- Y. Papaphilippou, P. Elleaume,
  Analytical considerations for reducing the effecitve emittance with variable dipole field strengths (PAC'05)
- A. Streun, *Minimum emittance superbend lattices* (PSI Internal Report, 2007)
- C.-x Wang, *Minimum emittance in storage rings with uniform or nonuniform dipoles* (PRST-AB, 2009)