Probing New Bosons & Nuclear Structure With Ytterbium Isotope Shifts

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based on work with

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Nuclear Shape and BSM Searches at Colliders, CERN, 13^{th} January 2025

Outline

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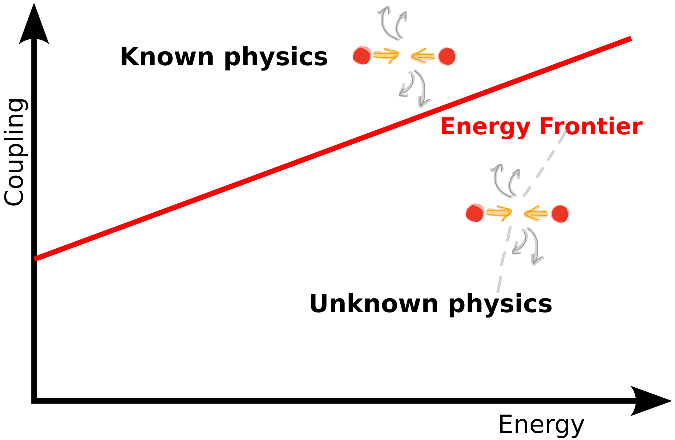
Where is the New Physics?



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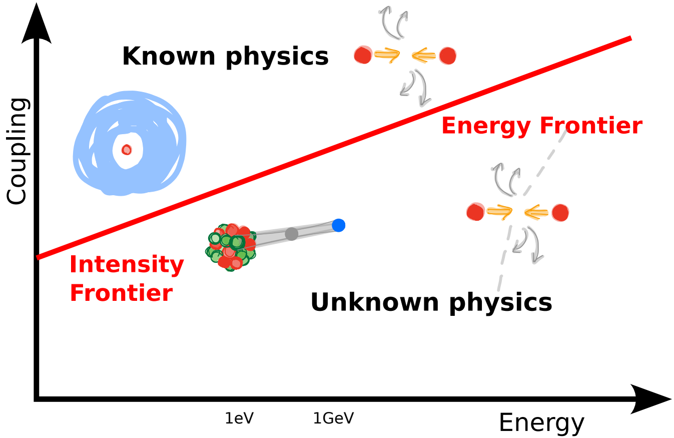


Why Isotope Shifts?

The most accurately measured numbers in physics are ratios of atomic clock transition frequencies:

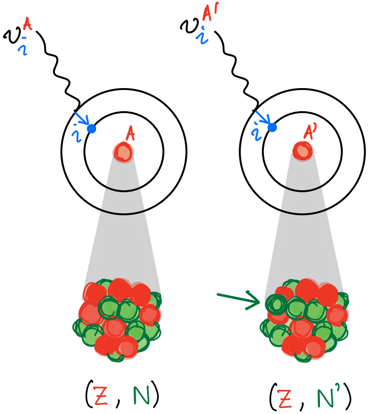
- $\nu_{\mathrm{Al}^+}/\nu_{\mathrm{Hg}^+} = 1.052871833148990438(55)$ (NIST; $\sigma_\nu/\nu \sim 5.2 \times 10^{-17}$)
  [Rosenband et al. Science 319, 1808 (2008)]
- $\nu_{\mathrm{Yb}}/\nu_{\mathrm{Sr}} = 1.207507039343337749(55)$ (RIKEN; $\sigma_\nu/\nu \sim 4.6 \times 10^{-17}$)
  [Nemitz et al. Nat. Photonics 10, 258 (2016)]
- $\nu_{\mathrm{E3}}/\nu_{\mathrm{E2}} = 0.932829404530965376(32)$ (PTB; $\sigma_\nu/\nu \sim 3.4 \times 10^{-17}$)
  [Lange et al. PRL 126 011102 (2021)]
- $\nu_{\mathrm{In}^+}/\nu_{\mathrm{Yb}^+} = 1.973773591557215789(9)$ (PTB; $\sigma_\nu/\nu \sim 4.4 \times 10^{-18}$)
  [Hausser et al. arXiv: 2402.16807 (2024)]

$\Rightarrow$ These are sensitive to "everything", but we cannot calculate the spectrum below around 1% accuracy.

So what can we do with these?

[slide by Julian Berengut]

Isotope Shifts



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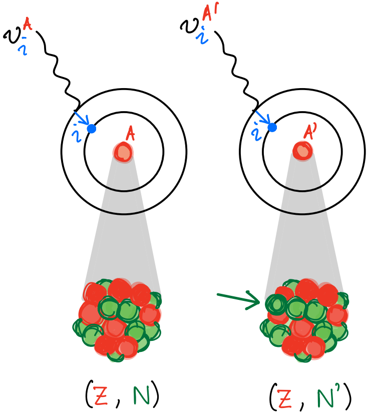


Isotope shifts:

$$\nu_i^{AA'} \equiv \nu_i^{A} - \nu_i^{A'}$$
$$= K_i \mu^{AA'} + F_i \delta\langle r^2\rangle^{AA'} + \ldots$$

- i: transition index
- AA': isotope pair index
- $K_i, F_i, \ldots$: electronic coeffs.
- $\mu^{AA'}, \delta\langle r^2\rangle^{AA'}, \ldots$: nuclear coeffs.
- Z: number of protons
- N, N': number of neutrons in A, A'

Isotope Shifts: Mass Shift & Field Shift

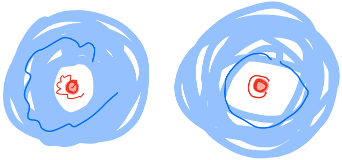
$$\nu_i^{AA'} = K_i \mu^{AA'} + F_i \delta \langle r^2 \rangle^{AA'} + \dots$$

Mass Shift

Nuclear motion in A vs. A'

$\Rightarrow$ Correction to e^- kin. energy

$$\propto \mu^{AA'} = \frac{1}{M^A} - \frac{1}{M^{A'}}$$



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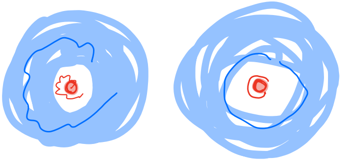
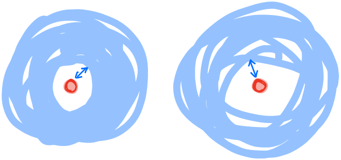
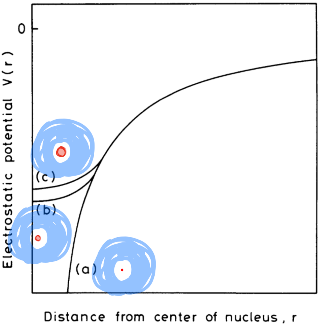


Field Shift

$$\propto \delta \langle r^2 \rangle^{AA'} = \langle r^2 \rangle^A - \langle r^2 \rangle^{A'}$$



Field Shift: Nuclear Size Effect



(a) Coulomb $V = -\frac{Ze}{4\pi r}$
(b) Finite size nucleus
(c) Larger nucleus

- **Inside the atom,** electron wavefct. affected by non-Coulombic nuclear potential, dep. on
  - Radial coordinate r
  - Nuclear **charge radius** $\langle r^2 \rangle = \frac{\int \rho_N(\mathbf{r}) r^2 \mathrm{d}\mathbf{r}^3}{\int \rho_N(\mathbf{r}) \mathrm{d}\mathbf{r}^3}$

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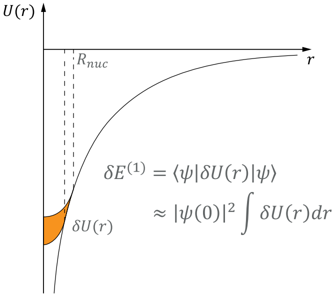

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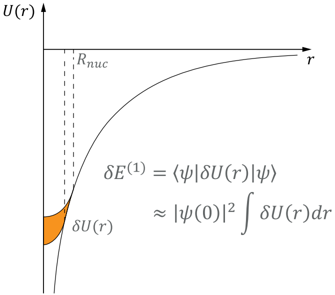

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$\Rightarrow$ Shift in $\langle r^2\rangle \Rightarrow$ Energy shift

$$\delta E_i \equiv F_i \delta\langle r^2\rangle^{AA'}$$

F_i : (Electronic) field shift constant

$\delta\langle r^2\rangle^{AA'}$: Charge radius variance

$$\delta\langle r^2\rangle^{AA'} = \langle r^2\rangle^{A} - \langle r^2\rangle^{A'}$$

Isotope Shifts: Mass Shift & Field Shift

$$\nu_i^{AA'} = K_i \mu^{AA'} + F_i \delta\langle r^2 \rangle^{AA'} + \dots$$

Mass Shift

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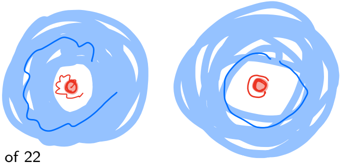
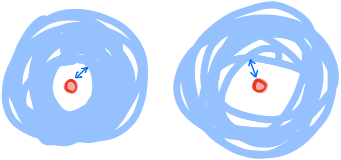


Field Shift

Nuclear charge distr. in A vs. A'

$\Rightarrow$ Difference in contact interactions between e^- & nuclei in A vs. A'

$$\propto \delta\langle r^2 \rangle^{AA'} = \langle r^2 \rangle^A - \langle r^2 \rangle^{A'}$$



Outline

The King-Plot: Trade Data for Nuclear Physics

[W. King, J. Opt. Soc. Am. 53, 638 (1963)]

Issue: Large uncertainty on charge radius variance $\delta\langle r^2\rangle^{AA'}$

$$\nu_1^{AA'} = K_1 \mu^{AA'} + F_1 \delta\langle r^2\rangle^{AA'}$$

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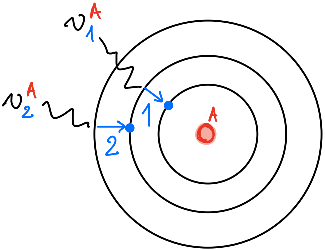

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⇒ Measure isotope shifts for 2 transitions

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$$\nu_2^{AA'} = K_2 \mu^{AA'} + F_2 \delta\langle r^2\rangle^{AA'}$$

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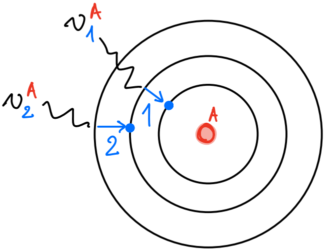

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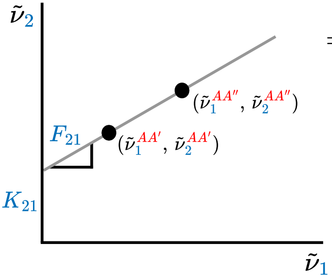
$\Rightarrow$ Eliminate charge radius variance $\delta\langle r^2\rangle^{AA'}$

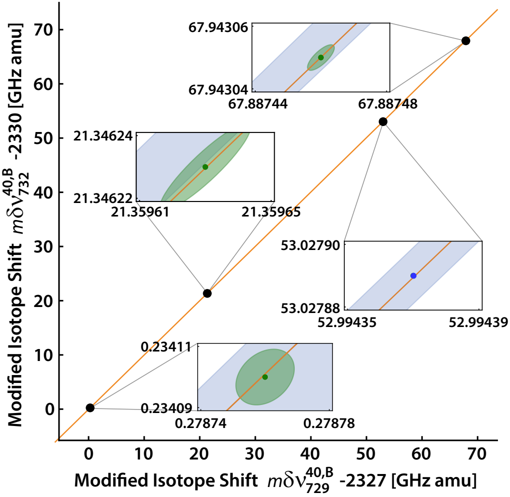
$$\tilde{\nu}_2^{AA'} = K_{21} + F_{21}\tilde{\nu}_1^{AA'}$$

$\tilde{\nu}_i^{AA'} \equiv \nu_i^{AA'}/\mu^{AA'}$ $\quad\Rightarrow$ data

$F_{21} \equiv F_2/F_1 \quad K_{21} \equiv K_2 - F_{21}K_1$ $\quad\Rightarrow$ fit



Example of a Linear King Plot: Ca^+ [arXiv:2311.17337]



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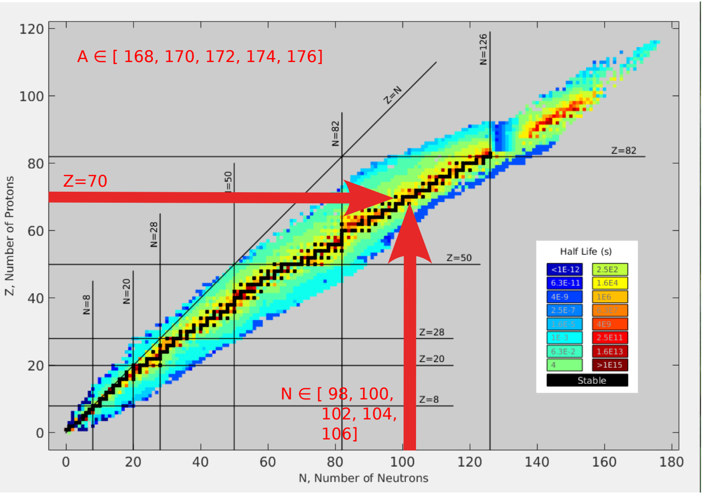
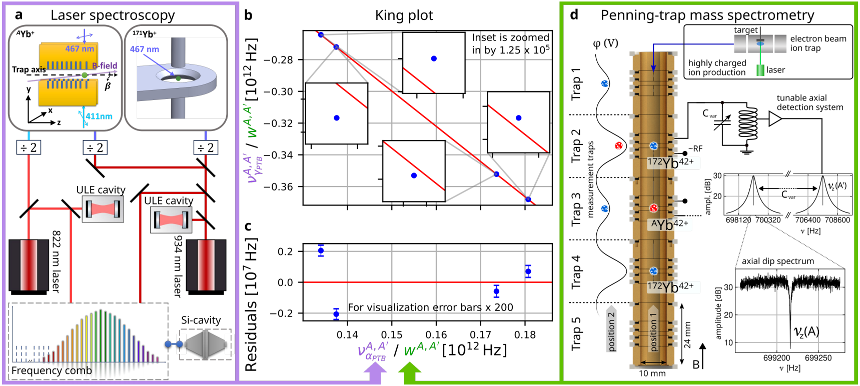
Ytterbium and its Stable Isotopes



PTB + MPIK = New Yb King Plot [arXiv:2403.07792]

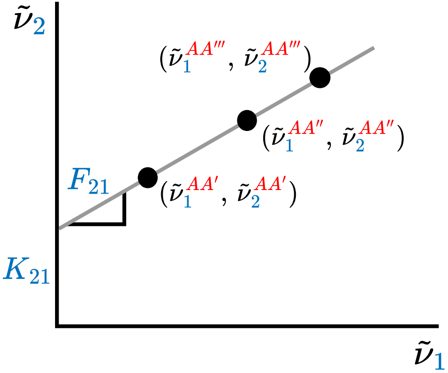


$$\text{Frequencies: } \frac{\delta\nu}{\nu} \sim \mathcal{O}(10^{-9}) \qquad \text{Mass ratios } \eta_A = \frac{m_A}{m_{172}} : \frac{\delta\eta}{\eta} \sim \mathcal{O}(10^{-12})$$

$\Rightarrow$ Hz-precision for isotope shifts

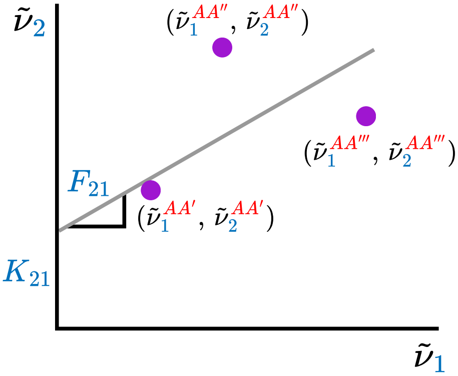
Observed King plot nonlinearity: $\sim 20.17(2)$ kHz

The King-Plot: Fit to Isotope Shift Data



$$\tilde{\nu}_2^{AA'} = K_{21} + F_{21}\tilde{\nu}_1^{AA'}$$
$$\tilde{\nu}_2^{AA''} = K_{21} + F_{21}\tilde{\nu}_1^{AA''}$$
$$\tilde{\nu}_2^{AA'''} = K_{21} + F_{21}\tilde{\nu}_1^{AA'''}$$

The King-Plot: Fit to Isotope Shift Data



$$\tilde{\nu}_2^{AA'} = K_{21} + F_{21}\tilde{\nu}_1^{AA'} + ?$$
$$\tilde{\nu}_2^{AA''} = K_{21} + F_{21}\tilde{\nu}_1^{AA''} + ?$$
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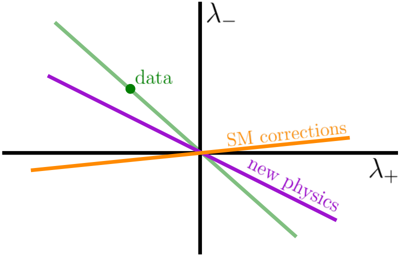
The Nonlinearity Decomposition Plot



- Plane of King linearity ($\mathbf{1} = (1,1,1,1)$)

$$\tilde{\boldsymbol{\nu}}_j \approx F_{j1}\tilde{\boldsymbol{\nu}}_1 + K_{j1}\mathbf{1}\,, \qquad j > 1.$$

- Project isotope-shift data onto $\tilde{\boldsymbol{\nu}}_1$, $\mathbf{1}$, Λ_+, Λ_- with $\Lambda_\pm \perp (\tilde{\boldsymbol{\nu}}_1, \mathbf{1})$:

$$\tilde{\boldsymbol{\nu}}_j = (\tilde{\boldsymbol{\nu}}_1, \mathbf{1}, \boldsymbol{\Lambda}_+, \boldsymbol{\Lambda}_-)\,(F_{j1}, K_{j1}, \lambda_+, \lambda_-)^T$$

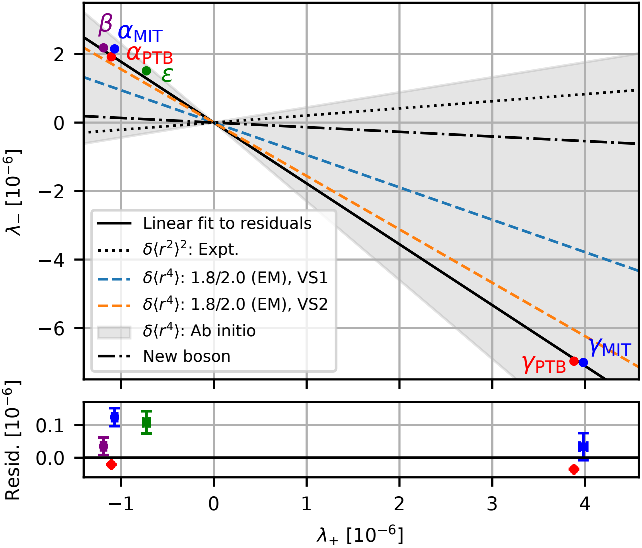
In presence of just one nonlinearity,

$$\tilde{\boldsymbol{\nu}}_j \approx F_{j1}\tilde{\boldsymbol{\nu}}_1 + K_{j1}\mathbf{1} + G^{(4)}_{j1}\boldsymbol{\delta}\widetilde{\langle \boldsymbol{r}^4\rangle}\,, \qquad j > 1.$$

slope: $\frac{\lambda_-}{\lambda_+} \equiv \frac{G^{(4)}_{j1}\boldsymbol{\delta}\widetilde{\langle \boldsymbol{r}^4\rangle}_-}{G^{(4)}_{j1}\boldsymbol{\delta}\widetilde{\langle \boldsymbol{r}^4\rangle}_+} = \frac{\boldsymbol{\delta}\widetilde{\langle \boldsymbol{r}^4\rangle}_-}{\boldsymbol{\delta}\widetilde{\langle \boldsymbol{r}^4\rangle}_+} \Rightarrow$ transition-universal

[arXiv:2004.11383, arXiv:2201.03578]

The Nonlinearity Decomposition Plot [arXiv:2403.07792]



Outline

Extracting Nuclear Physics from Isotope-Shifts

- **Assuming** $\delta\langle r^4\rangle$ **dominates,** what does the isotope-shift data tell us about the evolution of $\delta\langle r^4\rangle$ along the isotope chain?

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$\Rightarrow$ **"Put the King plot on it's head.":** Experimental data (under control)

King Plot (in transition space): Eliminate charge radius variance $\delta\langle r^2\rangle^{AA'}$

$$\begin{aligned} \nu_i^{AA'} &= F_i \delta\langle r^2\rangle^{AA'} && + K_i \mu^{AA'} && + \boldsymbol{G_i^{(4)}} \delta\langle r^4\rangle^{AA'} \\ \nu_1^{AA'} &= F_1 \delta\langle r^2\rangle^{AA'} && + K_1 \mu^{AA'} && + \boldsymbol{G_1^{(4)}} \delta\langle r^4\rangle^{AA'} \\ \hline \Rightarrow \nu_i^{AA'}/\mu^{AA'} &= F_{i1} \nu_1^{AA'}/\mu^{AA'} && + K_{i1} && + \boldsymbol{G_{i1}^{(4)}} \delta\langle r^4\rangle^{AA'}/\mu^{AA'} \end{aligned}$$

$\Rightarrow$ Fit F_{i1}, K_{i1}, $\boldsymbol{G_{i1}^{(4)} = G_i^{(4)} - F_{i1} G_1^{(4)}}$ (double-index electronic coefficient): AMBiT

Extracting Nuclear Physics from Isotope-Shifts

- **Assuming $\delta\langle r^4\rangle$ dominates,** what does the isotope-shift data tell us about the evolution of $\delta\langle r^4\rangle$ along the isotope chain?

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\nu_1^{AA'} &= F_1\delta\langle r^2\rangle^{AA'} + K_1\mu^{AA'} + \boldsymbol{G}_1^{(4)}\delta\langle r^4\rangle^{AA'} \\
\hline
\Rightarrow \nu_i^{AA'}/\mu^{AA'} &= F_{i1}\nu_1^{AA'}/\mu^{AA'} + K_{i1} + \boldsymbol{G}_{i1}^{(4)}\delta\langle r^4\rangle^{AA'}/\mu^{AA'}
\end{aligned}$$

⇒ Fit F_{i1}, K_{i1}, $\boldsymbol{G_{i1}^{(4)} = G_i^{(4)} - F_{i1}G_1^{(4)}}$ (double-index electronic coefficient): AMBiT

$\boldsymbol{\delta\langle r^4\rangle}$-Extraction (in isotope pair space): Eliminate mass shift coefficient K_i

$$\begin{aligned}
\nu_i^{AA'} &= F_i\delta\langle r^2\rangle^{AA'} + K_i\mu^{AA'} + \boldsymbol{G}_i^{(4)}\delta\langle r^4\rangle^{AA'} \\
\nu_i^{RR'} &= F_i\delta\langle r^2\rangle^{RR'} + K_i\mu^{RR'} + \boldsymbol{G}_i^{(4)}\delta\langle r^4\rangle^{RR'} \\
\hline
\nu_i^{AA'} - \frac{\mu^{AA'}}{\mu^{RR'}}\nu_1^{AA'} &= F_i\left(\delta\langle r^2\rangle^{AA'} - \frac{\mu^{AA'}}{\mu^{RR'}}\delta\langle r^2\rangle^{AA'}\right) + \boldsymbol{G}_i^{(4)}\underbrace{\left(\delta\langle r^4\rangle^{AA'} - \frac{\mu^{AA'}}{\mu^{RR'}}\delta\langle r^4\rangle^{RR'}\right)}_{\boldsymbol{Q^{AA'RR'}}}
\end{aligned}$$

⇒ Fit F_i, $\delta\langle r^2\rangle^{AA'}$: [Angeli, Marinova], $\boldsymbol{G}_i^{(4)}$: AMBiT

Extracting Nuclear Physics from Isotope-Shifts

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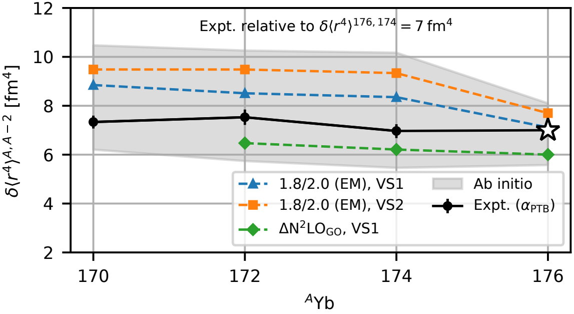
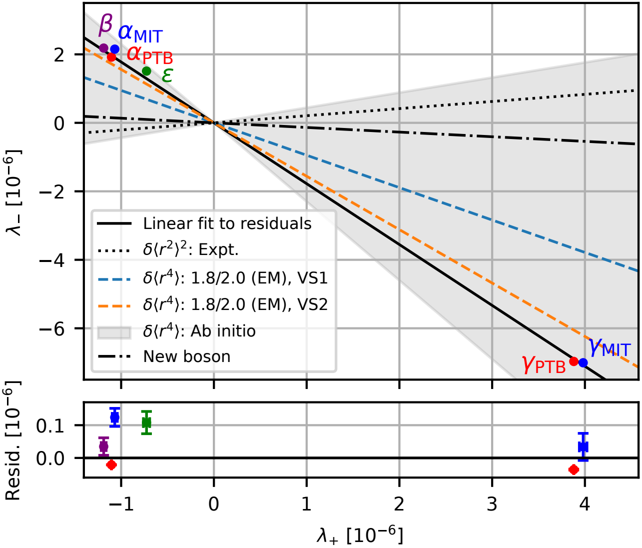


blue, **orange**, **green**: Calculations by group of Prof. Achim Schwenk
black: new spectroscopic method, fixed at $\star$

The Nonlinearity Decomposition Plot [arXiv:2403.07792]



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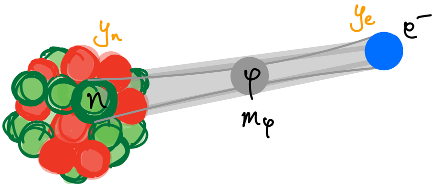
Dark Portals

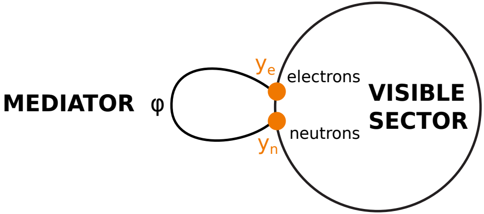


Question that can be addressed by King plot analyses:

Given a mediator mass m_φ in the **eV-GeV** range, ...

...how large can the coupling $y_e y_n$ be?

Dark Portals and Isotope Shift Measurements



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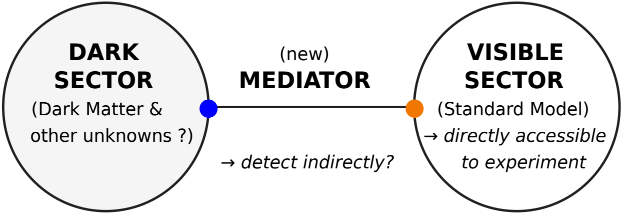

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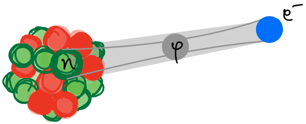
King-Plot Bounds on New Bosons [arXiv:1704.05068,2005.06144]



New effective Yukawa-potential

$$V_\phi(r) = -\alpha_{\mathrm{NP}}(A-Z)\frac{e^{-m_\phi r}}{r}$$

with $\alpha_{\mathrm{NP}} = (-1)^s \frac{y_e y_n}{4\pi}$, $s = 0, 1, 2$ (spin)

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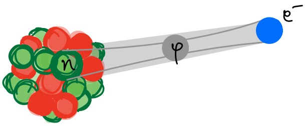

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$$\tilde{\nu}_2^{AA'} = K_{21}\tilde{\mu}^{AA'} + F_{21}\tilde{\nu}_1^{AA'} + G_{21}^{(4)}\delta\langle\widetilde{r^4}\rangle^{AA'} + \alpha_{\rm NP}X_{21}\tilde{\gamma}^{AA'}$$

$X_{21} = X_2 - F_{21}X_1$: NP electronic coefficient

$\tilde{\gamma}^{AA'} \equiv (A - A')/\mu^{AA'}$: NP nucl. coeff.

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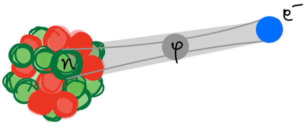

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$X_{21} = X_2 - F_{21}X_1$: NP electronic coefficient
$\tilde{\gamma}^{AA'} \equiv (A-A')/\mu^{AA'}$: NP nucl. coeff.

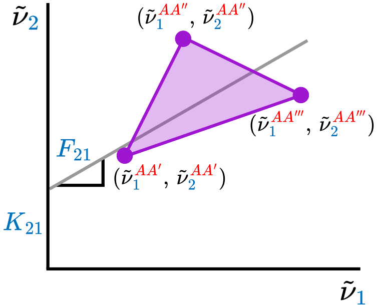
⇒ Extract $\alpha_{\rm NP}$ from fraction of volumes spanned by frequency vectors:

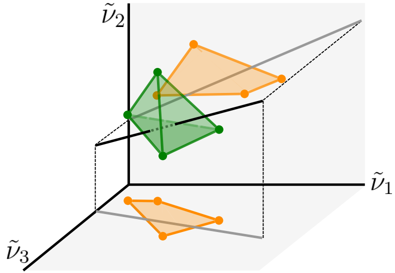
$$\alpha_{\rm NP} = \frac{Vol.}{Vol.|_{th,\alpha_{\rm NP}=1}} = \frac{2\det\left(\vec{\tilde{\nu}}_1, \vec{\tilde{\nu}}_2, \vec{\tilde{\nu}}_3, \vec{\tilde{\mu}}\right)}{\varepsilon_{ijk}\det\left(X_i\vec{\tilde{\gamma}}, \vec{\nu}_j, \vec{\nu}_k, \vec{\tilde{\mu}}\right)}$$

$\{\vec{\nu}_i\}$: data vect. in isotope-pair space, $\vec{\tilde{\mu}} \equiv (1,1,1,1)$, X_i, $\vec{\tilde{\gamma}}$: theory input

King-Plot Method in Presence of Nuclear Effects: The Generalised King Plot

[arXiv:2005.06144]





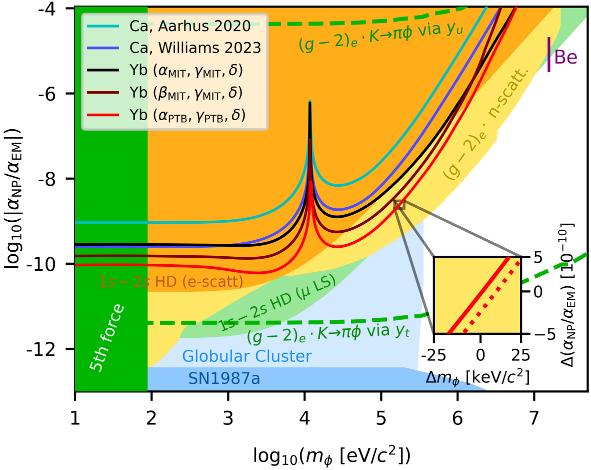
⇒ Test King linearity

⇒ Account for one King nonlinearity

⇒ Put bound on 2^{nd}

⇒ **King-plot method also works in presence of nuclear effects.**

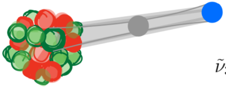
New Spectroscopy Bounds on New Physics

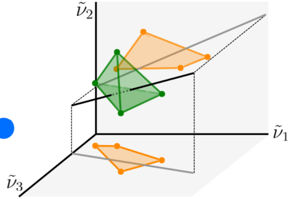


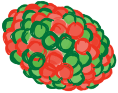
- $m_\phi \to 0$: $>$ size atom
- $m_\phi \to \infty$: not sensitive to contact interactions
- “Peaks” due to cancellations among electronic coefficients

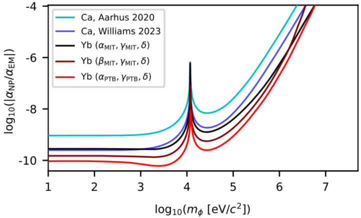
Conclusions

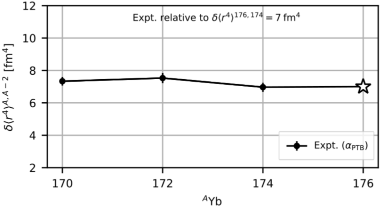
Atomic clocks are sensitive probes for











New mediators between *n* & e^- **Nuclear structure**

Check out our paper:

Yb King plot: `arXiv:2403.07792`

Stay tuned for:

- `kifit`: Global King-plot analysis

Thank you for your attention.

Backup slides

α_{NP} from Determinants

(No-Mass King-Plot:)

$$\begin{aligned}
\vec{\nu}_1 =& K_1\vec{\mu} + F_1\overrightarrow{\delta\langle r^2\rangle} + \alpha_{\mathsf{NP}} X_1\vec{\gamma} \\
\vec{\nu}_2 =& K_2\vec{\mu} + F_2\overrightarrow{\delta\langle r^2\rangle} + \alpha_{\mathsf{NP}} X_2\vec{\gamma} \\
\vec{\nu}_3 =& K_3\vec{\mu} + F_3\overrightarrow{\delta\langle r^2\rangle} + \alpha_{\mathsf{NP}} X_3\vec{\gamma} \\
\Rightarrow \det(\vec{\nu}_1, \vec{\nu}_2, \vec{\nu}_3) =& \alpha_{\mathsf{NP}} \det\left(\vec{K}, \vec{F}, \vec{X}\right) \det\left(\vec{\mu}, \overrightarrow{\delta\langle r^2\rangle}, \vec{\gamma}\right) \\
\Rightarrow \alpha_{\mathsf{NP}} =& \frac{Vol}{Vol|_{th,\alpha_{\mathsf{NP}}=1}} = \frac{\det(\vec{\nu}_1, \vec{\nu}_2, \vec{\nu}_3)}{\det\left(\vec{K}, \vec{F}, \vec{X}\right) \det\left(\vec{\mu}, \overrightarrow{\delta\langle r^2\rangle}, \vec{\gamma}\right)} \\
=& \frac{\det(\vec{\nu}_1, \vec{\nu}_2, \vec{\nu}_3)}{\frac{1}{2}\varepsilon_{ijk} \det(X_i\vec{\gamma}, \vec{\nu}_j, \vec{\nu}_k)}
\end{aligned}$$

Choose your King-Plot

Extraction of α_{NP} using the "determinant method" requires

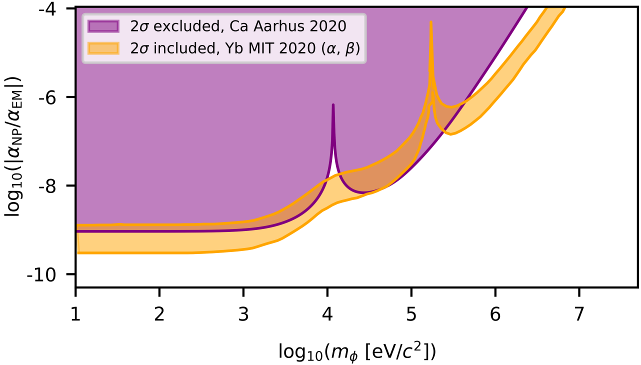
Type of King-Plot	Isotope-Pairs	Transitions	
Generalised King-Plot:	n	$n-1$	[PRR 2, 043444 (2020)]
No-Mass King-Plot:	n	n	[PRR 2, 043444 (2020)]

$n \geq 3$ (else cannot search for nonlinearities)

$$\alpha_{\mathrm{NP}} = \frac{V}{V|_{\mathrm{th},\alpha_{\mathrm{NP}}=1}} = \frac{(n-2)!\,\det\left(\vec{\tilde{\nu}}_1, \ldots, \vec{\tilde{\nu}}_{n-1}, \vec{\tilde{\mu}}\right)}{\varepsilon_{i_1,\ldots,i_{n-1}}\det\left(X_{i_1}\vec{\tilde{\gamma}}, \vec{\tilde{\nu}}_{i_2}, \ldots, \vec{\tilde{\nu}}_{i_{n-1}}, \vec{\tilde{\mu}}_{i_n}\right)}$$

$$\alpha_{\mathrm{NP}} = \frac{v}{v|_{\mathrm{th},\alpha_{\mathrm{NP}}=1}} = \frac{(n-1)!\,\det\left(\vec{\nu}_1, \vec{\nu}_2, \ldots, \vec{\nu}_n\right)}{\varepsilon_{i_1,i_2,\ldots,i_n}\det\left(X_{i_1}\vec{\gamma}, \vec{\nu}_{i_2}, \ldots, \vec{\nu}_{i_n}\right)}$$

Upper Bounds on $|\alpha_{\rm NP}|$ vs. New Mediator Mass m_ϕ



Nonlinear King plot relation:

$$\tilde{\nu}_2^{AA'} = K_{21}\tilde{\mu}^{AA'} + F_{21}\tilde{\nu}_1^{AA'} + G_{21}^{(2)}\delta\langle r^2\rangle^2 + G_{21}^{(4)}\delta\langle r^4\rangle + \ldots ?$$

X Coefficients

Overlap of new physics potential and electronic wavefunction

$$X_i = \int \mathrm{d}^3 r \frac{e^{-m_\phi r}}{r} \left[|\psi_b(r)|^2 - |\psi_a(r)|^2\right]$$

$|\psi(r)|^2$: electron density in absence of new physics,

a, b initial, final states

Requirement for searches for new light bosons:

- At least one of ψ_a or ψ_b should have good overlap with new potential.
- For tight bounds on α_{NP}, one X_i needs to be large.

Recipe for the Nonlinearity Decomposition Plot

[PRL 125, 123002 (2020), PRL 128, 163201 (2022)]

1. Arrange the isotope-shift data for all transitions $\tau \in \{\alpha, \beta, \gamma, \delta, \epsilon\}$ in n-vectors $\tilde{\boldsymbol{\nu}}_\tau$, where n is the number of isotope pairs (here 4):

$$\tilde{\boldsymbol{\nu}}_\tau = (\tilde{\nu}_\tau^{168,170}, \tilde{\nu}_\tau^{170,172}, \tilde{\nu}_\tau^{172,174}, \tilde{\nu}_\tau^{174,176})$$

2. Choose a reference transition, say δ.
3. Plane of King linearity is defined by the relations ($\mathbf{1} = (1, 1, 1, 1)$)

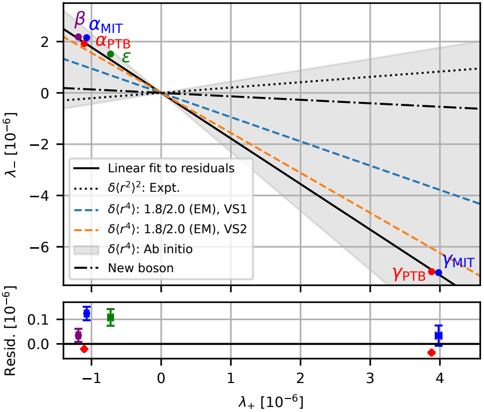
$$\tilde{\boldsymbol{\nu}}_\tau \approx F_{\tau\delta}\tilde{\boldsymbol{\nu}}_\delta + K_{\tau\delta}\mathbf{1}\,.$$

4. Define two ($n = 4$)−vectors $\Lambda_\pm$ that are orthogonal to $\tilde{\boldsymbol{\nu}}_\delta$, $\mathbf{1}$.
5. Project all isotope-shift data onto the four vectors $\tilde{\boldsymbol{\nu}}_\delta$, $\mathbf{1}$, Λ_+, Λ_-:

$$\tilde{\boldsymbol{\nu}}_\tau = \begin{pmatrix}\tilde{\boldsymbol{\nu}}_\delta & \mathbf{1} & \boldsymbol{\Lambda}_+ & \boldsymbol{\Lambda}_-\end{pmatrix} \begin{pmatrix}F_{\tau\delta} & K_{\tau\delta} & \lambda_+^{(\tau)} & \lambda_-^{(\tau)}\end{pmatrix}^T$$

6. Plot all points $(\lambda_+^{(\tau)}, \lambda_-^{(\tau)})$ in the same plane.

The Nonlinearity Decomposition Plot



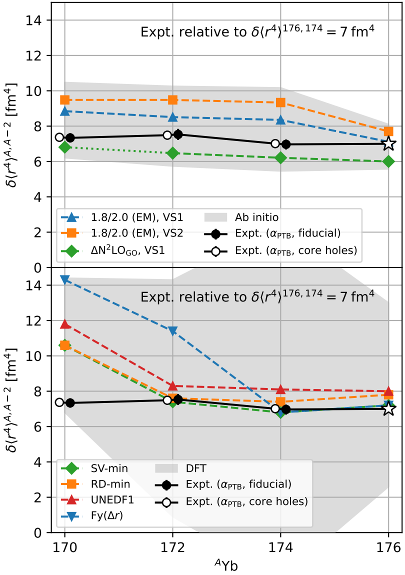
Notation	Transition	Refs.
$\alpha_{\mathrm{MIT,PTB}}$	${}^2S_{1/2} \to {}^2D_{5/2}$ E2 in Yb^+	MIT, t.w.
β	${}^2S_{1/2} \to {}^2D_{3/2}$ E2 in Yb^+	MIT
$\gamma_{\mathrm{MIT,PTB}}$	${}^2S_{1/2} \to {}^2F_{7/2}$ E3 in Yb^+	MIT, t.w.
δ	${}^1S_0 \to {}^3P_0$ in Yb	Kyoto
ϵ	${}^1S_0 \to {}^1D_2$ in Yb	Mainz

- $\delta\langle r^2\rangle^2$ estimated using Angeli & Marinova Tables of experimental nuclear ground state charge radii
- $\delta\langle r^4\rangle$: Calculations by group of Prof. Achim Schwenk, TU Darmstadt

In presence of just one nonlinearity, e.g. $G^{(4)}\delta\langle r^4\rangle$,

slope: $\frac{\lambda_-^{(\tau)}}{\lambda_+^{(\tau)}} = \frac{G_\tau^{(4)}\boldsymbol{\delta\langle r^4\rangle}_-}{G_\tau^{(4)}\boldsymbol{\delta\langle r^4\rangle}_+} = \frac{\boldsymbol{\delta\langle r^4\rangle}_-}{\boldsymbol{\delta\langle r^4\rangle}_+} \equiv \frac{\lambda_-}{\lambda_+} \Rightarrow$ transition-universal

$\delta\langle r^4\rangle$ Calculations: Ab initio vs. DFT



- Experimental $\delta\langle r^4\rangle^{AA'}$ values relative to $\delta\langle r^4\rangle^{176,174} = 7$ fm^4 extracted from isotope shifts from the α transition using atomic theory (fiducial, core holes)
- Above: ab initio calculations (t.w.)
- Below: density functional theory calculations (PRL.128.163201)
- Gray bands: estimated theory uncertainties