ISIS Neutron and Muon Source

Accelerator Physics

Lecture 8 & 9: Beam Instabilities

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12th February 2025

Contents

Aims of these Lectures

- Identify what beam instabilities are
- An overview of what drives beam instabilities
- The concepts of wakes and impedances
- Categorisations of beam instabilities

- These lectures provide a qualitative foundation to the complex topic of instabilities. Further details, and quantitative analysis, may be found in the references.

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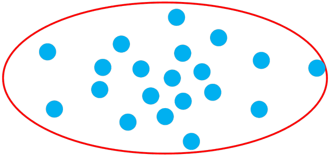
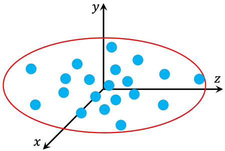
What are Beam Instabilities?



- A **beam instability** is diagnosed when a **moment** of the beam distribution exhibits exponential growth (e.g. mean horizontal position, x; vertical standard deviation, σ_x, etc.)
- This exponential growth may leads to emittance growth which can then lead to beam loss.

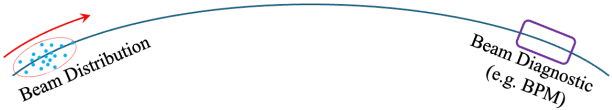
What are Beam Instabilities?

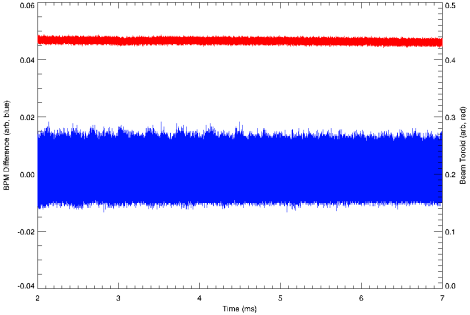


Particle distribution, $\psi(x, y, z, x', y', \delta)$

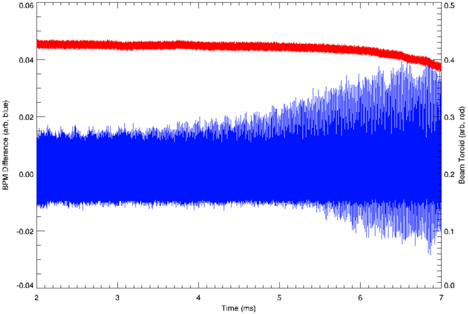
$$
\begin{aligned}
N &= \int_{-\infty}^{\infty} \psi \, dx\, dx' dy\, dy' dz\, d\delta \\
\langle x \rangle &= \frac{1}{N} \int_{-\infty}^{\infty} x\psi \, dx\, dx' dy\, dy' dz\, d\delta \\
\sigma_x &= \frac{1}{N} \int_{-\infty}^{\infty} (x - \langle x \rangle)^2 \psi \, dx\, dx' dy\, dy' dz\, d\delta
\end{aligned}
$$

What does it look like? - Stable Beam





What does it look like? - Unstable Beam



Why are Beam Instabilities Important?

- When accelerator performance is pushed, accelerators tend to reach an intensity limit.
- With study, understanding, simulation and mitigation often a new, higher limit is reached.
- This pattern can be seen with accelerators reaching to high-intensity or high-energy applications.
- This limit is usually defined by the onset of a beam instability.
- **Typical situation:** A dipolar instability occurs when the beam intensity is raised above a **threshold**. This threshold can be modified (to an extent) by accurate selection of machine parameters.

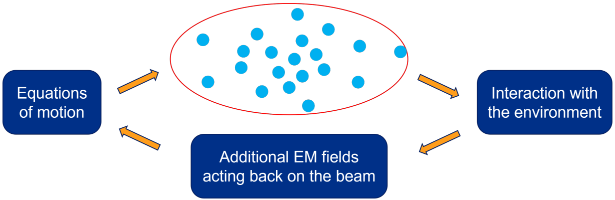
Why are Beam Instabilities Important?

- Understanding the **type** of instability, what **drives** it and how it changes with different beam and machine parameters, is essential.
- A proper understanding of the instability mechanism allows for appropriate measures to mitigate/suppress the instability.
- This may allow for an increase in beam intensity and/or better emittance conservation.
- An accelerator component found to drive the instability may be modified to limit their impact, or removed entirely.
- Further knowledge also allows for proper specification of an **active feedback system** to damp the instability.

Types of Beam Instabilities

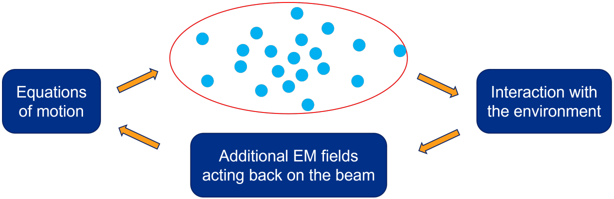
- Beam instabilities occur in both **linear** and **circular** machines.
- They are often characterised by the plane in which they occur: **longitudinal** (z, δ) or **transverse** (x, y, x', y').
- Another categorisation is with effects on different length scales: **short-range, single bunch** instabilities and **long-range, multibunch** instabilities and **coasting beam** instabilities.

Introduction to Wakes and Impedances



- Last term's lectures on particle dynamics were concerned primarily with single particle effects (transverse and longitudinal dynamics).
- This term we have seen the effect of treating the beam(s) as distributions of interacting charges (beam-beam and space charge lectures).

Introduction to Wakes and Impedances



Interaction with the environment:

- Beam self-fields (**indirect** and **direct space charge**),
- Fields from another bunch/beam (**beam-beam**),
- **Electron/ion cloud** production and accumulation.

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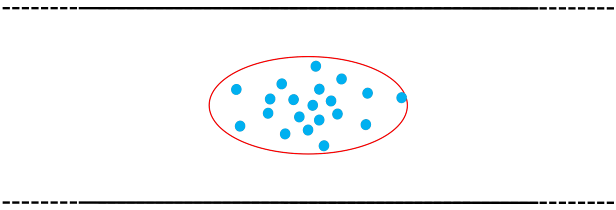
What are Wakes and Impedances?



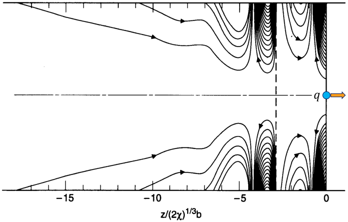
In order to obtain the additional EM fields from the interaction of the beam with its surroundings two, linked mathematical methods are commonly utilised

- **Wake fields** in the time domain.
- **Beam coupling impedances** in the frequency domain.

More about Wakes

A wake is a sustained track left by a moving body (e.g. a duck) in a medium (e.g. water)

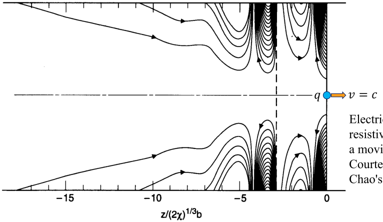
More about Wakes



Electric field lines in a resistive pipe generated by a moving point charge, q. Courtesy K. Bane (see Chao's Book figure 2.3).

- In the case of a particle beams, the wakes are transient electromagnetic fields generated by the particles' passage.
- These fields must satisfy **Maxwell's equations**.
- The presence of a vacuum chamber imposes boundary conditions that modify these electromagnetic fields.

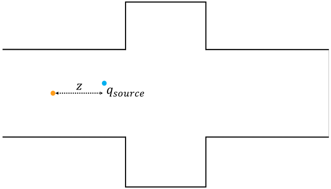
Wake Fields



Electric field lines in a resistive pipe generated by a moving point charge, q. Courtesy K. Bane (see Chao's Book figure 2.3).

- Fields generated by the beam can act on particles passing through that section later in time, affecting their dynamics.
- These fields are termed **wake fields**, and can be longitudinal or transverse:
  - Longitudinal - affect the energy of trailing beam particles.
  - Transverse - change the trailing particles' transverse momenta.

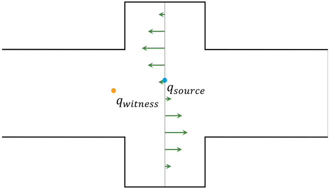
Wake Fields (general)



First, consider two particles travelling at $v \approx c$ in a perfectly conducting beampipe.

- With no changes in beampipe size, the source and witness particle (separated by a longitudinal distance z) feel no external forces.

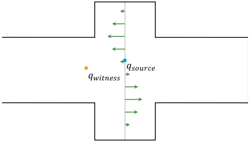
Wake Fields (general)



First, consider two particles travelling at $v \approx c$ in a perfectly conducting beampipe.

- With no changes in beampipe size, the source and witness particle (separated by a longitudinal distance z) feel no external forces.
- When the source particle encounters a discontinuity (e.g. cavity, device, change in cross-section) it leaves a wake field in the structure.

Wake Fields (general)



First, consider two particles travelling at $v \approx c$ in a perfectly conducting beampipe.

- With no changes in beampipe size, the source and witness particle (separated by a longitudinal distance z) feel no external forces.
- When the source particle encounters a discontinuity (e.g. cavity, device, change in cross-section) it leaves a wake field in the structure.
    - The source particle loses energy to the field.
    - The witness particle feels a force due to the field.

Wake Fields (general)

As we saw earlier, a pipe with finite conductivity causes fields to “drag” behind the source particle.

- The finite conductivity introduces a delay in the induced wall current.
- This produces a delayed electromagnetic field
  - Unlike a cavity, there’s no ringing, only a slow decay.
  - The witness particle feels a force due to the field as before.

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Wake Potentials and Wake Functions

- A complete description of the wake field will give the electric and magnetic field components as a function of time and position.
- Wake fields are dependent on the precise particle distribution, beampipe geometry and material properties.
- Typically a simplified description is called for, enabling analysis of their effect on beam dynamics: **wake functions** and **wake potentials**.
- These are evaluated for a given section of beampipe and provide the effect of a source particle on a witness particle as a function of the distance between them.

Wake Potentials

- The **wake potential** is the energy change induced by the wake field of a source particle on the witness particle.
- $$V = \int_{-L/2}^{L/2} \vec{F} d\vec{s} = \int_{-L/2}^{L/2} q_{source}(\vec{E} + \vec{v} \times \vec{B}) d\vec{s}$$
- The wake force, $\vec{F}$ is integrated over a period L (the length of the structure of interest) and is only a function of $z = s - vt$.
- The wake potential can be evaluated in the longitudinal or transverse planes.

Wake Functions

- The **wake function** is essentially the wake potential induced per unit charge.
-

$$\begin{aligned} W_{\parallel}(z) &= -\frac{1}{q_{source}\ q_{witness}} \int \vec{F} d\vec{s} \\ &= -\frac{\Delta E_{witness}}{q_{source}\ q_{witness}} \end{aligned}$$

- Here, the wake function $W_{\parallel}$ is the **longitudinal wake function** for the given component (units of VC^{-1}).

Wake Functions

- The **transverse wake function** is

$$W_x(z) = -\frac{1}{q_{source}\ q_{witness}} \frac{1}{\Delta x_{source}} \int \vec{F} d\vec{s}$$
$$= -\frac{E_0}{q_{source}\ q_{witness}} \frac{\Delta x'_{witness}}{\Delta x_{source}}$$

- There is a similar function for $W_y(z)$.
- For axisymmetric structures $W_x = W_y$ and is often just written $W_\perp$.
- The transverse wake function is in units of VC^{-1}m^{-1}.

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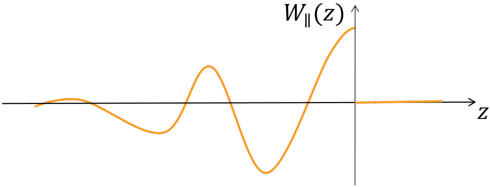
Properties of Wake Functions



- $W_{||}(z) = 0$ for $z > 0$ assuming $v \approx c$.
- Immediately behind the source particle, $z \to 0$, the witness particle should receive a retarding force implying $W_{||}(z \to 0) > 0$.
- The value of the wake at $z = 0$ is given by the fundamental theorem of beam loading.
- Therefore, $W_{||}(z)$ is discontinuous at $z = 0$.

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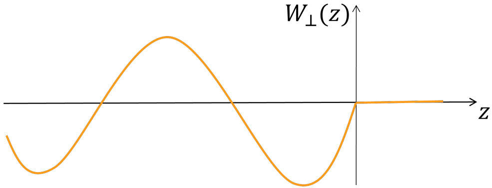
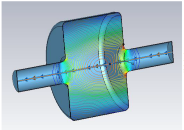
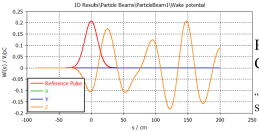
Properties of Wake Functions



- $W_\perp(z) = 0$ for $z > 0$ assuming $v \approx c$.
- The transverse wake is typically defined with $W_\perp(z \to 0) = 0$ unless space charge is included.
- Immediately behind the source particle the wake is negative indicating witness particles are deflected toward the source particle.
- The wake has a discontinuous derivative at $z = 0$.

Calculating Wake Functions





Pillbox cavity simulated in CST Studio

"CST Studio Suite Charged Particle Simulation Workflow & Solver Overview"

- In simple situations the wakes can be calculated analytically. However, wake potentials and wake functions are usually calculated numerically, using codes.
- Fields are calculated on a mesh using Maxwell's equations as the charge distribution moves through the component, with geometry and material properties specified.
- In this lecture we will not cover codes, or the process of extracting wake fields, but rather focus on their effects on beam dynamics.

What are Impedances?

- The wake function is an accelerator component may be thought of as its Green's function in the time domain (i.e. its response to a pulse excitation).
- Useful for macroparticle simulations, as it can easily be added to the single particle equations of motion.
- However, sometimes it's helpful to move to the frequency domain. The **beam coupling impedance** is the Fourier transform of the wake function

$$
\begin{aligned}
Z_{\|}(\omega) &= \int_{-\infty}^{\infty} W_{\|}(z) e^{-i\omega z/c} \frac{dz}{c} \\
Z_{\perp}(\omega) &= i \int_{-\infty}^{\infty} W_{\perp}(z) e^{-i\omega z/c} \frac{dz}{c}
\end{aligned}
$$

Physical Interpretation of Impedance

- Wake function describes the change in energy ($W_{\|}$) or transverse kick ($W_{\perp}$) resulting from the electromagnetic wake field of one particle acting on another as they both pass through an accelerator component.
- Beam coupling impedance, perhaps, requires a bit more careful thought to realise its usefulness.
- Let's start by considering the beam in the frequency domain (i.e. the frequency components of the beam current measured at some point in the accelerator).

$$\lambda(z') = \frac{1}{2\pi} \int \tilde{\lambda}(\omega) e^{i\omega z'/c}\, d\omega$$

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Physical Interpretation of Impedance

- Recall, the longitudinal wake function for a witness charge, e following a source "particle" of charge Ne through a section of accelerator is

$$\begin{aligned} W_{\parallel}(z) &= -\frac{\Delta E_{witness}(z)}{Ne^2} \\ \Delta E_{witness}(z) &= -Ne^2 W_{\parallel}(z) \end{aligned}$$

- In the case of a wake generated by a charge distribution (charge per unit length, $\lambda(z')$) this energy change becomes

$$\Delta E_{witness}(z) = -e^2 \int \lambda(z')\, W_{\parallel}(z - z')dz'$$

Physical Interpretation of Impedance

$$\Delta E_{witness}(z) = -e^2 \int \lambda(z')W_{\|}(z-z')dz'$$

- Substituting the expression for the beam spectrum into this equation and changing variables from z' to $z-z'$

$$\Delta E_{witness}(z) = \frac{e^2}{2\pi} \int \int \tilde{\lambda}(\omega)e^{i\omega z/c} W_{\|}(z')e^{-i\omega z'/c}\, dz' d\omega$$

- Notice the last part looks our definition for longitudinal impedance

$$\Delta E_{witness}(z) = \frac{e^2 c}{2\pi} \int \tilde{\lambda}(\omega) Z_{\|}(\omega) e^{i\omega z/c}\, d\omega$$

Physical Interpretation of Impedance

- With a Fourier transform of this (and rearranging slightly) we get

$$\int \frac{\Delta E_{witness}(z)}{e} e^{-i\omega z/c} \frac{dz}{c} = ec\tilde{\lambda}(\omega) Z_{\parallel}(\omega)$$

- The left hand side is a Fourier transform of an energy change per unit charge, or a voltage.
- The right hand side is simply the product of the "current spectrum", $\tilde{I}(\omega) = ec\tilde{\lambda}(\omega)$, and the impedance.
- Hence, with some appropriate definitions, the "voltage spectrum", $\tilde{V}(\omega)$, may be written

$$\tilde{V}(\omega) = \tilde{I}(\omega) Z_{\parallel}(\omega)$$

Physical Interpretation of Impedance

$$\tilde{V}(\omega) = \tilde{I}(\omega) Z_{||}(\omega)$$

- Here we can see that the picture is analogous to an electrical circuit.
- The impedance tells us, in the frequency domain, the voltage seen by the beam due to the beam current passing through the component.
- Note, the equation also shows that the impact of the impedance depends on the "overlap" of the impedance with the beam current spectrum.

Impedances (general)

$$Z(\omega) = Z_{\mathcal{R}}(\omega) + iZ_{\mathcal{I}}(\omega)$$

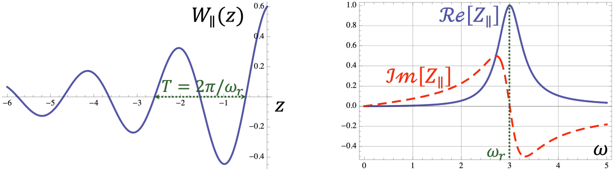
- In general, impedances are complex functions with real and imaginary parts.
- A particle beam can cover a wide frequency spectrum, from kHz, to the revolution frequency, up to many GHz dependent on the bunch length.
- Every component in an accelerator presents an impedance to the beam. What matters is how large the impedance is in the same frequency range of the beam.
- The impedance depends on the precise geometry and material make-up of the component.

Impedances (general)

	$Z_{\parallel}$	$Z_{\perp}$
Units	Ω	Ω/m
Symmetry	$Z_{\parallel,\mathcal{R}}(\omega) = Z_{\parallel,\mathcal{R}}(-\omega)$ Even	$Z_{\perp,\mathcal{R}}(\omega) = -Z_{\perp,\mathcal{R}}(-\omega)$ Odd
	$Z_{\parallel,\mathcal{I}}(\omega) = -Z_{\parallel,\mathcal{I}}(-\omega)$ Odd	$Z_{\perp,\mathcal{I}}(\omega) = Z_{\perp,\mathcal{I}}(-\omega)$ Even
Typ. scale	$\sim \Omega$	$\sim \mathrm{M}\Omega/\mathrm{m}$

- Negative frequencies are used to make calculations simpler.
- For a resistive cylindrical pipe $Z_{\perp} = 2cZ_{\parallel}/b^2\omega$.

Resonator Impedances

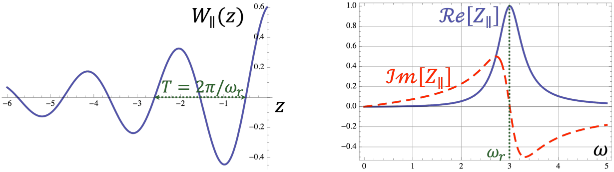


- A "cavity" can be modelled as an AC electrical circuit with

$$\omega_r = \frac{1}{\sqrt{LC}} \qquad Q = R\sqrt{\frac{C}{L}} = \frac{R}{L\omega_r} = RC\omega_r$$

- The resonant frequency, ω_r, is related to the oscillation of E_z, and the frequency of the mode excited.
- The decay time depends on how quickly the stored energy is dissipated (quality factor, Q).

Resonator Impedances



- If the current is modelled as $I = I_0 e^{i\omega t}$ then the impedance can be expressed as

$$Z_{\parallel,resonator}(\omega) = R\left[\frac{1 - iQ\left(\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2}\right]$$

- The resonator impedance has real (resistive) and imaginary (inductive/capacitive) parts.

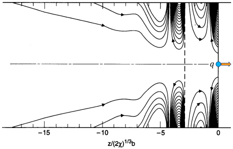
Resonator Impedances

- For high-Q cavities (narrowband resonators) this equation can be simplified near the resonance frequency to

$$Z_{\|,resonator}(\omega) \approx R\left[\frac{1 - i2Q\frac{\Delta\omega}{\omega_r}}{1 + \left(2Q\frac{\Delta\omega}{\omega_r}\right)^2}\right]$$

- High Q means the cavity resonates well, and the wake field is sustained for long periods and may produce **multibunch effects**.
- For low Q (~ 1, broadband resonators), the fields dissipate quickly and do not affect subsequent bunches.
- These may still produce **single bunch effects**.

Resistive Wall Impedance



Electric field lines in a resistive pipe generated by a moving point charge, q. Courtesy K. Bane (see Chao's Book figure 2.3).

- In a conductive beampipe, the particle beam induces an image current in a thin layer of the pipe.
- The wake decays as the figure above. On axis the wake function may be calculated

$$W_{\parallel}(z) = \frac{c}{4\pi b}\sqrt{\frac{Z_0}{\pi\sigma}}\frac{L}{|z|^{3/2}}$$

$$W_{\perp}(z) = \frac{c}{\pi b^3}\sqrt{\frac{Z_0}{\pi\sigma}}\frac{L}{|z|^{1/2}}$$

Resistive Wall Impedance

- This results in an impedance that is asymptotic at low frequency.

$$
\begin{aligned}
Z_{||}(\omega) &= (1-i)\frac{L}{2\pi\sigma\delta_s b} \\
Z_{\perp}(\omega) &= (1-i)\frac{2cL}{\pi\omega\sigma\delta_s b^3}
\end{aligned}
$$

- Where $\delta_s = \sqrt{2c/Z_0\sigma|\omega|}$ is the skin depth, b is the beampipe radius and σ is the conductivity of the beampipe material.
- Note the behaviour with beampipe radius.

Resistive Wall Impedance

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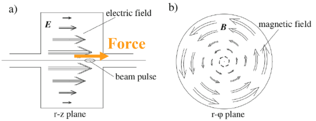
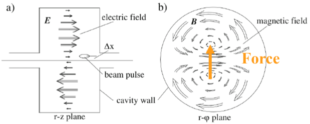
- Where $\delta_s = \sqrt{2c/Z_0\sigma|\omega|}$ is the skin depth, b is the beampipe radius and σ is the conductivity of the beampipe material.
- Note the behaviour with beampipe radius.

Impedance Summary

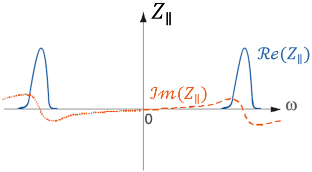
$Z_{\parallel}$ $\quad$ $Z_{\perp}$

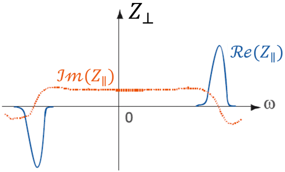
Resonator Object
Fields and Forces





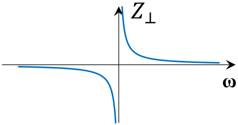
Resonator Object
Impedance





Resistive Wall

$$Z_{\parallel} \propto \sqrt{\omega}$$



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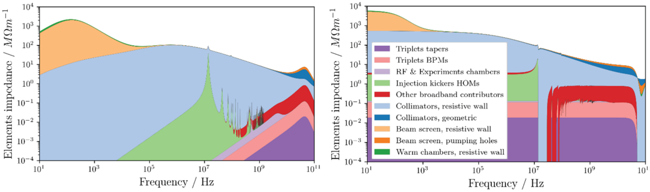
Impedance Model



Vertical Impedance of CERN LHC
left: $\mathcal{Re}(Z_y)$
right: $\mathcal{Im}(Z_y)$
"Building the Impedance Model of a Real Machine", B. Salvant et al., IPAC2019

- The impedance model summarises the main contributions from different elements.
- Enables prediction of beam instability thresholds.
- Helps to identify major impedance contributors (potential changes/upgrades).
- May help point to new impedance sources or inaccurate assumptions in existing models.

Beam Instabilities

- The range of phenomena associated with wake fields is quite large.
- Classifying the effects is not straightforward.
- Often categorised by: the plane in which they occur (**longitudinal** or **transverse**), and the length/time scale over which they occur (**single bunch** or **multibunch** instabilities).
- Understanding what instability occurs can help identify what drives it and how to cure/mitigate it.

Example Instabilities

Over the years many instabilities have been observed, categorised, explained and (sometimes) cured, e.g.:

- Negative mass instability (1959)
- Resistive wall instability (1960)
- Robinson instability (1964)
- Beam break-up instability (1966)
- Head-tail instability (1969)
- Microwave instability (1969)
- Beam-beam limit in colliders (1971)
- Potential well distortion (1971)
- Transverse mode coupling instability (1980)

Example Instabilities

Over the years many instabilities have been observed, categorised, explained and (sometimes) cured, e.g.:

- Hose instability (1987)
- Coherent synchrotron radiation instability (1990)
- Sawtooth instability (1993)
- Electron beam-ion instability (1996)
- Electron cloud instability (1997)
- Microbunching instability (2005)
- ...

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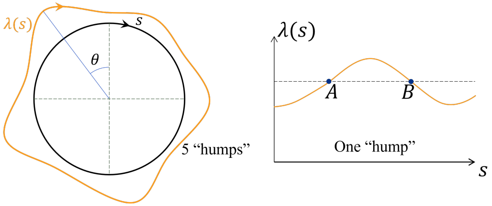
Negative Mass Instability



- Imagine an unbunched, coasting beam with a number of modulations in the line density $\lambda(s)$ around a ring.
- **Will these "humps" increase or decrease?**
- Focussing on one "hump" imagine particles at A and B.
- The **longitudinal space charge force** ($\propto -\partial\lambda/\partial s$) implies particle A decreases and B increases in energy.

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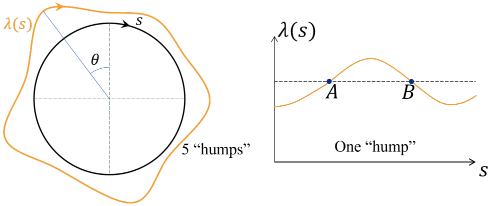
Negative Mass Instability



- Particle A is decelerated as it "sees" more charge ahead of it than behind.
- Particle B is accelerated as it "sees" a larger charge behind it pushing it forward.
- **Is this stable?**
- It depends on whether we're above or below **transition**.

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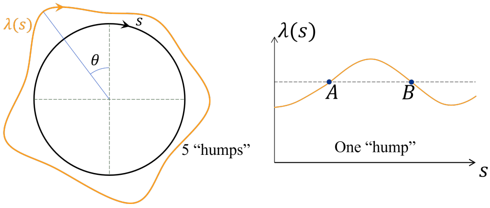

- $\gamma < \gamma_t$: An increase in energy increases ω_0. A and B move away from the "hump", smoothing it out. $\implies$ **STABLE**.
- $\gamma > \gamma_t$: An increase in energy decreases ω_0. A and B move toward the "hump" making it bigger. $\implies$ **UNSTABLE**.
- Hence described as the **negative mass instability**.

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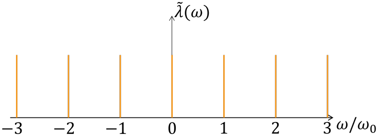
Transverse Beam Signals

$$\lambda(t) = \frac{q}{2\pi R} \sum_{n=-\infty}^{+\infty} e^{in\omega_0 t}$$



- A single particle on a central orbit produces a longitudinal signal.
- The spectrum, $\tilde{\lambda}(\omega)$, is a series of peaks at harmonics of the revolution frequency, ω_0.
- What about the transverse?

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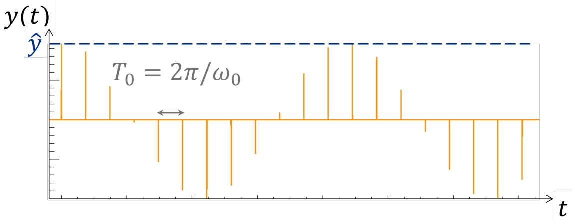
Transverse Beam Signals

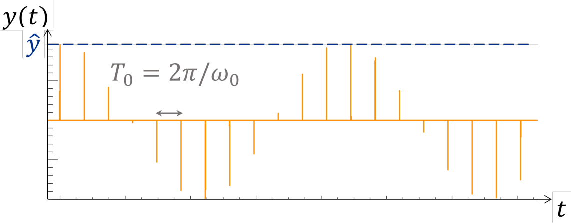


- Now imagine a particle oscillating about the axis.
- Transverse lectures have shown that the solution at a particular point in a lattice is

$$y(t) = \hat{y}\cos(\omega_\beta t + \phi) \tag{1}$$

- Where $\omega_\beta = Q\omega_0 = (k+q)\omega_0$.

Transverse Beam Signals



- Now imagine a particle oscillating about the axis.
- Transverse lectures have shown that the solution of the equation of motion is

$$y(t) = \hat{y}\cos(\omega_\beta t + \phi)$$

- Where $\omega_\beta = Q\omega_0 = (k + q)\omega_0$.

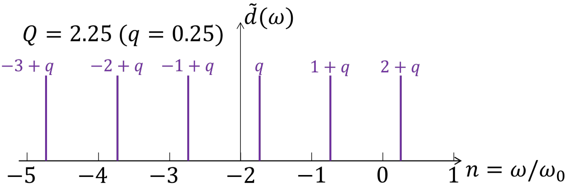
Transverse Beam Signals

- Computing its signal at a particular point in the ring

$$
\begin{aligned}
d(t) &= \hat{y}\cos(Q\omega_0 t + \phi)\frac{q}{2\pi R}\sum_{n=-\infty}^{+\infty} e^{in\omega_0 t} \\
&= \frac{q\hat{y}}{4\pi R}\left[e^{i(Q\omega_0 t+\phi)} + e^{-i(Q\omega_0 t+\phi)}\right]\sum_{n=-\infty}^{+\infty} e^{in\omega_0 t} \\
&= \frac{q\hat{y}}{2\pi R}\sum_{n=-\infty}^{+\infty}\cos\left[(n+Q)\omega_0 t + \phi\right]
\end{aligned}
$$

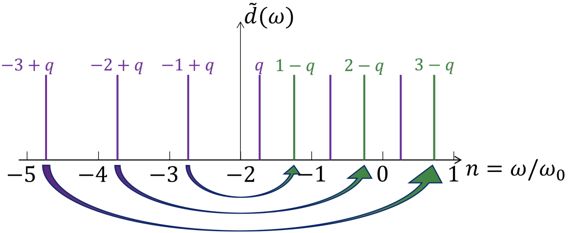
- What does is spectrum look like?

Transverse Beam Signals



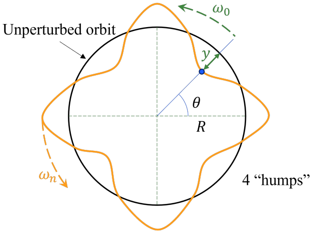
- The spectrum of the signal $d(t)$ is a series of equi-spaced lines of constant amplitude.
- These lines are at frequencies that correspond to $(n + Q)\omega_0$ where n is any integer.
- What would we see on an oscilloscope or network analyser?

Transverse Beam Signals



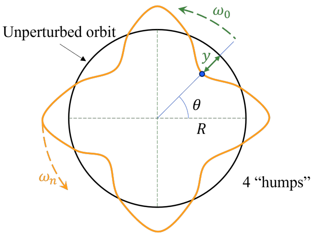
- On a network analyser we only see positive frequencies. The negative frequency lines are reflected about $\omega = 0$.
- In our case of $Q = 2.25$ the negative frequency lines become the lower **betatron sidebands** (green) and the positive frequency lines are the upper sidebands.
- Sidebands are $q\omega_0$ away from the revolution harmonic.

Transverse Coasting Beam Instability



- Imagine a scenario with particles arranged with a strict correlation between their longitudinal and transverse positions.
- The **mode** shown is $n = 4$, with four "humps" such that a snapshot gives $y = y_4 e^{-4i\theta}$.
- Each particle moves around the machine at ω_0, but the **pattern** rotates at $\omega_n \neq \omega_0$.

Transverse Coasting Beam Instability



- A position of particle at θ_0 at $t = 0$ will evolve as $y_{\theta_0}(t) = y_n e^{i(Q\omega_0 t - n\theta_0)}$
- After a time t the azimuth is $\theta = \theta_0 + \omega_0 t$
- Substitute into $y(t)$

$$y(\theta, t) = y_n e^{i[(Q+n)\omega_0 t - n\theta]}$$

Transverse Coasting Beam Instability

$$y(\theta, t) = y_n e^{i[(Q+n)\omega_0 t - n\theta]}$$

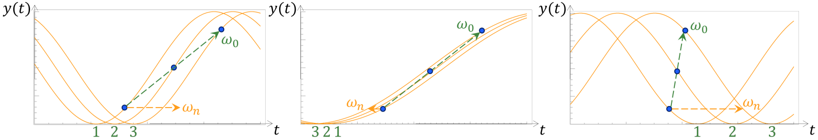
- For this pattern to be constant $(Q + n)\omega_0 t - n\theta = 0$
- As such, $\theta(t) = (1 + Q/n)\omega_0 t$
- The derivative of this, $\dot{\theta}$, is the rotation frequency of the mode ω_n

$$\omega_n = \dot{\theta} = \left(1 + \frac{Q}{n}\right)\omega_0$$

- Now we have three scenarios to consider:

$$n < -Q \qquad -Q < n < 0 \qquad n > 0$$

Transverse Coasting Beam Instability



- Snapshots at times t_0 (1), $t_0 + \Delta t$ (2) and $t_0 + 2\Delta t$ (3).

	$n < -Q$ $0 < \omega_n < \omega_0$	$-Q < n < 0$ $\omega_n < 0$	$n > 0$ $\omega_n < \omega_0$
Pattern moves	slower than particle	backwards	faster than particle
Wave speed	slow	backwards	fast

Growth Rate

- Only one mode n (single betatron sideband) relevant $\Rightarrow Z_\perp$ around frequency $(Q+n)\omega_0$.
- Do any modes grow? Assume that the Lorentz force is constant around the ring for a given y

$$
\begin{aligned}
F &= q(\vec{E} + \vec{v} \times \vec{B})_\perp \\
F(\theta, t) &= -i\frac{q\beta I Z_\perp}{2\pi R} y(\theta, t) \\
&= -i\frac{q\beta I Z_\perp}{2\pi R} y_n e^{i[(Q+n)\omega_0 t - n\theta]}
\end{aligned}
$$

- Substitute in $\theta(t) = \theta_0 + \omega_0 t$

$$
F(t) = -i\frac{q\beta I Z_\perp}{2\pi R} y_n e^{i[Q\omega_0 t - n\theta_0]} = -i\frac{q\beta I Z_\perp}{2\pi R} y(t)
$$

Growth Rate

$$F(t) = -i\frac{q\beta I Z_\perp}{2\pi R}y(t)$$

- The equation of motion is now

$$\begin{aligned}\ddot{y} + Q^2\omega_0^2 y &= \frac{F}{m_0\gamma} = -i\frac{q\beta I Z_\perp}{2\pi R m_0\gamma}y(t)\\ \ddot{y} + (Q\omega_0 + \Delta\Omega)^2 y &= 0 \qquad \text{with } \Delta\Omega = i\frac{q\beta I Z_\perp}{4\pi R m_0\gamma Q\omega_0}\end{aligned}$$

- Where we've made the assumption that $(\Delta\Omega)^2$ is negligible such that $(Q\omega_0 + \Delta\Omega)^2 \approx Q^2\omega_0^2 + 2\Delta\Omega Q\omega_0$

Growth Rate

- Notice the solution to this equation of motion is now

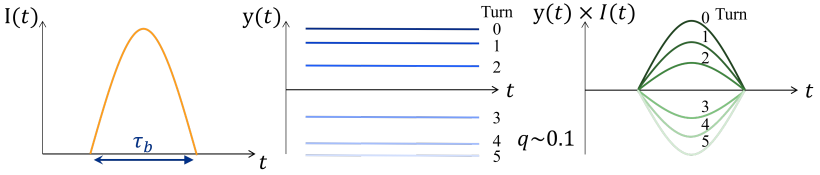
$$y(t) = y_n e^{i[(Q\omega_0 + \Delta\Omega)t - n\theta_0]}$$

- So the $\Delta\Omega$ is a frequency shift, growth or damping rate dependent on the nature of the impedance $Z_\perp$
- With $\omega_0 R = \beta c$ and $\gamma m_0 = E/c^2$

$$\Delta\Omega = i\frac{cIZ_\perp}{4\pi QE/q}$$

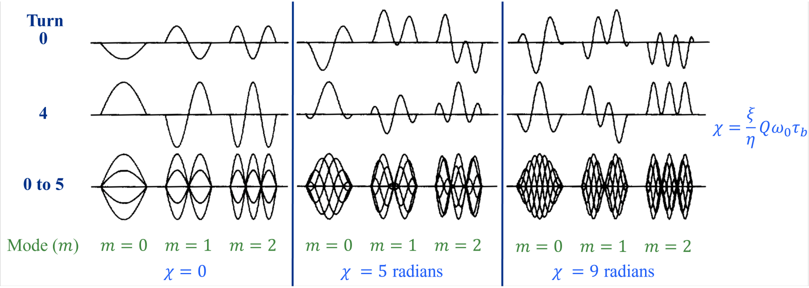
- **Unstable** if $\mathcal{I}m(\Delta\Omega) < 0 \Rightarrow \mathcal{R}e(Z_\perp[(Q+n)\omega_0]) < 0$
- As such, only **unstable** for $(Q+n) < 0$, i.e. **slow waves**.

Bunched Beams



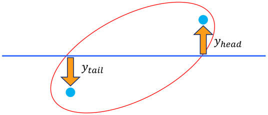
- For bunched beams, the line density may look something like this.
- All particles are performing synchrotron oscillations: their energy changes.
- Without chromaticity, all particles have the same tune (even with changing energy).
- As such, all particles oscillate in phase depending on their initial condition and a BPM difference signal looks like this.

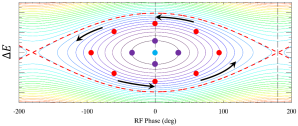
Bunched Beams with Chromaticity



- When chromaticity is non-zero, the Q varies as the particles perform their synchrotron oscillations.
- There is a betatron phase slip between the head and the tail, χ, providing extra "wiggles" along the bunch.
- This alters the observed BPM difference signal.

Head-tail Instability





- A severe limitation on single bunch intensity is the **head-tail instability**.
- Typically occurs for broadband impedances: short wakefields that only interact within the timescale of a single bunch.
- Transverse offsets at the head of the bunch create wakes which act back on the tail of the same bunch.
- The head and tail swap places which may then drive larger transverse oscillations leading to an instability.

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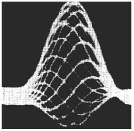
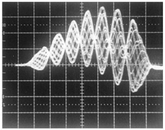
Head-tail Instability

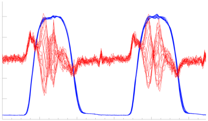
CERN PSB ~1974
(J. Gareyte & F. Sacherer)



CERN PS ~1999
(E. Métral)



ISIS ~2019
(R. Williamson)

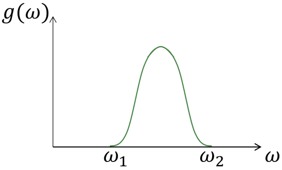


- This instability has been observed at many high intensity hadron accelerators.
- The mode that gets excited depends on the bunch spectrum, the impedance and the chromaticity.
- Instabilities, such as these, often limit the performance of the accelerator.
- What can we do?

Can we Cure Instabilities?

- We've found that instabilities are driven by impedances.
- To start mitigating an instability one must first find what drives it: what component(s) and at what frequency(ies).
- Redesigning components to **reduce impedances** at beam frequencies is one possibility.
- One can also **change the beam** to have a different frequency: rapidly accelerating, changing tune, bunch manipulation.
- If instabilities are still a problem they may also be eased by some mechanisms so far not included in our models:
    - **Spreads and non-linearities stabilize (Landau Damping).**
    - **Active feedback systems.**

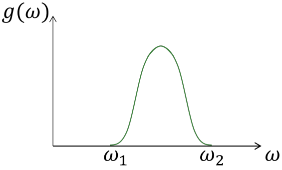
Landau Damping



- The beam can be modelled as N particles oscillating at their own frequency $\omega_1 < \omega < \omega_2$ with a normalised density $g(\omega)$ ($\int_{\omega_1}^{\omega_2} g(\omega)d\omega = 1$).
- The single particle response (**incoherent**) to an external driving excitation, $e^{i\Omega t}$ is

$$X = \frac{1}{\omega^2 - \Omega^2} e^{i\Omega t} = \frac{1}{(\omega - \Omega)(\omega + \Omega)} e^{i\Omega t}$$

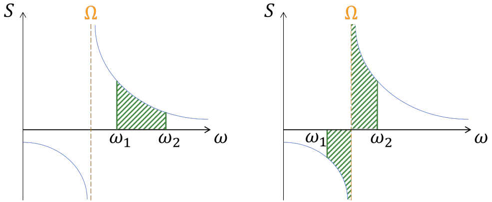
Landau Damping



- Assuming Ω is close to $\omega \sim \Omega_0$ the sum term $(\omega + \Omega) = 2\Omega_0$.
- Therefore, the **coherent response** is obtained by integrating these single particle responses

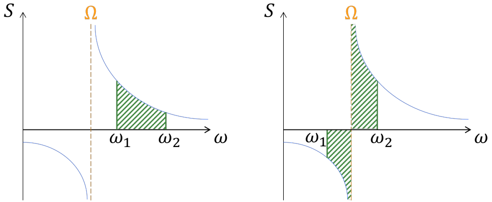
$$S = \frac{N}{2\Omega_0} \int_{\omega_1}^{\omega_2} i \frac{\frac{dg(\omega)}{d\omega}}{\omega - \Omega} d\omega \; e^{i\Omega t}$$

Landau Damping



- When the oscillation is outside the frequency range of the particles the coherent response is straightforwardly calculated
- When the oscillation is **within** the frequency range of the particles, how do we deal with the pole in the integral at $\omega = \Omega$?

Landau Damping



- The “trick” is to integrate “around” it in the complex plane.
- This leads to a **principal value (PV)** and a **residuum**.
- The residuum is a resistive term (in phase with the excitation) that absorbs energy.
- The PV is a reactive term (out of phase) that does not absorb energy.

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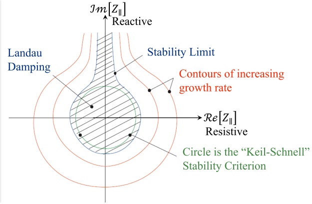
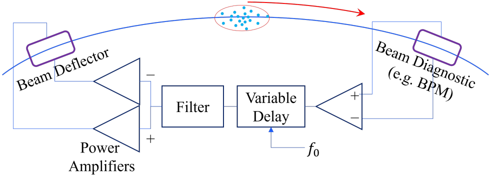
Landau Damping



- The resistive term leads to an area of stability such that coherent oscillations do not occur.
- In the longitudinal plane, for unbunched beams, a handy approximation for the stability limit is described by a circle. Known as the **"Keil-Schnell" criterion**.
- A similar situation occurs in the transverse plane due to a spread of betatron frequencies.

Active Damping Systems



- Position error is picked up by diagnostic.
- Convert this to an appropriate kick at a deflector.
- Betatron phase advance from pick-up to deflector of $\sim (2n+1)\pi/2$.
- Beam travel time and electronic delay time must be accounted for.

Summary

- **Instabilities** are an important factor in the design and operation of particle accelerators.
- **Wakefields** are generated by the interaction of the beam with its environment.
- The effects of wakefields can be accounted for (with certain assumptions) through **wake functions** or **impedances**.
- Impedances are often separated into longitudinal and transverse and vary as a function of frequency.
- Instabilities can be **mitigated** by changes to the impedance (e.g. component design), changes to the beam (e.g. Landau damping) or active damping as a last resort.

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References

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